A short review on the derivation of the nonlinear quantum Boltzmann equations

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Comm. Math. Sci., Vol. 5, pp. 55-71 (2007).

Abstract

In this review paper we describe the problem of deriving a Boltzmann equation for a system of N interacting quantum particles, under the appropriate scaling limits. We mainly follow the approach developped by the authors in previous works. From a rigorous viewpoint, only partial results are available, even for short times, so that the complete problem is still open.

1 Introduction

A large quantum system of N identical interacting particles can often be described in terms of a Boltzmann equation. This is an asymptotic model: the equation given from first principles is the N body Schrödinger equation. As such, the Boltzmann description only holds in suitable regimes, namely when the number of particles is *large*, and when the interaction potential between pairs of particles has a *small* effect. Concerning this last point, two quite different settings are relevant. In the so-called weak-coupling limit, the interaction potential itself is small, while the gas is dense: the typical distance between particles is of order one. In the low-density regime at variance, the elementary interaction potential is of order one, while the gas is rarefied: the typical distance between particles is large, hence the effect of the pairwise interactions is small.

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A precise definition of the scalings, and the form of the limiting kinetic equations, has been discussed by H. Spohn in [Sp2]. In the present text, we follow this reference to introduce the problem. We start from the N body Schrödinger equation, and we scale it along either the weak-coupling, or the low-density regime. Next, we follow the kinetic approach introduced by the authors in [BCEP1]: we transform the scaled N-particles Schrödinger equation into a hierarchy of kinetic equations. This step uses the Wigner transform and the BBGKY hierarchy. Then we try to show how, under these scaling limits, the one-particle Wigner function of the system is indeed expected to obey a Boltzmann equation with a suitable cross-section. This step heavily uses stationary phase considerations, in possibly large dimensions. The derivation we present here closely follows the works [BCEP1], [BCEP2], and [BCEP3].

We wish to stress that our arguments are partially formal: a complete, rigorous derivation of the quantum Boltzmann equation is still far from being achieved, even for short times.

To put the present text into perspective, we remind that a rigorous derivation of the classical Boltzmann equation in the low-density regime has been obtained in 1975 by Lanford for short times (see [L]). This result was later extended to all times in Reference [IP] (see also [CIP] for additional comments), for special situations yet. At the quantum level, the transition from the Schrödinger picture to the kinetic description is delicate in many ways. Most importantly, it brings a time reversible system to an irreversible one, a difficulty that is already present at the classical level. This fact is largely argued in the context of classical systems. We refer e.g. to [CIP] on that point.

We also remark that, in contrast with the quantum case, classical systems in the weak-coupling limit are described by a kinetic equation which is *diffusive* in velocity, namely by the Landau-Fokker-Planck equation (see for instance [Sp1] and [Ba]). Thus the domain of applicability of the Boltzmann equation is typically larger for quantum systems than for classical ones: in the former case, kinetic descriptions are relevant both for dilute gases (low density), and for dense, weakly interacting systems (weak coupling), while in the latter, only dilute gases are pertinent.

The present review text treats separately the weak coupling regime, and the low-density regime. This is a natural distinction. Another separation is in order yet. Indeed, since we deal with quantum systems, it is necessary to discuss the statistic independence of the particles under consideration. Namely, particles that follow the Maxwell-Boltzmann statistics[1] have a Wigner transform that may be taken as a tensor product. This gives a simple picture of the "molecular chaos" assumption, that lies at the core of the Boltzmann description of interacting particles. Bosonic particles on the other hand follow the Bose-Einstein statistics, while fermionic particles follow the Fermi-Dirac statistics. In these two cases, the molecular chaos assumption takes a more subtle form, which we discuss in section 4. This fact has a fairly important consequence. Namely, the Boltzmann equation that is appropriate in the Maxwell-Boltzmann situation is quadratic in the unknown, while it becomes cubic in the Fermi-Dirac or Bose-Einstein picture.

2 Setting of the problem

In this section, we give some quantitative statements describing the asymptotics from the scaled N-body Schrödinger equation to the Boltzmann equation. Our presentation distinguishes between the weak-coupling and low-density regimes, together with the Maxwell-Boltzmann versus Fermi-Dirac or Bose-Einstein statistics. Elements of proof are given in the next sections.

• *The weak coupling limit in the Maxwell-Boltzmann statistics*

We consider N identical quantum particles in $\mathbb{R}^3$. We assume that the mass of the particles, as well as $\hbar$, are normalized to unity. The interaction between particles is described by a two-body potential ϕ, and the total potential energy is taken as

$$U(x_1 \dots x_N) = \sum_{i<j} \phi(x_i - x_j). \tag{2.1}$$ 1.1

The associated Schrödinger equation reads

$$i\partial_t \Psi(X_N, t) = -\frac{1}{2}\Delta_N \Psi(X_N, t) + U(X_N)\Psi(X_N, t), \tag{2.2}$$ 1.2

where $\Delta_N = \sum_{i=1}^N \Delta_i$, Δ_i is the Laplacian with respect to the x_i variables, and X_N is a shorthand notation for $x_1 \dots x_N$.

Due to the fact that the particles are identical, the wave function Ψ is assumed to be symmetric in the exchange of particle, a property that is preserved along the time evolution induced by (2.2). This symmetry assumption will actually hold in the Fermi-Dirac or Bose-Einstein situation as well. As

[1] or particles obeying no statistics at all. Throughout this text, the reader may safely replace "Maxwell Boltzmann statistics" by "no statistics at all" if needed.

a consequence, we readily mention that all objects Ψ^ε_N, W^N, and f_j^N to be introduced later, are all symmetric as well.

