The Helmholtz equation with source term: High-frequency analysis, and radiation condition at infinity

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Abstract: This talk presents a review of results obtained in [BCKP], [CPR], and more recently in [C]. References [BCKP] and [CPR] are joints works with J.D. Benamou, T. Katsaounis, B. Perthame, respectively B. Perthame and O. Runborg.
The first two papers [BCKP] and [CPR] describe the propagation of the energy along the high-frequency Helmholtz equation with source term, using a Wigner function approach. Upon conveniently "averaging out" the high-frequency oscillations, we characterize how the energy radiated from the source is propagated along the rays of geometric optics.
The last and more recent work [C] analyses the radiation condition at infinity for these equations, a question that was left open in [BCKP] and [CPR]. In some sense we prove that the energy is only propagated along the *outgoing* rays of geometric optics: the *ingoing* rays carry no energy. To obtain this result, we need to compute in an accurate way the amplitude and the phase of the high-frequency oscillations (no "averaging" is performed). We also need appropriate assumptions on the refraction index. We refer to [WZ] for similar and independent results.

1 Introduction

The aim of this talk is twofolds.
First (see [BCKP] and [CPR]), we wish to describe the asymptotic propagation of energy as $\varepsilon \to 0$ ($\varepsilon > 0$), in the following prototype high-frequency Helmholtz equation

$$i\varepsilon\alpha_\varepsilon u^\varepsilon(x) + \frac{\varepsilon^2}{2}\Delta_x u^\varepsilon(x) + n^2(x)u^\varepsilon(x) = \frac{1}{\varepsilon^{d/2}} S\left(\frac{x}{\varepsilon}\right) . \tag{1}$$

Here, the unknown is $u^\varepsilon(x) \in \mathbf{C}$, the space variable is $x \in \mathbf{R}^d$ for some $d \geq 3$, and $\alpha_\varepsilon > 0$ is a small absorption parameter that goes to zero with ε (one may think $\alpha_\varepsilon \sim \varepsilon^N$ for some large integer N). Also, $n^2(x)$ is the refraction index, and S is a given source term. We further discuss the meaning of equation (1) below. Our aim is to prove that the Wigner measure associated with the sequence u^ε, which characterizes some "average value" of the oscillations of u_ε, tends to satisfy a transport equation with source term. The latter describes how the energy radiated from the source is propagated along the rays of geometric optics. Our analysis allows more general right-hand-sides than in (1), see section 3.
Second (see [C]), we introduce the (L^2-) rescaled function $w^\varepsilon(x) = \varepsilon^{d/2}u^\varepsilon(\varepsilon x)$, which measures the behaviour of u^ε at the microscopic scale (scale ε). It satisfies

$$i\varepsilon\, \alpha_\varepsilon\, w^\varepsilon(x) + \frac{1}{2}\Delta_x w^\varepsilon(x) + n^2(\varepsilon x)w^\varepsilon(x) = S\,(x) . \tag{2}$$

Our aim is to prove that w^ε converges to the outgoing solution to the Helmholtz equation, with coefficients frozen at the origin, i.e. $\lim_{\varepsilon\to 0} w^\varepsilon = w^{\text{out}}$, where w^{out} is defined as the solution to

$$i0^+ w^{\text{out}}(x) + \frac{1}{2}\Delta_x w^{\text{out}}(x) + n^2(0)w^{\text{out}}(x) = S\,(x) . \tag{3}$$

This result trivially holds when $n(x)$ does not depend on x, but is difficult to establish otherwise. It asserts that the energy in (2) is radiated in the outgoing direction, uniformly in ε. Note that the convergence $\lim w^\varepsilon = w^{\rm out}$ is actually needed to fully justify the analysis skteched in the first point above. Proving $\lim w^\varepsilon = w^{\rm out}$ leads us to compute in an accurate way the amplitude and the phase of the oscillations of w^ε (no averaging is performed this time). We show that the convergence $\lim w^\varepsilon = w^{\rm out}$ holds true under certain geometric assumptions on the refraction index $n^2(x)$. We also show that the above convergence fails for certain geometries.

Before going further, let us first explain the meaning of equations (1) and (2).

As is well-known, the Helmholtz equation is a degenerate version of the wave equation, corresponding to time harmonic solutions of the latter (i.e. solutions of the form $\exp(i\omega t)\times$ function of space only). From this point of view, both equations (1) and (2) modelize, in a time harmonic setting, the propagation of waves emmitted through the source term $\varepsilon^{-d/2}S(x/\varepsilon)$ (resp. $S(x)$), in a medium of variable index of refraction $n^2(x)$. The source is concentrated close to the origin, and creates the signal $u^\varepsilon(x)$ (resp. $w^\varepsilon(x)$) in the whole space $\mathbf{R}^d$. In both equations, the small absorption parameter $\alpha_\varepsilon > 0$ corresponds to a small damping term in the underlying wave equation: think e.g. of the damped wave equation $n^2(\varepsilon x)\partial^2_{tt}\widetilde{w}^\varepsilon(t,x) - 1/2\ \Delta_x \widetilde{w}^\varepsilon(t,x) + \varepsilon\alpha_\varepsilon\partial_t\widetilde{w}^\varepsilon(t,x) = -\exp(-it)S(x)$, where $\widetilde{w}^\varepsilon(t,x)$ is searched under the form $\exp(-it)w^\varepsilon(x)$, and similarly for the function u^ε. The absorption α_ε makes the Helmholtz equation elliptic, in that the solutions u^ε and w^ε are uniquely defined through (1) and (2) by an obvious energy estimate.
Throughout this paper, the given index of refraction $n^2(x)$ is assumed smooth, uniformly positive, and it goes to a constant at infinity[1]:

$$\begin{aligned} &\forall x \in \mathbf{R}^d\ , \quad 0 < n_0^2 \le n^2(x) \le n_1^2 < \infty\ , \qquad \text{and}\ \ n^2(x) \in C^\infty(\mathbf{R}^d)\ , \\ &n^2(x) = n_\infty^2 + O\left(\langle x\rangle^{-\rho}\right) \ \text{ as }\ x \to \infty\ , \end{aligned} \tag{4}$$

for some, possibly small, exponent $\rho > 0$ (here, we used the notation $\langle x\rangle := (1+x^2)^{1/2}$). In the language of Schrödinger operators, this means that the potential $n_\infty^2 - n^2(x)$ is a long range perturbation of a constant. Also, the given function $S(x)$ entering (2) is assumed smooth and fastly decaying at infinity (say $S \in \mathcal{S}(\mathbf{R}^d)$ - see the footnote).

Next, let us explain the difficulties in each of the two analysis sketched above.

In equation (1), the small parameter ε measures the wavelength of the oscillations of u^ε: as x varies by a quantity of order ε, the function u^ε typically varies by a value of order one. This fact is encoded in the rescaling of the Helmholtz operator $\varepsilon^2/2\,\Delta_x + n^2(x)$. On the other hand, it is readily seen that the source term $\varepsilon^{-d/2}S(x/\varepsilon)$ **has the same wavelength** ε as u^ε: the source is concentrated, at the scale ε, close to the origin. Hence the difficulty lies in describing the **resonant interaction** between the oscillations induced by the source $\varepsilon^{-d/2}S(x/\varepsilon)$, and those induced by the high-frequency Helmholtz operator $\varepsilon^2/2\,\Delta_x + n^2(x)$.

In equation (2) on the other hand, the rescaled function w^ε describes the microscopic behaviour of u^ε close to the origin (i.e. close to the source), after zooming in by the factor ε. Hence w^ε gives refined informations on u^ε close to the origin[2]. Now, the outgoing solution $w^{\rm out}$ is given

[1] Here and below, the smoothness assumption on n, as well as the smoothness and decay hypotheses on S, may be considerably relaxed. We refer to the articles [BCKP], [CPR], and [C] for the precise assumptions that are really needed for the analysis.

[2] Note however that it "forgets" the behaviour of u^ε outside a ball of radius $\gg \varepsilon$ centered at 0, so that describing the asymptotic behaviour of u^ε, or that of w^ε, is not quite the same.

by

$$w^{\rm out} = i \int_0^{+\infty} \exp\left(it\left(\frac{1}{2}\Delta_x + n^2(0)\right)\right) S\, dt \qquad \left(= \left(\frac{1}{2}\Delta_x + n^2(0) + i0^+\right)^{-1} S\right). \tag{5}$$

(Recall indeed, for $y \in \mathbf{R}$, the formula $(y + i0^+)^{-1} = i\int_0^{+\infty} \exp(ity)\, dt$, and replace $y = \frac{1}{2}\Delta_x + n^2(0)$). To explain the meaning of (5), let $\psi(t,x)$ be the solution to the Schrödinger equation $i\partial_t\psi = -1/2\,\Delta_x\psi - n^2(0)\psi$, with initial datum $\psi(0,x) = S(x)$. The function $\psi(t,x)$ is the value at time t that is obtained when propagating the source S along the flow of the Schrödinger operator $-1/2\,\Delta_x - n^2(0)$. From this point of view, formula (5) asserts $w^{\rm out}(t,x) = i\int_0^{+\infty} \psi(t,x)\, dt$, i.e. $w^{\rm out}$ is the integral along **positive times up to infinity** of the conveniently propagated value of the source S. This is actually why $w^{\rm out}$ is called "outgoing": the ingoing solution would correspond to "$+i0$" replaced by "$-i0$" in (3), or to "$+\infty$" replaced by "$-\infty$" in (5). A second, alternative, way to define $w^{\rm out}$, is to take the unique solution to $(\Delta_x/2 + n^2(0))w^{\rm out} = S$ that satisfies the **Sommerfeld radiation condition at infinity**. When $d = 3$, it takes the form

$$\frac{x}{\sqrt{2}|x|}\cdot\nabla_x w^{\rm out}(x) + in(0)w^{\rm out}(x) = O\left(\frac{1}{|x|^2}\right) \quad \text{as} \quad |x| \to \infty\ . \tag{6}$$

