Propagation of space moments In the Vlasov-Poisson Equation and further results

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Résumé. Pour une solution $f = f(t, x, v)$ du système de Vlasov-Poisson nous prouvons que, si la donnée initiale f^0 possède des moments dans la variable d'espace x d'ordre plus grand que 3, alors f possède également des moments d'ordre plus grand que 3 dans la variable $x - vt$ (propagation des moments d'espace d'ordre élevé). Nous prouvons également la propagation des moments d'ordre peu élevé dans les variables d'espace ou de vitesse. Enfin, nous établissons diverses estimations a priori pour des solutions du système de Vlasov-Poisson ayant une énergie cinétique infinie.

Abstract. We show that, if the initial data has moments in the space variable x higher than three, then the corresponding solution $f(t, x, v)$ of the Vlasov-Poisson System has also moments in $x - vt$ higher than three (propagation of high space moments). We also prove the propagation of low moments in the space or in the velocity variable, and state further a priori estimates for solutions of the Vlasov-Poisson System having infinite kinetic energy.

Key-words. Vlasov-Poisson System, propagation of moments, infinite kinetic energy.

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1 Introduction

We consider the three dimensional Vlasov-Poisson System (VPS). In this system, the function $f(t, x, v) \geq 0$ represents the microscopic density of particles located at the position $x \in \mathbb{R}^3$, with velocity $v \in \mathbb{R}^3$, at the time $t \in \mathbb{R}$, evolving in the self-consistent (repulsive or attractive) Coulomb potential it creates. The system reads,

$$\begin{cases} \partial_t f + v \cdot \nabla_x f + E \cdot \nabla_v f = 0 \ , \\ f(t=0,x,v) = f^0(x,v) \geq 0 \ , \\ E(t,x) = \pm \dfrac{1}{4\pi} \dfrac{x}{|x|^3} * \rho(t,x) \ , \\ \rho(t,x) = \int_{\mathbb{R}^3} f(t,x,v) \, dv \ . \end{cases} \tag{1.1}$$

Here, $\rho(t,x)$ represents the macroscopic density of particles located at the point x and at the time t, $E(t,x)$ is the self-consistent Coulombic or Newtonian force-field created by ρ, and the sign $-$ corresponds to a gravitational interaction (astrophysics), whereas $+$ describes an electrostatic interaction (semi-conductor devices). In fact, we will not distinguish between the repulsive and attractive cases in the sequel.

On the other hand, the Free-Transport equation is closely related to the VPS. It describes the free evolution of a particle-system with no interactions, and reads,

$$\begin{cases} \partial_t f + v \cdot \nabla_x f = 0 \ , \\ f(t=0,x,v) = f^0(x,v) \geq 0 \ . \end{cases} \tag{1.2}$$

This equation has the (unique) solution $f(t,x,v) = f^0(x - vt, v)$ and generates the macroscopic density $\rho^0(t,x) = \int_v f^0(x - vt, v) dv$. From a 'semi-group' point of view, (1.2) gives the C_0 group associated with (1.1), which is easily seen to be unitary in the spaces $L^p(\mathbb{R}^3_x \times \mathbb{R}^3_v)$ $(1 \leq p \leq \infty)$.

Our main results concerning the VPS are the following :

(i) We prove that, if the initial data f^0 satisfies $f^0 \in L^1 \bigcap L^\infty$ and $\int_{x,v} (|x|^m + |v|^\varepsilon) f^0 < \infty$ for some $m > 3$, $\varepsilon > 0$, then one can build a solution of (1.1) such that $\int |x - vt|^k \ f(t,x,v) \ dx \ dv \ \in L^\infty_{loc}(\mathbb{R}_t)$, for all $k < m$ (Theorem 5.1). This result, which we call *propagation of space moments* (of high order), is obvious on the Free-Transport equation above (even with $\varepsilon = 0$), and we show in fact that the nonlinearity in the VPS 'preserves' this property. We emphasize the fact that the solutions we build here have infinite kinetic energy.

(ii) The point (i) above leads to the restriction $m > 3$ and the case of the *low* moments remains. We show that one can propagate the velocity *and* the space moments of the initial data f^0 under the two possible sets of assumptions: $(|x|^m + |v|^p) \ f^0 \in L^1$ for some $m > 3$, $p > 0$, or for $m > 0$, $p > 3$ (Theorem 6.1). This result shows how one can also propagate the *low* moments in the VPS in one variable if we control enough moments in the other variable.

(iii) The first point allows to develop a theory of solutions to (1.1) with infinite kinetic energy, and we state here various results in this direction. The idea is that a priori estimates (regularizing effects) on the force field $E(t,x)$ can be obtained as soon as the initial data $f^0 \in L^1 \bigcap L^\infty$ has one additional moment (Theorem 2.1). In this sense, the initial kinetic energy $\int_{x,v} |v|^2 f^0$ is one particular moment, which does not need being finite. This kind of regularizing effect is well-known at the quantum level (See below). We also state decay estimates for the Repulsive VP System.

Global weak solutions to (1.1) were built in [11] under the natural assumptions $f^0 \in L^1 \bigcap L^\infty$, $\int_v |v|^2 f^0 < \infty$ (See also [1], [2], [13]). Also, global renormalized solutions were built in [7] for initial datas satisfying $f^0 \in L^1$ (mass), $f^0 \log^+ f^0 \in L^1$ ('entropy'), and $v^2 f^0 \in L^1$ (kinetic energy).

Besides, the construction of smooth solutions to the VPS was achieved in two different settings. On the one hand, [20] and [23] used compactly supported solutions and studied the characteristic curves of the natural ODE associated to the System; in this setting, they showed that smooth and compactly supported initial datas (say $C^1_c(\mathbb{R}^6)$) remain, say, C^1_c through time evolution, thanks to an appropriate decomposition of the phase-space (See [21] for refinements, [22] for a review paper on these methods). On the other hand, smooth solutions were also built by [16] for initial datas $f^0 \in L^1 \bigcap L^\infty$ having velocity moments of order higher than three, and they proved that these moments are propagated through time evolution.

All the above mentioned papers treat the case of solutions having finite kinetic energy ($\int_{x,v} v^2 f^0(x,v) < \infty$). However, solutions with infinite kinetic energy were recently built in [19], under the assumptions $f^0 \in L^1 \bigcap L^\infty$, $|x|^2 f^0 \in L^1$ (See [18] for results in the same direction concerning the Boltzmann equation): here, a new dispersive identity on the VPS is proved (See also [14]) that allows to propagate the second space moment, in the sense that $\int |x - vt|^2 f(t,x,v)\, dx\, dv \in L^\infty_{loc}(\mathbb{R}_t)$ (See (i) above). Surpringly, this gives rise to regularising effects in the VPS when the kinetic energy is not initially bounded, such as the estimate $\|E(t,x)\|_{L^2_x} \leq Ct^{-1/2}$ as t tends to 0 (See [19] for more details).

The present paper is a natural continuation of the works by [16] on the one hand, where the propagation of high moments in v was considered but not the x moments, and by [19] on the other hand, which concerns only the special case of the second space moment and the associated conservation law. To our knowledge, nothing was known concerning the general problem of the propagation of space moments in the VPS and the corresponding existence theory for solutions with infinite kinetic energy.

Indeed, we rely here on general x-moments in order to build solutions to the VPS (point (i) above): since we cannot derive, in general, any conservation law for these moments, only PDE arguments allow to prove the propagation of space moments. The main difficulty at this level stems from the fact that we deal here with solutions having infinite kinetic energy. In order to treat this important feature, the key point is that the VPS gives rise to a regularizing effect which implies, roughly speaking, that the potential energy immediately becomes finite as $t > 0$ under the assumptions we make here (See Theorem 2.1 below). Therefore, a natural singularity at $t = 0$ appears which contains most of the mathematical difficulties of our approach, and makes the main difference with the above papers. Notice that this kind of situations had been already pointed out in [19] for the case of the second space moment (the problem is much simpler in this case since we already have a conservation law).

Our method shows also how to propagate the *low* velocity moments which were not considered in [16] (point (ii) above).

Finally, the point (iii) above (Theorem 2.1 below) generalizes regularizing effects obtained in [5] and [19]. In fact, this Theorem contains the starting point of our approach: the major difficulty while dealing with the VPS is to bound the force field $E(t,x) = x/4\pi|x|^3 * \rho$ in the L^q_x spaces, which gives rise to the problem of bounding ρ in L^p for $p > 1$. In order to do so, one can use auxiliary *moments* of f^0, and try to propagate them through time evolution. Following this idea, we introduce in section 2 several preliminary lemmas in this direction, and deduce some a priori estimates on the force field $E(t,x)$. For instance, we show that the existence of a space moment at the time $t = 0$ gives a finite potential energy for $t > 0$ (See Theorem 2.1).

We would like to give another strong motivation for this work: the propagation of x-moments at the quantum level has been studied widely in the literature (See [6] and the references therein). But nothing was known at the classical level, which is mathematically more difficult because the impulsion v corresponds to $i\hbar\nabla$ at the quantum level, a stronger operator. Recall that the quantum analogue of the VPS is the Hartree Equation, or the Schrödinger-Poisson System (See e.g. [15], [19], [4]). Hence, the quantum analogue of the property (i) above concerning the space moments of the solution is,

$$x^m \phi \in L^2 \Rightarrow (x + it\hbar\nabla)^m \, \psi(t) \in L^2 \ , \ \ \forall t \in \mathbb{R} \ ,$$

where $\phi \in L^2$ is the initial data, and $\psi(t)$ is the corresponding solution to the Hartree Equation. This property is well-known, and one should notice that, although the x moments are propagated at the classical level under the assumption $|v|^\varepsilon f^0 \in L^1$ for some $\varepsilon > 0$, the Hartree equation allows to propagate the space moments of the initial data ϕ without any further assumption on the regularity of ϕ (i.e. we do not need to assume $\nabla^\varepsilon \phi \in L^2$ for some $\varepsilon > 0$). Also, the regularizing effect described in Theorem 2.1 below has a well-known (and stronger) quantum analogous, that is for instance,

$$x\phi \in L^2 \Rightarrow \psi(t) \in L^2 \bigcap L^6 \quad \text{for } t \neq 0$$

in three dimensions of space (See [6], [4]).