Next, we rescale the equation (2.2) according to the hyperbolic space-time scaling

$$x \longrightarrow \varepsilon x \ , \qquad t \longrightarrow \varepsilon t, \tag{2.3}$$ 1.3

which corresponds to looking at (2.2) over large times of the order $1/\varepsilon$, hence large distances of the order $1/\varepsilon$ (particles move with a "velocity" of order one). Simultaneously we also rescale the potential by

$$\phi \longrightarrow \sqrt{\varepsilon}\phi. \tag{2.4}$$ 1.z

The resulting equation reads, in these new units,

$$i\varepsilon\partial_t \Psi^\varepsilon(X_N, t) = -\frac{\varepsilon^2}{2}\Delta_N \Psi^\varepsilon(X_N, t) + U_\varepsilon(X_N)\,\Psi^\varepsilon(X_N, t), \tag{2.5}$$ 1.4

$$\text{where:} \qquad U_\varepsilon(x_1 \dots x_N) = \sqrt{\varepsilon}\sum_{i<j}\phi\left(\frac{x_i - x_j}{\varepsilon}\right). \tag{2.6}$$ 1.5

Naturally, the wave function $\Psi^\varepsilon(X_N, t)$ at time t is fully determined by Eq. (2.5) together with the initial datum $\Psi^\varepsilon(X_N, 0)$. The latter depends on the very statistics obeyed by the particles, and its value is specified later on (see (2.18)). We want to analyze the limit $\varepsilon \to 0$ in the above equations, while keeping

$$N = \varepsilon^{-3}. \tag{2.7}$$ 1.7

Both scalings (2.4) and (2.7) specify a weak coupling regime. Here, the gas of particles is dense (one particle per unit volume in the rescaled units), but the coupling between neighbouring particles is weak, of order $\sqrt{\varepsilon}$. The cumulated effect of all the interactions is of the size

$$\begin{aligned} &O(\text{time scale}) \times O(\text{density of obstacles}) \times O([\text{coupling}]^2) \\ &\quad = O(1/\varepsilon) \times O(1) \times O([\sqrt{\varepsilon}]^2) = O(1). \end{aligned} \tag{2.8}$$ ff

Note that the quadratic dependence upon the coupling constant in (2.8) is a standard fact in quantum mechanics. It is related with the so-called Fermi Golden Rule (see (2.22) below). Equivalently, it is a consequence of the Hamiltonian structure of the Schrödinger equation.

Following [BCEP1], and in order to tackle the asymptotics $\varepsilon \to 0$ in (2.5)-(2.6), we now adopt a kinetic approach. We introduce the Wigner transform of Ψ^ε, defined as (see [W], or the more recent reference [LP] for a general introduction to Wigner transforms)

$$\begin{aligned} W^N(X_N, V_N) = \\ \left(\frac{1}{2\pi}\right)^{3N} \int dY_N \ e^{iY_N \cdot V_N} \overline{\Psi}^\varepsilon \left(X_N + \frac{\varepsilon}{2} Y_N\right) \ \Psi^\varepsilon \left(X_N - \frac{\varepsilon}{2} Y_N\right). \end{aligned} \qquad (2.9) \quad \boxed{\texttt{1.8}}$$

As it is standard, W^N satisfies a transport-like equation, namely

$$(\partial_t + V_N \cdot \nabla_N) W^N(X_N, V_N) = \frac{1}{\sqrt{\varepsilon}} \big(T_N^\varepsilon W^N\big)(X_N, V_N). \qquad (2.10) \quad \boxed{\texttt{1.9}}$$

Here, $\partial_t + V_N \cdot \nabla_N = \partial_t + \sum_{i=1}^N v_i \cdot \nabla_{x_i}$ is the usual free stream operator. Also, the operator T_N^ε on the right-hand-side of (2.10) plays the role of a collision operator. It may be split into

$$\big(T_N^\varepsilon W^N\big)(X_N, V_N) = \sum_{0<k<\ell\leq N} \big(T_{k,\ell}^\varepsilon W^N\big)(X_N, V_N), \qquad (2.11) \quad \boxed{\texttt{1.10}}$$

where each $T_{k,\ell}^\varepsilon$ describes the "collision" of particle k with particle ℓ, through

$$\begin{aligned} \big(T_{k,\ell}^\varepsilon W^N\big)(X_N, V_N) = \frac{1}{i} \left(\frac{1}{2\pi}\right)^{3N} \int dY_N \, dV_N' \ \mathrm{e}^{iY_N \cdot (V_N - V_N')} \qquad (2.12) \quad \boxed{\texttt{1.11}} \\ \left[\phi\left(\frac{x_k - x_\ell}{\varepsilon} - \frac{y_k - y_\ell}{2}\right) - \phi\left(\frac{x_k - x_\ell}{\varepsilon} + \frac{y_k - y_\ell}{2}\right)\right] \ W^N(X_N, V_N'). \end{aligned}$$

Thus the total operator T_N^ε in (2.11) takes into account all possible "collisions" inside the N particles system. Equivalently, we may write[2] for $T_{k,\ell}^\varepsilon$

$$\begin{aligned} \big(T_{k,\ell}^\varepsilon W^N\big)(X_N, V_N) = -i \sum_{\sigma = \pm 1} \sigma \int \frac{dh}{(2\pi)^3} \ \widehat{\phi}(h) \ \mathrm{e}^{i \frac{h}{\varepsilon} (x_k - x_\ell)} \\ W^N\left(x_1, \ldots, x_k, \ldots, x_\ell, \ldots, x_N, v_1, \ldots, v_k - \sigma\frac{h}{2}, \ldots, v_\ell + \sigma\frac{h}{2}, \ldots, v_N\right). \end{aligned} \qquad (2.13) \quad \boxed{\texttt{1.12}}$$

Note that in (2.13), "collisions" may take place between *distant* particles ($x_k \neq x_\ell$). However, such distant collisions are penalized by the highly

[2] Here and below, $\widehat{f}(k) = \int e^{-ik \cdot x} f(x) \, dx$ denotes the Fourier transform of f.

oscillatory factor $\exp(ih(x_k - x_\ell)/\varepsilon)$. These oscillations turn out to play a crucial role throughout the analysis, and they explain why collisions tend to happen when $x_k = x_\ell$ in the limit $\varepsilon \to 0$ (see *e.g.* the commputation of $\mathcal{I}^{1,1,2}$ in section 3 below).

In order to transform (2.9) into a *hierarchy* of kinetic equations, we next introduce the *partial traces* of the Wigner transform W^N, denoted by f_j^N. They are defined through the following formula, valid for $j = 1.\dots, N-1$:

$$\begin{aligned} f_j^N(X_j, V_j) = \int dx_{j+1} \dots \int dx_N \int dv_{j+1} \dots \int dv_N \\ W^N(X_j, x_{j+1} \dots x_N; V_j, v_{j+1} \dots v_N) \end{aligned} \tag{2.14}$$ $\boxed{\texttt{1.13}}$

Obviously, we set $f_N^N = W^N$. The function f_j^N is the kinetic object that describes the state of the j particles subsystem at time t.