In other words $w^{\rm out}$ should behave roughly like $\exp(i2^{-1/2}n(0)|x|)/|x|$ at infinity in x. As a conclusion on these two ways of defining $w^{\rm out}$, it is fairly clear on formulae (5) or (6), that the "$i0^+$" in (3) induces **strong non-local effects**. This is the very reason why the statement (3) is far from obvious: when $\varepsilon \to 0$, there is a conflict between the "$+i\varepsilon\alpha_\varepsilon$" going to zero in (2), that brings back information **from infinity**, and the $n(\varepsilon x)$ that goes to $n(0)$ **for bounded values of** x only. The first point apparently leads to thinking that w^ε should satisfy the Sommerfeld radiation condition at infinity with the variable refraction index $n^2(\varepsilon x) \to_{|x|\to\infty} n_\infty$, i.e. w^ε should behave like $\exp(i2^{-1/2}n_\infty|x|)/|x|$ at infinity in x, while the second point leads to thinking that w^ε naturally goes to a solution of the Helmholtz equation with constant refraction index $n^2(0)$, i.e. w^ε should behave like $\exp(i2^{-1/2}n(0)|x|)/|x|$, two contradictory statements.

2 The rays of geometric optics, and the dispersive properties of the high-frequency Helmholtz operator

Before going to the specific analysis we have sketched before, we first give a short idea of the kind of bounds we have at hand on u^ε, resp. w^ε.

When the refraction index is constant, it is a well known fact that u^ε as well as w^ε are uniformly bounded in the weighted L^2 space $L^2(\langle x\rangle^{-(1+\delta)}\, dx)$, for any $\delta > 0$, uniformly in ε. The critical case $\delta = 0$ is excluded. The need for a weighted measure $\langle x\rangle^{-(1+\delta)}$ corresponds to the natural decay at infinity of solutions to the Helmholtz equation.

When the refraction index actually varies with x, the situation becomes more delicate. To obtain uniform estimates on u^ε or w^ε, one typically needs **dispersive properties** of the operator $\varepsilon^2\Delta_x/2 + n^2(x)$, or, alternatively, dispersive properties of the rays naturally attached to the operator $\varepsilon^2\Delta_x/2 + n^2(x)$. These are defined as the trajectories of the Hamilitonian $\xi^2/2 - n^2(x)$, i.e. the solutions to the following system of ODE's

$$\begin{aligned} &\frac{\partial}{\partial t}X(t,x,\xi) = \Xi(t,x,\xi)\ , && X(0,x,\xi) = x\ , \\ &\frac{\partial}{\partial t}\Xi(t,x,\xi) = \left(\nabla_x n^2\right)\left(X(t,x,\xi)\right)\ , && \Xi(0,x,\xi) = \xi\ , \end{aligned} \tag{7}$$

A standard assumption in this context is the **non-trapping condition at the zero energy**: it states that any ray $(X(t,x,\xi),\Xi(t,x,\xi))$ shot from a point x, with frequency vector ξ, having zero energy initially, goes to infinity as time increases, i.e.

$$\lim_{t\to\pm\infty}|X(t,x,\xi)|=+\infty, \qquad \forall(x,\xi) \text{ such that } \xi^2/2-n^2(x)=0. \tag{8}$$

Since the propagation of energy in the Helmholtz equation is essentially governed by such rays, this condition means, in essence, that the energy does not accumulate too much in bounded regions of space: it eventually escapes to infinity. In the case of a constant index of refraction, this condition is trivially satisfied, since rays are simply straight lines, going to infinity with constant speed.
Under the mere non-trapping condition, one has, in general, nothing better than

$$\|u^\varepsilon\|_{L^2(\langle x\rangle^{-(1+\delta)}dx)}\le C\varepsilon^{-1}, \tag{9}$$

for some C independent of ε, and similarly for w^ε. The reader may refer to the work by Agmon, Hörmander, [Ag], [AH], and, more recently, to Wang [Wa], Burq [Bu], for instance.
Recently, optimal estimates were established by B. Perthame and L. Vega in [PV1], [PV2] (where the weighted L^2 space are replaced by more precise homogeneous Besov-like spaces introduced in [AH]). Introducing a reinforced version of the non-trapping condition, they established an estimate that implies

$$\|u^\varepsilon\|_{L^2(\langle x\rangle^{-(1+\delta)}dx)}\le C, \tag{10}$$

and similarly for w^ε. We refer to [WZ] for similar uniform bounds.
We do not give more details on these questions, since the works [BCKP], [CPR], and [C] that we review here, anyhow establish the specific necessary bounds in a self-contained fashion. Let us simply retain from these considerations that throughout this paper, the non-trapping condition is assumed to hold (at least).

