L^2-solutions to the Schrödinger-Poisson System : Existence, Uniqueness, Time Behaviour, and Smoothing Effects.

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Abstract We study a system of infinitely many coupled Schrödinger equations with self-consistent Coulomb potential as the initial data has only a regularity of L^2-type. We first establish Strichartz' inequalities in the framework of vector-valued wave functions (density matrices). This allows us to prove a well-posedness result, and strong smoothing effects. Also, we state blow-up (resp. decay) estimates for the solution as time goes to zero (resp. infinity).

Key-words Schrödinger-Poisson System, Strichartz' estimates, density matrices, infinite kinetic energy, asymptotic behaviour.

1 Introduction

In the present paper, we are interested in the analysis of the so-called Schrödinger-Poisson System (SPS), which is a simple model used in studying the quantum transport in semi-conductor devices (See [ILZ],[MRS],[BM],[Ar],[Wa]). It can be written as a system of infinitely many coupled Schrödinger equations:

$$\forall j \in \mathbb{N} \begin{cases} \partial_t \psi_j(t,x) = \frac{i}{2}\Delta_x \psi_j(t,x) - iV(t,x)\psi_j(t,x) \ , \\ \psi_j(x)|_{t=0} = \phi_j(x) \ , \end{cases} \tag{1}$$

where

$$\begin{cases} V(t,x) = \dfrac{C}{|x|} *_x n(t,x) \ , \\ n(t,x) = \sum_j \lambda_j \ |\psi_j(t,x)|^2 \ , \end{cases} \tag{2}$$

and $\lambda = (\lambda_j)_{j\in\mathbb{N}}$ is an l^1 sequence of positive real numbers, $C = \pm \dfrac{1}{4\pi}$ (+ : repulsive, and - : attractive potential). This system appears in quantum mechanics and semi-conductor theory, and a more precise motivation is described at the end of this introduction.

The SPS (1)-(2) can be understood by considering the infinite vector $\psi(t,x) = (\psi_0(t,x), \psi_1(t,x), \psi_2(t,x), ...)$. The vector ψ is a wave function corresponding to a "mixed" quantum state : each "pure" wave function ψ_j is associated with the probability λ_j, thus generates the particle density $n_j := \lambda_j|\psi_j|^2$, so that the whole vector ψ generates the total density $n = \sum_j \lambda_j|\psi_j|^2$, and satisfies the (vector-valued) Schrödinger equation

$$\partial_t\psi = \frac{i}{2}\Delta_x\psi - iV(t,x)\,\psi\ ,$$

where V is the Coulomb potential created by n , and $\partial_t\psi$ denotes the vector $(\partial_t\psi_0, \partial_t\psi_1, ...)$. This last Schrödinger Equation is clearly the SPS.

From now on, we will say that the infinite vector ψ is a mixed quantum state, and each function ψ_j is a pure quantum state.

In the "pure" case $\lambda = (1, 0, 0, ...)$, ψ reduces to ψ_0 and the SPS is merely the Hartree-Equation (HE) :

$$\begin{cases} \partial_t\psi_0(t,x) = \frac{i}{2}\Delta_x\psi_0(t,x) - i(\dfrac{C}{r} *_x |\psi_0(t,x)|^2).\psi_0(t,x)\ , \\ \psi_0(t,x)|_{t=0} = \phi_0(x)\ . \end{cases} \tag{3}$$

Concerning the SPS, our goal in this paper is the following :

(i) the main tool to prove all the results stated here will be an intensive use of Strichartz-type inequalities. They are well-known in the case of pure quantum states (that is in the case of $\exp(it\Delta)$ acting on functions $\psi_0 \in L^2$), and we will first generalize them in the case of mixed quantum states (Theorem 2.1 below). All the results quoted in (ii)-(iv) below are more or less consequences of these inequalities.

(ii) we state an existence and uniqueness result for the SPS when the initial data $\phi = (\phi_j)_{j\in\mathbb{N}}$ only satisfies $\sum_j \lambda_j\|\phi_j\|^2_{L^2(\mathbb{R}^3)} < \infty$ (Theorem 2.2 below).

(iii) concerning the L^2 solutions obtained in (ii) we study the asymptotic behaviour of the solution $\psi(t,x)$ to the SPS as $t \to 0$ and $t \to \infty$, when the first moment of the initial data belongs to L^2 (Theorems 2.3-2.5 below). As $t \to 0$, we prove a smoothing effect, showing that $\psi(t,x)$ belongs to $L^2(\mathbb{R}^3)\bigcap L^6(\mathbb{R}^3)$ as soon as $t \neq 0$.

(iv) Once the points (i)-(iii) are proved, we prove several smoothing effects similar to (iii), under assumptions on the various moments of the initial data. For example, the solution to the SPS with initial data ϕ in L^2 becomes immediately C^∞ in the space variable if all the moments of ϕ belong to L^2 (Theorem 2.6).

(v) In order to complete this study, which is carried out in (i)-(iv) with rough initial datas, we prove in the Appendix the well-posedness of the SPS in every Sobolev space H^m ($m \geq 1$), i.e. when we deal with "smooth" initial datas.

The points (i)-(v) improve in the following way other results that can be found in the literature : as written above, the point (i) is a generalisation, for mixed quantum states, of well-known estimates. As in the pure case of the HE, they are the key for the study of the SPS with initial data in L^2. This type of inequalities were first obtained in [Str], and later in [GV3], [Ya]. A recent and spectacular development of the Strichartz' inequalities can be found in [Bo]. We refer also to [GV1], [Cz], [CP] for related results. Now, the point (ii) is an extension, in the case of mixed quantum states, of results in [Cz], [HO1], [HNT], [Ts], which develop an L^2 theory in the case of the HE and other Schrödinger Equations (pure quantum states). It also improves the H^2 or H^1 theories developed in [ILZ], [BM], or [Ar] in the case of the SPS (mixed quantum states - See also [Wa]). On the other hand, the decay estimates obtained in (iii) as $t \to 0$ are close to estimates obtained in [Pe] concerning the Vlasov-Poisson equation. To our knowledge, they are new concerning both the SPS and the HE with initial datas in L^2. As $t \to \infty$, we restate decay estimates obtained in [ILZ], [DF], [HT] in the H^2 or H^1 cases for the SPS and the HE, which hold then in the more general L^2 case. In fact, in the case of the HE, they can also be derived from results in [HO1]. Finally, points (iv)-(v) extend to the SPS results proved in [HO1], [HT] concerning the single HE.

Before going into the statement and proofs of the results, we now carry out the derivation of the SPS (See also [ILZ],[BM], [MRS],[Ar],[LP]).

We consider a (quantum) particle system in $\mathbb{R}^3$ evolving in the self-consistent attractive or repulsive Coulomb potential it creates.

The quantum formalism tells us this system is entirely described through its "density-matrix" $\rho(t,x,y) \quad ((x,y) \in \mathbb{R}^3 \times \mathbb{R}^3 \ , \ t \in \mathbb{R} \ )$, which is a hermitian ($\forall t \in \mathbb{R} \ , \ \rho(t,x,y) = \rho^*(t,y,x)$), positive, and trace-class operator acting on the Hilbert space $L^2(\mathbb{R}^3)$. Indeed, the knowledge of the density matrix allows to calculate the expectation of any quantum operator A associated with the system (that is, any hermitian, positive, and trace-class operator acting on $L^2(\mathbb{R}^3)$), thanks to the formula

$$< A > := Tr(\rho.A).$$

The function ρ satisfies the Von-Neumann-Heisenberg Equation (VNHE)

$$\begin{cases} i\partial_t \rho = [H, \rho] = H.\rho - \rho.H \ , \\ \rho|_{t=0} = \rho_0 \ , \end{cases} \tag{4}$$

where H is the Hamiltonian of the system,

$$H = \frac{-\Delta_x}{2} + V(t,x) \ , \tag{5}$$

and V is the self-consistent Coulomb potential of the system, created by the particle-density $n(x) := \rho(t,x,x)$,

$$V(t,x) = \frac{C}{r} * \rho(t,x,x) \ . \tag{6}$$

In fact, it seems that this equation needs to be solved in abstract spaces of trace-class operators ρ (See [Ar]), and this is the reason why many authors have used, as in [ILZ],[BM], [Wa], another (almost equivalent) model, that is the SPS, which gives a solution to the VNHE through the following (formal) manipulations : as ρ_0 enjoys the hermitian symmetry, it can be written as a tensor product

$$\rho_0(x,y) = \sum_j \lambda_j \phi_j(x).\phi_j^*(y) \ , \tag{7}$$

where the λ_j's are the eigenvalues of ρ_0, and the ϕ_j's are its eigenvectors. Moreover,

$$\left\{ \begin{array}{ll} (i) & \forall j \ , \ \lambda_j \geq 0 \ , \\ (ii) & \lambda = (\lambda_j)_{j\in\mathbb{N}} \in l^1 \ ; \ \sum_j \lambda_j = 1 \ , \end{array} \right. \tag{8}$$

In fact, (8)-(i) is a consequence of ρ_0 being a positive operator, (8)-(ii) is a consequence of ρ_0 being trace-class (which allows us to normalize λ as above).

Now, if we consider the SPS (1)-(2) with initial datas $\psi_j|_{t=0} = \phi_j$ and weights λ_j given by the decomposition (7), formal manipulations show that a solution $(\psi_j(t))_{j\in\mathbb{N}}$ to the SPS provides a natural solution to the VNHE,

$$\rho(t,x,y) := \sum_j \lambda_j \psi_j(t,x).\psi_j^*(t,y) \ .$$

These calculations complete the derivation of the SPS. As it is written above, many authors have studied the VNHE through the (almost equivalent) SPS, and the L^2 theory we develop here is also concerned with the SPS. In [Ar] however, we find a direct study of the VNHE, which is carried out in an H^1 case. To our knowledge, the problem of developing an L^2 theory directly on the VNHE is still open.

The end of this paper is organized as follows : in the second section, we collect the notations used throughout this article, and state all the results proved here; the third section is devoted to the proof of the Strichartz-type inequalities we will need while dealing with solutions in L^2; in the fourth section, we prove the existence and uniqueness result concerning the SPS when the initial data only satisfies $\sum_j \lambda_j \|\phi_j\|_{L^2}^2 < \infty$; sections 5 and 6 deal with the time behaviour of these L^2-solutions as the first moment of the initial data belongs to L^2, and we show there some smoothing effects ; section 7 is devoted to the proof of a more general smoothing effect, and appendix 8 is a study of the SPS in the Sobolev space H^m.

2 Definitions and main results

In this section, we introduce the notations and give our main results. These will be proved in the subsequent sections.

In the sequel $\lambda = (\lambda_j)_{j\in\mathbb{N}} \in l^1$ is the (fixed) sequence of nonnegative real numbers associated with the initial data ρ_0 (Cf Introduction).

Besides, the following notations will be used : $T(t)$ denotes the unitary group generated by $i\dfrac{\Delta}{2}$ in $L^2(\mathbb{R}^3)$ or, indifferently, in the space $L^2(\lambda)$ defined below (See [ILZ] and references therein, or [Pa] for a general course on evolution equations) ; $L^p(\mathbb{R}^3)$ will always be denoted by the single symbol L^p ; for $p \in [1,\infty]$, we denote by p' the conjugate exponent of p, defined by $1/p + 1/p' = 1$; z^* denotes the conjugate of the complex number z ; the function $\psi(t,x)$ will frequently appear as $\psi(t)$, $\psi(x)$, or even ψ ; finally, for any multi-index α, we denote by $|\alpha|$ its length, and write, for instance, x^α instead of $x_1^{\alpha_1}x_2^{\alpha_2}x_3^{\alpha_3}$.

Definition 2.1 *(i) let* $p \in [1,\infty]$*, then*
$L^p(\lambda) = \{\phi = (\phi_j)_{j\in\mathbb{N}}\ ;\ \ \|\phi\|^2_{L^p(\lambda)} = \sum_j \lambda_j \|\phi_j\|^2_{L^p} < \infty\}$.
(ii) let $m \in \mathbb{N}$*, then*
$H^m(\lambda) = \{\phi = (\phi_j)_{j\in\mathbb{N}}\ ;\ \ \|\phi\|^2_{H^m(\lambda)} = \sum_j \lambda_j \|\phi_j\|^2_{H^m} < \infty\ \}$.
(iii) let $T > 0$, $q,p \in [1,+\infty]$*, then*
$L^{q,p}_T(\lambda) = L^q([-T,T]; L^p(\lambda))$,
$L^{q,p}_{loc}(\lambda) = L^q_{loc}(\mathbb{R}^3; L^p(\lambda))$,
$L^{q,p}(\lambda) = L^q(\mathbb{R}; L^p(\lambda))$.
(iv) let $T > 0$, $q,p \in [1,+\infty]$*, then*
$X^{q,p}_T = L^{\infty,2}_T(\lambda) \bigcap L^{q,p}_T(\lambda)$ *with* $\|\psi\|_{X^{q,p}_T} := \|\psi\|_{L^{\infty,2}_T(\lambda)} + \|\psi\|_{L^{q,p}_T(\lambda)}$,
$Y^{q,p}_T = L^{1,2}_T(\lambda) + L^{q',p'}_T(\lambda)$ *with*
$\|\psi\|_{Y^{q,p}_T} = \inf_{\psi_1+\psi_2=\psi}(\|\psi_1\|_{L^{1,2}_T(\lambda)} + \|\psi_2\|_{L^{q',p'}_T(\lambda)})$.