Proceeding then as in the derivation of the BBGKY hierarchy for classical systems (see e.g.[CIP]), we readily transform the equation (2.10) satisfied by W^N into a hierarchy of equations for f_j^N $(1 \leq j \leq N)$, namely

$$\left(\partial_t + \sum_{k=1}^{j} v_k \cdot \nabla_k \right) f_j^N(X_j, V_j) = \frac{1}{\sqrt{\varepsilon}} T_j^\varepsilon f_j^N + \frac{N-j}{\sqrt{\varepsilon}} C_{j+1}^\varepsilon f_{j+1}^N, \tag{2.15}$$ $\boxed{\texttt{1.14}}$

with $f_{N+1}^N \equiv 0$ by convention. Eq. (2.10) is naturally recovered from (2.15) upon choosing $j = N$ in the latter equation. Here the new collision operator C_{j+1}^ε may be split into

$$C_{j+1}^\varepsilon = \sum_{k=1}^{j} C_{k,j+1}^\varepsilon, \tag{2.16}$$ $\boxed{\texttt{1.15}}$

and each $C_{k,j+1}^\varepsilon$ satisfies

$$\begin{aligned} C_{k,j+1}^\varepsilon f_{j+1}^N(X_j; V_j) = -i \sum_{\sigma = \pm 1} \sigma \int \frac{dh}{(2\pi)^3} \, dx_{j+1} \, dv_{j+1} \, \widehat{\phi}(h) \, e^{i \frac{h}{\varepsilon}(x_k - x_{j+1})} \\ f_{j+1}^N \left(x_1, x_2, \dots, x_{j+1}, v_1, \dots, v_k - \sigma \frac{h}{2}, \dots, v_{j+1} + \sigma \frac{h}{2} \right). \end{aligned} \tag{2.17}$$ $\boxed{\texttt{1.16}}$

The operator $C_{k,j+1}^\varepsilon$ describes the "collision" of particle k, belonging to the j-particle subsystem, with a particle outside the subsystem, conventionally denoted by the number $j+1$ (this numbering uses the fact that all particles are identical). The total operator C_{j+1}^ε takes into account all such collisions.

As usual (see e.g. [CIP]), equation Eq. (2.15) shows that the dynamics of the j-particle subsystem is governed by three effects: the free-stream operator, the collisions “inside” the subsystem (the T term), and the collisions with particles “outside” the subsystem (the C term).

To finish the specification of the problem, we finally need to select an initial value $\{f_j^0\}_{j=1}^N$ for the solution $\{f_j^N(t)\}_{j=1}^N$. The key point is that we assume $\{f_j^0\}_{j=1}^N$ is completely factorized: for all $j = 1, \ldots, N$, we suppose

$$f_j^0 = f_0^{\otimes j}, \tag{2.18}$$

1.17

where f_0 is a one-particle Wigner function, and f^0 is assumed to be a probability distribution. This is the point where the statistics enters. Assumption (2.18) is relevant for particles satisfying the Maxwell-Boltzmann statistics, but it totally excludes fermionic or bosonic behaviour. To be complete, we should also raise here a technical point. Strictly speaking, a quantum state whose Wigner transform is a general positive f_0, is not a wave function: it is rather a density matrix. As a consequence, and in view of the kind of initial data (2.18) we have in mind, the evolution equation we should start with is not the Schrödinger equation (2.5), but rather the associated Heisenberg equation for the density matrix. This is a harmless modification: in both cases the corresponding Wigner equation is anyhow Eq. (2.10) or, equivalently, Eq. (2.15).

In the limit $\varepsilon \to 0$, we expect that the j-particle distribution function $f_j^N(t)$, that solves the hierarchy (2.15) with initial data (2.18), tends to be factorized for all times: $f_j^N(t) \sim f(t)^{\otimes j}$ (molecular chaos). On top of that, the function $f(t)$ ($t \in [0, t_0)$ for some possibly small t_0), which is the limit of the one-particle distribution function $f_1^N(t)$, is expected to be the solution of the following Boltzmann equation

$$(\partial_t + v \cdot \nabla_x) f(t,x,v) = Q_{\mathrm{w}}(f,f)(t,x,v), \tag{2.19}$$

1.18

$$Q_{\mathrm{w}}(f,f)(t,x,v) = \int_{\mathbb{R}^3 \times \mathbb{S}_2} dv_1 \, d\omega \, B_{\mathrm{w}}(\omega, v - v_1) \; [f(t,x,v') f(t,x,v_1') - f(t,x,v) f(t,x,v_1)] \, . \tag{2.20}$$

1.19

Here, the index “w” refers to “weak-coupling”. Also, v' and v_1' denote the outgoing velocities after a collision with impact parameter $\omega \in \mathbb{S}^2$ and incoming velocities v and v_1. Explicitely:

$$v' = v - [v - v_1] \cdot \omega \quad \omega, \qquad v_1' = v_1 + [v - v_1] \cdot \omega \quad \omega. \tag{2.21}$$

omeg

Last, in Eq. (2.20), the factor $B_{\mathrm{w}}(\omega, v-v_1)$ is the cross-section. It depends on the microscopic interaction potential ϕ. In the weak-coupling limit, collisions take place at a small energy, and at a distance of order ε. For this reason, the cross section B_{w} is computed at low energy, and via the quantum rules. In other words, it agrees with the Born approximation of quantum scattering, namely

$$B_{\mathrm{w}}(\omega, v) = \frac{1}{8\pi^2} |\omega \cdot v| \, |\widehat{\phi}(\omega\,(\omega \cdot v))|^2. \qquad (2.22) \quad \boxed{\texttt{1.20}}$$

Note that the cross-section B_{w} is the only quantum factor in the purely classical equations (2.19)-(2.20). It retains the quantum features of the elementary "collisions".

• *The weak coupling limit in the Bose-Einstein or Fermi-Dirac statistics*

From a physical viewpoint, it certainly is more realistic to consider particles obeying the Fermi-Dirac or Bose-Einstein statistics, than considering the Maxwell-Boltzmann situation.

In this case, the starting point still is the scaled Schrödinger equation (2.5)-(2.6), or the equivalent hierarchy (2.15). The only new point is that we cannot take a totally decorrelated initial datum as in (2.18). Indeed, the Fermi-Dirac or Bose-Einstein statistics yield correlations even at time zero. In this perspective, the most uncorrelated states one can introduce, and that do not violate the Fermi-Dirac or Bose-Einstein statistics, are the so-called quasi-free states. They are described in section 4 below.