3 Asymptotic behaviour of u^ε: a Wigner function approach – [BCKP], [CPR]

In [BCKP][3], we compute the limit in ε of the so-called Wigner transform[4] of $u^\varepsilon(x)$, denoted by $f^\varepsilon(x,\xi)$. It is defined as

$$f^\varepsilon(x,\xi):=\int_{\mathbf{R}^d}\exp(-iy\cdot\xi)\,u^\varepsilon\left(x+\varepsilon\frac{y}{2}\right)\overline{u^\varepsilon\left(x-\varepsilon\frac{y}{2}\right)}\,dy. \tag{11}$$

Beyond this very definition, let us simply stress two facts. First it is well-known [LP] that $f(x,\xi)=\lim f^\varepsilon(x,\xi)$ exists and is a non-negative measure, provided u^ε is reasonably bounded in $L^2_{\rm loc}$ (which is the case here). Second, the asymptotic measure $f(x,\xi)$, which now depends on a space variable x and a frequency variable $\xi\in\mathbf{R}^d$, measures the energy carried by u^ε in the neighbourhood of the point x in space, close to the frequency $\xi\in\mathbf{R}^d$. Typically, $\int f(x,\xi)d\xi$ measures the asymptotic behaviour of $|u^\varepsilon(x)|^2$, and $\int\xi^2 f(x,\xi)d\xi$ measures the asymptotic behaviour of $|\varepsilon\nabla_x u^\varepsilon(x)|^2$. To give an example, if $u^\varepsilon(x)=a(x)\exp(i\phi(x)/\varepsilon)$ for some phase function

[3] The results in this paragraph hold under the condition $\lim w^\varepsilon=w^{\rm out}$, see (3). This condition trivially holds in the case $n(x)=$ cst. We refer to the next paragraph for assumptions ensuring $\lim w^\varepsilon=w^{\rm out}$ in the nonconstant coefficients case. Note also that the results in this paragraph are non-trivial even in the constant coefficients case.

[4] We use here a slightly different scaling than the one used in [BCKP]: u^ε should be replaced by $\varepsilon^{1/2}u^\varepsilon$ to fit the scaling of [BCKP]. This a harmless fact.

ϕ, the limiting measure f is $f(x,\xi) = |a(x)|^2\,\delta(\xi - \nabla_x\phi(x))$. Hence f captures in an 'averaged' way the 'amplitude' and the 'phase' of the oscillatory function u^ε.

As mentionned in the introduction, the difficulty in studying the high-frequency limit $\varepsilon \to 0$ in (1), lies in the description of the resonant interaction between the oscillations induced by the high-frequency Helmholtz operator $\varepsilon^2/2\,\Delta_x + n^2(x)$, that enhance frequencies such that $\xi^2/2 = n^2(x)$ (the other frequencies are anyhow *not* propagated through the medium), and those of the source $S(x/\varepsilon)$, that enhance concentration towards $x = 0$. The work [BCKP] completes this description, in that we prove the limiting measure $f = \lim f^\varepsilon$ satisfies the stationnary transport equation with source term

$$0^+ f(x,\xi) + \xi\cdot\nabla_x f(x,\xi) + \nabla_x n^2(x)\cdot\nabla_\xi f(x,\xi) = Q(x,\xi)\ . \tag{12}$$

On the more, the limiting source term Q in (12) has the value

$$Q(x,\xi) = \delta\left(\xi^2/2 - n^2(0)\right)\ \delta(x)\ |\widehat{S}(\xi)|^2. \tag{13}$$

Let us explain the meaning of (12) and (13). Clearly, the characteristics of the transport equation (12) are exactly the rays of geometric optics defined in (7). Hence equations (12) and (13) tell us that asymptotically, the source S shoots rays from the origin (this is the factor $\delta(x)$), with frequencies ξ that resonate with the high-frequency Helmholtz operator (this is the factor $\delta\left(\xi^2/2 - n^2(0)\right)$), up to a modulation term $|\widehat{S}(\xi)|^2$. On the more, the energy is exactly propagated along the rays $(X(t,x,\xi),\ \Xi(t,x,\xi))$. Finally, the term $0^+ f$ in (12) specifies a radiation condition at infinity for f. This condition is the trace, as $\varepsilon \to 0$ of the absorption coefficient $\alpha_\varepsilon > 0$ in (1). It gives f as the outgoing solution

$$f(x,\xi) = \int_0^{+\infty} Q\left(X(t,x,\xi), \Xi(t,x,\xi)\right)\ dt\ . \tag{14}$$

The reader is invited to compare formula (14) with (5), or to (17) below. Also, the reader should note that the non-trapping condition at the zero energy clearly plays an essential role in making the integral in (14) **converge**: without non-trapping, the trajectory $(X(t,x,\xi), \Xi(t,x,\xi))$ might spend an infinitely long time in a compact set, thus making the above integral diverge. Finally, let us stress that obtaining the radiation condition for f as the limiting effect of the absorption coefficient α_ε in (1), is actually one of the main difficulties in [BCKP].

