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The Vlasov-Poisson-Fokker-Planck System with infinite kinetic energy

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Abstract. We build solutions to the three dimensional Vlasov-Poisson-Fokker-Planck System having infinite kinetic energy. For that purpose, we first derive a new conservation law, which states the propagation of the second space moment for solutions to this system. The existence of such infinite kinetic energy solutions relies essentialy on the dispersive effects of the kinetic transport. We also show strong regularizing effects, such as an L^∞_x bound on the force field, combining dispersive properties and smoothing effects of the diffusion.

Key-words. Vlasov-Poisson-Fokker-Planck, propagation of moments, infinite kinetic energy, hypoellipticity.

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1 Introduction

This paper aims at building solutions to the three dimensional Vlasov-Poisson-Fokker-Planck (VPFP) System having infinite kinetic energy. This System reads:

$$\begin{cases} \partial_t f + v\cdot\nabla_x f + div_v(E-\beta v)\cdot f - \sigma\Delta_v f = 0 \ , \\ f(t=0,x,v) = f^0(x,v) \geq 0 \ , \\ E(t,x) = \pm\dfrac{1}{4\pi}\dfrac{x}{|x|^3} * \rho(t,x) \ , \\ \rho(t,x) = \int_{\mathbb{R}^3} f(t,x,v)\, dv \ . \end{cases} \tag{1.1}$$

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Here, $\beta \geq 0$ and $\sigma > 0$ are given constants. In this system, the function $f(t,x,v) \geq 0$ is the unknown microscopic density and describes the density of particles having the position $x \in \mathbb{R}^3$ and velocity $v \in \mathbb{R}^3$ at time $t \geq 0$ in the phase space. This function generates the macroscopic density $\rho(t,x)$, which is the rate of particles located at x in the physical space. In turn $\rho(t,x)$ induces the Coulombic or Gravitational force field $E(t,x)$ as above, implicitely given by the Poisson equation :

$$-\Delta_x V = \pm\rho = \omega\rho \ , \quad E = -\nabla_x V \ .$$

The sign $\omega = +1$ corresponds to the repulsive (Coulombic) interaction whereas the sign $\omega = -1$ describes the attractive (gravitational) interaction between the particles. Finally, the term $-\beta\ div_v v f$ corresponds to the friction effects in the fluid, and the term $-\Delta_v f$ describes grazing collisions between the particles : when colliding, the particles change velocities, and this effect gives a diffusion term in the velocity direction.

This paper includes the case $\beta = 0$ (frictionless fluid), but relies heavily on the assumption $\sigma > 0$, since it gives hypoellipticity for the linear VPFP operator (see below). The less regular case $\beta = 0$, $\sigma = 0$ corresponds to the Vlasov-Poisson equation, for which only weaker results can be obtained (the diffusion term in v vanishes in this case).

The main difficulty in order to treat this system, as well as the Vlasov-Poisson System, is to bound the self-consistent force field $E(t,x)$ in L^p_x spaces, since the linear theory of such systems is well known (i.e. the case where E is a given potential). Using the Sobolev inequalities $\|E(t,x)\|_{L^p_x} \leq C\|\rho(t,x)\|_{L^q_x}$ (for some p, q), one has therefore to bound the density $\rho(t,x)$ in L^q_x spaces in terms of the unknown f. The classical tool in this direction is the following interpolation inequality, which states that, for $m \geq 0$,

$$\|\rho(t,x)\|_{L^q_x} = \| \int_{\mathbb{R}^3} f(t,x,v)dv\|_{L^q_x} \leq C\|f(t)\|^{\theta}_{L^\infty_{x,v}} \ \ \||v|^m f(t,x,v)\|^{1-\theta}_{L^1_{x,v}} \tag{1.2}$$

for some value of $\theta \in [0,1]$, and for $q = 1 + m/3$ (See below). In particular, the control of the density improves as f has higher moments in the velocity variable. In three dimensions of space, we mention the two special cases $m = 2$ (finite kinetic energy) for which the corresponding force field $E(t,x)$ belongs to L^p_x for $3/2 < p \leq 15/4$, and $m > 6$ for which $E(t,x) \in L^p_x$ with $3/2 < p \leq \infty$.

Using this idea, P.L. Lions and B. Perthame proved ([LPe]) in the classical Vlasov-Poisson case $\beta = \sigma = 0$ that the assumption $f^0 \in L^1 \bigcap L^\infty_{x,v}$, $|v|^m f^0 \in L^1_{x,v}$ for some $m > 3$, implies the existence of a solution $f(t,x,v)$ satisfying $|v|^m f(t,x,v) \in L^1_{x,v}$ for all t (propagation of velocity moments), and here the force field is bounded in the corresponding L^p spaces thanks to (1.2). Uniqueness is not known under these conditions when $\sigma = 0$. Concerning also the Vlasov-Poisson case, K. Pfaffelmoser [Pf] and J. Schaeffer [Sc] proved in a completely different setting that, when f^0 is C^1 and compactly supported in (x, v), the corresponding solution $f(t, x, v)$ to the system remains C^1 and compactly supported in (x, v) for all times. Their proofs rely on a detailed study of the characteristic curves associated with the Vlasov-Poisson equation (See [Re2] for a review paper on these methods).

In the VPFP case $\sigma > 0$, F. Bouchut proved ([Bo1]) that existence and uniqueness holds for this system, globally in time, when the initial data satisfies $f^0 \in L^1 \bigcap L^\infty_{x,v}$ and $|v|^m f^0 \in L^1_{x,v}$ for some $m > 6$. In this case, he proves even the regularity $f \in C^0(\mathbb{R}^+_t\ ; L^1_{x,v})$ and $E \in L^\infty_{loc}(\mathbb{R}^+_t\ ; L^\infty_x)$. His proof relies on a technique similar to the one introduced in [LPe], but also makes an essential use of the regularizing effect of the diffusion term $-\Delta_v f$. In fact, the important point is the L^∞_x bound on the force field in this case, from which we deduce that the "regularity" of the initial datum is automatically preserved ; for example the assumption $|v|^m f^0 \in L^1_{x,v}$ implies $|v|^m f(t, x, v) \in L^\infty_{loc}(\mathbb{R}^+_t\ ; L^1_{x,v})$ for all $m > 6$. We would also like to quote the work of H. D. Victory, B. P. O'Dwyer [VO] and G. Rein, J. Weckler [RW] concerning classical solutions to the VPFP system.

On the other hand, it has been shown in [Bo2] that the VPFP system presents strong regularizing effects: for an initial datum $f^0 \in L^1 \bigcap L^\infty_{x,v}$ having merely finite kinetic energy $|v|^2 f^0 \in L^1_{x,v}$, the force field becomes immediately bounded, i.e. $\|E(t, x)\|_{L^\infty_x} \leq C t^{-\delta}$ for t close to 0, and for some $\delta > 0$, although its L^∞_x-norm is infinite at $t = 0$. We emphasize the importance of such an L^∞_x bound on the force field, since it allows to deduce many other bounds on $f(t, x, v)$, $\rho(t, x)$, and also on the flow of $E(t, x)$, and we refer to the famous paper of R.J. Di Perna and P.L. Lions [DPL1] concerning the matters of regularity of E. Finally, we also would like to quote the work of J.A. Carillo and J. Soler ([CS]) where the VPFP System is studied for an initial data a measure having finite kinetic energy $(\int_{x,v} |v|^2 f^0 < \infty)$.

All the above mentioned papers assume f^0 has moments in the velocity variable ($v^2 f^0 \in L^1$ or $v^m f^0 \in L^1$ for some $m > 2$), and in particular f^0 is assumed to have finite kinetic energy. Recently, B. Perthame ([Pe]) replaced the assumption of finite kinetic energy by $|x|^2 f^0 \in L^1_{x,v}$ in the Vlasov-Poisson case $\sigma = 0$. In that case, where the interpolation inequality (1.2) does not apply, and the force field $E(t, x)$ is a priori undefined, he proved regularizing effects analogous to the Schrödinger-Poisson equation, such as $\|E(t)\|_{L^2_x} \leq C t^{-1/2}$ near $t = 0$. His proof relies on a specific conservation law associated with the quantity $\int_{x,v} |x - vt|^2 f(t, x, v)$ and on the dispersive effects of the free transport operator $\partial_t + v \cdot \nabla_x$. More generally, the propagation of space moments in the Vlasov-Poisson equation is systematically studied in [Ca], where it is proved that the assumption $|x|^m f^0 \in L^1_{x,v}$ for general values of m allows to build a solution such that $|x - vt|^m f(t,x,v) \in L^1_{x,v}$ for all t, and the corresponding force field satisfies $\|E(t)\|_{L^p} \le Ct^{-\delta}$ near $t = 0$, for various values of p (even $p = \infty$). In that case also, essential use is made of the dispersive effects of the free transport. We mention that a similar work has been done for the Boltzmann equation under the assumption $|x|^2 f^0 \in L^1_{x,v}$ ([MP]).

It is therefore very natural to ask which effect, dispersive by transport or diffusive by hypoellipticity, is more important for the VPFP equation. The answer is as follows. The dispersive effect is stronger for "short times". And, in the spirit of [Pe], [Ca], we replace the assumption of finite kinetic energy $v^2 f^0 \in L^1$ by a finite second space moment $x^2 f^0 \in L^1$. Thanks to a specific conservation law involving the quantity $\int_{x,v} x^2 f^0(x,v)$, we first prove the basic existence result and the bound $\|E(t)\|_{L^p} \le Ct^{-\delta}$ for $3/2 < p \le 15/4$ and t close to 0. At this level, we recover the kind of regularizing effects first proved in [Pe], which proves that the transport term $\partial_t + v \cdot \nabla_x$ dominates the diffusion $-\sigma\Delta_v$. Then, just after the initial regularization, the diffusive effect wins, and we may use the arguments of [Bo1], [Bo2] to bound $\|E(t)\|_p$ for the higher values $15/4 < p \le \infty$. We observe that the effect of the diffusion dominates for this latter purpose.