As a consequence, the following steps are needed in order to pass to the limit in the hierarchy (2.15), and to identify the limiting Boltzmann equation. First, one should characterize the quasi-free states in term of their Wigner transform. Then, one should replace the initial condition (2.18) by the appropriate "quasi-free" initial data. Last, one should perform the asymptotic procedure on the resulting formulae.

It is expected that the one-particle distribution function $f_1^N(t)$ converges to the solution of the following *cubic* Boltzmann equation:

$$(\partial_t + v \cdot \nabla_x)\, f(t,x,v) = Q_{\mathrm{w},\theta}(f,f,f)(t,x,v), \qquad (2.23) \quad \boxed{\texttt{cb}}$$

$$\begin{aligned} Q_{\mathrm{w},\theta}(f,f,f)(t,x,v) = &\int_{\mathbb{R}^3\times\mathbb{S}^2} dv_1\, d\omega\; B_{\mathrm{w},\theta}(\omega, v-v_1) \\ &\big[f(t,x,v')f(t,x,v_1')(1+8\pi^3\theta f(t,x,v)\, f(t,x,v_1)) \\ &\quad -f(t,x,v)f(t,x,v_1)(1+8\pi^3\theta f(t,x,v')f(t,x,v_1'))\big]. \end{aligned} \qquad (2.24) \quad \boxed{\texttt{1.21}}$$

Here $\theta = +1$ or $\theta = -1$, depending on whether the Bose-Einstein or the Fermi-Dirac statistics is considered, respectively. The index "w,θ" refers to "weak coupling, with the bosonic or fermionic statistics". Finally, $B_{\mathrm{w},\theta}$ is the symmetrized or antisymmetrized cross-section derived from B_{w} (see (2.22)) in the natural way (see [BCEP2]).

As we see, the modification of the statistics transforms the quadratic Boltzmann equation (2.19)-(2.20) of the Maxwell-Boltzmann case, into a cubic form of the equation (fourth order terms cancel). Also, the statistics affects the form of the cross-section and B_{w} has to be (anti)symmetrized into $B_{\mathrm{w},\theta}$.

Note that the collision operator (2.24) has been introduced by Uehling and Uhlenbeck in 1933 on the basis of purely phenomenological considerations [UU].

• *The low-density limit*

There remains to consider the low-density limit. This regime is also called Boltzmann-Grad limit in the context of classical systems. Here, the starting point still is the unscaled Schrödinger equation (2.1)-(2.2). Contrary to the weak-coupling regime, we now scale it according to

$$t \to \varepsilon t, \quad x \to \varepsilon x, \quad \phi \to \phi, \quad N = \varepsilon^{-2}. \tag{2.25}$$ lde

In other words, the density of obstacles is ε, which is a rarefaction regime, but the potential is unscaled and keeps an $O(1)$ amplitude. In this case, the cumulated effect of the interactions has size

$$O(\text{time scale}) \times O(\text{density of obstacles}) \times O([\text{coupling}]^2) = \\ O(1/\varepsilon) \times O(\varepsilon) \times O(1) = O(1).$$

Another very important point is the following. Due to the fact that the density is vanishing, the particles are too rare to make the statistical correlations effective. As a consequence, we expect that the Maxwell-Boltzmann, Bose-Einstein, and Fermi-Dirac situations, all give rise to the same Boltzmann equation along the low-density limit.

As a matter of fact, the expected Boltzmann equation still is a quadratic Boltzmann equation in that case, namely

$$(\partial_t + v \cdot \nabla_x) f(t,x,v) = Q_\ell(f,f)(t,x,v), \tag{2.26}$$ ldb

$$Q_\ell(f,f)(t,x,v) = \\ \int_{\mathbb{R}^3 \times \mathbb{S}^2} dv_1\, d\omega\, B_\ell(\omega, v - v_1)\, \left[f(t,x,v') f(t,x,v_1') - f(t,x,v) f(t,x,v_1) \right]. \tag{2.27}$$ ldbb

Here, the index "ℓ" refers to "low-density". Also, v', v_1', and ω are as in (2.21). Last, the factor $B_\ell(\omega, v - v_1)$ is the cross-section. In the low-density limit, collisions take place at a large energy (contrary to the weak-coupling situation), and at a distance of order ε. For this reason, the cross section B_ℓ is computed at large energy, and via the quantum rules. In other words, it agrees with the *full* Born series expansion of quantum scattering, namely

$$B_\ell(\omega, v) = \frac{1}{8\pi^2} |\omega \cdot v| \, |\widehat{\phi}(\omega\,(\omega \cdot v))|^2 + \sum_{n \geq 3} B_\ell^{(n)}(\omega, v), \tag{2.28}$$ bbs

where each $B_\ell^{(n)}(\omega, v)$ is an explicitely known function, which is n-linear in ϕ (see [RS]). Note in passing that the convergence of the Born series expansion (2.28) is well-known for potentials satisfying a smallness assumption.

As is seen on these formulae, the only difference between the low-density and the weak-coupling regimes (at least for Maxwell-Boltzmann particles) lies in the very value of the cross-section. The two cross-sections B_{w} and B_ℓ are actually related through

$$B_\ell(\omega, v) = B_{\mathrm{w}}(\omega, v) + O([\phi]^3),$$

i.e. B_{w} and B_ℓ *coincide* up to third order in the potential. This very well reflects the fact that the weak-coupling regime involves only low-energy phenomena, while the low-density regime affects low to large energies.

In the next sections we briefly discuss the very few rigorous results concerning the above problems.

3 The weak-coupling limit for the Maxwell-Boltzmann statistics

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To analyze the asymptotic behaviour of the hierarchy (2.15), we adopt the same strategy as the one introduced by Lanford in [L] to treat the Boltzmann-Grad limit for classical systems. In other words, we study the asymptotic behavior of the solution $f_j^N(t)$, when expressed in terms of the series expansion obtained upon iterating the Duhamel formula. We write down on the other hand the hierarchy satisfied by the successive tensor products $f_j(t) := f(t)^{\otimes j}$, where $f(t)$ satisfies the Boltzmann equation (2.19)-(2.20) - this hierarchy is usually called "Boltzman hierarchy". We explicitely solve the Boltzmann hierarchy as a complete series expansion obtained upon iterating the Duhamel formula. We prove that the series expansion that expresses $f_j^N(t)$ converges, in a sense which is precised below, towards the analogous series expansion for $f_j(t) = f^{\otimes j}(t)$.