Notice that, in the case $\lambda = (1,0,0,...)$, the spaces $L^p(\lambda)$ reduce to the ordinary spaces $L^p(\mathbb{R}^3)$. The spaces $H^2(\lambda)$, $H^1(\lambda)$, $L^2(\lambda)$ were already used in [ILZ] and [BM] (See also [Wa]). Moreover, the introduction of the spaces $X^{q,p}_T$ and $Y^{q,p}_T$ is classical in the theory of the HE (see, e.g. [HO1]), and we only generalise here these spaces to study the general SPS. These last two spaces will mainly be useful in sections 6 and 7.

Definition 2.2 *We will say that a pair* (q,p) *is admissible, and write* $(q,p) \in \mathcal{A}$*, if the following holds :*
(i) $2 \le p < \frac{2n}{n-2} = 6$ *where* $n = dim(\mathbb{R}^n)$ $(n = 3)$.
(ii) $\frac{2}{q} = n(\frac{1}{2} - \frac{1}{p}) = 3(\frac{1}{2} - \frac{1}{p})$.

We are now able to state the main results of the present article. The first one is a generalisation of the well-known Strichartz' inequalities in the case of our weighted L^p spaces. The difficulty here comes from the definition of $L^{q,p}(\lambda)$, which has however to be used later on. An obvious extension of the classical Strichartz' inequality is for instance $\sum_j \lambda_j \|T(t)\phi_j\|^2_{L^{q,p}} \le C(q)\ \sum_j \lambda_j \|\phi_j\|^2_{L^2}$, which cannot be used for our purpose.

Theorem 2.1 *(Strichartz' inequalities for mixed quantum states) Let $T > 0$ and (q, p) be an admissible pair. Then the following holds :*
(i) there exists a constant $C(q)$, depending only on q, such that for all $\phi \in L^2(\lambda)$

$$\|T(t)\phi\|_{L_T^{q,p}(\lambda)} \leq C(q)\ \|\phi\|_{L^2(\lambda)}.$$

(ii) for all $(a, b) \in \mathcal{A}$, there exists $C(a, q)$ such that for all $f \in L_T^{q',p'}(\lambda)$

$$\| \int_0^t T(t-s)f(s)ds\|_{L_T^{a,b}(\lambda)} \leq\ C(a,q)\ \|f\|_{L_T^{q',p'}(\lambda)}\ ,$$

(iii) in particular, there exists a constant $C(q)$, depending only on q, such that for all $f \in Y_T^{q,p}$

$$\| \int_0^t T(t-s)f(s)ds\|_{X_T^{q,p}} \leq\ C(a,q)\ \|f\|_{Y_T^{q,p}}\ .$$

The second inequality will be of crucial importance while dealing with the non-linearity $V(\psi)\psi$. Indeed, our method will be to transform the Schrödinger-Poisson System into the integral equation :

$$\psi(t) = T(t)\phi - i\int_0^t T(t-s)f(s)ds := \alpha + \beta\ ,$$

where $f(s) = V(\psi(s))\psi(s)$. We are able to control the $L^2(\lambda)$ norm of α in terms of the single $L^2(\lambda)$ norm of ϕ. But to control the $L^2(\lambda)$ norm of β in terms of the $L^2(\lambda)$ norm of ψ is merely impossible. In particular, the application $\psi \to V(\psi)\psi$ is not locally Lipschitz in $L^2(\lambda)$ (while it is in every $H^m(\lambda)$ when $m \geq 1$). The main tool to avoid these problems will be to control the $L_T^{a,b}(\lambda)$ norms of β by some $L_T^{q',p'}(\lambda)$ norms of $V\psi$ (through Theorem 2.1), which will be possible, as we will see, in terms of some $L_T^{q,p}(\lambda)$ norms of ψ (See Lemma 4.1 below). In fact, the non-linearity β is, roughly speaking, Lipschitz in the spaces $L_{loc}^{q,p}(\lambda)$ with (q, p) admissible.

Remark 2.1. The Strichartz' inequalities for mixed quantum states are stated here under the assumption $\lambda = (\lambda_j)_{j\in\mathbb{N}} \in l^1$ with $\lambda_j \geq 0$ for all j, and in the case of the dimension $n = 3$. It is obvious, in view of the proof below, that it can be extended to all dimensions, and also under the weaker assumption $\lambda \in l^1$ (without assuming the positivity of the weights λ_j), just by modifying our weighted norms in the obvious way : $\|\phi\|_{L^2(\lambda)}^2 := \sum_j |\lambda_j|\ \|\phi_j\|_{L^2}^2$ (See [Cz] for the corresponding results one can get in the non-weighted L^p spaces).

In particular, the study that we carry out here in a three dimensional space could be done in any dimension, as in [HO1] in the case of the single HE. Since the physical meaning of the SPS is less obvious then, we will not work in a general dimension.

Also, it is clear from the proof given below that the constants appearing in Theorem 2.1 do not depend on the sequence $\lambda \in l^1$.

Theorem 2.2 *(Existence and uniqueness for the SPS posed in $L^2(\lambda)$).*
Let $\phi \in L^2(\lambda)$. Let $\sigma \in]\frac{3}{2}, 3[$, $p = \frac{2\sigma}{\sigma - 1} > 3$, and let $q = q(p)$ be such that (q, p) is admissible. Then, there exists a unique function

$$\psi \in C^0\Big(\mathbb{R}, L^2(\lambda)\Big) \bigcap L^{q,p}_{loc}(\lambda) \ ,$$

solution to the SPS with initial data ϕ. In this case, $V(\psi)\psi(t) \in L^{1,2}_{loc}(\lambda) + L^{q',p'}_{loc}(\lambda)$. Moreover, the following holds :
(i) $\forall t \in \mathbb{R}, \ \|\psi(t)\|_{L^2(\lambda)} = \|\phi\|_{L^2(\lambda)}$,
(ii) $\forall (a, b) \in \mathcal{A}, \quad \psi \in L^{a,b}_{loc}(\lambda)$,
(iii) Let ϕ_m be a sequence of initial datas in $L^2(\lambda)$ such that $\phi_m \stackrel{m\to\infty}{\longrightarrow} \phi$ in $L^2(\lambda)$. Then the corresponding sequence ψ_m of solutions to the SPS verifies

$$\forall (a, b) \in \mathcal{A}, \quad \psi_m \stackrel{m\to\infty}{\longrightarrow} \psi \in L^{a,b}_{loc}(\lambda).$$

Remark 2.2. In Theorem 2.2 and below, we deal with solutions to the SPS in the following sense : the assumption $V(\psi)\psi(t) \in L^{1,2}_{loc}(\lambda) + L^{q',p'}_{loc}(\lambda)$ allows to define solutions in a weak sense, e.g. in the distributional sense for each component of the vector-valued functions. We may also define solutions of the SPS in an integral sense (see [Pa]): $\forall t$, $\psi(t) = T(t)\phi - i\int_0^t T(t-s)V(\psi)\psi(s)ds$. We will see that these two different definitions are equivalent here, and we will therefore not make any distinction between them, except in the proof of Theorem 2.2 where we prove this equivalence.

Remark 2.3. Theorem 2.2 is very close to the corresponding existence and uniqueness result one can prove on the single HE stated in the usual space L^2. The difficulty in Theorem 2.2 is to generalize the Strichartz' inequalities in the weighted spaces $L^p(\lambda)$. Once this difficulty has been overcome, the situation becomes easier to handle, and we see that the SPS and the HE have a very similar structure. In fact, Theorems 2.4 (vii), 2.5, and 2.6 below extend to the SPS other results that are "classical" in the case of the Hartree Equation and other Schrödinger Equations (See [Cz],[HO1],[HNT],[Ts], and references therein), and their proofs are very similar to the ones in [HO1]. Besides, to our knowledge, the results and methods of Theorem 2.3 are new, both for the HE and the SPS.

Theorem 2.3 *(Asymptotic behaviour at t=0)*
Let $\phi \in L^2(\lambda)$ satisfy $x\ \phi \in L^2(\lambda)$. Let $\psi(t) \in C^0(\mathbb{R}; L^2(\lambda)) \bigcap L^{a,b}_{loc}(\lambda)$, for all $(a, b) \in \mathcal{A}$, be the corresponding solution of the SPS, and $n(t) = \sum_j \lambda_j |\psi_j(t)|^2$ the corresponding particle density. Let $T > 0$. Then there exist constants C depending only upon $\|\phi\|_{L^2(\lambda)}, \|x\ \phi\|_{L^2(\lambda)}, T, p$ such that :
(i) $\forall |t| \leq T, \ \forall p \in [2, 6], \quad \|\psi(t)\|_{L^p(\lambda)} \leq \dfrac{C}{|t|^{3(\frac{1}{2}-\frac{1}{p})}}$,

(ii) $\forall |t| \leq T,\ \forall p \in [1,3], \quad \|n(t)\|_{L^p} \leq \dfrac{C}{|t|^{3(1-\frac{1}{p})}},$

(iii) $\forall |t| \leq T,\ \forall p \in]3,\infty], \quad \|V(t)\|_{L^p} \leq \dfrac{C}{|t|^{(1-\frac{3}{p})}},$

(iv) $\forall |t| \leq T,\ \forall p \in]\frac{3}{2},\infty[, \quad \|\nabla V(t)\|_{L^p} \leq \dfrac{C}{|t|^{(2-\frac{3}{p})}}.$

Theorem 2.4 *(Asymptotic behaviour at $t = \infty$)*

Let ϕ, $\psi(t)$, and $n(t)$ be as in Theorem 2.3. Then there exist constants C depending only upon $\|\phi\|_{L^2(\lambda)}, \|x\ \phi\|_{L^2(\lambda)}, p$ such that :

(i) $\forall |t| \geq 1,\ \forall p \in [2,6], \quad \|\psi(t)\|_{L^p(\lambda)} \leq \dfrac{C}{|t|^{\frac{3}{2}(\frac{1}{2}-\frac{1}{p})}},$

(ii) $\forall |t| \geq 1,\ \forall p \in [1,3], \quad \|n(t)\|_{L^p} \leq \dfrac{C}{|t|^{\frac{3}{2}(1-\frac{1}{p})}},$

(iii) $\forall |t| \geq 1,\ \forall p \in [6,\infty],\ \forall \varepsilon > 0, \quad \|V(t)\|_{L^p} \leq \dfrac{C}{|t|^{(\frac{2}{3}-\frac{1}{p}-\varepsilon)}},\ (\varepsilon = 0 \text{ when } p = 6)$,

(iv) $\forall |t| \geq 1,\ \forall p \in]3,6],\ \forall \varepsilon > 0, \quad \|V(t)\|_{L^p} \leq \dfrac{C}{|t|^{(1-\frac{3}{p}-\varepsilon)}},$

(v) $\forall |t| \geq 1,\ \forall p \in]2,\infty[,\ \forall \varepsilon > 0, \quad \|\nabla V(t)\|_{L^p} \leq \dfrac{C}{|t|^{(1-\frac{1}{p}-\varepsilon)}},\ (\varepsilon = 0 \text{ when } p = 2)$,

(vi) $\forall |t| \geq 1,\ \forall p \in]\frac{3}{2},2[,\ \forall \varepsilon > 0, \quad \|\nabla V(t)\|_{L^p} \leq \dfrac{C}{|t|^{(2-\frac{3}{p}-\varepsilon)}},$

(vii) The pseudo-conformal law holds, that is, for all $t \in \mathbb{R}$,

$$\|(x + it\nabla)\psi(t)\|^2_{L^2(\lambda)} + t^2\ \|\nabla V(t)\|^2_{L^2} = \|x\ \phi\|^2_{L^2(\lambda)} + \int_0^t s\ \|\nabla V(s)\|^2_{L^2}\ ds.$$

Theorem 2.5 *(Regularity)*

Let ϕ and $\psi(t)$ be as in Theorem 2.3. Then the following holds :

(i) $\psi(t) \in C^0(\mathbb{R}^*; L^p(\lambda))$ *for* $2 \leq p \leq 6$,

(ii) $V(t) \in C^0(\mathbb{R}^*; L^p)$ *for* $3 < p \leq \infty$,

(iii) $\nabla V(t) \in C^0(\mathbb{R}^*; L^p)$ *for* $3/2 < p < \infty$,

(iv) $(x + it\nabla)\psi(t) \in C^0(\mathbb{R}; L^2(\lambda))$.

Remark 2.4. Time decays like in Theorem 2.3 have been obtained in [Pe] for the Vlasov-Poisson case, which is a limiting case when the SPS is rescaled with a vanishing Planck constant. Notice that these decays are the same in the case of the *free* Schrödinger Equation : the potential V does not modify the asymptotic behaviour of the solution as $t \to 0$.

Those of Theorem 2.4 extend, in the $L^2(\lambda)$-case, estimates that were obtained in [ILZ] for the SPS, under the assumption $\phi \in H^2(\lambda)$. We do not know whether they are optimal, and no better estimate is available, neither in the case of datas in $H^1(\lambda)$, nor in the case of the single HE with initial data in H^1. To our knowledge, only lower and upper bounds on the order of decay as $t \to \infty$ are available concerning the single HE (see [HO2]).