The main result of this paper is the following

Theorem 1
Let $f^0 \in L^1 \bigcap L^\infty_{x,v}$ satisfy $|x|^2 f^0 \in L^1_{x,v}$ and $|v|^\varepsilon f^0 \in L^1_{x,v}$ for some $\varepsilon > 0$. Assume $\sigma > 0$, and let $T > 0$ ($\beta = 0$ is allowed). Then, there exists a solution $f(t,x,v) \in L^\infty_{loc}([0;\infty[; L^1(\mathbb{R}^6_{x,v}))$ to the VPFP System with initial data f^0. Moreover, the following statement holds,

(i) Let $p \in [1;\infty]$, $q \in]3/2;\infty]$. Then, there exists a constant $C = C(T, \|f^0\|_{L^\infty}, \|(1+|x|^2)f^0\|_{L^1}, p, q)$ and exponents γ, $\delta > 0$ such that,

$$\|\rho(t,x)\|_{L^p_x} \le C\, t^{-\gamma}\ , \quad \|E(t,x)\|_{L^q_x} \le C\, t^{-\delta}\ ,$$

for all $t \in [0;T]$. Hölder estimates of the same kind hold for ρ, E, and ∇E.

(ii) We have the following conservation law, where $\phi(t) = (e^{\beta t} - 1)/\beta$,

$$\frac{d}{dt}\Big[\int_{\mathbb{R}^6} |x - \phi(t)v|^2 f(t,x,v)dxdv + \omega\phi(t)^2 \int_{\mathbb{R}^3} |E|^2(t,x)dx\Big] =$$
$$6\sigma\phi(t)^2 \|f^0\|_{L^1_{x,v}} + \omega(2\phi'(t) - 1)\phi(t) \int_{\mathbb{R}^3} |E|^2(t,x)dx\ .$$

More complete statements can be found in the text.

The end of the paper is organised as follows: in section 2, we recall the basic calculations concerning the solutions of the VPFP System, as well as the notations; in section 3, we derive the conservation law (ii) above; section 4 is devoted to the basic existence statement; and section 5 proves the regularizing effects (i).

2 Notations and basic facts

In this section, we recall the main basic calculations concerning the VPFP System. We should first indicate that the "splitting" appearing in the subsequent formulae was first used in the Vlasov-Poisson case ($\beta = 0$, $\sigma = 0$) by P.L. Lions and B. Perthame ([LPe]). It has found a systematic use in [Pe], [Bo1], [Bo2], [Ca] (See also [An]).

Indeed, the Duhamel formula for the hypoelliptic operator $Lf := v \cdot \nabla_x f - \beta \, div_v v f - \sigma \Delta_v f$ in $\mathbb{R}^{2N}$ (in the sequel, we shall only consider the physical case $N = 3$) gives the following splitting for a (say, regular) solution to the VPFP :

$$
\begin{aligned}
f(t,x,v) \quad &= \quad \int_{\mathbb{R}^{2N}} G(t,x,v,\xi,\gamma) f^0(\xi,\gamma) d\xi d\gamma \\
&\quad + \int_{s=0}^{t} \int_{\mathbb{R}^{2N}} \nabla_\gamma G(s,x,v,\xi,\gamma) E(t-s,\xi) f(t-s,\xi,\gamma) \, d\xi \, d\gamma \, ds \\
&:= \quad f^1(t,x,v) + f^2(t,x,v) \ , \qquad (2.1)
\end{aligned}
$$

where $G(t,x,\xi,\gamma)$ is the Green function associated to $\partial_t + L$. We will not specify the explicit value of the kernel G, which can be found in [Bo1], [Bo2]. Nevertheless, integrating (2.1) with respect to v gives the following formulae for the density ρ and the force field E. The important point here is that G acts essentially like a convolution by a Gaussian in the space variable, and this is the key tool for proving regularizing effects in the VPFP system. Indeed, let us define,

$$
\left\{
\begin{array}{l}
\mathcal{N}(x) := \dfrac{1}{(2\pi)^{N/2}} \exp(-\frac{x^2}{2}) \quad \in \mathcal{S} \ , \\
A_{j,k}(x) := \dfrac{\partial^2}{\partial x_j \partial x_k} (-\Delta)^{-1} \mathcal{N}(x) \quad \in L^p \quad , \ \forall \ 1 < p < \infty \ ,
\end{array}
\right. \qquad (2.2)
$$

and let $A(x)$ be the $N \times N$ matrix with coefficients $A_{j,k}(x)$. Then we have,

$$
\left\{
\begin{array}{ll}
\rho(t,x) & = \rho^1(t,x) + \rho^2(t,x) = \int_v f^1(t,x,v) \, dv + \int_v f^2(t,x,v) \, dv \ , \\
E(t,x) & = E^1(t,x) + E^2(t,x) \\
& \qquad = \pm \dfrac{x}{|S^{N-1}||x|^N} *_x \rho^1(t,x) \pm \dfrac{x}{|S^{N-1}||x|^N} *_x \rho^2(t,x) \ .
\end{array}
\right. \qquad (2.3)
$$

We also deduce from (2.1) the expressions,

$$
\left\{
\begin{array}{l}
\rho^1(t,x) = \dfrac{1}{[4\pi\sigma d(t)]^{N/2}} \mathcal{N}\left(\dfrac{x}{[4\sigma d(t)]^{1/2}} \right) *_x \displaystyle\int_{\mathbb{R}^N} f^0(x - \mu(t)v, v) dv \ , \\
\rho^2(t,x) = \displaystyle\int_0^t \dfrac{-\mu(s)}{[2\sigma d(s)]^{\frac{N+1}{2}}} \nabla \mathcal{N}\left(\dfrac{x}{[2\sigma d(s)]^{1/2}} \right) *_x M_{\mu(s)}(t-s,x) ds \ ,
\end{array}
\right. \qquad (2.4)
$$

and,

$$\left\{ \begin{array}{l} E^1(t,x) = \dfrac{1}{[4\pi\sigma d(t)]^{N/2}} \mathcal{N}(\dfrac{x}{[4\sigma d(t)]^{1/2}}) *_x E^0(\mu(t),x) \ , \\ E^2(t,x) = \displaystyle\int_0^t \dfrac{\mu(s)}{[2\sigma d(s)]^{N/2}} A(\dfrac{x}{[2\sigma d(s)]^{1/2}}) *_x M_{\mu(s)}(t-s,x)ds \ , \end{array} \right. \tag{2.5}$$

with,

$$\left\{ \begin{array}{l} E^0(\mu(t),x) = \pm \dfrac{x}{|S^{N-1}||x|^N} *_x \displaystyle\int_{\mathbb{R}^N} f^0(x-\mu(t)v,v)dv \ , \\ \rho^0(\mu(t),x) = \displaystyle\int_{\mathbb{R}^N} f^0(x-\mu(t)v,v)dv \ . \end{array} \right. \tag{2.6}$$

Here, we have used the following notations,

$$\mu(t) = \frac{1-e^{-\beta t}}{\beta} \quad , \quad d(t) = \int_0^t \mu(s)^2 ds \ , \tag{2.7}$$

as well as,

$$M_\lambda(t,x) = \int_{\mathbb{R}^N} E(t,x-\lambda v) \ f(t,x-\lambda v,v)dv \ . \tag{2.8}$$

More generally, we shall need the following,

$$\left\{ \begin{array}{l} \mu_\lambda(t) = \mu(t) + \lambda e^{-\beta t} \ , \ \ d_\lambda(t) = \int_0^t \mu_\lambda(s)^2 ds \ , \\ \rho_\lambda(t,x) = \int_{\mathbb{R}^N} f(t,x-\lambda v,v)dv \ , \end{array} \right. \tag{2.9}$$

and the corresponding decomposition,

$$\left\{ \begin{array}{l} \rho_\lambda = \rho_\lambda^1 + \rho_\lambda^2 \ , \\ \rho_\lambda^1(t,x) = \dfrac{1}{[4\pi\sigma d_\lambda(t)]^{N/2}} \mathcal{N}(\dfrac{x}{[4\sigma d_\lambda(t)]^{1/2}}) *_x \displaystyle\int_{\mathbb{R}^N} f^0(x-\mu_\lambda(t)v,v)dv \ , \\ \rho_\lambda^2(t,x) = \displaystyle\int_0^t \dfrac{-\mu_\lambda(s)}{[2\sigma d_\lambda(s)]^{\frac{N+1}{2}}} \nabla\mathcal{N}(\dfrac{x}{[2\sigma d_\lambda(s)]^{1/2}}) *_x M_{\mu_\lambda(s)}(t-s,x)ds \ . \end{array} \right. \tag{2.10}$$

We would like to point out the corresponding formulae in the case $\beta = 0$, $\sigma = 0$, which is the Vlasov-Poisson system. In that case, we obtain,

$$\left\{ \begin{array}{l} E' = E'^0 + E'^1 \ , \\ E'^0 = \dfrac{x}{|S^{N-1}||x|^N} *_x \displaystyle\int_{\mathbb{R}^N} f^0(x-vt,v)dv \ , \\ E'^1 = \displaystyle\int_0^t s(Ef)(t-s,x-vs,v)ds \ . \end{array} \right. \tag{2.11}$$

Formulae (2.11) differ from (2.4)-(2.5) through the convolution with a regularizing function in the space variable, as we already observed, but we also see that (2.4)-(2.5) involve in some sense a "decoupling" in time in formulae (2.11) (compare $(t-s, x-vs)$ in (2.11) with $(t-s, x-\mu(s)v)$ in (2.4)).