Let us come to the details. It is first easily proved, using computations similar to those performed below for f_j^N, that the solution to the Boltzmann hierarchy[3] associated with (2.19)-(2.20) is given by the following series expansion

$$\begin{aligned} f_j(t, X_j, V_j) \equiv f^{\otimes j}(t, X_j, V_j) = & \\ & \sum_{n \geq 0} \sum_{\ell_2=1}^{j+1} \cdots \sum_{\ell_n=1}^{j+n} \int_0^t dt_1 \cdots \int_0^{t_{n-1}} dt_n \, S(t-t_1) \, C_{\ell_1, j+1} \\ & S(t_1 - t_2) \, C_{\ell_2, j+2} \cdots S(t_{n-1} - t_n) \, C_{\ell_n, j+n} \, S(t_n) \, f_0^{\otimes (j+n)}. \end{aligned} \qquad (3.1) \quad \boxed{\texttt{fbol}}$$

Here, the operator $S(t)$ is the *free flow*, defined as

$$(S(t) f_j)(X_j, V_j) := f_j(X_j - V_j t, V_j). \qquad (3.2) \quad \boxed{\texttt{2.3}}$$

Also, the *classical* collision operator $C_{\ell,k+1}$, describing in an analogous fashion than the quantum object $C^{\varepsilon}_{\ell,k+1}$ a *classical* collision between particle $k+1$ and particle ℓ, is deduced from formula (2.20) and has the value

$$\begin{aligned} \left(C_{\ell,k+1} f_{k+1}\right)(t, X_k, V_k) := & \int_{\mathbb{R}^3 \times \mathbb{S}_2} dv_{k+1} \, d\omega \, B_{\mathrm{w}}(\omega, v_\ell - v_{k+1}) \\ & \big[f_{k+1} \left(X_k, x_\ell, v_1, \ldots, v_{\ell-1}, v'_\ell, v_{\ell+1}, \ldots, v_k, v'_{k+1}\right) \\ & - f_{k+1} \left(X_k, x_\ell, v_1, \ldots, v_{\ell-1}, v_\ell, v_{\ell+1}, \ldots, , v_k, v_{k+1}\right) \big], \end{aligned} \qquad (3.3) \quad \boxed{\texttt{cclass}}$$

where B_{w} has been defined in (2.22), and $v'_\ell = v_\ell - [v_\ell - v_{k+1}] \cdot \omega \, \omega$, $v'_{k+1} = v_{k+1} + [v_\ell - v_{k+1}] \cdot \omega \, \omega$, as in (2.21). This gives a complete series expansion expressing $f_j(t)$ in terms of the initial datum f_0.

In the similar spirit, we may write, for $1 \leq j \leq N$,

$$\begin{aligned} f_j^N(t) = \sum_{n=0}^{N-j} \frac{(N-j) \ldots (N-j-n)}{(\sqrt{\varepsilon})^n} \int_0^t dt_1 \ldots \int_0^{t_{n-1}} dt_n \; S^{\varepsilon}_{int}(t - t_1) C^{\varepsilon}_{j+1} & \\ S^{\varepsilon}_{int}(t_1 - t_2) C^{\varepsilon}_{j+2} \ldots S^{\varepsilon}_{int}(t_{n-1} - t_n) C^{\varepsilon}_{j+n} S^{\varepsilon}_{int}(t_n) f_0^{\otimes (j+n)}. & \end{aligned} \qquad (3.4) \quad \boxed{\texttt{2.1}}$$

Here $S^{\varepsilon}_{int}(t) f_j$ is the j-particle interacting flow, namely the solution to the initial value problem:

$$\begin{cases} (\partial_t + V_j \cdot \nabla_j) S^{\varepsilon}_{int}(t) f_j = \dfrac{1}{\sqrt{\varepsilon}} T^{\varepsilon}_j S^{\varepsilon}_{int}(t) f_j, \\ S^{\varepsilon}_{int}(0) f_j = f_j. \end{cases} \qquad (3.5) \quad \boxed{\texttt{2.2}}$$

[3] We do not write down the Boltzmann hierarchy here for sake of simplicity, and simply refer to [CIP] or [BCEP1] for details.

Then, we again expand $S^{\varepsilon}_{int}(t)$ as a perturbation of the free flow $S(t)$. We find

$$
\begin{aligned}
S^{\varepsilon}_{int}(t)f_j = S(t)f_j + \sum_{m\geq 0} \frac{1}{(\sqrt{\varepsilon})^m} \int_0^t d\tau_1 \int_0^{\tau_1} d\tau_2 \ldots \int_0^{\tau_{m-1}} d\tau_m \\
S(t-\tau_1)\, T^{\varepsilon}_j\, S(\tau_1-\tau_2)\, T^{\varepsilon}_j\, \ldots\, S(\tau_{m-1}-\tau_m)\, T^{\varepsilon}_j\, S(\tau_m)f_j.
\end{aligned} \qquad \boxed{\texttt{2.4}} \tag{3.6}
$$

Inserting (3.6) into (3.4), we obtain an explicit perturbative expansion that relates the value of $f_j^N(t)$ at time t, in terms of the initial datum f_0. This expression involves a series that contains a huge number of terms. It is similar to, though much more complicated than, equation (3.1). However, we expect that many of these terms are negligible in the limit. On top of that, we also expect that the other, non-vanishing contributions eventually converge towards the series expansion (3.1) (in some topology).

To give a flavour of the computations performed in [BCEP1] along these lines, let us now analyze *some* terms of the explicit expansion that expresses $f_j^N(t)$, and compare them with the analogous terms for $f_j(t)$.