Later, the analysis of [BCKP] was extended in [CPR] to more general oscillating/concentrating source terms. The paper [CPR] studies indeed the high-frequency analysis $\varepsilon \to 0$ in

$$i\varepsilon\alpha_\varepsilon u^\varepsilon(x) + \frac{\varepsilon^2}{2}\Delta_x u^\varepsilon(x) + n^2(x)u^\varepsilon(x) = \frac{1}{\varepsilon^q}\int_\Gamma S\left(\frac{x-y}{\varepsilon}\right)\ A(y)\ \exp\left(i\,\frac{\phi(x)}{\varepsilon}\right)\ d\sigma(y)\ . \tag{15}$$

(See also [CRu] for extensions - see [Fo] for the difficult case where n^2 has discontinuities). In (15), the function S again plays the role of a concentration profile like in (1), but the concentration occurs this time around a smooth submanifold $\Gamma \subset \mathbf{R}^d$ of dimension p instead of a point. On the more, the source term here includes additional oscillations through the (smooth) amplitude A and phase ϕ. One may think of an antenna that is close to a p dimensional surface, sending high-frequency waves in the whole space. In these notations $d\sigma$ denotes the induced euclidean surface measure on the manifold Γ, and the rescaling exponant q depends on the dimension of Γ together with geometric considerations, see [CPR]. Again, the important phenomenon lies in the resonant interaction between the high-frequency oscillations of the source, and the propagative modes of the medium dictated by the operator $\varepsilon^2/2\,\Delta_x + n^2(x)$. A statement similar to, but geometrically richer than, equation (12) is proved in [CPR]. In this case, the asymptotic source term Q has the value,

$$Q(x,\xi) = \int_\Gamma \delta(x-y)\,\delta(\xi^2/2 - n^2(y))\,\delta(\xi^t - \nabla\phi(y))\,|A(y)|^2\,|\widehat{S}(\xi)|^2\,d\sigma(y).$$

This time, the energy is radiated from the surface Γ (this is the factor $\delta(x-y)$), with frequencies ξ that both correspond to propagative modes of the medium (this is the $\delta(\xi^2/2 - n^2(y))$), **and** resonate with the oscillation $\exp(i\phi/\varepsilon)$ (this is the $\delta(\xi^t - \nabla\phi(y))$). Here, the subscript t on the vector ξ refers to the part of the vector ξ that is tangent, at the point $y \in \Gamma$, to the surface Γ. Again, the squared terms involving A and S are simply modulations in amplitude.

To end up this paragraph, let us mention that the analysis performed in [BCKP] relies at some point on the asymptotic behaviour of the scaled wave function $w^\varepsilon(x) = \varepsilon^{d/2}u^\varepsilon(\varepsilon x)$. Similarly, in [CPR] one needs to rescale u^ε around any point $y \in \Gamma$, setting $w^\varepsilon_y(x) := \varepsilon^{d/2}u^\varepsilon(y + \varepsilon x)$ for any such y. Actually, the analysis performed in theses papers partially relies on the **conjecture** that w^ε, solution to (2), indeed goes to the outgoing solution w^{out} defined before. This explains why we now describe the mechanism that governs the asymptotics of w^ε.

4 Asymptotic behaviour of w^ε: a wave-packet approach – [C]

The main theorem we want to discuss is the following
Theorem [C]
Let w^ε satisfy (2), with the source term S belonging to the Schwartz class[5] $\mathcal{S}(\mathbf{R}^d)$*, and n satisfying (4). More precisely, n is assumed to satisfy*

$$\forall \alpha \in \mathbf{N}^d, \quad \exists C_\alpha > 0, \quad \left|\partial_x^\alpha \left(n^2(x) - n_\infty^2\right)\right| \le C_\alpha\, \langle x\rangle^{-\rho-|\alpha|}\,. \tag{16}$$

Suppose that n satisfies the nontrapping condition at zero energy. Suppose finally that n satisfies the geometric assumption **(H)** *below. Then, we do have the convergence in the sense of distributions*[6]

$$w^\varepsilon \to w^{\mathrm{out}}\,.$$

If the geometric assumption **(H)** *is* not *fulfilled, the above convergence* fails.

Idea of proof

First step: reformulating of the problem, and giving the main ideas.
Before giving the geometric assumption **(H)**, we give a picture that helps understanding the behaviour of w^ε, and gives an idea of the proof. Our main strategy is to **transform the original question into a time-dependent problem**. In the spirit of formula (5) (see also (14)), we write w^ε as the integral over the whole time interval $[0, +\infty[$ of some propagated function.

Let us come o quantitative statements. We take a smooth test function ϕ, and evaluate the duality product $\langle w^\varepsilon, \phi\rangle := \int_{\mathbf{R}^d} w^\varepsilon(x)\phi(x)\,dx$. Recalling the link $w^\varepsilon(x) = \varepsilon^{d/2}u^\varepsilon(\varepsilon x)$, we recover $\langle w^\varepsilon, \phi\rangle = \langle u^\varepsilon, \phi_\varepsilon\rangle$, where we introduce the notation $f_\varepsilon(x) = \varepsilon^{-d/2} f\left(x/\varepsilon\right)$, for any function f. There remains to observe

$$u^\varepsilon = \frac{i}{\varepsilon}\int_0^{+\infty} \mathrm{e}^{-\alpha_\varepsilon t}\,\exp\left(i\frac{t}{\varepsilon}\left(\frac{\varepsilon^2}{2}\Delta_x + n^2(x)\right)\right) S_\varepsilon\;dt =: \frac{i}{\varepsilon}\int_0^{+\infty} \mathrm{e}^{-\alpha_\varepsilon t}\,U_\varepsilon(t) S_\varepsilon\,, \tag{17}$$