Remark 2.5. One can see that, in the $L^2(\lambda)$ case, the function $\psi(t)$ is "almost" in $H^1(\lambda)$, in the sense that it is in $L^2(\lambda) \bigcap L^6(\lambda)$ for all $t \neq 0$ (recall that $H^1 \subset L^2 \bigcap L^6$ in dimension 3), and that all the estimates obtained in the $H^2(\lambda)$ case hold true under our weaker assumptions. The only problem is that $\nabla\psi \notin L^2(\lambda)$. Indeed, if there was a $t_0 \neq 0$ such that $\nabla\psi(t_0) \in L^2(\lambda)$, we would get $\forall t$, $\psi(t) \in H^1(\lambda)$, just by solving the SPS with initial data $\psi(t_0) \in H^1(\lambda)$.

In fact, the main idea concerning the HE or the SPS is that the gain of one moment in the x variable for the initial data ϕ implies a gain of regularity for the solution $\psi(t)|_{t\neq 0}$, through the following formula (See sections 6-7 below):

$$\exp(-it\frac{x^2}{2t})\ (x+it\nabla)^\alpha\psi(t) = (it\nabla)^\alpha \exp(-it\frac{x^2}{2t})\ \psi(t).$$

This idea is a key ingredient in [HO1], [Cz] and others for the study of the HE, and allows for instance to prove the C^∞ smoothing effect written above (see Theorem 2.6). The operator $x + it\nabla$ is called the Galilei operator and corresponds to the Galilei invariance of the physical equations.

To conclude this remark, we can notice that, in our case, $(x+it\nabla)\psi \in L^2(\lambda)$, which can be understood as follows : since the particles have an infinite kinetic energy ($\nabla\psi \notin L^2(\lambda)$), they instantaneously go to infinity ($x\ \psi \notin L^2(\lambda)$). But this compensation phenomenon allows $(x + it\nabla)\psi$ to remain bounded in $L^2(\lambda)$.

Remark 2.6. The pseudo-conformal law of Theorem 2.4 was first introduced in [GV2] in the case of the HE, and is proved in the regular $H^2(\lambda)$ case for the SPS in [ILZ].

We now want to state a smoothing effect similar to Theorem 2.5, which is in fact more general. In order to do this, we need the following

Definition 2.3 *Let $k \in \mathbb{N}$. We define the space E_k by*

$$\begin{aligned} E_1 &= L^2(\lambda) \bigcap L^6(\lambda) , \\ E_2 &= L^2(\lambda) \bigcap L^\infty(\lambda) \bigcap C^0(\lambda) , \\ E_k &= E_2 \bigcap C^{k-2}(\lambda) \ for\ k \geq 3 . \end{aligned}$$

They are naturally endowed with the norms $\|u\|^2_{E_1} = \sum_j \lambda_j(\|u_j\|^2_{L^2} + \|u_j\|^2_{L^6})$, $\|u\|_{E_2}$ and $\|u\|_{E_k}$ being defined in the similar way. Here, C^k is the usual space of functions of class C^k.

Remark 2.6. In view of the classical Sobolev imbeddings, it is straightforward to check that $H^k(\lambda) \subset E_k$ with continuous imbedding, for every $k \in \mathbb{N}$.

We can now state the

Theorem 2.6 *(A general smoothing effect)*
Let $\phi \in L^2(\lambda)$ and $\psi(t) \in C^0(\mathbb{R}; L^2(\lambda))$ be the corresponding solution to the SPS. Let $J = x + it\nabla$. Suppose also there exists $k \in \mathbb{N} \bigcup \{+\infty\}$ such that, for every multi-index α satisfying $|\alpha| \leq k$, we have $x^\alpha \ \phi \in L^2(\lambda)$ (all the moments of ϕ up to order k belong to $L^2(\lambda)$). Then, the following holds :
(i) $\forall |\alpha| \leq k$, $J^\alpha \psi(t) \in C^0(\mathbb{R}; L^2(\lambda))$,

(ii) $\psi(t) \in C^0(\mathbb{R}^*; E_k)$, (ψ is "almost" in $H^k(\lambda)$) .

3 Proof of Strichartz' inequalities

This proof is close to those in [GV1], [Cz], or [Ya] concerning other Strichartz-type inequalities. It is performed in four steps. Note that, in the statement of Theorem 2.1, the real number $T > 0$ plays no particular role, so that in the sequel, we will take $T = \infty$, and work in the spaces

$$L^{q,p}_\infty(\lambda) := L^{q,p}(\lambda) := L^q(\mathbb{R}; L^p(\lambda)).$$

We begin with the following (easy) Lemma :

Lemma 3.1 *(i) Let $2 \leq p \leq \infty$, $\phi \in L^{p'}(\lambda)$ and $t \neq 0$. Then*

$$\|T(t)\phi\|_{L^p(\lambda)} \leq \frac{C}{|t|^{3(\frac{1}{2}-\frac{1}{p})}} \ \|\phi\|_{L^{p'}(\lambda)},$$

(ii) For $\phi, \ \psi \in L^2(\lambda)$, define the scalar product

$$< \phi|\psi >_{L^2(\lambda)} := \sum_j \lambda_j < \phi_j|\psi_j >_{L^2} .$$

Then, the adjoint of $T(t)$ with respect to $< .|. >_{L^2(\lambda)}$ is

$$T(t)^* = T(-t) .$$

Proof of Lemma 3.1.

To prove (i), we argue as usual

$$\begin{aligned} \|T(t)\phi\|^2_{L^p(\lambda)} &= \sum_j \lambda_j \|T(t)\phi_j\|^2_{L^p} \\ &\leq \sum_j \lambda_j \frac{C}{|t|^{3.2.(\frac{1}{2}-\frac{1}{p})}} \|\phi_j\|^2_{L^{p'}} \\ &\leq \left(\frac{C}{|t|^{3.(\frac{1}{2}-\frac{1}{p})}} . \|\phi\|_{L^{p'}(\lambda)}\right)^2 . \end{aligned}$$

The second part of the Lemma is obvious, since the same holds true for $T(t)$ acting on $L^2(\mathbb{R}^3)$. ■

We now notice that the proof of Strichartz inequalities in the case of $T(t)$ acting on $L^2(\mathbb{R}^3)$ is a consequence of the Lemma above stated in the simple case $L^2(\lambda) = L^2(\mathbb{R}^3)$, and of duality arguments combined with Riesz inequalities and interpolation arguments (See [Cz], [GV1],[Ya]). Hence, our main goal will be to check that the duality and interpolation arguments used in the non-weighted spaces L^p hold true in the weighted spaces $L^p(\lambda)$. This is stated in the following Lemma.

Lemma 3.2 *(i) Let $1 < p < \infty$. Then, the dual space to $L^p(\lambda)$ is identified with $L^{p'}(\lambda)$ through the inner product $< .|. >_{L^2(\lambda)}$,*

$$(\phi, \psi)_{(L^p(\lambda), L^{p'}(\lambda))} :=< \phi|\psi >_{L^2(\lambda)} ,$$

(ii) Let $1 < p, q < \infty$. Then, the dual space to $L^{q,p}(\lambda)$ is identified with $L^{q',p'}(\lambda)$ through

$$(\phi(t), \psi(t))_{(L^{q,p}(\lambda), L^{q',p'}(\lambda))} := \int_{\mathbb{R}} < \phi(t)|\psi(t) >_{L^2(\lambda)} dt ,$$

(iii) (Hölders' Inequality) Let $1 \le p \le \infty$. Then,

$$(\phi, \psi)_{(L^p(\lambda), L^{p'}(\lambda))} \le \|\phi\|_{L^p(\lambda)} \, \|\psi\|_{L^{p'}(\lambda)} ,$$

(iv) Let $1 \le p_0, q_0, p_1, q_1 < \infty$. Let p and q satisfy

$$\frac{1}{p} = \frac{1-\theta}{p_0} + \frac{\theta}{p_1} \quad ; \quad \frac{1}{q} = \frac{1-\theta}{q_0} + \frac{\theta}{q_1} \quad ; \quad \theta \in]0, 1[\, .$$

Then the interpolated space of order θ between $L^{q_0,p_0}(\lambda)$ and $L^{q_1,p_1}(\lambda)$ is

$$[L^{q_0,p_0}(\lambda), L^{q_1,p_1}(\lambda)]_\theta = L^{q,p}(\lambda) .$$

Remark 3.1. Here, we consider the complex interpolation theory, with the notations of [BL].

Proof of Lemma 3.2 : (iii) is straightforward. Besides, we notice that (i) implies (ii). Indeed, (i) implies that $L^p(\lambda)$ is a reflexive space with dual space $L^{p'}(\lambda)$ when $1 < p < \infty$. So it enjoys the Radon-Nykodym property. As a consequence (See [DU]), we obtain

$$(L^{q,p}(\lambda))' = \Big(L^q(\mathbb{R}; L^p(\lambda))\Big)' = L^{q'}\Big((L^p(\lambda))'\Big) = L^{q'}(L^{p'}(\lambda))$$

and (ii) is proved.
We now show (i) by noticing the following : let $d\mu$ be the measure

$$d\mu := \sum_j \lambda_j \delta_j$$

on $\mathbb{N}$, where δ_j is the Dirac measure at $j \in \mathbb{N}$. Then

$$L^p(\lambda) = l^2(\mathbb{N}, d\mu; L^p(\mathbb{R}^3)).$$

Thus, by the same argument as above, we obtain (i) since

$$(L^p(\lambda))' = \left(l^2(\mathbb{N}, d\mu; L^p(\mathbb{R}^3))\right)' = l^2(\mathbb{N}, d\mu; L^{p'}(\mathbb{R}^3)) = L^{p'}(\lambda).$$

We can now prove (iv). We have, under the assumptions of Lemma 3.2 (See [BL])

$$\begin{aligned} [L^{q_0,p_0}(\lambda), L^{q_1,p_1}(\lambda)]_\theta &= [L^{q_0}(\mathbb{R};\ L^{p_0}(\lambda))\ ,\ L^{q_1}(\mathbb{R};\ L^{p_1}(\lambda))]_\theta \\ &= L^q(\mathbb{R};\ [L^{p_0}(\lambda)\ ,\ L^{p_1}(\lambda)]_\theta), \end{aligned}$$

and

$$\begin{aligned} [L^{p_0}(\lambda)\ ,\ L^{p_1}(\lambda)]_\theta &= [l^2(\mathbb{N}, d\mu;\ L^{p_0}(\mathbb{R}^3))\ ,\ l^2(\mathbb{N}, d\mu;\ L^{p_1}(\mathbb{R}^3))]_\theta \\ &= l^2(\mathbb{N}, d\mu;\ [L^{p_0}(\mathbb{R}^3), L^{p_1}(\mathbb{R}^3)]_\theta) \\ &= l^2(\mathbb{N}, d\mu; L^p(\mathbb{R}^3)) \\ &= L^p(\lambda). \end{aligned}$$

And (iv) is proved. ■

We are now able to prove Theorem 2.1 through the following serie of Lemmas:

Lemma 3.3 *Let (q,p) be an admissible pair and $\psi \in L^{q',p'}(\lambda)$. Then,*

$$\Big\| \int_{-\infty}^{+\infty} \|T(t-s)\psi(s)\|_{L^p(\lambda)} ds \Big\|_{L^q} \leq C(q)\ \|\psi\|_{L^{q',p'}(\lambda)}.$$

Remark 3.2. Here and throughout the section, we will write Bochner integrals that may not make sense. In order to avoid such problems, all the subsequent majorisations should be obtained first on very smooth functions, then in the general case thanks to a density argument (see [Cz]). For instance, we should first deal with functions ψ whose components ψ_j belong to $\mathcal{S}$, with the additional condition $\sum_j \lambda_j \|\psi_j\|^2_{\mathcal{S}} < \infty$. Denote by $\mathcal{S}(\lambda)$ the space of all such functions. In fact, it is clear that $T(t)$ maps $\mathcal{S}(\lambda)$ to itself, and that $\mathcal{S}(\lambda)$ is dense in the weighted L^p spaces.

Proof of Lemma 3.3 : We have, using Lemma 3.1 (i),

$$A := \Big\| \int_{-\infty}^{+\infty} \|T(t-s)\psi(s)\|_{L^p(\lambda)} ds \Big\|_{L^q} \ \leq\ \Big\| \int_{-\infty}^{+\infty} \frac{C\ \|\psi(s)\|_{L^{p'}(\lambda)}}{|t-s|^{3(\frac{1}{2}-\frac{1}{p})}} ds \Big\|_{L^q}.$$

But, since (q,p) is admissible, we have $3\,(\frac{1}{2}-\frac{1}{p})=\frac{2}{q}$ and $2\leq p<6<\infty$, $2<q\leq\infty$.