The end of this paper is organised as follows : in section 2, we prove a general Lemma that we use throughout the paper, and deduce a priori estimates for solutions to the VPS ; section 3 is devoted to the local-in-time propagation of the space moments, and sections 4-5 show how to deal with arbitrary large time intervals (points (i)-(ii)); finally, section 6 deals with the propagation of low space and velocity moments, which are not considered in a first approach (point (iii)).

Our main results are Theorems 2.1, 5.1, 6.1.

2 Some a priori estimates

Before beginning our analysis of the VP equation, we first introduce some (commonly used) notations : L^p denotes either $L^p(\mathbb{R}^3)$ or $L^p(\mathbb{R}^3 \times \mathbb{R}^3)$, if the context makes it clear. We sometimes write also L^p_x instead of $L^p(\mathbb{R}^3_x)$, and $L^p_{x,v}$ instead of $L^p(\mathbb{R}^3_x \times \mathbb{R}^3_v)$. The corresponding norms appears frequently as $\|.\|_p$. For instance, we use the convention $\|\rho(t,x)\|_{L^p_x} \equiv \|\rho(t,x)\|_p \equiv \|\rho(t)\|_p$ (and the same for $E(t,x)$, ...). As usual p' stands for the conjugate exponent of p, that is : $1/p + 1/p' = 1$. Finally, we denote $C(a_1, \cdots, a_n)$ a positive constant depending only on the arguments.

Now, and as it was announced in the previous section, we first introduce several preliminary inequalities on the microscopic density $f(t,x,v)$, which will be used later. We begin with the

Definition 2.1 *Let $f(t,x,v) \geq 0$ be a solution to the VPS and $f^0(x,v)$ the corresponding initial data. We define, for $k \geq 0$:*
(i) (space moments)

$$M_k(t,x) := \int_{\mathbb{R}^3} |x-vt|^k f(t,x,v)dv \ , \quad M_k(t) := \int_{\mathbb{R}^6} |x-vt|^k f(t,x,v)dx\, dv \ ,$$

(ii) (velocity moments)

$$N_k(t,x) := \int_{\mathbb{R}^3} |v|^k f(t,x,v)dv \ , \quad N_k(t) := \int_{\mathbb{R}^6} |v|^k f(t,x,v)dx\, dv \ ,$$

(iii) the macroscopic density generated by the Free-Transport Equation (See Introduction) is: $\rho^0(t,x) := \int_{\mathbb{R}^3} f^0(x-vt,v)dv$.

We now state the

Lemma 2.1 *Let $f^0(x,v) \in L^1 \bigcap L^\infty$, $f^0 \geq 0$. Then, there are constants $C = C(\|f^0\|_{L^1 \bigcap L^\infty})$ such that :*

$$(1)\ \forall\ p \leq q\ , \qquad \|\rho^0(t,x)\|_{L^p_x} \leq C\ \|f^0(x,v)\|_{L^1_v(L^q_x)}\ ,$$

$$(2)\ \forall\ 1 \leq p \leq \tfrac{3+k}{3}\ , \qquad \|\rho^0(t,x)\|_{L^p_x} \leq C\ N_k(0)^{\frac{3}{kp'}}\ ,$$

$$(3)\ \forall\ p \leq q\ , \qquad \|\rho^0(t,x)\|_{L^p_x} \leq C\ t^{-\frac{3}{p'}}\ \|f^0\|_{L^1_x(L^q_v)}$$

$$(4)\ \forall\ 1 \leq p \leq \tfrac{3+k}{3}\ , \qquad \|\rho^0(t,x)\|_{L^p_x} \leq C\ t^{-\frac{3}{p'}}\ M_k(0)^{\frac{3}{kp'}}\ .$$

We see here how we can control the L^p norms of $\rho^0(t)$ ($t \neq 0$) in terms of some *moments* of f^0 (moments of L^p type in one variable, or more usually moments with weights). Notice that an exponent $\theta \in [0,1]$ should appear in the inequalities (1) and (3) above, giving for instance $\|f^0(x,v)\|^{\theta}_{L^1_v(L^q_x)}$ on the r.h.s. instead of $\|f^0(x,v)\|_{L^1_v(L^q_x)}$. For sake of simplicity, we do not write this exponent here, since it plays no role in the sequel. We do not prove the Lemma 2.1 which is an easy consequence of the following

Lemma 2.2 *Let $f(x,v) \in L^1 \bigcap L^\infty$ and set $M_k(t,x) := \int |x-vt|^k f(x,v)dv$, $M_k(t) := \int M_k(t,x)dx$, $N_k(t,x) := \int |v|^k f(x,v)dv$, and $N_k(t) := \int N_k(t,x)dx$ as in Definition 2.1 . Then, there are constants $C = C(\|f\|_{L^1 \bigcap L^\infty})$ such that :*

(1) $\forall\ p \geq q$, $\quad \|f(x-vt,v)\|_{L^p_x(L^q_v)} \leq \|f\|_{L^q_v(L^p_x)}$,

(2) $\forall\ 1 \leq p \leq \frac{3+k}{3+l}$, $\quad \|N_l(t,x)\|_{L^p_x} \leq C\ N_k(t)^{\frac{l}{k}+\frac{3}{kp'}}$,

(3) $\forall\ p \geq q$, $\quad \|f(x-vt,v)\|_{L^p_x(L^q_v)} \leq C\ t^{-3(\frac{1}{q}-\frac{1}{p})}\ \|f\|_{L^q_x(L^p_v)}$,

(4) $\forall\ 1 \leq p \leq \frac{3+k}{3+l}$, $\quad \|M_l(t,x)\|_{L^p_x} \leq C\ t^{-\frac{3}{p'}}\ M_k(t)^{\frac{l}{k}+\frac{3}{kp'}}$.

Proof of Lemma 2.2 We begin by proving the point (1). Indeed, for $p \geq q$,

$$\begin{aligned} \|f(x-vt,v)\|_{L^p_x(L^q_v)} &\leq \|f(x-vt,v)\|_{L^q_v(L^p_x)} \\ &\leq \|f(x,v)\|_{L^q_v(L^p_x)} . \end{aligned} \tag{2.1}$$

The point (2) is proved in [16], and (3) can be found in [5]. Now the point (4) is analogous to the proof of (2) in [16]. Indeed, if we set $M_l(t,x) = \int_v |x-vt|^l f(x,v)dv$, we have

$$\begin{aligned} M_l(t,x) &= \int_{|x-vt|\leq R} |x-vt|^l f(x,v)\ dv + \int_{|x-vt|\geq R} |x-vt|^l f(x,v)\ dv \\ &\leq \|f(x,v)\|_{L^q_v}\ (\int_{|x-vt|\leq R} R^{lq'} dv)^{1/q'} + R^{-k+l}\ M_k(t,x) \\ &\leq \|f(x,v)\|_{L^q_v}\ R^l\ (\frac{R}{t})^{3/q'}\ (\int_{|V|\leq 1} dV)^{1/q'} + R^{-k+l}\ M_k(t,x) . \end{aligned}$$

The point (4) is then obtained by letting $R^{k+\frac{3}{q'}} = M_k(t,x)\ t^{3/q'}\ \|f(x,v)\|^{-1}_{L^q_v}$ and $p' = \frac{kq'+3}{k-l}$ in the last inequality. ■

Remark 2.1 *As a direct consequence of Lemma 2.1-(4), the Riesz-Sobolev inequality (See [24]) immediatly implies,*

$$\forall\ 3/2 < p \leq \frac{3(3+k)}{6-k} \qquad (< \infty\ if\ k > 6)$$

$$\|E^0(t)\|_p \ \leq\ C\ t^{-(2-\frac{3}{p})} M_k^{\frac{1}{k}(2-\frac{3}{p})} . \tag{2.2}$$

Remark 2.2 *The points (1) and (2) in Lemma 2.1 give an L^∞ bound in time on $\|\rho^0(t)\|_p$, although estimates (3) and (4) give at the same time some decay of the latter as t goes to infinity and a regularizing effect on the L^p norms ($p > 1$) of $\rho^0(t)$ as t goes to 0. Both phenomenas behave in (3) as well as in (4) like $t^{-3/p'}$. One should also notice that this negative power of t in (4) does not depend on the value of k: a better regularity of f^0 in terms of its x-moments does not improve the decay of the macroscopic density.*

Concerning the blow-up of $\rho^0(t)$ at $t = 0$, we shall see in the subsequent sections that it gives rise to a difficulty when one wants to propagate the x-moments M_k of the initial data f^0 through the VPS. That is the reason why we will assume in sections 3-5 that f^0 has an additional moment of small order in the velocity variable. On the other hand, the decay estimate $\|\rho^0(t)\|_p \leq C\ t^{-3/p'}$ as $t \to \infty$, is in fact optimal, as we show it now.

Lemma 2.3 *(1) (Free Transport Equation) Let $f^0 \neq 0$ and $\rho^0(t)$ be as in Lemma 2.1. Then there exists a positive constant $C = C(f^0)$ such that,*

$$\|\rho^0(t)\|_p \geq C\ (1 + |t|)^{-\frac{3}{p'}}\ .$$

(2) (Repulsive VP System) Let $f^0 \in L^1 \bigcap L^\infty$, $f^0 \neq 0$, satisfy $(|v|^2 + |x|^2)f^0 \in L^1$. Let $f(t,x,v)$ be a corresponding solution of the Repulsive VPS which preserves energy. Then, there exists a positive constant $C = C(\|f^0\|_{L^1 \bigcap L^\infty}, \|(|v|^2 + |x|^2)f^0\|_{L^1})$ such that,

$$\|\rho(t,x)\|_{L^p_x} \geq C\ (1 + |t|)^{-\frac{3}{p'}}\ .$$

Proof of Lemma 2.3. The proof of these properties on the quantum level (Hartree equation or Free Schrödinger equation) can be found in [12] and [3]. We adapt them in the 'classical' context. We first prove the point (1). Let $R > 0$, we have,

$$\int_{|x| \leq Rt} \rho^0(t) dx \quad \leq \quad \|\rho^0(t)\|_p\ (Rt)^{3/p'}\ , \tag{2.3}$$

and, for $t \geq 1$,

$$\begin{aligned}
\int_{|x| \leq Rt} \rho^0(t) dx \quad &= \quad \int_{|X+Vt| \leq Rt} f^0(X,V) dX\ dV \\
&\geq \quad \int_{|X+Vt| \leq Rt\ ;\ |X| \leq R} f^0(X,V) dX\ dV \\
&\geq \quad \int_{|V| \leq R/2\ ;\ |X| \leq R} f^0(X,V) dX\ dV\ = const. > 0\ ,
\end{aligned} \tag{2.4}$$

for R sufficiently large. Estimates (2.3) and (2.4) give the result.