The regularisation in the x direction contained in the convolutions above is related to the hypoellipticity of the linear VPFP operator $v \cdot \nabla_x f - \beta \, div_v v f - \sigma \Delta_v f := Lf$ (note that this operator is still singular), and we refer to [Ho2] for a general theory for these operators. Also, the hypoellipticity of L has been strongly used in [DPL3], [BD].

Notations. *The following notations will be used throughout the paper:* $\| \cdot \|_p$ *denotes the* L^p *norm of a function; when the function depends on* (t, x)*,* $\|\rho(t, x)\|_p$ *or* $\|\rho(t)\|_p$ *means the* L^p *norm in the* x *variable; for* $1 \leq p \leq \infty$*,* p' *denotes the Hölder conjugate exponent of* p $(1/p + 1/p' = 1)$*. Also, we shall often estimate certain norms* $\|\rho(t)\|_p$ *on bounded intervals* $t \in [0, T]$ *where* $T > 0$ *is fixed; in these cases, any positive constant* C *or* $C(T)$ *is intended to depend on* T *and on the natural norms of the initial datum, unless the dependency of the constant is explicited. These norms are* $\|f^0\|_{L^1 \cap L^\infty}$*,* $\|x^2 f^0\|_1$*, (See Theorem 4.1 below), as it will be clear. Finally,* $\|u(x)\|_{C^{0,\alpha}}$ *is the usual Hölder norm of* u*, defined as,*

$$\|u(x)\|_{C^{0,\alpha}} = \sup_{x \neq y} \frac{|u(x) - u(y)|}{|x - y|^\alpha} .$$

3 A conservation law for the VPFP System

In this section, we derive a new conservation law for the second space moment of the density f in the three dimensional VPFP System. For that purpose, we shall assume here that f is a smooth solution to the VPFP, as in [RW]. We refer to [Pe], [Re1] for a similar identity on the three dimensional Vlasov-Poisson System ($\beta = 0$, $\sigma = 0$), which gives rise in fact to a Lyapunov functional in this special case.

Indeed, let us consider the linear equation

$$\partial_t f + v \cdot \nabla_x f + div_v(-\beta v f) = 0 . \tag{3.1}$$

For a prescribed initial datum, the general solution to (3.1) is,

$$f(t, x, v) = e^{3\beta t} \; f^0(x - \phi(t)v, e^{\beta t} v) \;\; \text{with} \;\; \phi(t) = \frac{e^{\beta t} - 1}{\beta} . \tag{3.2}$$

Now it is clear in (3.2) that the linear equation (3.1) propagates the velocity moments, in the sense that the assumption $|v|^m f^0 \in L^1$ implies $|v|^m f(t) \in L^1$ for all $t \geq 0$. The difficult point is that this property also holds on the nonlinear VPFP system ([Bo1]). Now we want to propagate the space moments. We are thus led to consider the following quantity,

$$X_2(t) = \int_{\mathbb{R}^6} |x - \phi(t)v|^2 f(t, x, v) \, dx \, dv . \tag{3.3}$$

It is clear in (3.2)-(3.3) that an initial datum satisfying $|x|^2 f^0 \in L^1$ gives rise to a finite $X_2(t)$, for all t, when f is a solution to (3.1) (in fact, $X_2(t)$ is constant in this case, up to a normalisation by $e^{-3\beta t}$). Now we derive a conservation law for this quantity when $f(t,x,v)$ is a solution of the nonlinear VPFP System. We have,

$$\begin{aligned}
\frac{d}{dt}X_2(t) &= \int_{\mathbb{R}^6}[-2\phi'(t)v]\cdot[x-\phi(t)v]f(t,x,v)dx\,dv \\
&\quad -\int_{\mathbb{R}^6}[x-\phi(t)v]^2\,[v\cdot\nabla_x f + div_v(E-\beta v)f - \sigma\Delta_v f]\,dx\,dv \\
&= (-2\phi'(t)+2)\int_{\mathbb{R}^6} v\cdot[x-\phi(t)v]f\,dx\,dv - 2\phi(t)\int_{\mathbb{R}^6} E\cdot[x-\phi(t)v]f\,dx\,dv \\
&\quad +2\beta\phi(t)\int_{\mathbb{R}^6} v\cdot[x-\phi(t)v]f\,dx\,dv + \sigma\int_{\mathbb{R}^6} f div_v(-2\phi(t)[x-\phi(t)v])\,dx\,dv\;,
\end{aligned}$$

Hence, using $\|f(t)\|_{L^1} = \|f^0\|_{L^1}$, we get,

$$\partial_t X_2(t) = 6\sigma\phi(t)^2\|f^0\|_{L^1} - 2\phi(t)\int_{\mathbb{R}^6} E\cdot[x-\phi(t)v]f\,dx\,dv\;. \tag{3.4}$$

Now, we transform the force term in (3.4), as in [Pe], [Re1], letting $\alpha := \pm 1/4\pi := \omega/4\pi$,

$$\begin{aligned}
\int_{\mathbb{R}^6} E\cdot xf dx\,dv &= \int_{\mathbb{R}^3} E\cdot x\rho dx = \alpha\int_{\mathbb{R}^6}\frac{x-y}{|x-y|^3}\rho(y)x\rho(x)dx\,dy \\
&= \frac{\alpha}{2}\int_{\mathbb{R}^6}\frac{1}{|x-y|}\rho(x)\rho(y)dx\,dy = \frac{1}{2}\int_{\mathbb{R}^3} V\rho dx = \frac{\omega}{2}\int_{\mathbb{R}^3}|E|^2 dx\;.
\end{aligned}$$

Here we have set $E = -\nabla_x V$, $-\Delta V = \omega\rho$. On the other hand, using the current density $j(x) = \int_v vf$, and the classical relation $\partial_t\rho + div_x j = 0$, we have,

$$\begin{aligned}
\int_{\mathbb{R}^6} E\cdot vf dx\,dv &= -\int_{\mathbb{R}^3}\nabla V j dx = -\int_{\mathbb{R}^3} V\partial_t\rho\,dx \\
&= -\omega\int_{\mathbb{R}^3}\nabla V\partial_t\nabla V\,dx = -\frac{\omega}{2}\partial_t\int_{\mathbb{R}^3}|E|^2 dx\;.
\end{aligned}$$

Collecting these relations in (3.4) gives the desired conservation law,

$$\frac{d}{dt}[X_2(t) + \omega\phi(t)^2\int_{\mathbb{R}^3}|E|^2(t)dx] = 6\sigma\phi(t)^2\|f^0\|_{L^1} + \omega(2\phi'(t)-1)\phi(t)\int_{\mathbb{R}^3}|E|^2(t)dx\;. \tag{3.5}$$

As a final remark, we would like to point out that this equality has been derived using in a first approach the explicit solution $g(t,x,v)$ to the linear problem (3.1) with initial datum $g^0(x,v) = |x|^2$. Our natural "candidate" $X_2(t)$ was then the integral $\int_{x,v} g(t)f(t)$ where f is the solution to the nonlinear VPFP problem. Naturally, we could have looked for the explicit solution $h(t)$ to the linear equation

$$\partial_t h + v\cdot\nabla_x h - \beta div_v vh - \sigma\Delta_v h = 0$$

with initial datum $h^0 = |x|^2$ (i.e. including the diffusion term). This gives exactly the same conservation law on the nonlinear system, and (essentially) the same value of $X_2(t)$.

4 The existence result

From the conservation (3.5) stated above for, say, regular solutions to the VPFP System, we build up in this section solutions to the VPFP for an initial datum having infinite kinetic energy. More precisely, the main result is the

Theorem 4.1 *Let $f^0 \in L^1 \bigcap L^\infty$, $f^0 \geq 0$ satisfy $|x|^2 f^0(x,v) \in L^1_{x,v}$ and $|v|^\varepsilon f^0(x,v) \in L^1_{x,v}$ for some $\varepsilon > 0$. Then, there exists a function $f(t,x,v)$, solution to the VPFP System in $\mathcal{D}'$, and having the following properties:*

(i) $f(t) \in L^\infty_{loc}([0,+\infty[; L^1 \bigcap L^\infty_{x,v})$.

(ii) $X_2(t) \leq C(T)$ for all $0 \leq t \leq T$.

(iii) $\int_{\mathbb{R}^6} |v|^\alpha f(t,x,v)\, dxdv \leq C(T, \||v|^\varepsilon f^0\|_1)$ for all $0 \leq t \leq T$ and $\alpha > 0$ small enough .

Consequently, we have the following estimates, valid for $0 \leq t \leq T$:

(iv) $\|E(t,x)\|_p \leq C(T) t^{-2+3/p}$ for all $3/2 < p \leq 15/4$.

(v) $\|\rho(t,x)\|_p \leq C(T) t^{-3/p'}$ for all $1 \leq p \leq 5/3$.

We observe that the blow-up exponents in (iv)-(v) are the same as those obtained in [Pe] in the case $\sigma = 0$. In that sense, the diffusion term $-\sigma\Delta_v$ plays no specific role in this section, and the transport operator $\partial_t + v\cdot\nabla_x$ dominates.

The technique of proof relies on interpolation inequalities, as in [LPe], [Pe], [Ca]. Also, the use of an ε-moment in the velocity variable is borrowed from [Pe], [Ca].