[5] This assumption may be considerably relaxed, see [C]
[6] Again, this convergence may be considerably strengthened, see [C]

where we defined the semi-classical propagator

$$U_\varepsilon(t) := \exp\left(i\frac{t}{\varepsilon}\left(\frac{\varepsilon^2}{2}\Delta_x + n^2(x)\right)\right) \tag{18}$$

associated with the semi-classical Schrödinger equation $i\varepsilon\partial_t\psi = -\varepsilon^2/2\,\Delta_x\psi - n^2(x)\psi$. We arrive in this way at the final formula

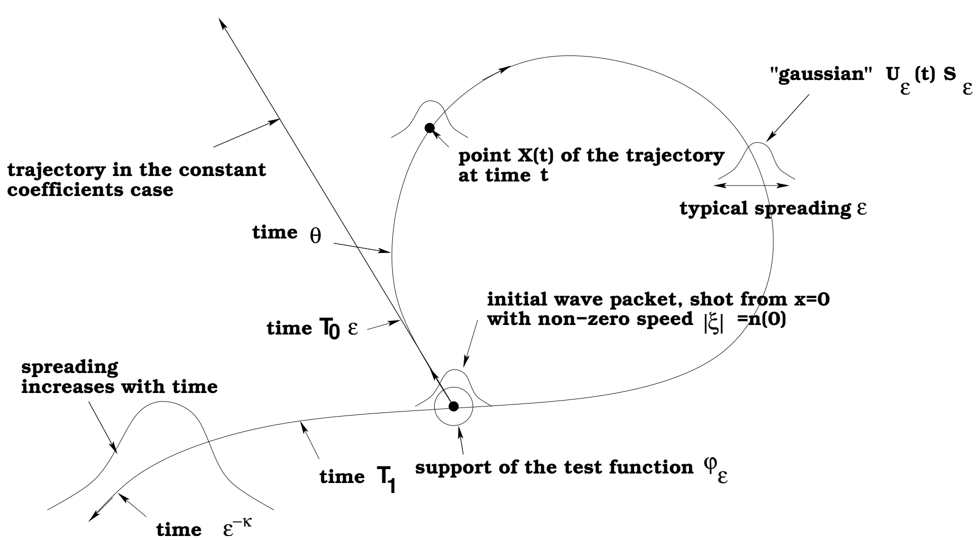
$$\langle w^\varepsilon, \phi\rangle = \frac{i}{\varepsilon}\int_0^{+\infty} \mathrm{e}^{-\alpha_\varepsilon t}\,\langle U_\varepsilon(t)S_\varepsilon, \phi_\varepsilon\rangle\,dt\,. \tag{19}$$

We now want to pass to the limit in this very integral. More precisely, we aim at proving that this term is asymptotic to

$$\langle w^{\mathrm{out}}, \phi\rangle = \frac{i}{\varepsilon}\int_0^{+\infty} \left\langle \exp\left(i\frac{t}{\varepsilon}\left(\frac{\varepsilon^2}{2}\Delta_x + n^2(0)\right)\right) S_\varepsilon, \phi_\varepsilon\right\rangle dt\,. \tag{20}$$

This is done upon distinguihsing various time scales in the integral, as we explain later. Before going to the details, we first give a (rough) interpretation of the above integrals (19) and (20).

In essence, the reader may think of S_ε as a gaussian centered at $x = 0$, with typical spreading of size ε. The gaussian S_ε is "shot" initially with some non-zero speed ξ satisfying $\xi^2/2 = n^2(0)$ (this is the zero energy condition). Similarly, the term $U_\varepsilon(t)S_\varepsilon$ in (19) may be thought of as a gaussian centered at the point $X(t, x, \xi)$, having speed $\Xi(t, x, \xi)$. For bounded values of time t, it has a typical spreading of size ε, that has however the tendency to grow as time increases. The factor ϕ_ε in (19) may be thought of as a gaussian centered at $x = 0$ as well. Finally, the scalar product $\langle U_\varepsilon(t)S_\varepsilon, \phi_\varepsilon\rangle$ may be thought of as the scalar product of the two above mentionned gaussians. This is the picture when computing w^ε through (19). In the case of w^{out}, the picture is the same, except that the trajectory $(X(t), \Xi(t))$ has to be replaced by the straight line with constant speed $(x + t\xi, \xi)$.
The next drawing illustrates our purpose.



With this picture in mind, the general idea is the following:
• For small times, the "gaussian" $U_\varepsilon(t)S_\varepsilon$ has a support that meets that of ϕ_ε, so that the scalar product $\langle U_\varepsilon(t)S_\varepsilon, \phi_\varepsilon\rangle$ is non-zero. On the more, the trajectory $X(t)$ is very close to the straight line $(x + t\xi, \xi)$ corresponding to the integral (20) that gives w^{out}. Hence small times should have a dominant contribution to the integral in (19).