Now we prove the point (2). To do this, we argue as in [12] and show in fact that, for R sufficiently large,

$$\liminf_{t\to\infty} \int_{|x|\le Rt} \rho(t,x)dx > 0 \ . \tag{2.5}$$

Combining (2.5) with (2.3) gives the result. Now suppose (2.5) is false. In this case we find an increasing sequence $t_k \to \infty$ such that $\int_{|x|\le Rt_k} \rho(t_k,x)dx \to 0$. Thus,

$$\begin{aligned} \int_{|x|\le Rt_k} v^2 f(t_k,x,v)dxdv &\le 2\int_{|x|\le Rt_k} (v-\frac{x}{t_k})^2 f(t_k,x,v) + 2\int_{|x|\le Rt_k} \frac{x^2}{t_k^2} f(t_k,x,v) \\ &\le \frac{C}{1+t_k} + 2R^2 \int_{|x|\le Rt_k} \rho(t_k,x)dx \ , \end{aligned} \tag{2.6}$$

where $C = C(\|f^0\|_{L^1\bigcap L^\infty}, \|x^2 f^0\|_{L^1})$. This last inequality is a consequence of the conservation law derived in [19], [14] which reads,

$$\partial_t \Big[\int_{x,v} |x-vt|^2 f(t,x,v)dx\ dv \ + t^2 \int_x |E(t,x)|^2 dx \Big] = +t \int_x |E(t,x)|^2 dx \ \le C \ ,$$

and implies that,

$$\int_{x,v} |x-vt|^2 f(t,x,v)dxdv \le C(1+t) \ . \tag{2.7}$$

Hence we get in (2.6),

$$\int_{|x|\le Rt_k} v^2 f(t_k,x,v)dxdv \to 0 \text{ as } k\to\infty \ . \tag{2.8}$$

¿From this we deduce,

$$\lim_{k\to\infty} \int_{|x|\ge Rt_k} v^2 f(t_k,x,v)dxdv \ = \ \lim_{k\to\infty} \int_{x,v} v^2 f(t_k,x,v)dxdv$$

$$= \ \lim_{k\to\infty} \int_{x,v} v^2 f^0 dxdv + \int_x E^2(t=0)dx - \int_x E^2(t_k)dx \ ,$$

thanks to the conservation of energy. But (2.7) and the conservation law stated above imply,

$$\int_x E^2(t_k,x)dx \le C\ t_k^{-1} \ ,$$

Thus,

$$\int_{|x|\ge Rt_k} v^2 f(t_k,x,v)dxdv \to_{k\to\infty} \int_{x,v} v^2 f^0 dxdv + \int_x E^2(t=0)dx \ . \tag{2.9}$$

We now observe that,

$$\int_{|x|\geq Rt_k} \rho(t_k,x)dx \to_{k\to\infty} \int_{x,v} f^0 dxdv \ , \tag{2.10}$$

and (2.7) implies,

$$\left(\int_{|x|\geq Rt_k} v^2 f(t_k,x,v)\ dxdv\right)^{1/2} \geq$$

$$\begin{aligned}
&\geq \ \left(\int_{|x|\geq Rt_k} \frac{x^2}{t_k^2} f(t_k,x,v)\ dxdv\right)^{1/2} - \left(\int_{|x|\geq Rt_k} \left(\frac{x}{t_k}-v\right)^2 f(t_k,x,v)\ dxdv\right)^{1/2} \\
&\geq \ R\left(\int_{|x|\geq Rt_k} f(t_k,x,v)\ dxdv\right)^{1/2} - \frac{C}{t_k^{1/2}} \ .
\end{aligned}$$

We pass to the limit in the above inequality and get, thanks to (2.9), (2.10),

$$\left(\int_{x,v} v^2 f^0(x,v)dxdv + \int_x E^2(t=0)dx\right)^{1/2} \geq R\ \|f^0\|_1^{1/2} \ , \tag{2.11}$$

which is a contradiction if R is large enough. This ends the proof of this Lemma. ■

We now want to derive some a priori estimates on the force field in the VPS from Lemma 2.1. We first need the following

Definition 2.2 *(1) Let $a \geq 6/5$. We define*

$$\begin{aligned}
p_{max}(a) &:= \frac{3a}{3-a} \quad (= \infty \ \textit{if}\ a \geq 3\ )\ , \\
p_{min}(a) &:= [p_{max}(a)]' = \frac{3a}{4a-3} \quad (= 1\ \textit{if}\ a \geq 3\ )\ , \\
I(a) &:= \ ]3/2;3[\bigcap [p_{min}(a);p_{max}(a)]\ .
\end{aligned}$$

(2) Let $m \geq 3/5$. We define

$$\begin{aligned}
q_{max}(m) &:= \frac{9+3m}{6-m} \quad (= \infty \ \textit{if}\ m \geq 6\ )\ , \\
q_{min}(m) &:= [q_{max}(m)]' = \frac{9+3m}{4m+3} \quad (= 1\ \textit{if}\ m \geq 6\ )\ , \\
J(m) &:= \ ]3/2;3[\bigcap [q_{min}(m);q_{max}(m)]\ .
\end{aligned}$$

Remark 2.3 *Notice that*

$$\begin{aligned}
&I(a) = \{2\} && \textit{iff} \quad a = 6/5\ , \\
&I(a) =]3/2;3[&& \textit{iff} \quad a \geq 3/2\ , \\
&J(m) = \{2\} && \textit{iff} \quad m = 3/5\ , \\
&J(m) =]3/2;3[&& \textit{iff} \quad m \geq 3/2\ .
\end{aligned}$$

Lemma 2.1 above allows us now to state the following

Theorem 2.1 *Let $f^0(x,v) \in L^1 \bigcap L^\infty$, $f(t,x,v)$ be a strong solution of the VPS (See [19] for the precise definition) with initial data f^0, and $E(t,x)$ be the corresponding force field.*

(1) Assume, in addition, $f^0 \in L^1_v(L^a_x)$ for some $a \geq 6/5$. Then, we have,

$$\|E(t)\|_p \leq C(T, \|f^0\|_{L^1 \bigcap L^\infty}, \|f^0\|_{L^1_v(L^a_x)}) \ ,$$

for all $0 \leq |t| \leq T$ and all $p \in I(a)$.

(2) Assume, in addition, $N_m(0) < \infty$ for some $m \geq 3/5$. Then, we have,

$$\|E(t)\|_p \leq C(T, \|f^0\|_{L^1 \bigcap L^\infty}, N_m(0)) \ ,$$

for all $0 \leq |t| \leq T$ and all $p \in J(m)$.

(3) Assume, in addition, $f^0 \in L^1_x(L^a_v)$ for some $a \geq 6/5$. Then, we have,

$$\|E(t)\|_p \leq t^{-(2-\frac{3}{p})} \, C(T, \|f^0\|_{L^1 \bigcap L^\infty}, \|f^0\|_{L^1_x(L^a_v)}) \ ,$$

for all $0 \leq |t| \leq T$ and all $p \in I(a)$.

(4) Assume, in addition, $M_m(0) < \infty$ for some $m \geq 3/5$. Then, we have,

$$\|E(t)\|_p \leq t^{-(2-\frac{3}{p})} \, C(T, \|f^0\|_{L^1 \bigcap L^\infty}, M_m(0)) \ ,$$

for all $0 \leq |t| \leq T$ and all $p \in J(m)$.

Remark 2.4 *The results of Theorem 2.1 give a priori estimates in the VPS as the initial data has infinite initial kinetic energy. This generalises the corresponding results obtained in [19] which only considers the case $M_2(0) < \infty$. Following the remark 2.2 above, we notice that $\|E(t)\|_p$ blows up at most like $t^{-(2-\frac{3}{p})}$ at $t = 0$ (cases (3) and (4)), or it is locally bounded (cases (1) and (2)).*

Remark 2.5 *A natural question is : do the estimates of Theorem 2.1 give stability results (and, in particular, existence results) for the VPS with initial data f^0 in the above spaces ?*

Unfortunately, the answer is negative : Let f^0_n be a sequence of initial data converging strongly to f^0 in the desired spaces, and $f^n(t,x,v)$ be the corresponding sequence of solutions to the VPS (say $f^0_n \in C^\infty_c$, and in this case $f^n(t,x,v) \in C^\infty(\mathbb{R}_t; C^\infty_c(\mathbb{R}^3_x \times \mathbb{R}^3_v))$). The compactness of the velocity averaging as stated in [9], [10], [8] allow to show the local strong convergence of

$\int_{|v|\le R} f^n(t,x,v)dv$ to $\int_{|v|\le R} f(t,x,v)dv$ for any $0<R<\infty$, and thus we can pass to the limit in the sense of distributions in

$$\partial_t f^n + v\cdot\nabla_x f^n + E^n\cdot\nabla_v f^n = 0\ ,$$

recalling that f^n converges weakly towards some f.

But we do not know if

$$\sup_n \int_{|v|\ge R} f^n dv \longrightarrow_{R\to\infty} 0\ ,$$

(for example in the L^1_x norm) under the only assumptions of Theorem 2.1. We can only prove it when $\|E^n(t)\|_p$ is bounded for some $p>3$. Thus, it is not clear whether

$$\rho^n(t,x) \longrightarrow \rho(t,x) = \int_v f(t,x,v)dv \quad in\ \mathcal{D}'\ ,$$

a loss of mass could occur and we cannot pass to the limit in Poisson's equation.

Proof of Theorem 2.1 . Following [16] and [19], we split the force field into two parts,

$$E(t,x) = E^0(t,x) + E^1(t,x)\ ,$$

where,

$$\begin{cases} E^0(t,x) := \dfrac{x}{4\pi|x|^3} *_x \rho^0(t,x) = \dfrac{x}{4\pi|x|^3} *_x \displaystyle\int_v f^0(x-vt,v) \\ E^1(t,x) := \displaystyle\int_0^t s \int_v (E\ f)(t-s,x-vs,v)\ dv\ ds\ . \end{cases}$$

As it will be clear below, $E^0(t)$ represents the short-time potential, and we have in some sense $E(t)\approx E^0(t)$ as $t\to 0$.