We begin with those terms of degree less than two in the potential.
The relevant terms are the following five:

$$\mathcal{I}_0 := S(t)f_j^0, \qquad \boxed{\texttt{2.5}} \tag{3.7}$$

$$\mathcal{I}_1 := \frac{N-j}{\sqrt{\varepsilon}} \int_0^t dt_1\; S(t-t_1)C^{\varepsilon}_{j+1}S(t_1)f^0_{j+1}, \qquad \boxed{\texttt{2.6}} \tag{3.8}$$

$$\mathcal{I}_2 := \frac{1}{\sqrt{\varepsilon}} \int_0^t d\tau_1\; S(t-\tau_1)T^{\varepsilon}_j S(\tau_1)f^0_j, \qquad \boxed{\texttt{2.7}} \tag{3.9}$$

$$\mathcal{I}_3 := \frac{N-j}{\varepsilon} \int_0^t d\tau_1 \int_0^{\tau_1} dt_1\; S(t-\tau_1)T^{\varepsilon}_j S(\tau_1-t_1)C^{\varepsilon}_{j+1}S(t_1)f^0_{j+1}, \qquad \boxed{\texttt{2.8}} \tag{3.10}$$

$$\mathcal{I}_4 = \sum_{r=1}^{j} \sum_{1\leq s<\ell\leq j+1} \mathcal{I}_4^{r,\ell,s}, \qquad \boxed{\texttt{2.9}} \tag{3.11}$$

$$\mathcal{I}_4^{r,\ell,s} := \frac{N-j}{\varepsilon} \int_0^t dt_1 \int_0^{t_1} d\tau_1\, S(t-t_1)C^{\varepsilon}_{r,j+1}S(t_1-\tau_1)T^{\varepsilon}_{\ell,s}S(\tau_1)f^0_{j+1}. \qquad \boxed{\texttt{2.10}} \tag{3.12}$$

It is possible to show (see [BCEP1]) that the terms $\mathcal{I}_i$, $i = 1, 2, 3$ are negligible in the limit $\varepsilon \to 0$. This phenomenon is mainly governed by *oscillations*, whose effect is to decrease the effective size, in ε, of the various terms (non-stationary phase). Somewhat more surprinsingly, an important role is also played by *cancellations* between terms whose effective size is a truly *diverging* power of ε. We do not give the details here.

For similar reasons, it is also possible to show (see [BCEP1]) that all the contributions to $\mathcal{I}_4$, but that given by $r = \ell$ and $s = j+1$, are equally vanishing. In other words, only the collision/recollision event *"particle ℓ hits particle $j+1$ through $C^\varepsilon_{\ell,j+1}$, then recollides it through $T^\varepsilon_{\ell,j+1}$"*, happens to give a non-zero contribution in this picture.

So, the only $O(1)$ term is $\mathcal{I}_4^{\ell,\ell,j+1}$, the collision-recollision event.

We compute this term for $\ell = j = 1$:

$$\mathcal{I}_4^{1,1,2} = -\frac{N-1}{\varepsilon} \sum_{\sigma,\sigma'=\pm 1} \sigma\sigma' \int_0^t dt_1 \int_0^{t_1} d\tau_1 \int dx_2\, dv_2\, \frac{dh}{(2\pi)^3}\, \frac{dk}{(2\pi)^3}$$

$$\widehat{\phi}(h)\, \widehat{\phi}(k)\, \mathrm{e}^{i\frac{h}{\varepsilon}\cdot\left(x_1-x_2-v_1(t-t_1)\right)} \mathrm{e}^{i\frac{k}{\varepsilon}\cdot\left(x_1-x_2-v_1(t-t_1)-(v_1-v_2-\sigma h)(t_1-\tau_1)\right)}$$
$$f_2^0\left(x_1 - v_1 t + \sigma\frac{h}{2}t_1 + \sigma'\frac{k}{2}\tau_1, x_2 - v_2 t_1 - \sigma\frac{h}{2}t_1 - \sigma'\frac{k}{2}\tau_1;\right.$$
$$\left. v_1 - \sigma\frac{h}{2} - \sigma'\frac{k}{2}, v_2 + \sigma\frac{h}{2} + \sigma'\frac{k}{2}\right). \tag{3.13}$$ 2.11

This term is apparently of size ε^{-4}. In order to perform its analysis, we need to take advantage of the fast oscillations. Rearranging terms, they read

$$\exp\left(i\,\frac{h+k}{\varepsilon}\cdot\big[x_1 - x_2 - v_1(t-t_1)\big]\right)\, \exp\left(-i\frac{k}{\varepsilon}\cdot\big[v_1 - v_2 - \sigma h\big]\,(t_1 - \tau_1)\right).$$

Hence, as seen by direct inspection (at least at an informal level), the oscillations induce two different phenomena. The first oscillatory exponential enforces the variable k to have the value $-h$, while the relative position of particles 1 and 2 at the time t_1 of the collision, which is precisely $x_1 - v_1(t-t_1) - x_2$, tends to vanish asymptotically (recall that particle 1 is "created" at time t and has position $x_1 - v_1(t-t_1)$ at time t_1, while particle 2 is "created" at time t_1, and has position x_2 at that time). This is all due to the fact that $\int \exp(iy\cdot x)\psi(x,y)dxdy = \psi(0,0)$ whenever ψ is smooth enough. Second, the difficult oscillatory term is the remaining $\exp(-ik(v_1 - v_2 - \sigma h)(t_1 - \tau_1)/\varepsilon)$. The previous argument now needs to be refined, since the space and velocity variables k, v_1, etc. entering this oscillation also are involved in the previously analyzed oscillatory term. The point is that this second exponential actually induces oscillations in the *independent time variable* $t_1 - \tau_1$: for that reason, the time variable τ_1 needs to be rescaled so that τ_1 becomes t_1. In other words, the collision occuring at time t_1 and the recollision occuring at time τ_1 eventually tend to happen *simultaneously*.

Technically, all these considerations lead us to the following change of variables, which is both physically and mathematically relevant:

$$t_1 - \tau_1 = \varepsilon s_1, \qquad \xi = (h+k)/\varepsilon, \tag{3.14}$$

2.12

i.e. $\tau_1 = t_1 - \varepsilon s_1$ and $h = -k + \varepsilon\xi$. This gives in (3.13) the equivalent value

$$\begin{aligned}
\mathcal{I}_4^{1,1,2} = &-(N-1)\,\varepsilon^3 \sum_{\sigma,\sigma'=\pm 1} \sigma\sigma' \int_0^t dt_1 \int dv_2 \, \frac{dk}{(2\pi)^3} \int_0^{t_1/\varepsilon} ds_1 \int dx_2 \, \frac{d\xi}{(2\pi)^3} \\
&\widehat{\phi}(-k+\varepsilon\xi)\;\widehat{\phi}(k)\;\mathrm{e}^{i\xi\cdot\left(x_1 - x_2 - v_1(t-t_1)\right)}\,\mathrm{e}^{-is_1 k\cdot(v_1 - v_2 - \sigma(-k+\varepsilon\xi))} \\
&f_2^0\left(x_1 - v_1 t + \sigma\frac{-k+\varepsilon\xi}{2}t_1 + \sigma'\frac{k}{2}[t_1 - \varepsilon s_1],\right. \\
&\qquad x_2 - v_2 t_1 - \sigma\frac{-k+\varepsilon\xi}{2}t_1 - \sigma'\frac{k}{2}[t_1 - \varepsilon s_1]; \\
&\qquad \left. v_1 - \sigma\frac{-k+\varepsilon\xi}{2} - \sigma'\frac{k}{2}, v_2 + \sigma\frac{-k+\varepsilon\xi}{2} + \sigma'\frac{k}{2}\right),
\end{aligned} \tag{3.15}$$