We recall here these Lemmas, whose proof can be found in the above mentioned papers,

Lemma 4.1 *Let $f(x,v) \geq 0$ belong to $L^1 \bigcap L^\infty$. Then, there exists a constant C, depending only on $\|f\|_{L^1\cap L^\infty}$, such that,*

(i) (moments in x) for all λ, μ, and $1 \leq p \leq (3+k)/3$. we have,

$$\| \int_{\mathbb{R}^3} f(x-\lambda v, v) dv \|_{L^p_x} \leq C\, |\lambda+\mu|^{-\frac{3}{p'}}\ \||x-\mu v|^k f\|_1^{\frac{3}{kp'}}\ .$$

(ii) (moments in v) for all λ, and $1 \leq p \leq (3+k)/3$. we have,

$$\| \int_{\mathbb{R}^3} f(x-\lambda v, v) dv \|_{L^p_x} \leq C\ \||v|^k f\|_1^{\frac{3}{kp'}}\ .$$

(iii) More generally, we have, for $0 \le j \le k$ and $1 \le p \le (3+k)/(3+j)$,

$$\| \int_{\mathbb{R}^3} |x-\mu v|^j f(x-\lambda v, v) dv \|_{L^p_x} \le C \, |\lambda+\mu|^{-\frac{3}{p'}} \, \| |x-\mu v|^k f\|_1^{\frac{3}{kp'}} \ ,$$

$$\| \int_{\mathbb{R}^3} |v|^j f(x-\lambda v, v) dv \|_{L^p_x} \le C \, \| |v|^k f\|_1^{\frac{3}{kp'}} \ .$$

As we can see, the control of a moment in x, as well as the control of a moment in v, allows to bound the density $\int_v f(x-\lambda v, v)$ in L^p, and this holds for the same values of p in both cases. But the moments in x give a factor $|\lambda+\mu|^{-\frac{3}{p'}}$ which blows up as $\lambda+\mu=0$. This explanes the negative powers of t in Theorem 4.1. Assuming that the initial datum in this Theorem has a velocity moment of given order would in fact decrease the exponents appearing in Theorem 4.1 (iv)-(v).

Now we come to the proof of the Theorem. We first prove the following intermediate result,

Lemma 4.2 *Let $f(t,x,v)$ be a regular solution to the VPFP System with initial datum f^0. Let $T>0$. Then, there exists a constant C depending only on $\|f^0\|_{L^1\cap L^\infty}$, $\|x^2 f^0\|_1$, and T, such that,*

$$\forall \, 0 \le t \le T \ \ , \ \ X_2(t) \le C \ .$$

Proof of Lemma 4.2. We integrate (3.5) with respect to t and get,

$$\begin{aligned} &X_2(t) + \omega\phi(t)^2 \textstyle\int_{\mathbb{R}^3} |E|(t)^2 dx = \\ &\quad = X_2(0) + \textstyle\int_0^t [6\sigma\phi(s)^2 \|f^0\|_1 + \omega(2\phi'(s)-1)\phi(s) \int_{\mathbb{R}^3} |E|(s)^2 dx] \, ds \ . \end{aligned}$$

Thus, for $0 \le t \le T$,

$$X_2(t) \le C(T) + \int_0^t (2\phi'(s)-1)\phi(s) \int_{\mathbb{R}^3} |E|(s)^2 dx \, ds + \phi(t)^2 \int_{\mathbb{R}^3} |E|(t)^2 dx \ . \tag{4.1}$$

Now, there is a $C>0$ such that $C^{-1}t \le (2\phi'(t)-1)\phi(t) \le Ct$ and $C^{-1}t \le \phi(t) \le Ct$ as $0 \le t \le T$. Thus,

$$X_2(t) \le C + C\int_0^t s \int_{\mathbb{R}^3} |E|(s,x)^2 dx \, ds + Ct^2 \int_{\mathbb{R}^3} |E|(t,x)^2 dx \ . \tag{4.2}$$

Now we bound the right hand side of (4.2) in terms of X_2. Indeed, the Riesz-Sobolev inequality for the singular convolution kernel $x/|x|^3$ ([St]), combined with Lemma 4.1-(i), gives,

$$\begin{aligned} \|E(t)\|_2 &\le C\|\rho(t)\|_{6/5} \le C\mu(t)^{-1/2} X_2(t)^{1/4} \\ &\le Ct^{-1/2} X_2(t)^{1/4} \ . \end{aligned} \tag{4.3}$$

Now the constants C appearing in (4.2)-(4.3) satisfy,

$$C = C(T, \sup_{t\in[0;T]} |f(t)\|_{L^1\cap L^\infty}) = C(T, \|f^0\|_{L^1\cap L^\infty}) \tag{4.4}$$

thanks to the maximum principle for the VPFP System (See [Bo1], [Bo2]). Combining estimates (4.2) and (4.3) gives,

$$X_2(t) \leq C(T) + C(T)\int_0^t X_2(s)^{1/2} ds + C(T)X_2(t)^{1/2} \ , \tag{4.5}$$

for some $C(T)$ as in (4.4). Now $C(T)$ is completely determined through the initial data and the given time T, and we observe that $X_2(t) - C(T)\ X_2(t)^{1/2} \geq X_2(t)/2$ for large values of $X_2(t)$ in front of $C(T)$. Since we are only interested in the large values of $X_2(t)$, we can write (4.5) in the form,

$$\frac{1}{2}X_2(t) \leq C(T) + C(T)\int_0^t X_2(s)^{1/2} ds \ .$$

From this we deduce,

$$X_2(t) \leq C(T) \quad \text{for all} \quad 0 \leq t \leq T \tag{4.6}$$

for some constant $C(T)$, thanks to Gronwall's Lemma. ■

Now we come to the

Proof of Theorem 4.1. Lemma 4.2 gives a constant $C = C(T, \|(1+x^2)f^0\|_1, \|f^0\|_\infty)$ such that $X_2(t) \leq C$ on $[0, T]$. From this we deduce, using another time Lemma 4.1 together with Riesz-Sobolev inequality, the following bound for $t \in [0, T]$,

$$\|\rho(t)\|_p \leq Ct^{-3/p'} \qquad \forall 1 \leq p \leq 5/3 \ , \tag{4.7}$$

$$\|E(t)\| \leq Ct^{-2+3/p} \qquad \forall 3/2 < p \leq 15/4 \ . \tag{4.8}$$

We shall need in fact a slightly stronger version of (4.7). Taking $t \in [0, T]$, $s \in [0, t]$, and $p \in [1, 5/3]$, we write (See (2.10) for the definitions),

$$\begin{aligned} \|\rho_{\mu(s)}(t-s)\|_p &= \| \int_{\mathbb{R}^3} f(t-s, x-\mu(s)v, v)\ dv\|_p \\ &\leq C\ [\mu(s) + \phi(t-s)]^{-3/p'} X_2(t-s)^{3/2p'} \ , \end{aligned}$$

thanks to Lemma 4.1-(i). Hence,

$$\|\rho_{\mu(s)}(t-s)\|_p \leq Ct^{-3/p'} \ . \tag{4.9}$$

Now estimates (4.7)-(4.9) have been proved for regular solutions to the VPFP system. The next step is to pass to the limit in this system.

According to the assumptions of Theorem 4.1, we take a sequence f^0_n of regular intial datas converging to f^0 in $L^1((1+x^2+|v|^\varepsilon)\ dx\ dv)$, as well as in $L^\infty_{x,v}$, and we call $f^n(t,x,v)$ the corresponding sequence of regular solutions to the VPFP (and the associated $\rho^n(t,x)$, $E^n(t,x)$, $V^n(t,x)$).

Since we control the $L^1((1+x^2)dxdv)\bigcap L^\infty$-norm of the initial datas f^0_n uniformly, we have the uniform estimates (4.7)-(4.9) for the sequence $f^n(t,x,v)$, thanks to Lemma 4.2. Classically, the averaging Lemma as stated in [GPS], [GLPS], [DPL2] or [DPLM] allows then to pass to the limit in the VPFP equation, including the nonlinear term $E^n f^n$ (say in the distributional sense - see [DPL2]). The only difficult term is the Poisson equation $-\Delta V^n = \rho^n$, since it is not obvious whether the density $\rho^n = \int_v f^n(t,x,v)$ should converge to $\rho = \int_v f(t,x,v)$ (recall that only quantities of the type $\int_{|v|\le R} f^n$, R being finite, converge to $\int_{|v|\le R} f^n$; a "loss of mass" could occur when we take $R=\infty$ - See [Ca]). In order to show that $\rho^n \to \rho$, we need therefore to show that the sequence $f^n(t,x,v)$ is tight is the v direction. At this level, we see the need of an $L^1(|v|^\varepsilon dxdv)$-assumption on the initial data. For this reason, we introduce in Lemma 4.3 below a quantity analogous to $X_2(t)$. This Lemma provides a uniform bound on some velocity moment of the sequence f^n, and it implies the tightness we are now seeking. Indeed, applying Lemma 4.3 to the sequence f^n gives

$$\sup_n \int_x \int_{|v|\ge R} f^n(t,x,v) \to 0 \quad \text{as} \quad R\to\infty\ ,$$

and this implies in turn, together with the averaging Lemma, the convergence $\rho_n \to \rho$, say in $\mathcal{D}'$. This completes the proof of Theorem 4.1. ■

It remains to state the

Lemma 4.3 *Let f be a (regular) solution to the VPFP System, with initial data $f^0 \in L^1((1+x^2+|v|^\varepsilon)dxdv)\bigcap L^\infty$ for some $\varepsilon>0$. For $\alpha>0$, define*

$$V_\alpha(t) := \int_{\mathbb{R}^6} < v >^\alpha\ f(t,x,v)\ dx\ dv \quad \textit{where} \quad < v > := (1+v^2)^{1/2}\ .$$

Then, for α small enough with respect to ε, we have

$$V_\alpha(t) \le C(T, \|(1+x^2+|v|^\varepsilon)f^0\|_1, \|f^0\|_\infty) \quad \textit{for all} \quad 0\le t\le T\ .$$

Proof of Lemma 4.3. We proceed as in [Ca]. In a first approach, we write,

$$\begin{aligned}
\frac{d}{dt}V_\alpha(t) &= -\int_{\mathbb{R}^6} < v >^\alpha\ [v\cdot\nabla_x f + div_v(E-\beta v)f - \sigma\Delta_v f]\ dx\ dv \\
&\le C\int_{\mathbb{R}^6} [< v >^{\alpha-1}\ |E| f + \beta < v >^\alpha\ f + \sigma < v >^{\alpha-2}\ f]\ dx\ dv \\
&\le C\ \|E(t)\|_{3+\alpha}\ V_\alpha^{\frac{\alpha+2}{\alpha+3}} + C\ V_\alpha + C V_{\alpha-2} && (4.10)\\
&\le C\ t^{-(2-\frac{3}{3+\alpha})}\ V_\alpha^{\frac{\alpha+2}{\alpha+3}} + C\ V_\alpha\ . && (4.11)
\end{aligned}$$

Here (4.10) is a consequence of the Hölder inequality in x together with Lemma 4.1-(iii) for the quantity $\int_v < v >^{\alpha-1} f$, and (4.11) is simply a consequence of estimate (4.8).