Thus except for the special case $(q,p)=(\infty,2)$, for which the results are however obvious, we can use the Hardy-Riesz-Sobolev inequality (See [St]) and get

$$A \;\leq\; C(q)\;\|\psi(s)\|_{L^{q',p'}(\lambda)}\;, \tag{9}$$

and the proof is complete. ∎

Lemma 3.4 *Let (q,p) be an admissible pair. The following results hold true :*

(i) $\|\int_0^t T(t-s)\psi(s)ds\|_{L^{q,p}(\lambda)}\leq C(q)\;\|\psi(s)\|_{L^{q',p'}(\lambda)}$,

(ii) $\|\int_0^t T(t-s)\psi(s)ds\|_{L^{\infty,2}(\lambda)}\leq C(q)\;\|\psi(s)\|_{L^{q',p'}(\lambda)}$,

(iii) $\|\int_0^t T(t-s)\psi(s)ds\|_{L^{q,p}(\lambda)}\leq C(q)\;\|\psi(s)\|_{L^{1,2}(\lambda)}$.

Proof of Lemma 3.4. Thanks to Lemma 3.3, (i) is obvious. We have now obtained the statement of Theorem 2.1 (ii) in the simple case $(a,b)=(q,p)$. We now want to prove a slight generalisation of (ii) and (iii). Let $K(t,s)\in L^\infty(\mathbb{R}^2)$, with, for convenience, $\|K\|_{L^\infty}=1$. We prove

(ii') $\|\int_{s\in\mathbb{R}}K(t,s)T(t-s)\psi(s)ds\|_{L^{\infty,2}(\lambda)}\leq C(q)\;\|\psi(s)\|_{L^{q',p'}(\lambda)}$,

(iii') $\|\int_{s\in\mathbb{R}}K(t,s)T(t-s)\psi(s)ds\|_{L^{q,p}(\lambda)}\leq C(q)\;\|\psi(s)\|_{L^{1,2}(\lambda)}$.

Now, (ii') is immediate. Indeed, thanks to Lemmas 3.1 (ii), and 3.2 (iii),
$\|\int_{s\in\mathbb{R}}K(t,s)T(t-s)\psi(s)ds\|^2_{L^2(\lambda)}=$

$$\begin{aligned}
&= \; <\int_{s\in\mathbb{R}}K(t,s)T(t-s)\psi(s)ds|\int_{u\in\mathbb{R}}K(t,u)T(t-u)\psi(u)du>_{L^2(\lambda)}\\
&= \; \int_{s\in\mathbb{R}}<K(t,s)\psi(s)|\int_{u\in\mathbb{R}}K(t,u)T(s-u)\psi(u)du>_{L^2(\lambda)}\\
&\leq \; \|K(t,s)\psi(s)\|_{L^{q',p'}(\lambda)}\;\|\int_{u\in\mathbb{R}}K(t,u)T(s-u)\psi(u)du\|_{L^{q,p}(\lambda)}\\
&\leq \; \|\psi(s)\|_{L^{q',p'}(\lambda)}\;\|\int_{u\in\mathbb{R}}K(t,u)T(s-u)\psi(u)du\|_{L^{q,p}(\lambda)}\;.
\end{aligned}$$

Hence, thanks to Lemma 3.3,

$$\|\int_{s\in\mathbb{R}}K(t,s)T(t-s)\psi(s)ds\|^2_{L^2(\lambda)}\leq C(q)\;\|\psi(s)\|^2_{L^{q',p'}(\lambda)}.$$

Therefore (ii'), thus (ii), are proved.

Next, we prove (iii'). We avoid here the special case $q = \infty$, for which the result follows from (ii'). We write

$$\| \int_{s\in\mathbb{R}} K(t,s)T(t-s)\psi(s)ds\|_{L^{q,p}(\lambda)} = \\ = \sup\{A(\phi);\ \phi(t) \in L^{q',p'}(\lambda),\ \|\phi(t)\|_{L^{q',p'}(\lambda)} = 1\} ,$$

$$A(\phi) := \int_{t\in\mathbb{R}} < \int_{s\in\mathbb{R}} K(t,s)T(t-s)\psi(s)ds|\phi(t) >_{L^2(\lambda)} dt .$$

The term $A(\phi)$ can be upper bounded thanks to Hölder's inequality in the weighted spaces (Lemma 3.2 (iii)),

$$\begin{aligned} A(\phi) &= \int_{t\in\mathbb{R}}\int_{s\in\mathbb{R}} K(t,s) < T(t-s)\psi(s)|\phi(t) >_{L^2(\lambda)} \\ &= \int_{s\in\mathbb{R}} < \psi(s)| \int_{t\in\mathbb{R}} K(t,s)T(s-t)\phi(t) >_{L^2(\lambda)} \\ &\le \|\psi(s)\|_{L^{1,2}(\lambda)} \| \int_{t\in\mathbb{R}} K(t,s)T(s-t)\phi(t)dt \|_{L^{\infty,2}(\lambda)} \\ &\le \|\psi(s)\|_{L^{1,2}(\lambda)} \|\phi(t)\|_{L^{q',p'}(\lambda)}, \end{aligned}$$

thanks to (ii'). This completes the proof of (iii'), hence of (iii). ■

We can now prove Theorem 2.1 (ii) for *any* admissible pair (a, b), just by interpolating the different inequalities in the Lemma 3.4 :

Proof of Theorem 2.1 (ii). Let (a, b) be an admissible pair. We first treat the case $q \le a < \infty$.

Then it is enough to suppose that

$$\begin{cases} q < a < \infty , \\ 2 < b < p , \end{cases}$$

(the case $(a, b) = (\infty, 2)$ has already been proved), and we interpolate the inequalities in Lemma 3.4 (i)-(ii) with

$$\frac{1}{a} = \frac{\theta}{q} + \frac{1-\theta}{\infty} \quad ; \quad \frac{1}{b} = \frac{\theta}{p} + \frac{1-\theta}{2}.$$

Indeed, Hölder's inequality in space (Lemma 3.2 (iii)), then in time, reads
$\| \int_0^t T(t-s)\psi(s)ds\|_{L^{a,b}(\lambda)} \le$

$$\begin{aligned} &\le \Big\| \, \| \int_0^t T(t-s)\psi(s)ds\|^{\theta}_{L^p(\lambda)} \, \| \int_0^t T(t-s)\psi(s)ds\|^{1-\theta}_{L^2(\lambda)} \Big\|_{L^a} \\ &\le \| \int_0^t T(t-s)\psi(s)ds\|^{\theta}_{L^{q,p}(\lambda)} \, \| \int_0^t T(t-s)\psi(s)ds\|^{1-\theta}_{L^{\infty,2}(\lambda)} \\ &\le C(q) \, \|\psi(s)\|_{L^{q',p'}(\lambda)} , \end{aligned}$$

thanks to Lemma 3.4 (i) and (ii). And Theorem 2.1 (ii) is proved in this case.

Next, we treat the case $2 < a < q$.

We can then suppose

$$\begin{cases} 2 < a < q < \infty \ , \\ 2 < p < b \ . \end{cases}$$

Lemma 3.4 (i) and (iii) written for the admissible pair be (a, b) tells us that the linear operator

$$F \ : \ \psi \longrightarrow \int_0^t T(t-s)\psi(s)ds$$

maps

$$\begin{cases} L^{a',b'}(\lambda) & \longrightarrow & L^{a,b}(\lambda) & \textit{with norm} \ \leq \ 1 \ , \\ L^{1,2}(\lambda) & \longrightarrow & L^{a,b}(\lambda) & \textit{with norm} \ \leq \ C(a). \end{cases} \tag{10}$$

Then we choose $\theta \in]0, 1[$ such that

$$\frac{1}{q'} = \frac{\theta}{a'} + \frac{1-\theta}{1} \quad ; \quad \frac{1}{p'} = \frac{\theta}{b'} + \frac{1-\theta}{2}.$$

Interpolating the spaces $L^{a',b'}(\lambda)$ and $L^{1,2}(\lambda)$ involved in (9) as recalled in Lemma 3.2 (iv), we get

$$F \ : \ L^{q',p'}(\lambda) \longrightarrow L^{a,b}(\lambda) \quad \textit{with norm} \ \leq \ C(a)^{\theta}.$$

This ends the proof of Theorem 2.1 (ii).

In order to end this section, we state now the

Proof of Theorem 2.1 (i). It is an easy calculation, since we can use the usual Strichartz' inequality and write

$$\begin{aligned} \|T(t)\phi\| L^{q,p}(\lambda)^2 &= \|(\sum \lambda_j \|T(t)\phi_j\|_{L^p}^2 \ )^{1/2}\|_{L^q}^2 \\ &= \| \sum_j \lambda_j \|T(t)\phi_j\|_{L^p}^2 \ \|_{L^{q/2}} \qquad (q \geq 2) \\ &\leq \ \sum_j \lambda_j \|T(t)\phi_j\|_{L^{q,p}}^2 \\ &\leq \ \sum_j \lambda_j \|\phi_j\|_{L^2}^2 = \|\phi\|_{L^2(\lambda)}^2 \ . \end{aligned}$$

■

4 Proof of Theorem 2.2

In order to separate the difficulties, we begin by defining the short range and long range part of the potential (respectively denoted by V_1 and V_2), as follows :

$$\begin{aligned} V(\psi) &= C/r * \sum_j \lambda_j \; |\psi_j(x)|^2 \\ &:= K_1(r) * \sum_j \lambda_j \; |\psi_j(x)|^2 + K_2(r) * \sum_j \lambda_j \; |\psi_j(x)|^2 \\ &:= V_1(\psi) + V_2(\psi), \end{aligned}$$

where $K_1 := C \;\; \chi(r)/r$ and $K_2 := C \;\; (1-\chi(r))/r$ and χ is a C^∞ function satisfying : $\chi(r) = 1$ for $0 \leq r \leq 1$ and $\chi(r) = 0$ for $r \geq 2$.

Now we want to pass to the limit in the SPS in order to deduce an $L^2(\lambda)$ theory from the available $H^1(\lambda)$ theory. The following Lemma gives a control of the nonlinearities $V_k(\psi)\psi$ $(k = 1, 2)$ in *some* space $L_{loc}^{q'_k,p'_k}(\lambda)$, where $(q_k, p_k) \in \mathcal{A}$. It allows in turn to use Strichartz' inequalities of Theorem 2.1. Of course, V_1 and V_2 behave differently, and the main idea is that Theorem 2.1 gives a control of $\int_0^t T(t-s) \; V_k(\psi)(s) \; \psi(s)$ $(k = 1, 2)$ in *any* space $L_{loc}^{a,b}(\lambda)$, in term of *one* single $L_{loc}^{q'_k,p'_k}(\lambda)$-norm of $V_k(\psi)\psi$.

Lemma 4.1 *Let $\psi(t)$ and $\phi(t)$ be two solutions of the SPS with respective initial datas ψ_0 and $\phi_0 \in H^1(\lambda)$. Let $T > 0$, $\sigma \in]\frac{3}{2}, 3[$, $p = \frac{2\sigma}{\sigma-1} > 3$, and let $q = q(p)$ be such that (q, p) is admissible. Let q' and p' be the conjugated exponents of q and p. Let $M = Max(\|\psi_0\|_{L^2(\lambda)}; \|\phi_0\|_{L^2(\lambda)})$. Then the following holds :*
(i) (short range potential)

$$\|V_1(\psi(t)) \; \psi(t) - V_1(\phi(t)) \; \phi(t)\|_{L_T^{q',p'}(\lambda)} \leq C(p) \; M^2 \; T^{1-\frac{2}{q}} \|\psi(t) - \phi(t)\|_{L_T^{q,p}(\lambda)},$$

(ii) (long range potential)

$$\|V_2(\psi(t)) \; \psi(t) - V_2(\phi(t)) \; \phi(t)\|_{L_T^{1,2}(\lambda)} \leq C \; M^2 \; T \; \|\psi(t) - \phi(t)\|_{L_T^{\infty,2}(\lambda)},$$

(iii) in particular,

$$\|V(\psi(t)) \; \psi(t) - V(\phi(t)) \; \phi(t)\|_{Y_T^{q,p}} \leq C \; M^2 \; \max(T, T^{1-\frac{2}{q}}) \; \|\psi(t) - \phi(t)\|_{X_T^{q,p}}.$$

In other words, each function $\psi \to V_k(\psi)\psi$ $k = 1, 2$ is roughly speaking locally Lipschitz from some space $L_T^{a,b}(\lambda)$ into $L_T^{a',b'}(\lambda)$ with (a, b) admissible, and the corresponding Lipschitz constant reads $C(M) \; T^\alpha$ with $\alpha > 0$, so that $T^\alpha \to 0$ as $T \to 0$. This will be fundamental for the sequel. The technical difficulty is that the relevant admissible pairs are not the same in the case of the short range potential ($(a, b) = (q, p)$ with (q, p) defined above) and of the long range potential ($(a, b) = (\infty, 2)$).

Besides, note that, as $\|\psi(t)\|_{L^2(\lambda)}$ and $\|\phi(t)\|_{L^2(\lambda)}$ are constant with respect to t, M is also an upper bound of these quantities, for all t.