2.13

and the reader should keep in mind that the weak-coupling regime implies $(N-j)\varepsilon^3 \sim 1$ in (3.15). In the limit $\varepsilon \to 0$, the above formula gives the asymptotic value

$$\begin{aligned}
\mathcal{I}_4^{1,1,2} \underset{\varepsilon\to 0}{\sim} &- \sum_{\sigma,\sigma'=\pm 1} \sigma\sigma' \int_0^t dt_1 \int dv_2 \, \frac{dk}{(2\pi)^3} \\
&|\widehat{\phi}(k)|^2 \left(\int_0^{+\infty} e^{-is_1 k\cdot(v_1 - v_2 + \sigma k)}\, ds_1\right) \\
&f_2^0\left(x_1 - v_1 t - (\sigma - \sigma')\frac{k}{2}t_1, x_1 - v_1(t-t_1) - v_2 t_1 + (\sigma-\sigma')\frac{k}{2}t_1;\right. \\
&\qquad \left. v_1 + (\sigma - \sigma')\frac{k}{2}, v_2 - (\sigma-\sigma')\frac{k}{2}\right).
\end{aligned} \tag{3.16}$$

2.14

In other words, the asymptotic process $\varepsilon \to 0$ tends to produce in (3.15) a Dirac mass at $\xi = 0$ and $x_2 = x_1 - v_1(t - t_1)$ on the one hand, and an oscillatory integral $\int_0^{+\infty} ds_1 \cdots$ on the other hand, which translates the fact that $\tau_1 = t_1 - \varepsilon s_1$, *i.e.* that the collision and recollision event happen at the same time. As we shall see, this oscillatory integral also allows to recover conservation of kinetic energy along the collisions.

In [BCEP1], we completely, and rigorously, justify formula (3.16). In particular the emergence of the oscillatory integral $\int_0^{+\infty} ds_1 \cdots$, can be fully explained. The main ingredient is that the oscillatory factor $\exp(is_1 \cdots)$ in (3.15) has size $s_1^{-3/2}$ as s_1 becomes large, uniformly in ε.

Next, we turn to identifying the limiting value obtained in (3.16). To do so, we observe the equality (in the distributional sense)

$$\mathrm{Re} \int_0^{\infty} e^{-is_1 k\cdot(v_1 - v_2 + \sigma k)}\, ds_1 = \pi \delta(k \cdot (v_1 - v_2 + \sigma k)). \tag{3.17}$$ 2.15

Using formula (3.17) we realize that the contribution $\sigma = -\sigma'$ in (3.16) gives rise to the *gain term*

$$\int_0^t dt_1 \int_{\mathbb{R}^3 \times \mathbb{S}^2} dv_2\, d\omega\ B_{\mathrm{w}}(\omega, v_1 - v_2) \\ f_2^0(x_1 - v_1(t - t_1) - v_1' t_1, x_2 - v_2(t - t_1) - v_2' t_1; v_1', v_2'), \tag{3.18}$$ 2.16

where B_{w} has been defined in (2.22), and $v_1' = v_1 - [v_1 - v_2] \cdot \omega\, \omega$, $v_2' = v_2 + [v_1 - v_2] \cdot \omega\, \omega$ as in (2.21). In this picture, the variable k measures the momentum transferred during the collision, and the Dirac mass $\delta(\cdots)$ in (3.17) expresses nothing else than the conservation of the energy during the collision. The momentum conservation is automatically satisfied. Similarly, the term $\sigma = \sigma'$ in (3.16) yields the *loss term*

$$\int_0^t dt_1 \int_{\mathbb{R}^3 \times \mathbb{S}^2} dv_2\, d\omega\, B_{\mathrm{w}}(\omega, v_1 - v_2) f_2^0(x_1 - v_1 t, x_1 - v_1(t - t_1) - v_2 t_1; v_1, v_2). \tag{3.19}$$ 2.17

We finally remark that the imaginary part of the time integral in the left hand side of (3.17) does not give any contribution. This uses a cancellation effect.

We have now proved

$$\lim_{\varepsilon \to 0} \mathcal{I}^{1,1,2} = \int_0^t dt_1 S(t - t_1)\, C_{1,2}\, S(t_1)\, f_0^{\otimes 2}, \tag{3.20}$$

in accordance with (3.1).

Let us draw a preliminary conclusion. Up to now, we have studied those terms entering the full perturbative series expansion of $f_j^N(t)$, that are of degree less than two in the potential. Two important facts come out of this analysis. First, only collision-recollision terms have a non-vanishing contribution, *i.e.* terms of the form

$$\varepsilon^{-4} \int_0^t dt_1 \int_0^{t_1} d\tau_1 \; S(t-t_1) C^{\varepsilon}_{\alpha,\beta} S(t_1-\tau_1) T^{\varepsilon}_{\alpha,\beta} S(\tau_1) f^0_{j+1}, \tag{3.21}$$

crc

for any possible values of the particles names α and β. These terms correspond to particles α, β "colliding" (through the T term) at time τ_1, then immediately "recolliding" (through the C term) at time t_1 (in (3.21) we have replaced the true prefactor $(N-j)/\varepsilon$ by ε^{-4} for simplicity). All other terms involving $S(t-t_1)C^{\varepsilon}_{\alpha,\beta}S(t_1-\tau_1)T^{\varepsilon}_{\alpha',\beta'}$ with $(\alpha,\beta) \neq (\alpha',\beta')$ do vanish. Second, we can also explicitely compute the limiting value of (3.21): it agrees with the gain term and loss term of the physically expected Boltzmann equation. Hence, in a sense, our quantum system agrees with the Boltzmann evolution *up to the second order in the potential*.