We now observe on (4.11) that the exponent $2 - \frac{3}{3+\alpha} > 1$ prevents the use of Gronwall's Lemma. At this stage, we are led to improve the estimate on the field E in (4.8), and we want to prove indeed that $\|E(t)\|_{3+\alpha} \in L^1_{loc}(t)$ for possibly small α. We decompose the field according to the introduction (See (2.5)),

$$E(t) = E^1(t) + E^2(t) \ ,$$

and estimate separately each term. First, we have,

$$\begin{aligned} \|E^1(t)\|_{3+\alpha} &\leq C \ \|E^0(\mu(t), x)\|_{3+\alpha} \\ &\leq C \ \|\rho^0(\mu(t), x)\|_{\frac{3(3+\alpha)}{6+\alpha}} \ , \end{aligned}$$

thus,

$$\|E^1(t)\|_{3+\alpha} \leq C \ \|\rho^0(\mu(t), x)\|^{(1-\theta)}_{(3+\varepsilon)/3} \ \|\rho^0(\mu(t), x)\|^{\theta}_{5/3} \ , \tag{4.12}$$

$$\text{with} \quad (1-\theta)\frac{3}{3+\varepsilon} + \frac{3\theta}{5} = \frac{6+\alpha}{3(3+\alpha)} \ . \tag{4.13}$$

Now we estimate each term in the right-hand-side of (4.12), using Lemma 4.1-(ii) for the first factor and 4.3-(i) for the second. This decreases the negative powers of t obtained in (4.11). Indeed, we obtain,

$$\|E^1(t)\|_{3+\alpha} \ \leq \ Ct^{-3\,\theta/[5/3]'} \ , \tag{4.14}$$

because $N_\varepsilon(0) < \infty$. But (4.14) gives,

$$\begin{aligned} \frac{3\theta}{(5/3)'} &= \frac{6\theta}{5} \\ &\to_{\alpha\to 0} \ \frac{3/(3+\varepsilon) - 2/3}{3/(3+\varepsilon) - 3/5} < 1 \end{aligned}$$

Thus, $\|E^1(t)\|_{3+\alpha} \in L^1_{loc}$ for sufficiently small α.

We proceed in a similar way for the other term. First we write,

$$\|E^2(t)\|_{3+\alpha} \ \leq \ C \int_0^t \mu(s)\ d(s)^{-3/2a'} \ \|M_{\mu(s)}(t-s, x)\|_b \ ,$$

thanks to Young's inequality in x, as we choose a and b such that,

$$\frac{1}{b} - \frac{1}{a'} = \frac{1}{3+\alpha} \quad , \quad a' \in]1; \infty[\ .$$

Thus, observing that $C^{-1}t \le \mu(t) \le Ct$ and $C^{-1}t^3 \le d(t) \le Ct^3$ for $t \in [0;T]$ and some $C > 0$, we get

$$\begin{aligned}
\|E^2(t)\|_{3+\alpha} &\le C\int_0^t s^{1-9/2a'} \,\| \int_v E(t-s, x-\mu(s)v)\, f(t-s, x-\mu(s)v, v)\|_b \, ds \\
&\le C\int_0^t s^{1-9/2a'} \,\mu(s)^{-3/c}\, \|E(t-s)\|_c \,\| \int_v f^{c'}(t-s, x-\mu(s)v, v)\|_{b/c'}^{1/c'} \, ds \\
&\le C\int_0^t s^{1-9/2a'} \,\mu(s)^{-3/c}\, \|E(t-s)\|_c \,\|\rho_{\mu(s)}(t-s)\|_{b/c'}^{1/c'} ds\,, \qquad (4.15)
\end{aligned}$$

thanks to Hölder in v and using the bound $\|f(t)\|_\infty \le C(T)$, as in [LPe]. In view of the estimates (4.7)-(4.9), we choose c such that $c \in]3/2; 15/4]$ and $b/c' \in [1; 5/3]$. Once this has been done, we replace the terms E and ρ in (4.15) according to (4.7)-(4.9), and obtain,

$$\|E^2(t)\|_{3+\alpha} \le C\int_0^t s^{1-9/2a'} \,\mu(s)^{-3/c}\, (t-s)^{-2+3/c}\, (t^{-3/[b/c']'})^{1/c'} \, ds \qquad (4.16)$$

Now we choose $a' \approx \infty$ (but $< \infty$), $b \approx 3+\alpha$ ($< 3+\alpha$), and $c' \approx 3$ (< 3), so that the different constraints on the exponents are satisfied. After collecting the exponents, we get in (4.16),

$$\|E^2(t)\|_{3+\alpha} \le Ct^{-\eta}\,,$$

for some $\eta > 0$ which can be chosen arbitrarily small. Therefore, $\|E^2(t)\|_{3+\alpha} \in L^1_{loc}$ for possibly small α.

Collecting the estimates on the field gives,

$$\frac{d}{dt}V_\alpha(t) \le g(t)V_\alpha(t)$$

for some $g(t) \in L^1_{loc}$, and Lemma 4.3 is proved. ■

5 Regularizing effects

In the preceding section we established the existence of a "singular" solution to the VPFP, and proved estimates on the field and on the density. These estimates read $\|E(t)\|_p \le C\, t^{-2+3/p}$ for $p \in]3/2; 15/4]$ and $\|\rho(t)\|_q \le C\, t^{-3/p'}$ for $q \in [1; 5/3]$, locally in time (See (4.7)-(4.9)). This was performed by using essentially the behaviour of the transport operator $\partial_t + v \cdot \nabla_x$. Indeed we did not use the convolutions by a Gaussian in formulae (2.2)-(2.6), which are typical for the diffusion term $-\sigma\Delta_v$: we did only estimate terms like $\rho^o(t,x) = \int_v f^0(x-\mu(t)v, v)$ with $\mu(t) \approx t$ (compare with $\rho^o(t,x) = \int_v f^0(x-tv, v)$ when there is no diffusion term).

The aim of this section is to establish such estimates as $15/4 < p \le \infty$ and $5/3 < q \le \infty$, using this time the effect of the diffusion. In particular, we show that the field E becomes immediately L^∞_x, which is a key estimate (See Introduction). In fact, even Hölder estimates can be obtained on ρ and E, and we shall use as in [Bo1], [Bo2], the following convention for Hölder norms,

$$\|u\|_{C^{0,\alpha}} = \|u\|_q \ , \quad \text{as} \quad -\frac{\alpha}{3} = \frac{1}{q} \in]-\frac{1}{3}; 0[\ .$$

This kind of regularizing effects has already been studied by F. Bouchut in the finite kinetic energy case ([Bo1], [Bo2]), and we will follow the same approach.

For our purpose we introduce, the

Definition 5.1 *For α, $\delta \geq 0$ and $1 \leq q, p \leq \infty$, let*

$$\begin{aligned} K_{p,\alpha}(t) &:= \sup_{s\in[0,t]} s^\alpha \|E(s)\|_p \\ S_{q,\delta}(t) &:= \sup_{s\in[0,t]} \sup_{u\in[0,s]} (s-u)^\delta \|\rho_{\mu(u)}(s-u)\|_q \end{aligned}$$

The same notations are used for negative values of p and q.

Using these notations, the results of the preceding section read, $K_{p,2-3/p}(t) \leq C$ and $S_{q,3/q'}(t) \leq C$ for $3/2 < p \leq 15/4$, $1 \leq q \leq 5/3$ and $0 \leq t \leq T$, and we want to look at greater values of p and q.

Now, the main result of this section reads,

Theorem 5.1 *Let f^0, $f(t)$ be as in Theorem 4.1. Then, for all $0 \leq t \leq T$, there exists a C as in Theorem 4.1 such that,*

(i) $\forall 15/4 < p \leq \infty \ , \quad \forall \alpha > \alpha(p) \ , \quad K_{p,\alpha}(t) \leq C \ ,$

(ii) $\forall 5/3 < q \leq \infty \ , \quad \forall \delta > \delta(q) \ , \quad S_{q,\delta}(t) \leq C \ .$

Here, the coefficients $\alpha(p)$ and $\delta(q)$ are given by,

$\alpha(p) = 12/5 - 9/2p$ *for* $15/4 < p \leq 100/9$,

$\alpha(p) = 33/10 - 29/2p$ *for* $100/9 < p \leq \infty$,

$\delta(q) = 99/10 - 29/2q$ *for* $5/3 < q \leq \infty$.