We now estimate each term E^0 and E^1 separately.

First step. We first consider E^0. Under the assumption of the point (1) in Theorem 2.1, we write, thanks to the Riesz-Sobolev inequality, combined with Lemma 2.1 (1),

$$\begin{aligned} \|E^0(t)\|_p &\le C\ \|\rho^0(t)\|_{\frac{3p}{3+p}}\ , \quad \text{for } 3/2<p<\infty \\ &\le C\ \|f^0\|_{L^1_v(L^a_x)}\ , \end{aligned}$$

and this last inequality holds as soon as $3p/(3+p)\le a$, that is $p\le p_{max}(a)$. Here, C denotes a constant as in Theorem 2.1 (1) (in fact, it is independant of t, T).

The case (2) is treated similarly, as well as (3) and (4), where one obtains an additional factor $t^{-(2-\frac{3}{p})}$.

Second step. We now consider E^1. In order to treat this term, we state an easy and fundamental Lemma, that will be used throughout this paper, and which we borrow from [16].

Lemma 2.4 *Let $p/a' \geq 1$ and $E(t,x)$, $E^1(t,x)$ as above. Then the following holds,*

$$\|E^1(t)\|_p \leq \int_0^t s^{1-\frac{3}{a}}\ \|E(t-s)\|_a\ \|f^0\|_\infty^{\frac{a'-1}{a'}}\ \|f(t-s,x-vs,v)\|^{1/a'}_{L^{p/a'}_x(L^1_v)}\ ds\ .$$

Proof of Lemma 2.4 . We write,

$$\| \int_0^t s \int_v (E\ f)(t-s,x-vs,v)\ dvds\|_{L^p_x} \leq$$

$$\leq \int_0^t s^{1-\frac{3}{a}}\ \|E(t-s)\|_a\ \|\Big(\int_v f^{a'}(t-s,x-vs,v)\ dv\Big)^{1/a'}\ \|_{L^p_x}\ ds\ ,$$

which gives Lemma 2.4, using the fact that $\|f(t)\|_\infty \leq \|f^0\|_\infty$, for all t. ∎

We now come back to the proof of Theorem 2.1. In each case (1)-(4), we apply Lemma 2.4, choosing $p = a'$, (that is $p' = a$), which allows to "cancel" the factor $\|f(t-s,x-vs,v)\|^{1/a'}_{L^{p/a'}_x(L^1_v)}$ in this Lemma, thanks to the conservation of the L^1-norm.

Let us begin with the case (1). Using a constant C which depends on f^0 as mentionned in Theorem 2.1, we get,

$$\begin{aligned}\|E^1(t)\|_p\ &\leq\ C\ \int_0^t s^{1-\frac{3}{p'}}\ \Big(\|E^0(t-s)\|_{p'} + \|E^1(t-s)\|_{p'}\Big)\ ds\\ &\leq\ C\ \int_0^t s^{1-\frac{3}{p'}}\ ds + C\ \int_0^t s^{1-\frac{3}{p'}}\ \|E^1(t-s)\|_{p'}\ ds\ , \qquad (2.12)\end{aligned}$$

because the term $\|E^0(t)\|_{p'}$ in (2.12) can be upper bounded thanks to the first step, which requires the restrictions $3/2 < p' \leq p_{max}(a)$. Moreover, since $3/2 < p' < 3$, we can choose a real $q > 1$ such that $s^{1-\frac{3}{p'}} \in L^q_{loc}$. Hence,

$$\|E_1(t)\|_p\ \leq\ C\ \Big(1 + [\int_0^t \|E^1(s)\|^{q'}_{p'}]^{1/q'}\Big)\ . \qquad (2.13)$$

We now come back to Lemma 2.4, we change p into p' and take $a = p$, and we obtain,

$$\begin{aligned}\|E^1(t)\|_{p'}\ &\leq\ C\ \int_0^t s^{1-\frac{3}{p}}\ \Big(\|E^0(t-s)\|_p + \|E^1(t-s)\|_p\Big)\ ds\\ &\leq\ C + C\ \int_0^t s^{1-\frac{3}{p}}\ \|E^1(t-s)\|_p\ ds\ , \qquad (2.14)\end{aligned}$$

thanks to the first step, combined again with the restrictions $3/2 < p \leq p_{max}(a)$, $p < 3$. Hence,

$$\|E^1(t)\|_{p'}\ \leq\ C\ \Big(1 + [\int_0^t \|E^1(s)\|^{q'}_p]^{1/q'}\Big)\ , \qquad (2.15)$$

for the same choice of q as in (2.13). As in [19], estimates (2.13) together with (2.15) give, thanks to Gronwall's Lemma, $\|E^1(t)\|_p \in L^\infty_{loc}(\mathbb{R}_t)$. Now, the point (1) of Theorem 2.1 is proved.

The second point can be obviously treated in exactly the same way. Now, the proof of the last two assertions follows the same two steps. In these cases, the first step gives,

$$\|E^0(t-s)\|_p \leq C\ |t-s|^{-(2-\frac{3}{p})}\ , \qquad \forall\ 3/2 < p \leq p_{max}(a) \text{ or } q_{max}(m)\ .$$

Hence,

$$\int_0^t s^{1-\frac{3}{p}}\ \|E^0(t-s)\|_p\ ds \in L^\infty_{loc}(\mathbb{R}_t)\ , \quad \forall\ p \in I(a) \text{ or } J(m)\ ,$$

which allows to adapt easily the second step above. Now, the Theorem 2.1 is proved. ■

3 Propagation of high space-moments for small time intervals.

With the regularity of the force field obtained in the previous section (Theorem 2.1), a natural question is now : is it possible to propagate the initial moment of f^0 through time evolution ? In the case where $|v|^m f^0 \in L^1$ (case (2) in Theorem 2.1), it has been shown in [16] that solutions to the VPS can be built which satisfy $|v|^k f(t) \in L^\infty_{loc}(\mathbb{R}_t; L^1_{x,v})$ for all $0 \leq k < m$ (propagation of the velocity moments). Their proof needed the additional assumption $m > 3$. Our goal is now to treat the case of moments in the space variable, (case (4) in Theorem 2.1), and to build solutions to the VPS such that $|x - vt|^m f(t) \in L^\infty_{loc}(\mathbb{R}_t; L^1_{x,v})$ (See Introduction). The corresponding question in cases (1) and (3) above will not be treated here (moments of L^p-type in the space or velocity variable).

In this section as well as in the next one, we make the following assumptions on the initial data, and use the following conventions :

(H1) $f^0 \in L^1 \bigcap L^\infty$, $\qquad (|x|^m + |v|^\varepsilon)\ f^0 \in L^1$ for some $m > 3$ and $\varepsilon > 0$.

(H2) We will always work on a bounded time interval $|t| \leq T$, where $T > 0$ is fixed. In fact, as the VPS is time reversible, we even restrict ourselves to the (fixed) time interval $[0; T]$.

(H3) We often omit the explicit dependence of the various constants with respect to the parameters of the problem. Unless explicitly mentioned, any constant C depends *a priori* on T and on the norms of the initial data appearing in (H1) (that is $\|f^0\|_{L^1 \bigcap L^\infty}$, $\||x|^m f^0\|_{L^1}$, $\||v|^\varepsilon f^0\|_{L^1}$).

Moreover, the whole calculations below should first be written on very smooth functions, and we always deal with C^∞ and compactly supported functions (See Remark 2.5). We will skip the corresponding limiting argument as it is obvious here : indeed, we show below a uniform bound of the type $|v|^\varepsilon f(t,x,v) \in L^\infty_{loc}(\mathbb{R}_t; L^1_{x,v})$ (Lemma 3.1), and this prevents a 'loss of mass' as mentioned in the Remark 2.5.

(H4) We use the following notation for the behaviour of a function $g(t)$ near $t=0$,

$$|g(t)| \le C\, t^{-\alpha+0} \;\Leftrightarrow\; \exists\, C\ ,\ \exists\, \beta < \alpha\ ,\ s.t.\ \forall\, |t| \le T\ ,\ |g(t)| \le C\, |t|^{-\beta}\ ,$$

where C depends *a priori* on f^0 and T (See (H3)).

The main result of this section is the following

Theorem 3.1 *Let $3 < k < m$. Then there exists a time $t_k > 0$ such that*

$$M_k(t) \in L^\infty([0;t_k])\ .$$

Remark 3.1 *More precisely, we should write, $\|M_k(t)\|_{L^\infty([0;t_k])} \le C$ where C and t_k depend on f^0, T as in (H3).*

This Theorem is proved at the end of this section.

Using the PDE, one can write the following (formal) calculation,

$$\begin{aligned} \partial_t M_k(t) &\le k\, t \int_x |E(t,x)|\, M_{k-1}(t,x)dx \\ &\le k\, t\, \|E(t)\|_p\, \|M_{k-1}(t,x)\|_{p'}\ , \end{aligned} \tag{3.1}$$

and, thanks to Lemma 2.2, we get (See Theorem 2.1 and its proof for the notations),

$$\partial_t M_k(t) \;\le\; C\, t^{1-\frac{3}{p}} \left(\|E^0(t)\|_p + \|E^1(t)\|_p\right) M_k(t)^{1-\frac{1}{k}+\frac{3}{kp}}\ . \tag{3.2}$$

At this point, we come back to the proof of Theorem 2.1 (4) and observe that it gives in fact the following refined estimates for $3/2 < p < 3$,

$$\begin{cases} t^{1-\frac{3}{p}}\, \|E^1(t)\|_p \le C\, t^{-1+0}\ , \qquad (\le C\, t^{-1+(2-\frac{3}{p})}\ )\ , \\ t^{1-\frac{3}{p}}\, \|E^0(t)\|_p \le C\, t^{-1}\ , \end{cases}$$

Thus, for any possible choice of k and p, a factor t^{-1} appears in (3.2), which prevents any direct use of Gronwall's Lemma. Therefore, our very first aim in this section will be to replace the worse term, $t^{1-\frac{3}{p}}\ \|E^0(t)\|_p$, in (3.2) by t^{-1+0}. This is possible through the assumption $|v|^\varepsilon f^0 \in L^1$ in (H1). The propagation of the space moments $\int_{x,v} |x|^k\ f^0(x,v)dxdv = M_k(0)$ on small time intervals will be deduced from this first step.