• For very large times, the trajectory $X(t)$ escapes to infinity, thanks to the non-trapping condition. Hence the "support" of $U_\varepsilon(t)S_\varepsilon$ ceases to meet that of ϕ_ε, and the contribution of very large times to (19) should be small. The difficulty, however, lies in controlling the growth of the spreading of the "gaussian" $U_\varepsilon(t)S_\varepsilon$ as time increases (too large an increase might bring back energy into the support of ϕ_ε).
• For moderate times, the trajectory $X(t)$ might self-intersect, thus giving rise to a non-zero contribution to the integral in (19). On the other hand, the claimed convergence $\lim w^\varepsilon = w^{\text{out}}$ asserts that such self-intersections should *not* contribute to the integral (19) asymptotically. One needs thus an assumption that imposes, in essence, that such self-intersection should be "rare".

This motivates the introduction of the following geometric condition.

(H) geometric assumption

We suppose that the set

$$\Big\{(\xi, \eta, t) \in \mathbf{R}^{2d+1} \text{ such that } \eta^2/2 = n^2(0)\ ,\ X(t, 0, \xi) = 0\ ,\ \Xi(t, 0, \xi) = \eta\Big\}$$

is a smooth submanifold in $\mathbf{R}^{2d+1}$*, with a dimension* $k < d - 1$*.*

We refer to [C] for the precise statements. Let us simply mention that this assumption is generically satisfied.

Second step: passing to the limit in the time integral (19)
We now detail the limiting procedure skteched above. We choose two (large) cutoff parameters in time, denoted by T_0 and T_1, and we analyze the contributions to the time integral (19) that are due to the three regions $0 \le t \le T_0\,\varepsilon$, $T_0\,\varepsilon \le t \le T_1$, and $t \ge T_1$. We also choose a (small) exponent $\kappa > 0$, and we occasionally treat separately the contributions of very large times $t \ge \varepsilon^{-\kappa}$. Associated with these truncations, we take once and for all a smooth cutoff function χ defined on $\mathbf{R}$, such that $\chi(z) \equiv 1$ when $|z| \le 1$, $\chi(z) \equiv 0$ when $|z| \ge 2$, $\chi(z) \ge 0$ for any z. To be complete, there remains to finally choose a (small) cutoff parameter in energy $\delta > 0$. Accordingly we distinguish in the L^2 scalar product $\langle U_\varepsilon(t)S_\varepsilon, \phi_\varepsilon\rangle$ between energies close to (or far from) the zero energy, which is critical for our problem. In other words, we set the self-adjoint operator $H_\varepsilon := -\varepsilon^2/2\,\Delta_x - n^2(x)$, together with its associated truncation $\chi_\delta\,(H_\varepsilon) := \chi\,(H_\varepsilon/\delta)$. This object is perfectly well defined using standard functional calculus for self-adjoint operators. All these parameters allow to decompose the right-hand-side of (19) into small, moderate, and large times, as well as zero and non-zero energies. We now study each of the subsequent terms.

• **The contribution of small times** is

$$\frac{1}{\varepsilon}\int_0^{2T_0\,\varepsilon} \chi\left(\frac{t}{T_0\,\varepsilon}\right)\ \mathrm{e}^{-\alpha_\varepsilon t}\ \langle U_\varepsilon(t)S_\varepsilon, \phi_\varepsilon\rangle\ dt\ .$$

This term gives the dominant contribution in (19), provided the cutoff parameter T_0 is taken large enough. This (easy) analysis essentially boils down to manipulations on the time dependent Schrödinger operator $i\partial_t + \Delta_x/2 + n^2(\varepsilon x)$, for *finite times* t of the order $t \sim T_0$ at most.

• **The contribution of moderate and large times, away from the zero energy**, is

$$\frac{1}{\varepsilon}\int_{T_0\,\varepsilon}^{+\infty} (1-\chi)\left(\frac{t}{T_0\,\varepsilon}\right)\ \mathrm{e}^{-\alpha_\varepsilon t}\ \Big\langle U_\varepsilon(t)\,(1-\chi_\delta)\,(H_\varepsilon)\,S_\varepsilon, \phi_\varepsilon\Big\rangle\ dt\ .$$

This term has a vanishing contribution, provided T_0 is large enough. This easy result relies on a non-stationnary phase argument in time, recalling that $U_\varepsilon(t) = \exp(-itH_\varepsilon/\varepsilon)$ and the energy H_ε is larger than $\delta > 0$.

• **The contribution of very large times, close to the zero energy** is

$$\frac{1}{\varepsilon}\int_{\varepsilon^{-\kappa}}^{+\infty} \mathrm{e}^{-\alpha_\varepsilon t} \left\langle U_\varepsilon(t)\chi_\delta\left(H_\varepsilon\right) S_\varepsilon, \phi_\varepsilon \right\rangle dt \,.$$

This term has a vanishing contribution as $\varepsilon \to 0$. To prove so, we use results proved by X.P. Wang [Wa]: these essentially assert that for any $s \geq 0$, the operator $\langle x\rangle^{-s}\, U_\varepsilon(t)\chi_\delta(H_\varepsilon)\, \langle x\rangle^{-s}$ has the natural size $\langle t\rangle^{-s}$ as time goes to infinity, provided the critical zero energy is non-trapping. In the above mentionned picture, this means very roughly that the semiclassical operator $U_\varepsilon(t)\chi_\delta(H_\varepsilon)$ sends rays initially close to the origin, at a distance of the order t from the origin, and at most an energy of the order t^{-s} is brought back into bounded regions of space. Hence the corresponding contribution in (19) has size $\sim \varepsilon^{-1}\displaystyle\int_{\varepsilon^{-\kappa}}^{\infty} t^{-s} \sim \varepsilon^{s\kappa-2} \to 0$, provided s is large and κ is small enough.