Proof of Lemma 4.1. We first prove (i). We notice that $K_1 \in L^\sigma$. Then, thanks to Hölder's and Young's inequalities, we have

$$\forall v, w \in L^2 \ , \ \forall u \in L^p \ , \ \|(K_1 * u.v)\ w\|_{L^{p'}} \leq C\ \|u\|_{L^p}\ \|v\|_{L^2}\ \|w\|_{L^2}\ , \tag{11}$$

$$\forall u, v \in L^2 \ , \ \forall w \in L^p \ , \ \|(K_1 * u.v)\ w\|_{L^{p'}} \leq C\ \|u\|_{L^2}\ \|v\|_{L^2}\ \|w\|_{L^p}\ . \tag{12}$$

Then, for a fixed $k \in \mathbb{N}$, we may estimate :

$$\begin{aligned}
A_k &:= \|(K_1 * \sum_j \lambda_j |\psi_j|^2)\psi_k - (K_1 * \sum_j \lambda_j |\phi_j|^2)\phi_k\|_{L^{p'}} \\
&\leq \|(K_1 * \sum_j \lambda_j (\psi_j - \phi_j)\psi_j^*)\ \psi_k\|_{L^{p'}} + \|(K_1 * \sum_j \lambda_j \phi_j (\psi_j - \phi_j)^*)\ \psi_k\|_{L^{p'}} \\
&\quad + \|(K_1 * \sum_j \lambda_j |\phi_j|^2)\ (\psi_k - \phi_k)\|_{L^{p'}} \\
&:= a + b + c.
\end{aligned}$$

Thanks to (11), we get

$$\begin{aligned}
a &\leq \sum_j \lambda_j \|K_1 * ((\psi_j - \phi_j).\psi_j^*)\ \psi_k\|_{L^{p'}} \\
&\leq C(p)\ \sum_j \lambda_j \|\psi_j - \phi_j\|_{L^p} \|\psi_j\|_{L^2} \|\psi_k\|_{L^2} \\
&\leq C(p)\ \|\psi - \phi\|_{L^p(\lambda)}\ \|\psi\|_{L^2(\lambda)}\ \|\psi_k\|_{L^2}.
\end{aligned}$$

The same estimates give

$$b \leq C(p)\ \|\phi\|_{L^2(\lambda)}\ \|\psi - \phi\|_{L^p(\lambda)}\ \|\psi_k\|_{L^2}.$$

And, thanks to (12)

$$c \leq C(p)\ \|\phi\|^2_{L^2(\lambda)}\ \|\psi_k - \phi_k\|_{L^p}.$$

A summation over k and Hölder's inequality in time give

$$\begin{aligned}
&\|V_1(\psi(t))\ \psi(t) - V_1(\phi(t))\ \phi(t)\|_{L_T^{q',p'}(\lambda)} = \|(\textstyle\sum_k \lambda_k A_k^2)^{1/2}\|_{L_T^{q'}} \\
&\leq C(p)\ \Big\|(\sum_k \lambda_k \|\phi\|^2_{L^2(\lambda)}\ \|\psi - \phi\|^2_{L^p(\lambda)}\ \|\psi_k\|^2_{L^2} + \lambda_k \|\phi\|^4_{L^2(\lambda)}\ \|\psi_k - \phi_k\|^2_{L^p})^{1/2}\ \Big\|_{L_T^{q'}} \\
&\leq C(p)\ \Big\|\ (\|\phi\|^2_{L^2(\lambda)}\ \|\psi - \phi\|^2_{L^p(\lambda)}\ \|\psi\|^2_{L^2(\lambda)} + \|\phi\|^4_{L^2(\lambda)}\ \|\psi - \phi\|^2_{L^p(\lambda)}\ )^{1/2}\ \Big\|_{L_T^{q'}} \\
&\leq C(p)\ M^2\ \|\ \|\psi - \phi\|_{L^p(\lambda)}\ \|_{L_T^{q'}} \\
&\leq C(p)\ M^2\ T^{1-\frac{2}{q}}\ \|\psi - \phi\|_{L_T^{q,p}(\lambda)}\ ,
\end{aligned}$$

and the point (i) is proved. Next, we note that the proof of (ii) is exactly the same, thanks to the inequality

$$\forall u, v, w \in L^2 \ , \ \|(K_2 * u.v)\ w\|_{L^2} \leq C\|u\|_{L^2}\ \|v\|_{L^2}\ \|w\|_{L^2}\ ,$$

because $K_2 \in L^\infty$. And the proof of Lemma 4.1 is complete. ■

Proof of Theorem 2.2.
First step. We begin with the existence statement. Let $T > 0$ to be chosen later. And let $\phi_m \in H^1(\lambda)$ be a sequence of initial datas satisfying $\phi_m \overset{m\to\infty}{\longrightarrow} \phi \in L^2(\lambda)$. Let $\psi_m(t) \in H^1(\lambda)$ be the corresponding solutions to the SPS. Let also $M = \sup_{m,t}(\|\psi_m(t)\|_{L^2(\lambda)}) < \infty$.

We show that $\psi_m(t)$ is a Cauchy sequence in the spaces $L_T^{a,b}(\lambda)$ for every admissible pair (a, b), and for some $T > 0$ sufficiently small. Indeed, for all $m \in \mathbb{N}$ and $t \in \mathbb{R}$, we can write

$$\psi_m(t) = T(t)\phi_m - i\int_0^t T(t-s)V_1(\psi_m)(s)\ \psi_m(s)ds\ - i\int_0^t T(t-s)V_2(\psi_m)(s)\ \psi_m(s)ds.$$

Hence, for all $m, k \in \mathbb{N}$ and $t \in \mathbb{R}$, we have

$$\begin{aligned}
\psi_m(t) - \psi_k(t) &= T(t)(\phi_m - \phi_k) \\
&\quad -i\int_0^t T(t-s)(V_1(\psi_m)(s)\ \psi_m(s) - V_1(\psi_k)(s)\ \psi_k(s))ds \\
&\quad -i\int_0^t T(t-s)(V_2(\psi_m)\ \psi_m(s) - V_1(\psi_k)\ \psi_k(s))ds \\
&:= \alpha + \beta + \gamma\ .
\end{aligned}$$

From now on, let (a, b) be *any* admissible pair. We estimate the $L_T^{a,b}(\lambda)$ norm of α, β, and γ, thanks to Theorem 2.1 and Lemma 4.1 :

$$\|\alpha\|_{L_T^{a,b}(\lambda)} \le C(a)\ \|\phi_m - \phi_k\|_{L^2(\lambda)}\ , \tag{13}$$

$$\|\beta\|_{L_T^{a,b}(\lambda)} \le C(a,q)\ \|V_1(\psi_m)(t)\ \psi_m(t) - V_1(\psi_k)(t)\ \psi_k(t)\|_{L_T^{q',p'}(\lambda)}\ , \tag{14}$$

where (q, p) is the admissible pair defined in Lemma 4.1. Note here the importance of Theorem 2.1 (ii) : we can control *any* $L_T^{a,b}(\lambda)$ norm of β in terms of *one* single $L_T^{q',p'}(\lambda)$ norm. Hence (14) implies, together with Lemma 4.1 (i),

$$\|\beta\|_{L_T^{a,b}(\lambda)} \le C(a,q)\ M^2\ T^{1-\frac{2}{q}}\ \|\psi_m(t) - \psi_k(t)\|_{L_T^{q,p}(\lambda)}\ . \tag{15}$$

Finally, Theorem 2.1 (ii) and Lemma 4.1 (ii) implie

$$\begin{aligned}
\|\gamma\|_{L_T^{a,b}(\lambda)} &\le C(a)\ \|V_2(\psi_m)(t)\ \psi_m(t) - V_2(\psi_k)(t)\ \psi_k(t)\|_{L_T^{1,2}(\lambda)} \\
&\le C(a)\ M^2\ T\ \|\psi_m(t) - \psi_k(t)\|_{L_T^{\infty,2}(\lambda)}\ .
\end{aligned} \tag{16}$$

Collecting the inequalities (13),(15), and (16) gives

$$\|\psi_m(t)-\psi_k(t)\|_{L_T^{a,b}(\lambda)} \le C(a,q)\Big(\|\phi_m-\phi_k\|_{L^2(\lambda)} + M^2\,T^{1-\frac{2}{q}} \quad (17)$$
$$\|\psi_m(t)-\psi_k(t)\|_{L_T^{q,p}(\lambda)} + M^2\,T\,\|\psi_m(t)-\psi_k(t)\|_{L_T^{\infty,2}(\lambda)}\Big)\,.$$

The inequality (17) holds for *any* admissible pair (a,b). Therefore, if we write it in the special case $(a,b)=(q,p)$ and then in the case $(a,b)=(\infty,2)$ we get, after summation

$$\|\psi_m(t)-\psi_k(t)\|_{X_T^{q,p}} \le C(q)\,\|\phi_m-\phi_k\|_{L^2(\lambda)} \quad (18)$$
$$+C(q)\,M^2\,T^{1-\frac{2}{q}}\,\|\psi_m(t)-\psi_k(t)\|_{X_T^{q,p}}\,,$$

where we have taken $T\le 1$ in order to upper bound T by $T^{1-\frac{2}{q}}$ in (17) (recall that $X_T^{q,p}=L_T^{q,p}(\lambda)\bigcap L_T^{\infty,2}(\lambda)$). Finally, the inequality (18) shows that there is a $T_0=T_0(q,M)$ sufficiently small such that

$$\|\psi_m(t)-\psi_k(t)\|_{L_{T_0}^{q,p}(\lambda)\,\bigcap\,L_{T_0}^{\infty,2}(\lambda)} \le 2\,C(q,M)\,\|\phi_m-\phi_k\|_{L^2(\lambda)}.$$

Therefore, ψ_m is Cauchy in $L_{T_0}^{q,p}(\lambda)\bigcap L_{T_0}^{\infty,2}(\lambda)$. Also, inequality (17) shows in turn, for a fixed a, that there is a $T_a=T(a,q,M)$ sufficiently small such that

$$\|\psi_m(t)-\psi_k(t)\|_{L_{T_a}^{a,b}(\lambda)} \le 2\,C(a,q,M)\,\|\phi_m-\phi_k\|_{L^2(\lambda)}.$$

We now have proved the existence of a limit ψ such that :

$$\psi_m \overset{m\to\infty}{\longrightarrow} \psi \ \textit{in}\ L_{T_0}^{q,p}(\lambda),\ L_{T_a}^{a,b}(\lambda),\ \textit{and}\ L_{T_0}^{\infty,2}(\lambda). \quad (19)$$

As the sequence ψ_m belongs to $C^0([-T_0,T_0];L^2(\lambda))$ and its $L^2(\lambda)$ norm is independent of t, we get

$$\psi\in C^0([-T_0,T_0];L^2(\lambda)) \quad , \quad \|\psi(t)\|_{L^2(\lambda)}=\|\phi\|_{L^2(\lambda)}. \quad (20)$$

We now remark that $T_0=T_0(q,M)$ and $T_a=T(a,q,M)$, and as above mentioned (See (20)), the upper bound M (depending on $\|\phi\|_{L^2(\lambda)}=\|\psi|_{t=0}\|_{L^2(\lambda)}$) can be chosen constant during the time evolution, so that one can reiterate the argument (with initial data $\psi(T_0)$, $\psi(2T_0)$, ...) and cover the whole real line. Therefore the statements (19), (20) hold true for all $T_0>0$, and

$$\psi_m \overset{m\to\infty}{\longrightarrow} \psi \ \ \textit{in}\ L_{loc}^{q,p}(\lambda),\ L_{loc}^{a,b}(\lambda)\ \textit{as}\ (a,b)\in\mathcal{A},\ C^0(\mathbb{R};L^2(\lambda)).$$

The last point we would like to make clear before ending this part of the proof is the equations satisfied by the limit ψ.

We have, for all $m, j \in \mathbb{N}$ and for all $t \in \mathbb{R}$

$$\partial_t \psi_m^j = \frac{i}{2}\Delta\psi_m^j - iV_1(\psi_m)\ \psi_m^j - iV_2(\psi_m)\ \psi_m^j \tag{21}$$

where the ψ_m^j are the components of the vector ψ_m (and the same notation for the limit ψ). And the convergence of ψ_m^j to ψ^j in $C^0(\mathbb{R}; L^2(\lambda))$ implies that

$$\psi_m^j \overset{m\to\infty}{\longrightarrow} \psi^j \quad , \quad \Delta\psi_m^j \overset{m\to\infty}{\longrightarrow} \Delta\psi^j \quad in \quad \mathcal{D}'. \tag{22}$$

Also, $\psi_m \overset{m\to\infty}{\longrightarrow} \psi$ in $C^0(L^2(\lambda)) \bigcap L^{q,p}_{loc}(\lambda)$. So, by Lemma 4.1 (i), we get

$$V_1(\psi_m)\ \psi_m \overset{m\to\infty}{\longrightarrow} V_1(\psi)\ \psi \quad in \quad L^{q',p'}_{loc}(\lambda) \tag{23}$$

hence this convergence holds in $\mathcal{D}'$ for each component. Next, by Lemma 4.1 (ii),

$$V_2(\psi_m)\ \psi_m \overset{m\to\infty}{\longrightarrow} V_2(\psi)\ \psi \quad in \quad L^{1,2}_{loc}(\lambda) \tag{24}$$

hence this convergence holds in $\mathcal{D}'$ for each component.