We begin with the

Lemma 3.1 *Let f^0 satisfy (H1). Then, for $\alpha > 0$ with α/ε small enough, we have :*

(1) $\|E^0(t)\|_{3+\alpha} \in L^1_{loc}(\mathbb{R}_t)$,

(2) $N_\alpha(t) := \int_{x,v} |v|^\alpha \, f(t,x,v) \, dx \, dv \ \in L^\infty_{loc}(\mathbb{R}_t)$.

Proof of Lemma 3.1 . It is proved in [19] that (1) implies (2) (See also Section 6), so that we only prove (1). We have,

$$\|E^0(t)\|_{3+\alpha} \le C \, \|\rho^0(t)\|_{\frac{3(3+\alpha)}{6+\alpha}} ,$$

and we show that this last term belongs to $L^1_{loc}(\mathbb{R}_t)$, thanks to an interpolation of $L_x^{\frac{3(3+\alpha)}{6+\alpha}}$ between $L^{1+\alpha}_x$ and $L^{5/3}_x$. Indeed, the assumption $|v|^\varepsilon f^0 \in L^1_{x,v}$ allows to show that $\|\rho^0(t)\|_{1+\alpha}$ is bounded (in time) for α small enough :

$$\|\rho^0(t)\|_{\frac{3(3+\alpha)}{6+\alpha}} \ \le \ \|\rho^0(t)\|^\theta_{5/3} \, \|\rho^0(t)\|^{1-\theta}_{1+\alpha} , \tag{3.3}$$

where,

$$1+\alpha < \frac{9+3\alpha}{6+\alpha} < 5/3 , \qquad \frac{\theta}{5/3} + \frac{1-\theta}{1+\alpha} = \frac{6+\alpha}{9+3\alpha} .$$

The estimate (3.3) implies, *via* Lemma 2.1,

$$\|\rho^0(t)\|_{\frac{3(3+\alpha)}{6+\alpha}} \le C \, t^{-\frac{6\theta}{5}} \, M_2(0)^{\frac{3\theta}{5}} \, N_{3\alpha}(0)^{\frac{1-\theta}{1+\alpha}} ,$$

and we now use the fact that $N_{3\alpha}(0) = \int_{x,v} |v|^{3\alpha} \, f^0(x,v) dx dv < \infty$ for $\alpha \le \varepsilon/3$.

Moreover, $M_2(0) = \int_{x,v} |x|^2 \, f^0(x,v) dx dv < \infty$ and, $6\theta/5 < 1$.

Indeed, this last inequality is given by,

$$\theta = \frac{\frac{1}{1+\alpha} - \frac{6+\alpha}{9+3\alpha}}{\frac{1}{1+\alpha} - \frac{1}{5/3}} = 5 \, \frac{3 - 4\alpha - \alpha^2}{18 - 21\alpha - 9\alpha^2} < 5/6 ,$$

for all $0 < \alpha < 1$.

This gives,

$$\|\rho^0(t)\|_{\frac{3(3+\alpha)}{6+\alpha}} \le C \, t^{-1+0} .$$

Hence Lemma 3.1 is proved. ∎

We deduce now from the previous result the

Lemma 3.2 *Let $0 \le k < m$. Let $k' \le m$ satisfy*

$$\frac{3(k+3)}{6+k} < \frac{3+k'}{3}$$

(in particular, $k' = k$ is allowed for $3 < k < m$). Then, there exists an exponent $\beta > 0$ such that,

$$\|E(s)\|_{3+k} \leq C\ s^{-(2-\frac{3}{3+k})+0}\ M_{k'}(s)^{\beta}\ .$$

Proof of Lemma 3.2 . Let $p := 3 + k$. Choose $k' \leq m$ such that,

$$\frac{3(3+k)}{6+k} = \frac{3p}{3+p} < \frac{3+k'}{3}\ .$$

Now, observe that the quantity $3p/(3+p)$ increases with p, and choose a $\gamma > 0$ satisfying,

$$\frac{3(p+\gamma)}{3+(p+\gamma)} \leq \frac{3+k'}{3}\ .$$

We write then, following the same idea as in the proof of Lemma 3.1,

$$\begin{aligned} \|E(s)\|_p &\leq C\ \|\rho(s)\|_{\frac{3p}{3+p}} \\ &\leq C\ \|\rho(s)\|^{\theta}_{\frac{3(p+\gamma)}{3+(p+\gamma)}}\ \|\rho(s)\|^{1-\theta}_{\frac{3+\alpha}{3}}\ , \end{aligned}$$

where $\alpha > 0$ is a small number (See Lemma 3.1). Now we let $q := \frac{3(p+\gamma)}{3+(p+\gamma)}$, and obtain, applying Lemma 2.1,

$$\|E(s)\|_p \leq C\ s^{-\frac{3\theta}{q'}}\ M_{k'}(s)^{\frac{3\theta}{k'q'}}\ N_{\alpha}(s)^{\frac{3(1-\theta)}{3+\alpha}}\ , \tag{3.4}$$

where,

$$\frac{3+\alpha}{3} < \frac{3p}{3+p} < \frac{3(p+\gamma)}{3+(p+\gamma)} = q\ , \qquad \frac{1-\theta}{(3+\alpha)/3} + \frac{\theta}{q} = \frac{3+p}{3p}\ ,$$

$$q \leq \frac{3+k'}{3}\ .$$

But we check that,

$$\frac{3\theta}{q'} < 2 - \frac{3}{p}\ ,$$

because,

$$\frac{3\theta}{q'} = 3\ [\frac{2p-3}{3p} - \frac{(1-\theta)\alpha}{3+\alpha}] < 2 - \frac{3}{p}\ .$$

We now observe that Lemma 3.1 implies $N_{\alpha}(s) \in L^{\infty}_{loc}$ in time, so that (3.4) gives,

$$\|E(s)\|_p \leq C\ s^{-(2-\frac{3}{p})+0}\ M_{k'}(s)^{\beta}\ ,$$

with $\beta := \frac{3\theta}{k'q'}$. This ends the proof of Lemma 3.2. ■

Proof of Theorem 3.1 . We make the following change of notation,

$$\tilde{M}_k(t) := \sup_{s\in[0;t]} M_k(s)\ ,$$

and let $k \geq 1$, $p = 3 + k$. This allows to write,

$$\begin{aligned}
\partial_t \tilde{M}_k(t) &\leq \partial_t M_k(t) \\
&\leq k\, t \int_{x,v} |E(t)|\, f(t)\, |x - vt|^{k-1} dx dv \\
&\leq C\, t\, \|E(t)\|_p\, \|\tilde{M}_{k-1}(t,x)\|_{p'} \\
&\leq t\, C\, t^{-(2-\frac{3}{p})+0}\, \tilde{M}_{k'}(t)^{\beta}\, t^{-\frac{3}{p}}\, \tilde{M}_k(t)^{\frac{k+2}{k+3}} ,
\end{aligned}$$

and the last inequality is a consequence of Lemmas 2.2 and 3.2. We have chosen here k' as in Lemma 3.2. Thus, choosing now $k' = k$,

$$\partial_t \tilde{M}_k(t) \leq C\, t^{-1+0}\, \tilde{M}_{k'}(t)^{\beta}\, \tilde{M}_k(t)^{\frac{k+2}{k+3}} \tag{3.5}$$

$$\partial_t \tilde{M}_k(t) \leq C\, t^{-1+0}\, \tilde{M}_k(t)^{\delta} , \tag{3.6}$$

for some exponent $\delta > 0$. We conclude thanks to Gronwall's Lemma. ■

We have now proved that the moments $M_k(t)$ can be propagated through time evolution in VPS, for small time intervals. We now want to propagate these quantities for arbitrary large times, and in fact the previous method does clearly not apply, since we do not control the exponent δ in the Gronwall's-like inequality (3.6). Another series of estimates is needed, which are proved in the next section. As in [16], this work is much more delicate.

4 Propagation of high space-moments for large time intervals.

We still use the assumptions and notations (H1)-(H4), as in the previous section.

We fix some k such that $3 < k < m$, and we want to find a solution to the VPS satisfying $M_k(t) \in L^\infty([t_k; T])$. Here, $t_k > 0$ represents the (small) time interval such that $M_k(t) \in L^\infty([0; t_k])$ as in Theorem 3.1.

As a last notation, we assume throughout this section

(H5) $0 < t_0 \leq t_k/2$ is a fixed time, whose value will be chosen later.

The main result of this section is the

Theorem 4.1 *Assume $E(t,x) \in L^\infty_{loc}(\mathbb{R}_t; L^{3/2}_x)$. Then, for all $3 < k < m$, we have*

$$M_k(t) \in L^\infty_{loc}(\mathbb{R}_t) .$$

Remark 4.1 *We show in section 5 how to relax the assumption $E \in L^\infty_{loc}(\mathbb{R}_t; L^{3/2}_x)$, which does not hold in general (only the weaker space $L^{3/2,\infty}_x$ can be obtained).*

The proof of this Theorem is given at the end of this section. We first state the following fundamental estimate,

Lemma 4.1 *Let $3 < k < m$. Then, for all $t_k \le t \le T$, we have,*

$$\| \int_0^{t_0} s \int_v (E\ f)(t-s, x-vs, v)\ dv\ ds \|_{3+k} \le C\ t_0^{\gamma}\ (1 + M_k(t))^{\beta}\ ,$$

where γ, $\beta > 0$ are some exponents (whose value depends on k), and C depends on f^0, T (See H3), and k.