Also, we have,

(iii) $\forall 0 < \alpha < 1/3 \ , \quad \forall \gamma > 66/5 \ , \ \|\rho(t)\|_{C^{0,\alpha}} + \|\nabla_x E(t)\|_{C^{0,\alpha}} \leq C\, t^{-\gamma} \ .$

(iv) $\forall 0 < \alpha < 1 \ , \quad \forall \gamma > 187/40 \ , \ \|E(t)\|_{C^{0,\alpha}} \leq C\, t^{-\gamma} \ .$

The end of this section is devoted to the proof of this Theorem. Note that, in this Theorem, the constants C depend also on α, δ, ...

According to the splitting (2.4)-(2.5), we shall also need the

Definition 5.2

$$\begin{aligned} K^{(1)}_{p,\alpha}(t) &:= \sup_{s\in[0,t]} s^\alpha \|E^1(s)\|_p \\ S^{(1)}_{q,\delta}(t) &:= \sup_{s\in[0,t]} \sup_{u\in[0,s]} (s-u)^\delta \|\rho^1_{\mu(u)}(s-u)\|_q \ . \end{aligned}$$

Also, $K^{(2)}_{p,\alpha}(t)$ and $S^{(2)}_{q,\delta}(t)$ are defined in the similar way, for positive and negative values of p and q.

Remark 1 *We observe that $K_{p,\alpha}$ and $S_{q,\delta}$ are decreasing functions of α and δ, in the sense that,*

$$K_{p,\alpha}(t) \le C(T) K_{p,\beta}(t) \quad \textit{for} \;\; \alpha \ge \beta \; .$$

Our first (and easier) task, is to bound $K^{(1)}$ and $S^{(1)}$. This is done in the following Lemma. We see on these terms how the convolution in the x variable allows to bound these terms for large values of p and q. Naturally, such regularizing effects cannot be obtained in the Vlasov-Poisson case $\sigma = 0$. Note that these terms correspond to the "free" evolution of the system, i.e. $S^{(1)}$ represents the density generated by the solution to the linear VPFP System (where $E = 0$) with initial data f^0.

Lemma 5.1 *Under the assumptions of Theorem 5.1, we have,*
(i) $\forall 15/4 < p \le \infty$, $K^{(1)}_{p,12/5-9/2p} \le C$,
(ii) $\forall 5/3 < q \le \infty$, $S^{(1)}_{q,39/10-9/2q} \le C$.
The same estimates hold true for $-1/3 < p, q < 0$.

Proof of Lemma 5.1 . For the sake of simplicity, here and in the rest of this section we will normalise all the constants appearing in (2.4)-(2.5) to unity. Also, we restrict ourselves to the case q, $p \ge 0$. With this convention, we write,

$$\begin{aligned}
\|E^1(t)\|_p &= \|d(t)^{-3/2} \exp(-x^2/d(t)^{1/2}) *_x E^0(\mu(t), x)\|_p \\
&\le C \|d(t)^{-3/2} \exp(-x^2/d(t)^{1/2})\|_a \|E^0(\mu(t), x)\|_b \\
&\qquad \text{with} \quad 1/b - 1/a' = 1/p \quad \text{and} \quad a' \in [1; \infty] \\
&\le C d(t)^{-3/2a'} \; \mu(t)^{-2+3/b} \\
&\qquad \text{thanks to Lemma 4.1 and Riesz-Sobolev Inequality} \\
&\le C t^{-3/2b-2+9/2p} \; ,
\end{aligned}$$

and we choose the optimal value $b = 15/4$. Concerning the density, we write,

$$\begin{aligned}
\|\rho^1_{\mu(s)}(t-s)\|_q &\le C \|d(t)^{-3/2} \exp(-x^2/d(t)^{1/2})\|_a \|\rho^0_{\mu(s)}(\mu(t-s), x)\|_b \\
&\qquad \text{with} \quad 1/b - 1/a' = 1/q \; .
\end{aligned}$$

We observe that,

$$\begin{aligned}
\rho^0_{\mu(s)}(\mu(t-s), x) &= \int_{\mathbb{R}^3} f^0(x - \mu_{\mu(s)}(t-s)v, v) \, dv \; , \\
\mu_{\mu(s)}(t-s) &= \mu(t) \; .
\end{aligned}$$

Therefore,

$$\begin{aligned}\|\rho^1_{\mu(s)}(t-s)\|_q &\leq Cd(t)^{-3/2a'}\|\rho^0(\mu(t),x)\|_b \\ &\leq Ct^{-3/2b-3+9/2q} \ ,\end{aligned}$$

and we choose the optimal value $b = 5/3$. We indicate at the end of this section how to treat the case $q,\, p < 0$. The proof is complete. ■

Now we want to study the behaviour of the non-linear terms $K^{(2)}_{p,\alpha(p)}$ and $S^{(2)}_{q,\delta(q)}$. In order to do this, we first establish two Lemmas which will be enough to get the conclusion of Theorem 5.1 . Two analogous Lemmas can be found in [Bo2], for the case of the VPFP System with finite kinetic energy. In this more regular case, the quantity $\mathcal{S}_q(t) := \sup_{\lambda\geq 0}\|\int_v f(t,x-\lambda v,v)\|_q$ is used, and it satisfies an estimate of the type $Ct^{-\delta}$ for some $\delta > 0$. In our case however we could not get anything better than $\mathcal{S}_q(t) = \infty$. This is the reason why we used the quantity $S_{q,\delta(q)}$ defined above (it corresponds to the choice $\lambda = \mu(t)$ in $\mathcal{S}_q(t)$), and a compensation phenomenon allows $S_{q,\delta(q)}(t)$ to remain bounded.

The desired Lemmas read as follows.

Lemma 5.2 *Let $r > 3/2$, $1 \leq p < \infty$, $\delta \geq 0$.*
Take $q \in [1;\infty]$ satisfying $\quad 1/q \in]1/pr' + 2/3r - 4/9; 1/pr'[$,
and define $\quad \gamma = \alpha + \delta/r' - 9/2(1/q - 1/pr' - 2/3r + 4/9)$.
Then we have the following estimate,

$$K^{(2)}_{q,\gamma}(t) \leq C\ K_{r,\alpha}(t)\ S_{p,\delta}(t)^{1/r'} \ .$$

The same result holds true for $-1/3 < q < 0$.

Lemma 5.3 *Let $r > 6$, $1 \leq p \leq \infty$, $\delta \geq 0$.*
Take $q \in [1;\infty]$ satisfying $\quad 1/q \in]1/pr' + 2/3r - 1/9; 1/pr']$,
and define $\quad \gamma = \alpha + \delta/r' - 9/2(1/q - 1/pr' - 2/3r + 1/9)$.
Then we have the following estimate,

$$S^{(2)}_{q,\gamma}(t) \leq C\ K_{r,\alpha}(t)\ S_{p,\delta}(t)^{1/r'} \ .$$

The same result holds true for $-1/3 < q < 0$ with the additionnal restriction $1/q < 1/pr'$.

These Lemmas give the way to estimate $S_{q,\delta}$ and $K_{p,\alpha}$ for great values of p and q from a bootstrap argument. It shows also how the coefficients α and δ describing the singularity at $t = 0$ get worse as p and q grow.

Proof of Lemma 5.2 . We argue as in the proof of Lemma 5.1 , using the convolution with a kernel which belongs to any L^p $(1 < p < \infty)$. As in the proof of Lemma 5.1, we restrict ourselves to the case $q \geq 0$, and we shall indicate at the end of this section how to treat the convolutions in the case $q < 0$. We write,

$$\begin{aligned}
\|E^2(t)\|_q &= \| \int_0^t \frac{\mu(s)}{d(s)^{3/2}} A(\frac{x}{d(s)^{1/2}}) *_x M_{\mu(s)}(t-s,x)\|_q ds \\
&\leq \int_0^t \mu(s)\ d(s)^{-3/2a'}\ \|M_{\mu(s)}(t-s,x)\|_{pr'} ds \\
&\quad \text{with } 1/pr' - 1/a' = 1/q \quad \text{and} \quad a' \in]1;\infty[\ , \\
&\leq \int_0^t \mu(s)\ d(s)^{-3/2a'}\ \|E(t-s)\|_r\ \mu(s)^{-3/r} \|\rho_{\mu(s)}(t-s,x)\|_p^{1/r'} ds\ ,
\end{aligned}$$

thanks to Hölder's inequality in v (See (4.15)). Hence,

$$\|E^2(t)\|_q \leq C \int_0^t s^{1-9/2a'-3/r}\ (t-s)^{-\alpha-\delta/r'}\ K_{r,\alpha}(t)\ S_{p,\delta}(t)^{1/r'}\ ds\ . \tag{5.1}$$

We estimate the right-hand side of (5.1) below,

First case : $\alpha + \frac{\delta}{r'} < 1$.

In that case, the factor $(t-s)^{-\alpha-\delta/r'}$ in (5.1) is integrable. Also, one easily checks that the condition $1/q > 1/pr' + 2/3r - 4/9$ implies $1 - 9/2a' - 3/r > -1$, so that the s-factor in (5.1) is indeed integrable. Therefore, (5.1) gives,

$$\|E^2(t)\|_q \leq C\ t^{-\gamma}\ K_{r,\alpha}(t)\ S_{p,\delta}(t)^{1/r'}\ . \tag{5.2}$$

Second case: $\alpha + \frac{\delta}{r'} \geq 1$.