Finally, note that the density $n_m := \sum_j \lambda_j |\psi_m|^2$ also converges to $n := \sum_j \lambda_j |\psi|^2$ in $C^0(L^1)$. Thus, the statements (21)-(24) clearly imply that the Schrödinger equation holds for each ψ_j :

$$\partial_t \psi^j = \frac{i}{2}\Delta\psi^j - iV_1(\psi)\ \psi^j\ - iV_2(\psi)\ \psi^j = \frac{i}{2}\Delta\psi^j - iV(\psi)\ \psi^j$$

But,we can also pass to the limit in the integral equation, and obtain alternatively

$$\begin{aligned}\psi(t) &= T(t)\phi - i\int_0^t T(t-s)V_1(\psi)\ \psi(s)ds\ - i\int_0^t T(t-s)V_2(\psi)\ \psi(s)ds \\ &= T(t)\phi - i\int_0^t T(t-s)V(\psi)(s)\ \psi(s)ds\ .\end{aligned}$$

This concludes the first part of the proof (existence).

Second step. We now prove uniqueness. Let ψ be a solution to the SPS in the class $C^0(\mathbb{R}; L^2(\lambda)) \bigcap L^{q,p}_{loc}(\lambda)$. We first prove that ψ, which is a solution to the SPS considered as a Partial Differential Equation, is also a solution to the SPS in the integral form used above (see the Remark 2.2).

Indeed, Lemma 4.1 implies that $V_1(\psi)\ \psi$ belongs to $L^{q',p'}_{loc}(\lambda)$ and $V_2(\psi)\ \psi$ belongs to $L^{1,2}_{loc}$ (See (23) and (24) above). Then, one checks that

$$\forall j\ ,\ \partial_s T(t-s)\psi_j(s) = -i\ T(t-s)V_1(\psi(s))\psi_j(s) - i\ T(t-s))V_2(\psi(s))\psi_j(s)\ ,$$

holds, e.g. in the distributional sense. Hence, by integration on s, one sees that ψ is also a solution to the integral equation

$$\psi(t) = T(t)\phi - i\int_0^t T(t-s)V_1(\psi(s))\psi(s)\ - i\int_0^t T(t-s)V_2(\psi(s))\psi(s).$$

This last integration on s requires an easy regularization argument, and we skip the proof.

We now check the uniqueness on this integral formulation, first locally in time, hence globally, as in the "existence" part of the proof. Indeed, the estimates used above show that two solutions of the SPS that coincide at $t=0$ also coincide on $[-T_0, T_0]$ for sufficiently small $T_0 > 0$ (estimate (18)), and one can reiterate the argument thanks to (20).

It remains to prove the continuous depence on the initial data, but it is an easy consequence of the methods used in the "existence" part of the proof, and we skip it. ■

5 Proof of Theorem 2.3

We need a preliminary remark. In the case of the L^p spaces, one has the classical result (See [Cz]),

$$\|T(t)\phi\|_{L^p} \leq \frac{C(\|\phi\|_{L^2}, \|x\ \phi\|_{L^2})}{|t|^{3(\frac{1}{2}-\frac{1}{p})}}\ ,\ p \in [2,6].$$

One can easily extend this result in the weighted $L^p(\lambda)$ spaces, and get

$$\|T(t)\phi\|_{L^p(\lambda)} \leq \frac{C(\|\phi\|_{L^2(\lambda)}, \|x\ \phi\|_{L^2(\lambda)})}{|t|^{3(\frac{1}{2}-\frac{1}{p})}}\ ,\ p \in [2,6].$$

Then, let $T>0$ and $p \in]2,6[$, the case $p=2$ being obvious in the sequel, and denote C a constant $C(\|\phi\|_{L^2(\lambda)}, \|x\ \phi\|_{L^2(\lambda)}, T, p)$. The case $p=6$ will be treated in section 6 below. With the notations of Theorem 2.3, we have

$$\psi(t) = T(t)\phi + \int_0^t T(t-s)(\frac{C}{r} * (\sum_j \lambda_j \psi_j \psi_j^*)).\psi(s)ds.$$

Therefore, for $t>0$,

$$\begin{aligned}\|\psi(t)\|_{L^p(\lambda)} &\leq \frac{C}{t^{3(\frac{1}{2}-\frac{1}{p})}} + \int_0^t \frac{C}{(t-s)^{3(\frac{1}{2}-\frac{1}{p})}}\ \|\frac{1}{r} * (\sum_j \lambda_j \psi_j \psi_j^*).\psi(s)\|_{L^{p'}(\lambda)} \\ &\leq C\ \frac{1}{t^{3(\frac{1}{2}-\frac{1}{p})}} + C\int_0^t \frac{1}{(t-s)^{3(\frac{1}{2}-\frac{1}{p})}}\ \|\psi(s)\|_{L^a(\lambda)}\ \|\psi(s)\|_{L^2(\lambda)}\ \|\psi(s)\|_{L^2(\lambda)}\ ds\ ,\end{aligned}$$

with $\frac{1}{a} = \frac{2}{3} - \frac{1}{p}$, thanks to Hölder and Riesz inequalities. Hence, using Hölder's inequality in time

$$\begin{aligned}\|\psi(t)\|_{L^p(\lambda)} \leq \quad & \frac{C}{t^{3(\frac{1}{2}-\frac{1}{p})}} \\ & + Ct^{-3(\frac{1}{2}-\frac{1}{p})+\frac{1}{\mu}}\ M^2\ (\int_0^t \|\psi(s)\|^{\alpha}_{L^a(\lambda)}\ ds)^{\frac{1}{\alpha}}. \qquad (25)\end{aligned}$$

Here, we have set

$$M = \|\phi\|_{L^2(\lambda)} = \|\psi(t)\|_{L^2(\lambda)} \ ; \ \ \frac{1}{\alpha} = \frac{3}{2p} - \frac{1}{4} \ \in]0, \frac{1}{2}[$$

$$\frac{1}{\mu} = \frac{5p-6}{4p} \ \in]\frac{1}{2}, 1[\ ; \ \ t \leq T \ .$$

But one checks that $\frac{2}{\alpha} = 3(\frac{1}{2} - \frac{1}{a})$, and $\alpha \in]2, \infty[$, $a \in]2, 6[$, so that (α, a) is admissible. Therefore ψ belongs to $L^{\alpha,a}_{loc}(\lambda)$.

Thus (25) implies

$$\|\psi\|_{L^p(\lambda)} \leq \frac{C}{t^{3(\frac{1}{2}-\frac{1}{p})}}$$

This proves the part (i) of Theorem 2.3 for $p \in [2, 6[$. The case $p = 6$ will be treated in section 6 below. The other statements in Theorem 2.3 are direct consequences of this estimate through Hölder and Riesz inequalities. ■

6 Proof of Theorem 2.4.

We divide the proof in two steps. First we prove that the pseudo-conformal law of Theorem 2.4 (vii) holds as an inequality (see below). In a second step, it is then possible to deduce the decay estimates of Theorem 2.4. In section 7 below, we will prove that the function $J\psi(t) := (x + it\nabla)\psi(t)$ (see the Remark 2.5) belongs to $C^0(\mathbb{R}; L^2(\lambda))$. We will deduce then Theorem 2.5, and also the pseudo-conformal law of Theorem 2.4 (vii) (as an equality).

First step. Proof of the pseudo-conformal law as an inequality. Let $\phi_m \in H^1(\lambda)$ be a sequence of initial datas which converge to ϕ in $L^2(\lambda)$ and such that $x\ \phi_m$ converges also to $x\ \phi$ in $L^2(\lambda)$. Let $\psi_m(t)$ be the corresponding solutions of the SPS. We have (See [ILZ]), for all t

$$\begin{aligned} &\|(x + it\nabla)\psi_m(t)\|^2_{L^2(\lambda)} + t^2\ \|\nabla V(\psi_m)(t)\|^2_{L^2} = \\ &\|x\ \phi_m\|^2_{L^2(\lambda)} + \int_0^t s\ \|\nabla V(\psi_m)(s)\|^2_{L^2}\ ds. \end{aligned} \qquad (26)$$

Now, we show that we can pass to the limit in each term, and we begin by $t^2\ \|\nabla V(\psi_m)(t)\|^2_{L^2}$. But, thanks to Hölder and Riesz inequalities, $\|\nabla V(\psi_m(t)) - \nabla V(\psi(t))\|_{L^2} =$

$$\begin{aligned} &= \ \|C\ \frac{x}{r^3} *_x \sum_j \lambda_j (|\psi^j_m(t)| - |\psi^j(t)|)\ (|\psi^j_m(t)| + |\psi^j(t)|)\|_{L^2} \\ &\leq \ C\ \|\sum_j \lambda_j (|\psi^j_m(t)| - |\psi^j(t)|)\ (|\psi^j_m(t)| + |\psi^j(t)|)\|_{L^{6/5}} \\ &\leq \ C\ \|\psi_m(t) - \psi(t)\|_{L^2(\lambda)}\ \|\psi_m(t) + \psi(t)\|_{L^3(\lambda)}. \end{aligned} \qquad (27)$$

Therefore, (27) and estimate (i) of Theorem 2.3 give, for all $0 < \varepsilon < t < T < +\infty$,

$$\|\nabla V(\psi_m(t)) - \nabla V(\psi(t))\|_{L^2} \quad \leq \quad C\ \|\psi_m(t) - \psi(t)\|_{L^2(\lambda)}, \tag{28}$$

where C is a constant $C(\varepsilon, T, \|\phi\|_{L^2(\lambda)}, \|x\ \phi\|_{L^2(\lambda)})$. But, as a consequence of the assumption $\phi_m \stackrel{m\to\infty}{\longrightarrow} \phi$, $\psi_m(t)$ converges to $\psi(t)$ in $C^0(\mathbb{R}; L^2(\lambda))$ (See Theorem 2.2 (iii)). Hence,

$$\nabla V(\psi_m(t)) \stackrel{m\to\infty}{\longrightarrow} \nabla V(\psi(t))\ in\ C^0([\varepsilon, T]; L^2) \quad \forall\ 0 < \varepsilon < T < +\infty. \tag{29}$$

Next, we treat the integral term of (26). We have, using Theorem 2.3 (iv),

$$\begin{aligned} |\int_0^\varepsilon s\ \|\nabla V(\psi_m(s))\|_{L^2}^2 ds| \quad &\leq \quad \int_0^\varepsilon s\ \frac{C(\|\phi\|_{L^2(\lambda)}, \|x\ \phi\|_{L^2(\lambda)}, T)}{s}\ ds \\ &\leq \quad \varepsilon\ C(\|\phi\|_{L^2(\lambda)}, \|x\ \phi\|_{L^2(\lambda)}, T). \end{aligned} \tag{30}$$

Therefore (30) together with (29) implies that the integral term in (26) converges in $C^0(\mathbb{R})$ to $\int_0^t s\ \|\nabla V(\psi_m)(s)\|_{L^2}^2$.

Thus, by taking the limit in (26) on each compact set $[\varepsilon, T]$, thanks to (29)-(30), we can conclude that, for all $t \in \mathbb{R}$,

$$(x + it\nabla)\psi(t) \in L^2(\lambda)\ ,$$

and also

$$\begin{aligned} \|(x + it\nabla)\psi(t)\|_{L^2(\lambda)}^2 + t^2\ \|\nabla V(\psi)(t)\|_{L^2}^2 \quad \leq \quad &\|x\ \phi\|_{L^2(\lambda)}^2 \\ &+ \int_0^t s\ \|\nabla V(\psi)(s)\|_{L^2}^2 ds. \end{aligned} \tag{31}$$

We will show in section 7 below that this inequality is in fact an equality.

Second step. Decay estimates as $t \to \infty$. We write, thanks to (31), and using the same arguments as in the second step

$$\|(x + it\nabla)\psi(t)\|_{L^2(\lambda)}^2 + t^2\ \|\nabla V(\psi)(t)\|_{L^2}^2 \leq \|(x + i\nabla)\psi(1)\|_{L^2(\lambda)}^2$$

$$+\|\nabla V(\psi)(1)\|_{L^2}^2 + \int_1^t s\ \|\nabla V(\psi)(s)\|_{L^2}^2 ds\ .$$

This implies, thanks to Theorem 2.3 (iv) together with (31) stated for $t = 1$,

$$t^2\ \|\nabla V(\psi)(t)\|_{L^2}^2 \quad \leq \quad \|(x + i\nabla)\psi(1)\|_{L^2(\lambda)}^2 + \|\nabla V(\psi)(1)\|_{L^2}^2 + \int_1^t s\ \|\nabla V(\psi)(s)\|_{L^2}^2 ds\ ,$$

hence,

$$t^2\ \|\nabla V(\psi)(t)\|_{L^2}^2 \quad \leq \quad C(\|\phi\|_{L^2(\lambda)}, \|x\ \phi\|_{L^2(\lambda)}) + \int_1^t s\ \|\nabla V(\psi)(s)\|_{L^2}^2 ds\ . \tag{32}$$

Therefore Gronwall's Lemma gives in (32)

$$\forall t \geq 1 \ , \quad \|\nabla V(\psi)(t)\|_{L^2} \leq \frac{C(\|\phi\|_{L^2(\lambda)}, \|x\ \phi\|_{L^2(\lambda)})}{t^{\frac{1}{2}}} \ . \tag{33}$$

And, combining (33) with Theorem 2.3 (iv) for $p = 2$, we obtain

$$\forall t \in \mathbb{R} \ , \quad \|\nabla V(\psi)(t)\|_{L^2} \leq \frac{C(\|\phi\|_{L^2(\lambda)}, \|x\ \phi\|_{L^2(\lambda)})}{|t|^{\frac{1}{2}}}.$$

This proves the statement (v) in Theorem 2.4 for $p = 2$. Also, we obtain, thanks to (31)

$$\forall t \in \mathbb{R} \ , \ \|(x + it\nabla)\psi(t)\|^2_{L^2(\lambda)} \leq C(\|\phi\|_{L^2(\lambda)}, \|x\ \phi\|_{L^2(\lambda)})\ (1 + |t|) \ . \tag{34}$$

As usual (See [Cz]), we consider $\psi_g(t) := \exp(-\frac{ix^2}{2t})\psi(t)$. We notice the

Lemma 6.1 *The following formula holds true :*
$it\nabla\psi_g(t) = \exp(-\frac{ix^2}{2t})(x + it\nabla)\psi(t)$.
In particular, for $t \neq 0$*, the assumption* $(x + it\nabla)\psi(t) \in L^2(\lambda)$ *is equivalent to* $\nabla\psi_g(t) \in L^2(\lambda)$.