Remark 4.2 *Here and in the sequel, we will often omit the distinction between $M_k(t)$ and $\tilde{M}_k(t)$ (See the proof of Theorem 3.1).*

Proof of Lemma 4.1 . Let $q := 3 + k$, and,

$$A(t) := \| \int_0^{t_0} s \int_v (E\ f)(t-s, x-vs, v) dv ds \|_{3+k}\ .$$

A first application of Lemmas 2.1 and 2.2 gives,

$$A(t) \quad \le \quad \int_0^{t_0} s^{(1-\frac{3}{p})}\ \|E(t-s)\|_p\ t^{-\frac{3(q-p')}{qp'}}\ M_{k'}(t-s)^{\frac{3(q-p')}{k'qp'}}\ ds\ , \qquad (4.1)$$

whenever k' and p, which will be chosen later, satisfy,

$$1 \le \frac{q}{p'} \le \frac{3+k'}{3}\ , \qquad p \in]3/2; 3[\ .$$

Now we observe that Theorem 2.1 (4) gives,

$$\|E(t)\|_p \in L^{\infty}([\frac{t_k}{2}; T])\ , \qquad \forall\ 3/2 < p < 3\ .$$

Hence, we get in (4.1),

$$A(t) \quad \le \quad C\ t_0^{2-\frac{3}{p}}\ M_{k'}(t)^{\frac{3(q-p')}{k'qp'}}\ , \qquad (4.2)$$

where,

$$C = t_k^{-\frac{3(q-p')}{qp'}}\ \sup\{\|E(t-s)\|_p\ /\ s \in [0; t_0]\ ,\ t \in [t_k; T]\ \}\ < \infty\ ,$$

because $t - s \ge t_k/2 > 0$. Thus, (4.2) gives,

$$A(t) \quad \le \quad C\ t_0^{\gamma}\ M_{k'}(t)^{\beta}\ , \qquad (4.3)$$

for some exponents γ, $\beta > 0$. We write, as in the proof of Theorem 3.1,

$$\partial_t M_{k'}(t) \quad \le \quad C\ t^{1-\frac{3}{3+k'}}\ \|E(t)\|_{3+k'}\ M_{k'}(t)^{\frac{k'+2}{k'+3}}\ . \qquad (4.4)$$

We now integrate (4.4) over the time interval $[t_{k'};t]$, (we avoid the time $t=0$) where $t_{k'}$ is such that $M_{k'}(s) \in L^\infty_{loc}([0;t_{k'}])$ (See Theorem 3.1). In fact, we can even assume $t_{k'} = t_k$. Hence,

$$M_{k'}(t)^{\frac{1}{3+k'}} \leq C\ M_{k'}(t_k)^{\frac{1}{3+k'}}\ +\ C \sup_{s\in[t_k;t]} (\|E(s)\|_{3+k'})\ ,$$

and,

$$M_{k'}(t) \quad\leq\quad C\ (1 + \sup_{s\in[t_k;t]} (\|E(s)\|_{3+k'})\ )^{3+k'}\ . \tag{4.5}$$

Now (4.5) together with (4.3) give,

$$A(t) \quad\leq\quad C\ t_0^\gamma\ (1 + \sup_{s\in[t_k;t]} (\|E(s)\|_{3+k'})\ )^{\tilde\beta}\ , \tag{4.6}$$

where $\tilde\beta > 0$ is another exponent related to β. We now use Lemma 2.1 (4) in order to majorise $\|E(s)\|_{3+k'}$ on the time interval $[t_k;t]$, and obtain,

$$\begin{aligned} \|E(s)\|_{3+k'} &\leq \|\rho(s)\|_{\frac{3(3+k')}{6+k'}} \\ &\leq C\ M_k(s)^{\frac{1}{k}\ (2-\frac{3}{3+k'})}\ . \end{aligned} \tag{4.7}$$

This holds for $s \in [t_k;t]$ (away from $s=0$), and k is the exponent of Lemma 4.1. Indeed, the coefficients p, k' and k have to satisfy $(9+3k')/(6+k') \leq (3+k)/3$, $p \in]3/2;3[$, and we recall that $(3+k)/p' < (3+k')/3$ (See (4.1)) . Thus, choosing p' close enough to 3, we obtain a k' arbitrary close to k (but $> k$), whereas the assumption $k > 3$ implies $(9+3k)/(6+k) < (3+k)/3$. With such a choice of k' and p', we obtain (4.7) above.

Now (4.6) and (4.7) give,

$$A(t) \leq C\ t_0^\gamma\ (1 + M_k(t)^\beta)\ ,$$

for some exponents γ and $\beta > 0$. This ends the proof of Lemma 4.1. ■

We are now able, thanks to Lemma 4.1, to prove Theorem 4.1.

Proof of Theorem 4.1 . In fact, the major difficulty for our purpose was to obtain the local-in-time propagation of the moments $M_k(t)$, since undesirable factors t^{-1} were obtained in a first approach (See Theorem 3.1). The way from this local property towards the global-in-time propagation follows now the same ideas as in [16]. Indeed, let $3 < k < m$. We write,

$$\begin{aligned} \|E(t)\|_{3+k} &\leq \|E^0(t)\|_{3+k} + \|E^1(t)\|_{3+k} \\ &\leq \|E^0(t)\|_{3+k} + \|\int_0^{t_0} s \int_v (E\ f)(t-s, x-vs, v)\|_{3+k} + \|\int_{t_0}^t \cdots \|_{3+k} \qquad (4.8) \\ &:= A(t) + B(t) + D(t)\ , \qquad (4.9) \end{aligned}$$

and we upper bound each term of this sum.

First, Lemma 2.1 gives,

$$\begin{aligned} A(t) &\leq C\ \|\rho^0(t)\|_{\frac{3(3+k)}{6+k}} \\ &\leq C\ M_m(0)^{\frac{3+2k}{m(3+k)}} , \end{aligned}$$

for all $k \leq m$ and $t \in [t_k; T]$.

Secondly, we obtain through Lemma 4.1,

$$B(t) \leq C\ t_0^{\gamma}\ (1 + M_k(t)^{\beta}) ,$$

for all $3 < k < m$ and $t \in [t_k; T]$.

Finally, we estimate the term $D(t)$. In order to do so, we apply Lemma 2.4 with the choice $a = 3/2$ ($a' = 3$), $p = 3 + k$, and then estimate the quantity $\|f(t-s, x-vs, v)\|_{L_x^{p/a'}(L_v^1)}$ thanks to Lemma 2.1 . We get,

$$\begin{aligned} D(t) &\leq \int_{t_0}^{t} s^{-1}\ \|E(t-s)\|_{3/2}\ t^{-\frac{3(p-3)}{3p}}\ M_k(t-s)^{\frac{3(p-3)}{3kp}} ds \\ &\leq C\ |\log(t_0)|\ M_k(t)^{\frac{1}{k+3}} . \end{aligned}$$

Collecting these inequalities in (4.9), we obtain,

$$\|E(t)\|_{3+k} \leq C\ (1 + t_0^{\gamma}\ M_k(t)^{\beta}\ +\ |\log(t_0)|\ M_k(t)^{\frac{1}{k+3}}\ ) , \tag{4.10}$$

for all $t \in [t_k; T]$.

We now choose $t_0^{\gamma} := M_k(t)^{-\beta}$ in (4.10), which is only possible for large values of $M_k(t)$ (recall the restriction $t_0 \leq t_k/2$). Obviously, it is the only interesting case. We obtain therefore,

$$\|E(t)\|_{3+k} \leq C\ (1 + M_k(t)^{\frac{1}{3+k}}\ \log(M_k(t))\ ) , \tag{4.11}$$

and we write as in (4.4) and (3.5)

$$\begin{aligned} \partial_t M_k(t) &\leq t^{1-\frac{3}{3+k}}\ \|E(t)\|_{3+k}\ M_k(t)^{\frac{k+2}{k+3}} \\ &\leq C\ (1 + M_k(t)^{\frac{1}{k+3}} \log(M_k(t))\ )\ M_k(t)^{\frac{k+2}{k+3}} , \end{aligned} \tag{4.12}$$

for all $t \in [t_k; T]$. Gronwall's Lemma gives the result. ■

5 Conclusion : Propagation of high space-moments.

We now prove the main theorem of this paper, and show how to relax the additional assumption $E(t,x) \in L^{\infty}_{loc}(\mathbb{R}_t; L_x^{3/2})$ made in Theorem 4.1 .

Theorem 5.1 *Let $f^0 \in L^1 \bigcap L^\infty$. Assume $|x|^m f^0 \in L^1$ for some $m > 3$, and $|v|^\varepsilon f^0 \in L^1$ for some $\varepsilon > 0$. Then, there exists a solution $f(t)$ to the VPS such that, for all $3 < k < m$ and $|t| \le T$, we have,*

$$M_k(t) = \int_{x,v} |x - vt|^k f(t,x,v) \;\le\; C(T, \|f^0\|_{L^1 \bigcap L^\infty}, \||x|^m f^0\|_{L^1}, \||v|^\varepsilon f^0\|_{L^1}) \, .$$

Remark 5.1 *As in Remark 2.2, one can also bound $E(t)$ and $\rho(t)$ in L^p spaces for the solutions given by Theorem 5.1. This is an obvious consequence of Lemma 2.1.*

Proof of Theorem 5.1 . We argue as in [16]. Let χ_R be a C^∞ function, $\chi_R \equiv 0$ for $|x| \ge R+1$, $\chi_R \equiv 1$ on $|x| \le R$ $(R > 0)$. We introduce

$$\begin{cases} F_R(t,x) := \dfrac{(1-\chi_R(x))\, x}{4\pi|x|^3} * \rho(t,x) \, , \\[2ex] E_R(t,x) := \dfrac{\chi_R(x)\, x}{4\pi|x|^3} * \rho(t,x) \, , \end{cases}$$

and we write,

$$\partial_t f + v \cdot \nabla_x f + F_R \cdot \nabla_v f \;=\; -E_R \cdot \nabla_v f \, . \tag{5.1}$$

We introduce the flow $\Phi_s^t(x,v) := (X_s^t(x,v); V_s^t(x,v))$ defined as the solution of,

$$\begin{cases} \partial_t X_s^t(x,v) = V_s^t(x,v) & ; \quad X_s^s(x,v) = x \, , \\ \partial_t V_s^t(x,v) = F_R(t, X_s^t(x,v)) & ; \quad V_s^s(x,v) = v \, . \end{cases}$$

Classically, we now rewrite (5.1) as,

$$f(t,x,v) \;=\; f^0(\Phi_t^0(x,v)) - \int_0^t (\nabla_v E_R f)(t-s, \Phi_t^{t-s}(x,v))\, ds \, . \tag{5.2}$$