In that case, we argue as in [Bo2]. We want to modify the factor $(t-s)^{-\alpha-\delta/r'}$ into an integrable term. For that purpose, we consider the VPFP System with initial data $f(\varepsilon, x, v)$ ($\varepsilon > 0$). We note $f^\varepsilon(t,x,v)$ the corresponding solution (and we assume it is unique, since we deal here with regular solutions to the VPFP). We note also $E_\varepsilon(t,x)$ the corresponding force field (and $E^1_\varepsilon(t,x)$, $E^2_\varepsilon(t,x)$ the associated splitting.). With these notations, we observe,

$$\begin{aligned}
E_\varepsilon(t-s,x) &= E(t-s+\varepsilon)\ , \\
\rho^\varepsilon_{\mu(s)}(t-s) &= \rho_{\mu(s)}(t-s+\varepsilon)\ .
\end{aligned}$$

Hence we write, as in (5.1),

$$\begin{aligned}
\|E^2_\varepsilon(t)\|_q &\leq C \int_0^t s^{1-9/2a'-3/r}\ \|E_\varepsilon(t-s)\|_r\ \|\rho^\varepsilon_{\mu(s)}(t-s)\|_p^{1/r'}\ ds \\
&\leq C \int_0^t s^{1-9/2a'-3/r}\ (t+\varepsilon-s)^{-\alpha-\delta/r'}\ K_{r,\alpha}(t+\varepsilon)\ S_{p,\delta}(t+\varepsilon)^{1/r'}\ ds
\end{aligned} \tag{5.3}$$

Now we take $\varepsilon = t/2$ in (5.3), and we get the result. ■

Remark 2 *Performing the operation* $\varepsilon = t/2$ *above changes the interval* $[0;T]$ *into* $[0;3T/2]$*. Also, the estimate in Lemma 5.2 above should write,*

$$K^{(2)}_{q,\gamma}(t) \leq C\ K_{r,\alpha}(3t/2)\ S_{p,\delta}(3t/2)^{1/r'}\ .$$

For simplicity, we did not keep track of these technical points.

Proof of Lemma 5.3 . We proceed as above, and write,

$$\|\rho^2_{\mu(s)}(t-s)\|_q = \| \int_{u=0}^{t-s} \frac{\mu_{\mu(s)}(u)}{d_{\mu(s)}(u)^2} \nabla \mathcal{N}\left(\frac{x}{d_{\mu(s)}(u)^{1/2}}\right)\ *_x M_{\mu_{\mu(s)}(u)}(t-s-u,x)\|_q$$

$$\leq\ C \int_{u=0}^{t-s} \mu_{\mu(s)}(u)^{1-3/r}\ d_{\mu(s)}(u)^{-1/2-3/2a'}\ \|E(t-s-u)\|_r\ \|\rho_{\mu_{\mu(s)}(u)}(t-s-u,x)\|_p\ , \tag{5.4}$$

with $1/pr' - 1/a' = 1/q$ and $a' \in [1;\infty]$. Now we observe that $\mu_{\mu(s)}(u) = \mu(s+u)$, so that

$$\rho_{\mu_{\mu(s)}(u)}(t-s-u,x) = \rho_{\mu(s+u)}(t-(s+u),x)\ ,$$

and we get in (5.4),

$$\begin{aligned}&\|\rho^2_{\mu(s)}(t-s)\|_q \leq \\ &\quad \leq \int_{u=0}^{t-s} \mu_{\mu(s)}(u)^{1-3/r}\ d_{\mu(s)}(u)^{-1/2-3/2a'}\ (t-s-u)^{-\alpha-\delta/r'}\ K_{r,\alpha}(t)\ S_{p,\delta}(t)^{1/r'}\ .\end{aligned}$$

It is easy to check that,

$$\frac{\mu_{\mu(s)}(u)^{1-3/r}}{d_{\mu(s)}(u)^{1/2+3/2a'}} \leq \frac{\mu(u)^{1-3/r}}{d(u)^{1/2+3/2a'}}\ .$$

Hence,

$$\|\rho^2_{\mu(s)}(t-s)\|_q \leq \int_{u=0}^{t-s} u^{-1/2-3/r-9/2a'}\ (t-s-u)^{-\alpha-\delta/r'}\ K_{r,\alpha}(t)\ S_{p,\delta}(t)^{1/r'}\ ,$$

and we conclude as in the proof of Lemma 5.2 . ■

We can now come to the

Proof of Theorem 5.1. Lemma 5.3 shows that one can bound $S^{(2)}_{q,\gamma}$ with the help of $S_{p,\delta}$ for exponents $q \geq p$. Since we already know how to bound $S^{(1)}_{q,\gamma}$ for any q, this allows to bound $S_{q,\gamma}$ thanks to $S_{p,\delta}$ for values $q \geq p$. We have indeed the obvious estimate $S \leq S^{(1)} + S^{(2)}$ (and the same for K). Before going further, we first look for the optimal values of the parameters, in the sense that we want to get the greatest value of q combined with the lowest value of the associated γ. Indeed, we recall that γ describes the rate at which our solution "blows-up" at $t = 0$.

In order to get this optimal choice, we observe that the L^p norm of the force field $\|E(t)\|_p$ should blow up at least like $t^{-\alpha}$, with $\alpha \geq 12/5 - 9/2p$, since it is the case for the first term $E^1(t)$ in the splitting (See Lemma 5.1-(i)). So we cannot hope any better α than $\alpha \geq 12/5 - 9/2p$ in the statement of Lemma 5.3. Therefore, using the explicit value of γ in this Lemma, we get,

$$\gamma \geq \frac{12}{5} - \frac{9}{2r} + (\delta + \frac{9}{2p})\frac{1}{r'} + \frac{3}{r} - \frac{1}{2} - \frac{9}{2q} \ .$$

Here, we use $1/q \leq 1/pr'$, and get,

$$\gamma \geq \frac{12}{5} - \frac{9}{2r} + \frac{\delta}{r'} + \frac{3}{r} - \frac{1}{2} := \gamma_{min}(q) \ . \tag{5.5}$$

Now $\gamma_{min}(q)$ is the lowest possible value of γ associated with a fixed q. The best possible choice of the parameters in then obtained by minimizing $\gamma_{min}(q)$ under the constraint $\frac{1}{q} \in]\frac{1}{pr'} + \frac{2}{3r} - \frac{1}{9}; \frac{1}{pr'}]$. And this last condition gives the two constraints,

$$\frac{1}{r'} \geq \frac{p}{q} \quad \text{and} \quad \frac{1}{r'} > \frac{5/9 - 1/q}{2/3 - 1/p} \ . \tag{5.6}$$

Equating the right-hand sides of (5.6) gives the optimal relations between the parameters,

$$q = \frac{6}{5}p \quad \text{and} \quad r' \approx \frac{6}{5} \ (\text{ but } r' < 6/5 \ ) \ , \tag{5.7}$$

and in this case $\gamma \approx \gamma_{min}(q)$ ($\gamma > \gamma_{min}(q)$).

The same considerations give the following optimal choice of the parameters in Lemma 5.2,

$$q = 3p \quad \text{and} \quad r' \approx 3 \ (\text{but } r' < 3 \ ) \ . \tag{5.8}$$

After these preliminary considerations, we decompose the proof of Theorem 5.1 as follows.

First Step. We bound $K_{r,\alpha(r)}$ for some $r > 6$.

Let $r > 6$, $r \approx 6$, whose value will be fixed later. According to (5.7), we apply Lemma 5.3 with $p = 5/3$, $\delta(p) = 6/5$, $q = 5r'/3$. This gives

$$\begin{cases} S^{(2)}_{5r'/3,\delta} \leq C \ K_{r,\alpha} \ S^{1/r'}_{5/3,6/5} \\ \delta = \alpha + 6/5r' - 9/2(-2/3r + 1/9) \ . \end{cases} \tag{5.9}$$

We already observe in (5.9) that $S_{5/3,6/5} \leq C$, thanks to (4.7).

On the other hand, Lemma 5.1 implies

$$S^{(1)}_{5r'/3,\delta(5r'/3)} \leq C \; . \tag{5.10}$$

(See Theorem 5.1 for the definition of $\delta(q)$). It is easy to check on the formula (5.9) that $\delta \geq \delta(\frac{5r'}{3})$. Finally, it is clear that the quantities $S_{q,\delta}$ are decreasing functions of δ, according to the Remark 1 above. Therefore, (5.9) together with (5.10) gives,

$$S_{5r'/3,\delta} \leq C \; K_{r,\alpha(r)} \quad , \tag{5.11}$$

and δ is given by (5.9). We translate this bound on the density into a bound on the force field, thanks to Lemma 5.2. More precisely, we take, according to (5.8), $R \approx 3/2$, $R > 3/2$, and $q = r$, and write Lemma 5.2 in the form,

$$\begin{cases} K^{(2)}_{r,\gamma} \leq C \; K_{R,2-3/R} \; S^{1/R'}_{5r'/3,\delta} \leq C \; K_{R,2-3/R} \; K^{1/R'}_{r,\alpha(r)} \\ \gamma = 2-3/R + \delta/R' - 9/2(1/r - 3/5r'R' - 2/3R + 4/9) \; . \end{cases} \tag{5.12}$$

Now, in view of Lemma 5.2, this majorisation needs the following constraint,

$$\frac{1}{r} \in \left] \frac{3}{5r'R'} + \frac{2}{3R} - \frac{4}{9}; \frac{3}{5r'R'} \right[\; . \tag{5.13}$$

The upper bound in (5.13) is equivalent to $1/r < 3/(5R' + 3)$, and this last quantity is $\approx 1/6$, and $> 1/6$ as $r \approx 6$ and $R \approx 3/2$.

The lower bound in (5.13) is equivalent to

$$\frac{1}{r} > \frac{1 + 7R/3}{24R - 9} \; ,$$

and this last quantity is $\approx 1/6$, $< 1/6$ under the same assumptions.