The most difficult terms are the last two that we describe now.

• **The contribution of large times, close to the zero energy** is

$$\frac{1}{\varepsilon}\int_{T_1}^{\varepsilon^{-\kappa}} \mathrm{e}^{-\alpha_\varepsilon t} \left\langle U_\varepsilon(t)\chi_\delta\left(H_\varepsilon\right) S_\varepsilon, \phi_\varepsilon \right\rangle dt \,.$$

The treatment of this term is similar in spirit to, though much harder than, the analysis performed in the previous term: using only information on the localization properties of $U_\varepsilon(t)$ $\chi_\delta\left(H_\varepsilon\right)$ S_ε and ϕ_ε, we prove that this term has a vanishing contribution, provided T_1 is large enough. To do so, we use ideas of Bouzouina and Robert [BR], to establish a version of the Egorov theorem that holds true for *polynomially large times* in ε. In other words, this step requires quite a precise control of the spreading of $U_\varepsilon(t)S_\varepsilon$, since here the trajectory $X(t)$ is not far enough from the origin to allow for a rough estimate of the spreading (contrary to the preceding term). The conclusion of this step for any time $T_1 \leq t \leq \varepsilon^{-\kappa}$, the term $\langle U_\varepsilon(t)\chi_\delta\left(H_\varepsilon\right) S_\varepsilon, \phi_\varepsilon\rangle$ is equal to zero (this is the expected orthogonality of the supports of $U_\varepsilon(t)\chi_\delta\left(H_\varepsilon\right) S_\varepsilon$ resp. ϕ_ε), plus a remainder term due to the increase of the spreading. For any integer N, the latter has size $\varepsilon^{-1}\displaystyle\int_{T_1}^{\varepsilon^{-\kappa}} \varepsilon^N t^{N^2} dt$ (note the polynomial increase in time), which is a vanishing contribution, provided N is large, and κ is small.

• **The contribution of moderate times close to the zero energy** is

$$\frac{1}{\varepsilon}\int_{T_0\,\varepsilon}^{T_1} (1-\chi)\left(\frac{t}{T_0\,\varepsilon}\right) \mathrm{e}^{-\alpha_\varepsilon t} \left\langle U_\varepsilon(t)\chi_\delta\left(H_\varepsilon\right) S_\varepsilon, \phi_\varepsilon \right\rangle dt \,.$$

This is the most difficult term: contrary to all preceding terms, it cannot be analyzed using only geometric informations on the support of the relevant functions. On this time scale the supports of $U_\varepsilon(t)\chi_\delta\left(H_\varepsilon\right) S_\varepsilon$ resp. ϕ_ε may intersect (see the above drawing). This might create a dangerous accumulation of energy at the intersection point (i.e. at the origin). For that reason, we need a precise evaluation of the semi-classical propagator $U_\varepsilon(t)$, for times up to the order $t \sim T_1$. This is done using the elegant wave-packet approach of M. Combescure and D. Robert [CRo] (see also [Ro]): projecting S_ε over the standard gaussian wave packets, we can compute $U_\varepsilon(t)S_\varepsilon$ in a quite explicit fashion, with the help of classical quantities like, typically, the linearized flow of the Hamiltonian $\xi^2/2 - n^2(x)$. This gives us an integral representation with a complex valued phase function. Then, one needs to insert a last (small) cutoff parameter in time, denoted $\theta > 0$. For small times, using the above mentioned representation formula, we first prove that the term

$$\frac{1}{\varepsilon}\int_{T_0\,\varepsilon}^{\theta} (1-\chi)\left(\frac{t}{T_0\,\varepsilon}\right) \mathrm{e}^{-\alpha_\varepsilon t} \left\langle U_\varepsilon(t)\chi_\delta\left(H_\varepsilon\right) S_\varepsilon, \phi_\varepsilon \right\rangle dt \,,$$

vanishes asymptotically, provided θ is small, and T_0 is large enough. To do so, we use that for small enough θ, the propagator $U_\varepsilon(t)$ acting on S_ε resembles the free Schrödinger operator $\exp\left(it[\Delta_x/2 + n^2(0)]\right)$. Then, for later times, we prove that the remaining contribution

$$\frac{1}{\varepsilon}\int_\theta^{T_1} \mathrm{e}^{-\alpha_\varepsilon t} \left\langle U_\varepsilon(t)\chi_\delta\left(H_\varepsilon\right) S_\varepsilon, \phi_\varepsilon \right\rangle dt \ ,$$

is small. This uses stationary phase formulae in the spirit of [CRR], and this is where the geometric assumption **(H)** enters.

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