On the other hand, observing that $\Phi_t^s(x,v) = (X_t^s(x,v), V_t^s(x,v))$ defines a diffeomorphism on $\mathbb{R}^6$, we have the formula,

$$\begin{aligned} [\frac{\partial}{\partial v} E_R f](t-s, \Phi_t^{t-s}(x,v)) &= (\frac{\partial}{\partial v} X_{t-s}^t)(\Phi_t^{t-s}(x,v)) \cdot \frac{\partial}{\partial x}[E_R f(t-s, \Phi_t^{t-s}(x,v))] \\ &\quad + (\frac{\partial}{\partial v} V_{t-s}^t)(\Phi_t^{t-s}(x,v)) \cdot \frac{\partial}{\partial v}[E_R f(t-s, \Phi_t^{t-s}(x,v))] \, , \end{aligned} \tag{5.3}$$

where the symbols $\frac{\partial}{\partial v}$ and $\frac{\partial}{\partial x}$ in (5.3) denote the Jacobian matrix with respect to v or x. For sake of simplicity, we introduce the notation,

$$\begin{aligned} \frac{\partial x_1}{\partial V}(x,v) &:= \frac{\partial}{\partial v}(X_{t-s}^t)(\Phi_t^{t-s}(x,v)) \, , \\ \frac{\partial v_1}{\partial V}(x,v) &:= \frac{\partial}{\partial v}(V_{t-s}^t)(\Phi_t^{t-s}(x,v)) \, . \end{aligned}$$

With this notation, we rewrite (5.3) as,

$$
\begin{aligned}
&[\frac{\partial}{\partial v}E_R f](t-s,\Phi_t^{t-s}(x,v)) = \frac{\partial}{\partial x}[\frac{\partial x_1}{\partial V}\cdot(E_R f)(t-s,\Phi_t^{t-s}(x,v))] \\
&-[\frac{\partial}{\partial x}\frac{\partial x_1}{\partial V}]\cdot(E_R f)(t-s,\Phi_t^{t-s}(x,v)) + \frac{\partial}{\partial v}[\frac{\partial v_1}{\partial V}\cdot(E_R f)(t-s,\Phi_t^{t-s}(x,v))] \\
&\qquad\qquad -[\frac{\partial}{\partial v}\frac{\partial v_1}{\partial V}]\cdot(E_R f)(t-s,\Phi_t^{t-s}(x,v)) \ . \qquad (5.4)
\end{aligned}
$$

Now we use (5.4) in (5.2), integrate the resulting formula with respect to v and perform the convolution with the kernel $x/4\pi|x|^3$. This gives,

$$
\begin{aligned}
E(t,x) \;&=\; \frac{x}{4\pi|x|^3} *_x \int_v f^0(\Phi_t^0(x,v))dv + \int_0^t\int_v (E_R\ f)(t-s,\Phi_t^{t-s}(x,v))\ \frac{\partial x_1}{\partial V}dvds \\
&\quad -\frac{x}{4\pi|x|^3} *_x \int_0^t\int_v (E_R\ f)(t-s,\Phi_t^{t-s}(x,v))\ \frac{\partial}{\partial x}\ \frac{\partial x_1}{\partial V}dvds \\
&\quad -\frac{x}{4\pi|x|^3} *_x \int_0^t\int_v (E_R\ f)(t-s,\Phi_t^{t-s}(x,v))\ \frac{\partial}{\partial v}\ \frac{\partial v_1}{\partial V}dvds \ , \qquad (5.5)
\end{aligned}
$$

that is,

$$
E(t,x) \;=\; a(t,x)+b(t,x)+c(t,x)+d(t,x) \ . \qquad (5.6)
$$

Our aim is now to reproduce the proof of Theorem 4.1 in the present case, where the flow $(x,v)\to(x-vt,v)$ has been replaced by the unknown flow $(x,v)\to(X_t^0,V_t^0)$.

It is now easy to see that the flow (X_s^t,V_s^t), which is C^∞ in (x,v), tends to $(x+v(t-s),v)$ in $C^0(|t|,|s|\le T;C^2(\mathbb{R}^6))$ as $R\to\infty$, uniformly in x, v ($T>0$ is fixed). This assumption is a direct consequence of the estimates,

$$
\|F_R\|_\infty\ ,\ \|DF_R\|_\infty\ ,\ \|D^2F_R\|_\infty \le C/R^2\ ,
$$

where C depends only on $\|f^0\|_{L^1\bigcap L^\infty}$.

Taking this remark into account allows to majorize a, b, c, and d in (5.2) for $t\in[t_k;T]$, as in the proof of Theorem 4.1.

Estimates for a. We have,

$$
\begin{aligned}
\|a(t,x)\|_{L_x^{3+k}} \;&\le\; C\ \|\int_v f^0(\Phi_t^0(x,v))dv\|_{\frac{3(3+k)}{6+k}} \\
&\le\; C\ t^{-\delta}\ \int_{x,v}|x-vt|^m f^0(\Phi_t^0(x,v))dxdv\ , \\
&\qquad \text{thanks to Lemma 2.1, where } \delta>0 \text{ is some exponent,} \\
&\le\; C\ \int_{X,V}|X_0^t(X,V)-t\ V_0^t(X,V)|^m\ f^0(X,V)dXdV\ , \\
&\qquad \text{because } t\ge t_k>0 \quad , \\
&\le\; C\ (\int_{X,V}|X|^m\ f^0(X,V)dXdV+\eta)\ ,
\end{aligned}
$$

where $\eta > 0$ is an arbitrary small number, thanks to the convergence of the force field F_R and of the flow (X_s^t, V_s^t) that we already observed.

Estimates for c and d. We write, as in Lemma 2.4 ,

$$\|c(t,x)\|_{L_x^{3+k}} \leq \int_0^t \| \int_v (E_R\ f)(t-s, \Phi_t^{t-s}(x,v))\ \frac{\partial}{\partial x}\frac{\partial x_1}{\partial V} dv \|_{\frac{3(3+k)}{6+k}} ds \tag{5.7}$$

$$\leq C \int_0^t \Big\| \ \|E_R(t-s, X_t^{t-s}(x,v))\|_{L_v^a}\ \|f(t-s, \Phi_t^{t-s}(x,v))\|_{L_v^{a'}}\ \Big\|_{\frac{3(3+k)}{6+k}}\ \|\frac{\partial}{\partial x}\frac{\partial x_1}{\partial V}\|_\infty\ ds\ .$$

We estimate each factor of the right-hand-side in (5.7),

$$\begin{aligned}\|E_R(t-s, X_t^{t-s}(x,v))\|_{L_v^a}^a &= \int_v |E_R|^a (t-s, X_t^{t-s}(x,v)) dv \\ &= \int_X |E_R|^a (t-s, X)\ |\frac{\partial X}{\partial v}|^{-1}\ dX\ ,\end{aligned} \tag{5.8}$$

where we have set $X := X_t^{t-s}(x,v)$, the x-variable being fixed here. Now $X \to x - vs$ in $C^0(|s|, |t| \leq T; C^2(\mathbb{R}^6))$ as $R \to \infty$ (See above). Thus,

$$|\partial X/\partial v|^{-1} \to s^{-3} \ \ as\ \ R \to \infty\ , \tag{5.9}$$

in $C^0(s,t;C^1)$, so that the change of variables $v \to X$ is indeed allowed for large values of R (when $s \neq 0$). We get,

$$\|E_R(t-s, X_0^s)\|_{L_v^a} \leq C\ (s(1-\eta))^{-3/a}\ \|E_R(t-s)\|_a\ ,$$

for $\eta > 0$ arbitrary small.

Furthermore, we observe that $3(3+k)/(6+k) \in]3/2; 3[$, so that one can choose $a' = 3(3+k)/(6+k)$ in (5.7), and in this way the second term of the estimate (5.7) becomes constant.

Finally, $\frac{\partial}{\partial x}\frac{\partial x_1}{\partial V} \to 0$ as $R \to \infty$, and we have the refined estimate $|\frac{\partial}{\partial x}\frac{\partial x_1}{\partial V}(x,v)| \leq C\eta|s|$, as $R \to \infty$, uniformly for $|s|$, $|t| \leq T$ and x, $v \in \mathbb{R}^3$. Collecting all these informations we obtain,

$$\|c(t,x)\|_{L_x^{3+k}} \leq \eta\ ,$$

$\eta > 0$ being an arbitrary small number. The same can be proved in a similar way for the term $\|d(t,x)\|_{L_x^{3+k}}$.

Estimates for b. We need an estimate analogous to that of Lemma 2.4, which would hold for the modified flow (X_s^t, V_s^t) and with E replaced by E_R. Once we have this estimate, we can end the proof of Theorem 5.1 by arguing as in Theorem 4.1, thanks to the regularity $E_R(t) \in L^\infty_{loc}(\mathbb{R}_t, L_x^{3/2})$. Indeed, for all $\alpha > 0$ and $t > 0$ we have,

$$\begin{aligned}\|E_R(t)\|_{L^{3/2}} &\leq C\|\rho(t)\|_{L^1 \bigcap L^{1+\alpha}} \\ &\leq CN_{3\alpha}(t) \\ &\leq C\ ,\end{aligned}$$

thanks to Lemma 3.1 . But we write, as in (5.7)-(5.9),

$$\|b(t,x)\|_{3+k} = \int_0^t \| \int_v (E_R\, f)(t-s, \Phi_t^{t-s}(x,v)) \frac{\partial x_1}{\partial V}\, dv \|_{3+k}\, ds$$

$$\leq \int_0^t \|E_R(t-s, X_t^{t-s}(x,v))\|_{L_v^a} \, \|f(t-s, \Phi_t^{t-s}(x,v))\|_{L_x^{3+k}(L_v^{a'})} \, \|\frac{\partial x_1}{\partial V}\|_\infty \, ds$$

$$\leq \; C \int_0^t s^{1-3/a} \, (1+\eta) \, \|E_R(t-s)\|_a \, \|f(\cdots)\|_\infty^{1-\theta} \; \|f(\cdots)\|^\theta_{L_x^{p/a'}(L_v^1)} \quad (5.10)$$

where $\theta = 1/a'$, $p = 3+k$, and $\eta > 0$ is small. It remains to estimate, thanks to Lemma 2.1,

$$\|f(\cdots)\|^{1/a'}_{L_x^{p/a'}(L_v^1)} \;\leq\; t^{-\delta} \left[\int_{x,v} |x-vt|^{k'} \, f(\cdots)\, dx\, dv \right]^{\frac{3(p-a')}{k'pa'}} , \quad (5.11)$$

where $1 \leq p/a' \leq (3+k')/3$, and $\delta > 0$ is some exponent. Since $t \geq t_k > 0$, this exponent is unimportant. Moreover,

$$\begin{aligned}
\int_{x,v} |x-vt|^{k'} \, f(\cdots)\, dxdv \;&=\; \int_{X,V} |X^t_{t-s} - tV^t_{t-s}|^{k'} \, f(t-s,X,V)\, dXdV \\
&\leq\; \int_{X,V} (\eta + |X - V(t-s)|)^{k'} \, f(t-s,X,V)\, dXdV \\
&\leq\; M_{k'}(t-s) + \eta \, , \qquad (5.12)
\end{aligned}$$

for $\eta > 0$ arbitrary small, thanks to the convergence of the flow Φ^t_s as $R \to \infty$. Collecting the inequalities (5.10)-(5.12), we get,

$$\|b\|_{3+k} \;\leq\; C \int_0^t [s^{1-\frac{3}{a}} \, (1+\eta) \, \|E_R(t-s)\|_a \, M_{k'}(t-s)^{\frac{3(p-a')}{k'pa'}} + \eta] \, ds \, . \quad (5.13)$$