Therefore, (5.13) holds, and (5.12) is valid for $r > 6$, $r \approx 6$ close enough in function of R, and $R > 3/2$, $R \approx 3/2$. Now we observe in (5.12) that $\gamma \approx \alpha(r)/3 + 1 \leq \alpha(r)$ because $\alpha(r) \approx 33/20 \geq 3/2$.

Therefore, combining (5.12) with $K^{(1)}_{r,\alpha(r)} \leq C$ (See Lemma 5.1) and with $K_{R,2-3/R} \leq C$ (See (4.8)) gives,

$$K_{r,\alpha(r)} \leq C \; K^{1/R'}_{r,\alpha(r)} \; ,$$

thus,

$$K_{r,\alpha(r)} \leq C \; .$$

Second Step. We bound $S_{\infty,\delta}$ for any $\delta > \delta(\infty)$.

We write Lemma 5.3 with $p = q = \infty$, $r > 6$ ($r \approx 6$) as in the first step, and $\delta > \delta(\infty)$ ($\delta \approx \delta(\infty)$). We obtain,

$$\begin{cases} S^{(2)}_{\infty,\gamma} \leq C \; K_{r,\alpha} \; S^{1/r'}_{\infty,\delta} \\ \gamma = \alpha + \delta/r' + 3/r - 1/2 \; . \end{cases} \tag{5.14}$$

On the other hand, we already know

$$S^{(1)}_{\infty,\delta(\infty)} \leq S^{(1)}_{\infty,\,39/10} \leq C\ . \tag{5.15}$$

Finally, one easily checks that $\gamma \leq \delta$ since $\gamma \approx \delta(\infty)$. Therefore, combining (5.14) and (5.15) gives, as in the first step,

$$S_{\infty,\delta} \leq C S^{1/r'}_{\infty,\delta}\ ,$$

thus,

$$S_{\infty,\delta} \leq C(T)\ , \tag{5.16}$$

This ends the second step.

Third Step. We bound $K_{q,\alpha}$ for any $\alpha > \alpha(q)$ and $1 \leq q < \infty$.

Thanks to the second step, we know that $S_{p,\delta} \leq C$ for all $\delta > \delta(p)$ (interpolate the bound on $S_{\infty,\delta}$ with the bound on $S_{5/3\,,\,6/5}$ in (4.7)). Therefore, writing Lemma 5.2 with the choice $r \approx 3/2$ ($r > 3/2$), $q \approx pr'$ ($q > pr'$) gives,

$$\begin{cases} K^{(2)}_{q,\gamma} \leq C\ K_{r,\alpha(r)}\ S^{1/r'}_{p,\delta} \\ \gamma = \alpha(r) + \delta/r' + 9/2(1/q - 1/pr' - 2/3r + 4/9)\ . \end{cases} \tag{5.17}$$

Using Lemma 5.1 gives the desired bound on K, since $\gamma \approx \alpha(q)$ in (5.17). The case $q = \infty$ is recovered from the Hölder estimates below by interpolation.

Fourth Step. Hölder estimates.

We notice that we can apply exactly the same method in order to estimate the Hölder norms of ρ and E. For this we only need to replace in the above calculations the convolutions inequalities in L^p, by the convolutions inequalities in $C^{0,\alpha}$. They read (See [Bo2], [Ho1]),

$$\|u *_x a\|_{C^{0,\alpha}} \leq C\|u\|_p \tag{5.18}$$

for,

$$1 \leq p \leq \infty\ \ ,\ \ 1 \leq l < \infty\ \ ,\ \ \alpha = N(1/l - 1/p) \in]0;1[$$

$$a \in C^1(\mathbb{R}^N - \{0\})\ \ ,\ \ |a(x)| \leq C/|x|^{N/l'}\ \ ,\ \ |\nabla a(x)| \leq C/|x|^{1+N/l'}\ .$$

In our case, we use these convolution inequalities for the kernel $a(x) = \mathcal{N}(x) \in \mathcal{S}$, or $a(x) = A(x)$. When $a = A$, we observe indeed that, for all $0 \leq \beta < 3$, we have, $x^\beta A(x) \in L^\infty$, (use the Fourier transform), and this allows the use of (5.18) for $a = A$. Also, these considerations justify Lemmas 5.1-5.2-5.3 in the case of negative exponents.

We write now Lemma 5.3 for $q \approx -9$ ($q < -9$) (this corresponds to the space $C^{0,\beta}$ with $\beta \approx 1/3$), $r \approx \infty$, $\alpha \approx 33/10$, $p = \infty$, $\delta \approx 99/10$. This gives,

$$S_{q,\gamma} \leq C \ , \quad \text{for } \ \gamma \approx 66/5 \ ,$$

and it proves Theorem 5.1-(iii). Part (iv) is obtained in the same way. We get the estimates on ∇E by using the explicit formula for ∇E and observing that $\nabla A \in L^p$ $(1 < p < \infty)$.

The proof of Theorem 5.1 is now complete. ■

References

[An] H. Andréasson, *Global existence of smooth solutions in three dimensions for the Vlasov-Poisson-Boltzmann equation arising in semi-conductor physics*, Nonlin. Anal., T.M.A., Vol. 28, n. 7, 1193-1211 (1996).

[Bo1] F. Bouchut, *Existence and Uniqueness of a Global Smooth Solution for the Vlasov-Poisson-Fokker-Planck System in Three Dimensions*, J. Funct. Anal., Vol. 111, 239-258 (1993).

[Bo2] F. Bouchut, *Smoothing Effect for the Non-linear Vlasov-Poisson-Fokker-Planck System*, J. Diff. Eq., Vol. 122, 225-238 (1995).

[BD] F. Bouchut, J. Dolbeault *On long-time asymptotics of the Vlasov-Fokker-Planck and Vlasov-Poisson-Fokker-Planck System with Coulombic and Newtonian potentials*, Diff. Int. Eq., Vol. 8, n. 3, 487-514 (1995).

[CS] J.A. Carillo, J. Soler, *On the Vlasov-Poisson-Fokker-Planck equation with measures in Morrey spaces as initial data*, J. Math. Anal. Appl., Vol. 207, N. 2, p. 475-495 (1997).

[Ca] F. Castella, *Propagation of Space-Moments in the Vlasov-Poisson Equation and Further Results*, to appear in Ann. I.H.P., Anal. NonLinéaire (1998).

[DPL1] R.J. Di Perna, P.L. Lions, *Ordinary Differential Equations, Transport Theory and Sobolev Spaces*, Inv. Math., Vol. 98, 511-547 (1989).

[DPL2] R.J. Di Perna, P.L. Lions, *Global weak solutions of Vlasov-Maxwell Systems*, Comm. Pure Appl. Math., Vol. 42, 729-757 (1989).

[DPL3] R.J. Di Perna, P.L. Lions, *On the Fokker-Planck-Boltzmann Equation*, Comm. Math. Phys., Vol. 120, 1-23 (1988).

[DPLM] R.J. Di Perna, P.L. Lions, Y. Meyer, *L^p regularity of velocity averages*, Ann. I.H.P., Vol. 8, n. 3-4, 271-287 (1991).

[GPS] F. Golse, B. Perthame, R. Sentis, *Un résultat de compacité pour les équations de transport*, C. R. Acad. Sci. Paris, Série I 301, 341-344 (1985).

[GLPS] F. Golse, P.L. Lions, B. Perthame, R. Sentis, *Regularity of the moments of the solution of a transport equation*, J. Funct. Anal., Vol. 76, 110-125 (1988).

[Ho1] L. Hörmander, *The Analysis of Linear Partial Differential Operators*, Vol. I, Springer Verlag, New-York-Berlin (1983).

[Ho2] L. Hörmander, *Hypoelliptic second order differential equations*, Acta Math., Vol. 119, 147-171 (1967).

[IR] R. Illner, G. Rein, *Time decay of the solutions of the Vlasov-Poisson system in the plasma physical case*, Math. Meth. Appl. Sci., Vol. 19, N. 17, p. 1409-1413 (1996).

[LPe] P.L. Lions, B. Perthame, *Propagation of moments and regularity for the 3-dimensional Vlasov-Poisson system*, Invent. Math., Vol. 105, 415-430 (1991).

[PM] B. Perthame, S. Mischler, *Solutions of the Boltzmann equation with infinite energy*, SIAM J. Math. Anal., Vol. 28, N. 5, 1015-1027 (1997).

[Pe] B. Perthame, *Time decay, Propagation of Low Moments and Dispersive Effects for Kinetic Equations*, Comm. PDE, Vol. 21, 659-686 (1996).

[Pf] K. Pfaffelmoser, *Global classical solutions of the Vlasov-Poisson system in three dimensions for general initial data*, J. Diff. Eq., 95, 281-303 (1992).

[Re1] G. Rein, *Growth estimates for the solutions of the Vlasov-Poisson system in the plasma physics case*, Math. Nachr., 191, 269–278 (1998).

[Re2] G. Rein, *Selfgravitating systems in Newtonian theory - the Vlasov-Poisson system in three dimensions*, Banach Center Publ., Vol. 41, Part I, 179-194 (1997).

[RW] G. Rein, J. Weckler, *Generic global classical solutions of the Vlasov-Fokker-Planck-Poisson system in three dimensions*, J. Diff. Eq., Vol. 95, 281-303 (1992).

[Sc] J. Schaeffer, *Global existence of smooth solutions to the Vlasov-Poisson system in three dimensions*, Comm. PDE., Vol. 16, N. 8-9, 1313-1335 (1991).

[St] E.M. Stein, *Singular integrals and differentiability properties of functions*, Princeton University Press, Princeton Mathematical Series 30 (1970).

[VO] H.D. Victory, B.P. O'Dwyer, *On classical solutions of Vlasov-Poisson-Fokker-Planck systems*, Indiana Univ. Math. J., Vol. 39, 105-157 (1990).