Thanks to this lemma, (34) writes

$$\|\nabla\psi_g(t)\|_{L^2(\lambda)} \leq \frac{C(\|\phi\|_{L^2(\lambda)}, \|x\ \phi\|_{L^2(\lambda)})}{t^{\frac{1}{2}}}\ (1 + \frac{1}{t})^{\frac{1}{2}} \ . \tag{35}$$

Then, Gagliardo-Nirenberg's inequality gives

$$\|\psi_g(t)\|_{L^p(\lambda)} = \|\psi(t)\|_{L^p(\lambda)} \leq C(p)\ \|\nabla\psi_g(t)\|^a_{L^2(\lambda)}\ \|\psi_g(t)\|^{(1-a)}_{L^2(\lambda)} \ , \tag{36}$$

with $a = 3(\frac{1}{2} - \frac{1}{p})$, and $p \in [2, 6]$. Notice that Gagliardo-Nirenberg's inequality in the weighted spaces $L^p(\lambda)$ is a straightforward consequence of the usual one.

Finally, (35) and (36) give the asymptotic behaviour as $t \to 0$ and $t \to \infty$ of $\|\psi(t)\|_{L^p(\lambda)}$ for $p \in [2, 6]$. Now, Hölder's and Riesz' Inequalities give the results of Theorems 2.3 (ii)-(iv), 2.4 (ii). Now it remains to state the decay estimates of Theorem 2.4 for $V(t)$ and $\nabla V(t)$ as $t \to \infty$. Again, Riesz' Inequality gives the estimates

$$\begin{cases} \|\nabla V(t)\|_p \leq C\ |t|^{-(1-\frac{3}{2p})} \ , \ \forall\ p \in]3/2; \infty[\ , \ \forall\ |t| \geq 1 \ , \\ \|V(t)\|_p \leq C\ |t|^{-\frac{1}{2}(1-\frac{3}{p})} \ , \ \forall\ p \in]3; \infty[\ , \ \forall\ |t| \geq 1 \ . \end{cases}$$

Now, we write (Hölder),

$$\begin{aligned} \|\nabla V(t)\|_p &\leq \|\nabla V(t)\|_2^{\theta}\ \|\nabla V(t)\|_q^{(1-\theta)} \\ &\leq C\ |t|^{-\frac{\theta}{2}}\ |t|^{-(1-\theta)(1-\frac{3}{2q})} \ , \end{aligned}$$

and we choose successively $q < \infty$ but close to ∞ in the case $p > 2$, then $q > 3/2$ but close to $3/2$ in the case $p < 2$, and get Theorem 2.4 (v)-(vi), including the case $p = 2$ which is given in (33). Finally, we write (Gagliardo-Nirenberg),

$$\|V(t)\|_p \leq C \, \|\nabla V(t)\|_{\frac{3p}{3+p}} \ ,$$

and use Theorem 2.4 (v)-(vi) in order to get Theorem 2.4 (iii)-(iv). This method holds as $p \neq \infty$. In the case $p = \infty$, we should write,

$$\|V(t)\|_\infty \leq C \, \|\nabla V(t)\|_{3+\varepsilon}^{\theta} \, \|V(t)\|_r^{(1-\theta)} \ ,$$

and choose $\varepsilon > 0$ close to 0, $r \in]3, \infty[$, θ close to 1.

7 Proof of Theorems 2.5 and 2.6

The proof is performed in two steps. As it was announced at the beginning of section 6, we first prove that $J\psi(t) = (x + it\nabla)\psi(t) \in C^0(\mathbb{R}; L^2(\lambda))$. Then we deduce that the pseudo-conformal law of Theorem 2.4 (vii) holds as an equality, and also prove Theorem 2.5 as an easy consequence of the continuity of $J\psi$. In a second step, similar arguments allow to prove Theorem 2.6.

First step. Proof of the continuity of $J\psi(t)$. Let ϕ_m and ψ_m be as in section 6. One can easily check the following formula, where the ψ_m^j are the components of the vector ψ_m,

$$\partial_t J\psi_m = \frac{i}{2}\Delta J\psi_m - i(\frac{C}{r} * \sum_j \lambda_j |\psi_m^j|^2) \; J\psi_m + (\frac{2C}{r} * Im(\sum_j \lambda_j \psi_m^{j\,*} \; J\psi_m^j))\psi_m.$$

Hence this equation gives, for all $k, m \in \mathbb{N}$, the integral formulation

$$J\psi_m - J\psi_k = A + B + D \ , \tag{37}$$

where A, B, D are given by

$$\begin{aligned}
A &:= T(t)(x\phi_m - x\phi_k) \ , \\
B &:= -i \int_0^t T(t-s)\Big((\frac{C}{r} * \sum_j \lambda_j |\psi_m^j|^2) \; J\psi_m - (\frac{C}{r} * \sum_j \lambda_j |\psi_k^j|^2) \; J\psi_k \Big)(s) ds \ , \\
D &:= \int_0^t T(t-s)\Big((\frac{2C}{r} * Im(\sum_j \lambda_j \psi_m^{j\,*} \; J\psi_m^j))\psi_m \\
&\quad -(\frac{2C}{r} * Im(\sum_j \lambda_j \psi_k^{j\,*} \; J\psi_k^j))\psi_k \Big)(s) ds \ .
\end{aligned}$$

We now prove that the sequence $J\psi_m$ is Cauchy in the space $X_T^{q,p} := C^0([-T;T]; L^2(\lambda)) \bigcap L_T^{q,p}(\lambda)$ for $0 < T < 1$ sufficiently small, where the admissible pair (q, p) is as in Theorem 2.2 (see definition 2.2). In order to do this, we need the following Lemma.

Lemma 7.1 *Let $T > 0$, and σ, p, q be as in Lemma 4.1. Let $u, v, w \in C^0(\mathbb{R}; L^2(\lambda))$. Let $M(u,v) = \sup_{t\in[-T;T]}(\|u(t)\|_{L^2(\lambda)} + \|v(t)\|_{L^2(\lambda)})$. Then the following holds :*
(i) $\|(1/r * \sum_j \lambda_j u_j v_j) \cdot w\|_{Y_T^{q,p}} \leq C(q)\ M(u,v)\ \max(T^{1-\frac{2}{q}}, T)\ \|w\|_{X_T^{q,p}}$,
(ii) $\|(1/r * \sum_j \lambda_j u_j v_j) \cdot w\|_{Y_T^{q,p}} \leq C(q)\ M(u,w)\ \max(T^{1-\frac{2}{q}}, T)\ \|v\|_{X_T^{q,p}}$.

Proof of Lemma 7.1. It is a straightforward adaptation of the proof of Lemma 4.1. ■

We come back to the first step of our proof. First we note that

$$\|A\|_{X_T^{q,p}} \leq C(q) \|x\ \phi_m - x\ \phi_k\|_{L^2(\lambda)}\ .$$

Let now $M := \sup_{t\in[-1;1]}(\|J\psi_m(t)\|_{L^2(\lambda)} + \|J\psi_k(t)\|_{L^2(\lambda)} + \|\psi_m(t)\|_{L^2(\lambda)} + \|\psi_k(t)\|_{L^2(\lambda)})$. Section 6 (see (34)) shows that M is well-defined and depends only upon $\|\phi\|_{L^2(\lambda)}$, $\|x\ \phi\|_{L^2(\lambda)}$. We are able to majorize B and D as in the proof of Theorem 2.2 (see estimates (14)-(18)). Indeed, thanks to Lemma 7.1 and Theorem 2.1 (iii),

$$\|B\|_{X_T^{q,p}} \leq C(q)\ M^2\ T^{1-\frac{2}{q}}\ (\ \|J\psi_m - J\psi_k\|_{X_T^{q,p}}\ +\ \|\psi_m - \psi_k\|_{X_T^{q,p}}\ )\ ,$$

$$\|D\|_{X_T^{q,p}} \leq C(q)\ M^2\ T^{1-\frac{2}{q}}\ (\ \|J\psi_m - J\psi_k\|_{X_T^{q,p}}\ +\ \|\psi_m - \psi_k\|_{X_T^{q,p}}\ )\ .$$

Therefore, if C denotes any constant depending only upon $\|\phi\|_{L^2(\lambda)}$, $\|x\ \phi\|_{L^2(\lambda)}$, (37) implies

$$\begin{aligned} \|J\psi_m - J\psi_p\|_{X_T^{q,p}} \ &\leq\ C\ \|x\ \phi_m - x\ \phi_k\|_{L^2(\lambda)} \\ &\quad + C\ T^{1-\frac{2}{q}}\ (\|J\phi_m - J\phi_k\|_{X_T^{q,p}} + \|\psi_m - \psi_k\|_{X_T^{q,p}})\ . \end{aligned} \tag{38}$$

Now, for $T = T(\|\phi\|_{L^2(\lambda)}\ ,\ \|x\ \phi\|_{L^2(\lambda)})$ sufficiently small, we conclude as in section 4 that $J\psi_m$ is Cauchy in $X_T^{q,p}$, since ψ_m is also Cauchy in this space.

Now, starting the same argument on $[t-T'; t+T']$, we find therefore a T' depending only on $\|\psi(t)\|_{L^2(\lambda)}$, $\|J\ \psi(t)\|_{L^2(\lambda)}$ such that $J\psi_m$ is Cauchy in $C^0([t-T'; t+T']; L^2(\lambda)) \bigcap L^q([t-T'; t+T']; L^p(\lambda))$. Since we already know by section 6 that $\|J\ \psi(t)\|_{L^2(\lambda)} < \infty$ for all t, this allows us to cover the whole real line and show

$$J\psi(t) \in C^0(\mathbb{R}; L^2(\lambda)) \bigcap L^{q,p}_{loc}(\lambda)\ .$$

Indeed, any compact interval of time is a finite union of small intervals $[t-T'; t+T']$.

Also, given a $t \in \mathbb{R}$, we can take the limit in (26) as in the first step of section 6, but this time in $C^0(\mathbb{R}; L^2(\lambda))$, and we get the equality in the pseudo-conformal law.

The standard use of ψ_g (see Lemma 6.1 above) gives now Theorem 2.5. Indeed, the assumption $J\psi(t) \in C^0(\mathbb{R}; L^2(\lambda))$ implies $\psi_g \in C^0(\mathbb{R}^*; H^1(\lambda))$. Thanks to

Gagliardo-Nirenberg's inequality (36), we get $\psi_g \in C^0(\mathbb{R}^*; L^2(\lambda) \bigcap L^6(\lambda))$, hence Theorem 2.5.

Second step. Proof of Theorem 2.6. Theorem 2.6 is now proved in the case where $k = 1$. We now deduce the general case from the following

Lemma 7.2 *Let ϕ be as in Theorem 2.6 ($k \neq \infty$), $\phi_m \in H^1(\lambda)$ such that $x^\alpha \phi_m$ converges to $x^\alpha \phi$ in $L^2(\lambda)$ for all $|\alpha| \leq k$. Let ψ_m be the corresponding sequence of solutions to the SPS. Finally, let $T > 0$. Then the following holds :*
(i) $\|J^\alpha \psi_m\|_{X_T^{q,p}} \leq C(T, q, \|(1+x^2)^{k/2}\phi\|_{L^2(\lambda)})$,
(ii) $J^\alpha \psi_m$ is Cauchy in $X_T^{q,p}$.

Proof of Lemma 7.2. The Lemma is already proved if $k = 1$. Suppose it is proved up to order k. Let now α, $|\alpha| = k+1$. Coming back to the SPS, we easily get

$$J^\alpha \psi_m \qquad = T(t) x^\alpha \phi_m + A + B + D \ , \tag{39}$$

where

$$\begin{aligned} A &= -i \int_0^t T(t-s) \Big(\frac{C}{r} * \sum_j \lambda_j [(-1)^\alpha (J^\alpha \psi_m^j)^* \psi_m^j + \psi_m^{j\,*} J^\alpha \psi_m^j] \Big) \psi_m(s) ds \ , \\ B &= -i \int_0^t T(t-s) \Big(\frac{C}{r} * \sum_j \lambda_j \psi_m^{j\,*} \psi_m^j \Big) J^\alpha \psi_m(s) ds \ , \\ D &= -i \sum_{a,b,c} \int_0^t T(t-s) \Big(\frac{C}{r} * \sum_j \lambda_j (-1)^a (J^a \psi_m^j)^* J^b \psi_m^j \Big) J^c \psi_m(s) ds \ . \end{aligned}$$

In D, the sum is taken over all multi-index $a, b, c \neq \alpha$ such that $a + b + c = \alpha$.