Proof of the Theorem. Collecting the estimates on a, c and d in (5.6) gives, for $t \in [t_k, T]$,

$$\begin{aligned}
\|E(t)\|_{3+k} \;&\leq\; C + C\| \int_0^t \int_v E_R f(t-s, \Phi_t^{t-s}(x,v)) \frac{\partial x_1}{\partial V}\, dv\, ds \|_{3+k} \\
&\leq\; C + C\| \int_0^{t_0} \int_v \cdots \|_{3+k} + C\| \int_{t_0}^t \int_v \cdots \|_{3+k} \, , \qquad (5.14)
\end{aligned}$$

where t_0 is as in the proof of Theorem 4.1 . Now we use the estimate (5.13) for the two terms $\int_0^{t_0} \cdots$ and $\int_{t_0}^t \cdots$, with the choice $a > 3/2$ in the first integral and $a = 3/2$ in the second integral. This gives, as in the proof of Theorem 4.1,

$$\|E(t)\|_{3+k} \;\leq\; C + C \int_0^{t_0} s^{1-\frac{3}{a}} \, \|E_R(t-s)\|_a \, M_{k'}(t-s)^{\frac{3(p-a')}{k'pa'}} \, ds + C |\log t_0| M_k(t)^{\frac{1}{k+3}} \, . \quad (5.15)$$

Moreover, the term $\int_0^{t_0} \cdots$ in (5.15) is estimated as in Lemma 4.1 with E replaced by E_R. Thus, (5.15) gives,

$$\|E(t)\|_{3+k} \;\leq\; C + C\, t_0^{\gamma}\, (1 + M_k(t)^{\beta}) + C\, |\log t_0|\; M_k(t)^{\frac{1}{k+3}} \;,$$

(See (4.10)-(4.12)), and we conclude as in the proof of Theorem 4.1 . Our proof is now complete. ■

6 Propagation of low space and velocity moments.

As in [16], we come up against the difficulty of propagating the moments of order ≤ 3 in the VPS. Thus, at this level, only the moments in v or x of order > 3 can be propagated. We end this paper by showing how this restriction can be removed.

Theorem 6.1 *(1) Let $f^0 \in L^1 \bigcap L^\infty$. Assume $|x|^m f^0 \in L^1$ for some $m > 3$, and $|v|^p f^0 \in L^1$ for some $p > 0$.*

Then, there exists a solution $f(t,x,v)$ of the VPS such that,

$$\forall\; k \in [0; p[\;, \quad N_k(t) = \int_{x,v} |v|^k\; f(t,x,v)\; dxdv \in L^\infty_{loc}(\mathbb{R}_t)\;,$$

$$\forall\; k \in [0; m[\;, \quad M_k(t) = \int_{x,v} |x - vt|^k\; f(t,x,v)\; dxdv \in L^\infty_{loc}(\mathbb{R}_t)\;.$$

(2) Let $f^0 \in L^1 \bigcap L^\infty$. Assume $|v|^m f^0 \in L^1$ for some $m > 3$, and $|x|^p f^0 \in L^1$ for some $p > 0$.

Then, there exists a solution $f(t,x,v)$ of the VPS such that,

$$\forall\; k \in [0; p[\;, \quad M_k(t) \in L^\infty_{loc}(\mathbb{R}_t)\;,$$

$$\forall\; k \in [0; m[\;, \quad N_k(t) \in L^\infty_{loc}(\mathbb{R}_t)\;,$$

Remark 6.1 *As we can see, the existence of some moments in the x variable allows to propagate the low moments in v, and the converse holds as well.*

On the other hand notice that, as in Remark 2.2, we can bound $E(t)$ and $\rho(t)$ in some L^p spaces through Lemma 2.1.

Proof of Theorem 6.1 . We first show the point (1). The result concerning M_k has already been proved and we concentrate on N_k. We assume $p \geq 1$, and want to propagate $N_k(t)$ for $1 \leq k < p$. If $0 \leq p < 1$, the method below applies, replacing $N_k(t)$ by $\int_{x,v} (1 + |v|^2)^{k/2}\; f(t,x,v)$. Moreover, we can also assume $p \leq 3$, since the propagation of moments of order > 3 has been shown in [16]. Thus, we write as in [19],

$$\partial_t N_k(t) \ \leq \ C \ (\|E^0(t)\|_{3+k} + \|E^1(t)\|_{3+k}) \ N_k(t)^{\frac{k+2}{k+3}} \ . \tag{6.1}$$

Let us now show that $\|E^i(t)\|_{3+k} \in L^1_{loc}(\mathbb{R}_t)$ $(i = 0, 1)$. If this holds, (6.1) immediately gives the result thanks to Gronwall's Lemma. We first majorise $\|E^1(t)\|_{3+k}$.

Lemmas 2.4 and 2.1 together give, for the values $q = 3 + k$, $a' \in]3/2; 3[$, and $1 \leq q/a' \leq (3 + K)/3$ ($3 < K < m$ is now fixed),

$$\|E^1(t)\|_{3+k} \ \leq \ C \ \int_0^t s^{1-\frac{3}{a}} \ \|E(t-s)\|_a \ t^{-3\frac{q-a'}{qa'}} \ M_K(t-s)^{\frac{3(q-a')}{Kqa'}} \ ds \ . \tag{6.2}$$

Moreover, Theorem 2.1 gives,

$$\|E(t-s)\|_a \leq C \ |t-s|^{-(2-\frac{3}{a})} \ ,$$

and, thanks to the propagation of the moment $M_K(t)$, we have,

$$M_K(t) \in L^\infty_{loc}(\mathbb{R}_t) \ .$$

Therefore, we obtain in (6.2),

$$\|E^1(t)\|_{3+k} \leq C \ [\int_0^t s^{1-\frac{3}{a}} \ (t-s)^{-(2-\frac{3}{a})} \ ds] \ t^{-\frac{3(q-a')}{qa'}} \ .$$

Hence, $\|E^1(t)\|_{3+k} \in L^1_{loc}(\mathbb{R}_t)$ (choose $a' \in]3/2; 3[$ close to 3 and use $q < p \leq 3 < K$).

It remains to show that the same holds for $\|E^0(t)\|_{3+k}$. We argue as in the proof of Lemma 3.1, and we notice that the assumption $k < p \leq 3$ allows one to write,

$$\begin{aligned} \|E^0(t)\|_{3+k} \ &\leq \ C \ \|\rho^0(t)\|_{\frac{9+3k}{6+k}} \\ &\leq \ C \ \|\rho^0(t)\|^{\theta}_{\frac{3+m}{3}} \ \|\rho^0(t)\|^{1-\theta}_{\frac{3+k}{3}} \ , \end{aligned} \tag{6.3}$$

where,

$$\frac{3+k}{3} < \frac{9+3k}{6+k} < \frac{3+m}{3} \ , \qquad \frac{\theta}{(3+m)/3} + \frac{1-\theta}{(3+k)/3} = \frac{6+k}{9+3k} \ , \qquad \theta \in]0; 1[\ .$$

Now Lemma 2.1 gives in (6.3),

$$\|E^0(t)\|_{3+k} \leq C \ t^{-\frac{3m\theta}{3+m}} \ M_m(0)^{\frac{3\theta}{3+m}} \ N_k(0)^{\frac{3(1-\theta)}{3+k}} \ ,$$

and it remains to show that $\frac{3m\theta}{3+m} < 1$. But,

$$\begin{aligned}\frac{3m\theta}{3+m} &= \frac{3m}{3+m}\left[\frac{3/(3+k)\ -\ (6+k)/(9+3k)}{3/(3+k)\ -\ 3/(3+m)}\right]\\ &= m\,\frac{3-k}{3(m-k)} < 1\ \ (m>3)\ .\end{aligned}$$

This ends the proof of the first point.

The second one is very similar, and can actually be treated in a much easier way. Again we restrict ourselves to the case $1 \le p \le 3$, and we write, for $1 \le k < p$,

$$\partial_t M_k(t) \ \le\ C\, t^{1-\frac{3}{3+k}}\, M_k(t)^{\frac{k+2}{k+3}}\, (\|E^0(t)\|_{3+k} + \|E^1(t)\|_{3+k})\ . \tag{6.4}$$

Lemma 2.1 gives, thanks to $k < 3 < m$,

$$\|E^0(t)\|_{3+k} \le C\, \|\rho^0(t)\|_{\frac{3(3+k)}{6+k}} \le C\, N_m(0)\ .$$

Moreover, Lemma 2.4 and 2.1 together give, for $k < 3 < K < m$, $a \in]3/2; 3[$,

$$\|E^1(t)\|_{3+k} \ \le\ C \int_0^t s^{1-\frac{3}{a}}\, \|E(t-s)\|_a\, N_K(t-s)^{\frac{3(p-a')}{Kpa'}}\ . \tag{6.5}$$

The propagation of the moments $N_K(t)$ gives in (6.5),

$$\|E^1(t)\|_{3+k} \in L^\infty_{loc}(\mathbb{R}_t)\ ,$$

and we conclude thanks to Gronwall's Lemma in (6.5). ■

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