We prove (i). Let $0 < T_0 < 1$, $T_0 \leq T$, to be chosen later. We have, thanks to Theorem 2.1 (i),

$$\|T(t) x^\alpha \phi_m\|_{X_{T_0}^{q,p}} \leq C(q) \|x^\alpha \phi_m\|_{L^2(\lambda)} \ .$$

Besides, thanks to the induction hypothesis, Lemma 7.1 and Theorem 2.1 (iii) give

$$\|D\|_{X_{T_0}^{q,p}} \leq C(T, q, \|(1+x^2)^{k/2}\phi\|_{L^2(\lambda)}).$$

Finally, we can again majorise A and B as we did for D, and write for instance

$$\|A\|_{X_{T_0}^{q,p}} \leq T_o^{1-\frac{2}{q}} \ C(T, q, \|\phi\|_{L^2(\lambda)}) \ \|J^\alpha \psi_m\|_{X_{T_0}^{q,p}} \ ,$$

and the same for B. Collecting the inequalities in (39), we get

$$\begin{aligned} \|J^\alpha \psi_m\|_{X_{T_0}^{q,p}} \ \leq \ & C(q) \|x^\alpha \phi_m\|_{L^2(\lambda)} + T_o^{1-\frac{2}{q}} \ C(T, q, \|\phi\|_{L^2(\lambda)}) \ \|J^\alpha \psi_m\|_{X_{T_0}^{q,p}} \\ & + C(T, q, \|(1+x^2)^{k/2}\phi\|_{L^2(\lambda)}) \ . \end{aligned} \tag{40}$$

Now we choose $T_0^{1-\frac{2}{q}} = 1/2C(T, q, \|\phi\|_{L^2(\lambda)})$ in (40) and get

$$\|J^\alpha \psi_m\|_{X^{q,p}_{T_0}} \leq C(q)\|x^\alpha \phi_m\|_{L^2(\lambda)} + C(T, q, \|(1+x^2)^{k/2}\phi\|_{L^2(\lambda)}) .$$

Starting back from T_0, and reiterating the argument on $[0, 2T_0]$ with the *same* T_0 gives now, with obvious notations

$$\begin{aligned}\|J^\alpha \psi_m\|_{X^{q,p}([0;2T_0])} &\leq C(q)\|J^\alpha \psi_m(T_0)\|_{L^2(\lambda)} + C(T, q, \|(1+x^2)^{k/2}\phi\|_{L^2(\lambda)}) \\ &\leq C(q)\|x^\alpha \phi_m\|_{L^2(\lambda)} + C(T, q, \|(1+x^2)^{k/2}\phi\|_{L^2(\lambda)}) .\end{aligned}$$

Hence we can cover the whole interval [-T;T] and get Lemma 7.2 (i).

Part (ii) of this Lemma is obtained through the same manipulations and makes use of the point (i). We refer also to the first step of this section. ■

Now, Lemma 7.2 (ii) clearly shows the point (i) in Theorem 2.6.

The point (ii) is given by considering the function ψ_g. Let $t \neq 0$ and α satisfy $|\alpha| \leq k$. The point (i) of Theorem 2.6, combined with Lemma 6.1 above, implies

$$\nabla^\alpha \psi_g(t) = (\frac{1}{it})^\alpha \exp(\frac{-ix^2}{2t}) J^\alpha \psi(t) \in C^0(\mathbb{R}^*; L^2(\lambda)) ,$$

hence

$$\psi_g(t) \in C^0(\mathbb{R}^*; H^k(\lambda)) .$$

Therefore, with the notation of Theorem 2.6, we get

$$\psi_g(t) \in C^0(\mathbb{R}^*; E_k) .$$

But for any k, we note that $\|\psi_g(t)\|_{E_k} = \|\psi(t)\|_{E_k}$, because $\psi(t) = \exp(\frac{ix^2}{2t})\psi_g(t)$. Therefore, the point (ii) of Theorem 2.6 is proved.

8 Appendix

In the previous sections, we have developed a theory for the SPS with rough initial datas, and observed in this case strong smoothing properties. We show here how one can get a similar theory for smooth initial datas. We begin with the following

Lemma 8.1 *(i) Let $u, v, w \in H^1(\lambda)$. Then,*

$$\begin{aligned}\|(1/r * \sum_j \lambda_j u_j v_j) w\|_{H^1(\lambda)} &\leq C\|u\|_{L^2(\lambda)}\|v\|_{L^2(\lambda)}\|w\|_{H^1(\lambda)} , \\ \|(1/r * \sum_j \lambda_j u_j v_j) w\|_{H^1(\lambda)} &\leq C\|u\|_{L^2(\lambda)}\|v\|_{H^1(\lambda)}\|w\|_{L^2(\lambda)} ,\end{aligned}$$

(ii) Let $m \geq 1$ *and* $u, v, w \in H^m(\lambda)$. *Then,*

$$\|(1/r * \sum_j \lambda_j u_j v_j) w\|_{H^m(\lambda)} \leq C \|u\|_{H^m(\lambda)} \|v\|_{H^m(\lambda)} \|w\|_{H^m(\lambda)} ,$$

(iii) Let $m \geq 1$. *Then, the nonlinearity* $V(\psi)\psi = (1/r * \sum_j \lambda_j |\psi_j|^2)\psi$ *is locally Lipschitz in* $H^m(\lambda)$.

Proof of Lemma 8.1. Thanks to the Sobolev imbedding $H^m(\lambda) \subset L^2(\lambda) \bigcap L^6(\lambda)$, a straightforward adaptation of the proof of Lemma 4.1 above gives

$$\|(1/r * \sum_j \lambda_j u_j v_j) w\|_{L^2(\lambda)} \leq C \|u\|_{L^2(\lambda)} \|v\|_{L^2(\lambda)} \|w\|_{H^1(\lambda)} , \tag{41}$$

$$\|(1/r * \sum_j \lambda_j u_j v_j) w\|_{L^2(\lambda)} \leq C \|u\|_{L^2(\lambda)} \|v\|_{H^1(\lambda)} \|w\|_{L^2(\lambda)} . \tag{42}$$

We write then

$$\nabla\Big((1/r * \textstyle\sum_j \lambda_j u_j v_j) w\Big) = (1/r * \sum_j \lambda_j \nabla(u_j) v_j) w + (1/r * \sum_j \lambda_j u_j \nabla(v_j)) w$$
$$+ (1/r * \textstyle\sum_j \lambda_j u_j v_j) \nabla w ,$$

and, thanks to (41), (42), we get the desired estimates on $\|\nabla(1/r * \sum_j \lambda_j u_j v_j) w\|_{L^2(\lambda)}$. Therefore, (i) is proved.

Then, (ii) becomes immediate by induction on m. Indeed, (ii) is proved for $m = 1$ and, for any multiindex α, we have

$$\nabla^\alpha\Big((1/r * \sum_j \lambda_j u_j v_j) w\Big) = \sum_{\beta,\gamma,\delta} (1/r * \sum_j \lambda_j \nabla^\beta u_j \nabla^\gamma v_j) \nabla^\delta w ,$$

where the sum is taken over all β, γ, δ such that $\beta + \gamma + \delta = \alpha$. We note that $\nabla^\beta u \in H^{m-|\beta|}(\lambda)$, $\nabla^\gamma v \in H^{m-|\gamma|}(\lambda)$, $\nabla^\delta w \in H^{m-|\delta|}(\lambda)$. But, the condition $\beta + \gamma + \delta = \alpha$ implies $|\beta| + |\gamma| + |\delta| = |\alpha| = m \geq 1$. Thus, at least *one* of the integers $m - |\beta|$, $m - |\gamma|$, or $m - |\delta|$ is greater than 1. We conclude thanks to the Lemma 8.1 (i), by writing, say in the case $m - |\delta| \geq 1$,

$$\begin{aligned} \|(1/r * \sum_j \lambda_j \nabla^\beta u_j \nabla^\gamma v_j) \nabla^\delta w\|_{L^2(\lambda)} &\leq C\ \|\nabla^\beta u\|_{L^2(\lambda)} \|\nabla^\gamma v\|_{L^2(\lambda)} \|\nabla^\delta w\|_{H^1(\lambda)} \\ &\leq C \|u\|_{H^m(\lambda)} \|v\|_{H^m(\lambda)} \|w\|_{H^m(\lambda)} . \end{aligned}$$

A summation over the multi-index gives the result (ii).

Now, (iii) is an easy consequence of (ii). Thus, Lemma 8.1 is proved. ■

Lemma 8.1 (iii) immediately gives the *local-in-time* well-posedness of the SPS in $H^m(\lambda)$ for $m \geq 1$ (see, e.g. [Pa]). But this can be improved, since we have the

Theorem 8.1 *Let $m \geq 1$, and $\phi \in H^m(\lambda)$. Then the SPS with initial data ϕ has a unique global solution*

$$\psi(t) \in C^0(\mathbb{R}; H^m(\lambda)) \ .$$

Moreover, the following regularity holds for all k such that $m - 2k \geq -1$,

$$\psi(t) \in C^0(\mathbb{R}; H^m(\lambda)) \bigcap C^1(\mathbb{R}; H^{m-2}(\lambda)) \bigcap ... \bigcap C^k(\mathbb{R}; H^{m-2k}(\lambda)) \ .$$

Proof of Theorem 8.1. In order to prove the first part of the Theorem, it suffices to show that the local solution $\psi(t)$ built through Lemma 8.1 is in fact global. Thus, we now prove that $\|\psi(t)\|_{H^m(\lambda)}$ is bounded on bounded intervals.

This result is proved in [Ar] when $m = 1$ and in [ILZ] when $m = 2$. Now, we prove it by induction on m. We write,

$$\psi(t) = T(t)\phi + \int_0^t T(t-s)V(\psi)\psi(s)ds \ ,$$

hence, since the group $T(t)$ is unitary on any space $H^m(\lambda)$ (it preserves the $L^2(\lambda)$ norm and it commutes with the derivation), we get,

$$\|\psi(t)\|_{H^m(\lambda)} \quad \leq \quad \|\phi\|_{H^m(\lambda)} + \int_0^t \|V(\psi)\psi(s)\|_{H^m(\lambda)} ds \ . \tag{43}$$

It remains to bound the integral term in (43). But, following Lemma 8.1, we get for $m \geq 2$,

$$\|V(\psi)\psi(s)\|_{H^m(\lambda)} \quad \leq \quad C \ \|\psi(s)\|^2_{H^{m-1}(\lambda)} \ \|\psi(s)\|_{H^m(\lambda)} \ . \tag{44}$$

(It suffices to count the orders of derivation as in the proof of Lemma 8.1 (ii)). Moreover, the induction on m gives that the term $\|\psi(s)\|_{H^{m-1}(\lambda)}$ in (44) is bounded on bounded intervals. Thus, if $t \in [-M; M]$, (43) together with (44) gives

$$\|\psi(t)\|_{H^m(\lambda)} \leq \|\phi\|_{H^m(\lambda)} + C \sup_{[-M;M]} (\|\psi(s)\|_{H^{m-1}(\lambda)})^2 \int_0^t \|\psi(s)\|_{H^m(\lambda)} ds \ ,$$

and we conclude thanks to Gronwall's Lemma.

The second part of Theorem 8.1 is obtained by induction on k. Indeed, we first write, thanks to Lemma 8.1, together with the assumption $\psi \in C^0(\mathbb{R}; H^m(\lambda))$,

$$\partial_t \psi = \frac{i}{2} \Delta\psi - iV(\psi)\psi \in C^0(\mathbb{R}; H^{m-2}(\lambda)) \ ,$$

(the loss of two derivatives is in fact due to the Laplacian). Then, we set $u := \partial_t \psi$ and write

$$\partial_{t,t}\psi \quad = \quad \frac{i}{2}\Delta u + A(t) + B(t) \ , \tag{45}$$

where $A(t)$, $B(t)$ are given through

$$A(t) := -i\Big(\frac{2C}{r} * Re(\sum_j \lambda_j u_j \psi_j^*)\Big)\psi ,$$
$$B(t) := -i(\frac{C}{r} * \sum_j \lambda_j |\psi_j|^2) u .$$

Lemma 8.1, together with the assumptions $\psi \in C^0(\mathbb{R}; H^m(\lambda))$, $u \in C^0(\mathbb{R}; H^{m-2}(\lambda))$, implies

$$\partial_{t,t}\psi \in C^0(\mathbb{R}; H^{m-4}(\lambda)) .$$

And one can reiterate the argument, applying again Lemma 8.1 in order to get that any time derivative $\psi^{(k)}(t)$ is continuous as soon as $\psi^{(k-1)}(t) \in H^1(\lambda)$. Thus, the induction can be carried out for values of k such that $m - 2(k-1) \geq 1$. This gives the second part of the Theorem. ■

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