Naturally, this result is far from being conclusive: there are examples, like e.g. the pathologies of the Broadwell model quoted in [CIP], for which the agreement fails at the fourth order only.

Now, [BCEP1] proves more than agreement up to second order. We indeed consider the subseries (of the full series expansion expressing $f_j^N(t)$) formed by *all* the collision-recollision terms. In other words, we sum up *all* terms of the form (3.21) and consider the subseries of $f_j^N(t)$ given by

$$\sum_{n\geq 1} \sum_{\alpha_1,\dots,\alpha_n,\beta_1,\dots,\beta_n} \varepsilon^{-4n} \int_0^t dt_1 \int_0^{t_1} d\tau_1 \; S(t-t_1) C^{\varepsilon}_{\alpha_1,\beta_1} S(t_1-\tau_1) T^{\varepsilon}_{\alpha_1,\beta_1}$$
$$\cdots \int_0^{\tau_{n-1}} dt_n \int_0^{t_n} d\tau_n \, S(\tau_{n-1}-t_n) C^{\varepsilon}_{\alpha_n,\beta_n} S(t_n-\tau_n) T^{\varepsilon}_{\alpha_n,\beta_n} S(\tau_n) f^0_{j+n+1}. \tag{3.22}$$

crcb

Here the sum runs over all possible choices of the particles number α's and β's. We establish in [BCEP1] that the subseries (3.22) is indeed convergent for short times, uniformly in ε. Moreover, we prove that it approaches the corresponding *complete* series expansion obtained by solving iteratively the Boltzmann equation (2.19)-(2.20), with cross-section given by (2.22), namely the expansion given by (3.1). Technically, our analysis proves that each term $S(t_{i-1}-t_i)C^{\varepsilon}_{\alpha_i,\beta_i}S(t_i-\tau_i)T^{\varepsilon}_{\alpha_i,\beta_i}$ in (3.22) goes to the corresponding $S(t_{i-1}-t_i)C_{\alpha_i,\beta_i}$ in (3.1) as $\varepsilon \to 0$. Besides, each variable τ_i in (3.22) eventually needs to be rescaled as $\tau_i = t_i - \varepsilon s_i$, and all integrals $\int_0^{t_i} d\tau_i$ eventually become $\int_0^{+\infty} ds_i$, giving rise to oscillatory integrals of the form (3.17) that allow to recover the natural conservation of kinetic energy along each classical collision.

This is a much stronger convergence result towards the Boltzmann equation than mere "convergence up to second order in the potential". Technically, it is proved using a summation argument that we obtain through the stationary phase in large dimensions with a uniform control with respect to the dimension.

However, this does not completely finishes the proof yet: the true series expansion of $f_j^N(t)$ contains many more terms than those we retain in (3.22). Unfortunately, a rigorous proof of the term-by term convergence for the full series expansion giving $f_N^j(t)$ is still missing. Even more difficult seems to find a uniform bound on this series. Thus a mathematical justification of the quantum Boltzmann equation is a still an open, challenging and difficult problem.

4 The Bose-Einstein and Fermi-Dirac statistics

s4

The weak-coupling limit is more difficult to analyze when considering the case of Bosons and Fermions. Indeed, the statistics then modifies the structure of the states, and a complete factorization of the initial datum as in (2.18) is not compatible with Bose-Einstein or Fermi-Dirac statistics.

Systems of independent particles obeying the Bose-Einstein or Fermi-Dirac statistics are usually called quasi-free. Their reduced density matrices satisfy the following property (the integer j denotes the number of particles):

$$\rho_j(x_1 \dots x_j; y_1 \dots y_j) = \sum_{\pi \in \mathcal{P}_j} \theta^{s(\pi)} \prod_{i=1}^{j} \rho(x_i; y_{\pi(i)}). \tag{4.1}$$

3.1

Here $\rho(x, y)$ is the kernel of a one-particle density matrix, $\mathcal{P}_j$ is the group of the permutations of j elements, and, to each permutation π, we associate its signature $s(\pi)$ which is 1 if π is even, and -1 if π is odd. As usual, $\theta = 1$ in the bosonic case, while $\theta = -1$ in the fermionic case. Condition (4.1) implies that the Wigner function of a quasi-free state is given by the following sum over all permutations

$$f_j(x_1, v_1, \dots, x_j, v_j) = \sum_{\pi \in \mathcal{P}_j} \theta^{s(\pi)} f_j^{\pi}(x_1, v_1, \dots, x_j, v_j) \tag{4.2}$$

3.2

where each f_j^π has the value

$$f_j^\pi(x_1, v_1, \ldots, x_j, v_j) = \int dy_1 \ldots dy_j dw_1 \ldots dw_j \; e^{i(y_1 \cdot v_1 + \cdots + y_j \cdot v_j)}$$

$$\prod_{k=1}^{j} e^{-\frac{i}{\varepsilon} w_k \cdot (x_k - x_{\pi(k)})} \; e^{-\frac{i}{2} w_k \cdot (y_k + y_{\pi(k)})} \; f\left(\frac{x_k + x_{\pi(k)}}{2} + \varepsilon \frac{y_k - y_{\pi(k)}}{4}, w_k\right), \quad (4.3) \qquad \boxed{\texttt{3.3}}$$

and f is a given one-particle Wigner function. Note in passing that the Maxwell-Boltzmann case treated in the previous section corresponds, in this picture, to only retaining the contribution due the permutation $\pi =$ Identity in (4.3).

Plugging in the hierarchy (2.15) an initial datum satisfying (4.3), we can follow the same procedure as we did in section 3 for the Maxwell-Boltzmann statistics: we write the full perturbative series expansion expressing $f_j^N(t)$ in terms of the initial datum (see (3.4) and (3.6)), and try to analyze its asymptotic behaviour.

As in the previous section, we first restrict our attention to those terms of degree less than two in the potential.

The analysis up to second order is performed in [BCEP2]. We actually recover here Eq. (2.24) with the suitable $B_{\mathrm{w},\theta}$. The number of terms to control is much larger than in section 3, due to the sum over all permutations that enters the definition (4.2) of the initial state. Also, the asymptotics is much more delicate. In particular, we stress the fact that the initial datum (4.2)-(4.3) brings its own highly oscillatory factors in the process, contrary to the Maxwell-Boltzmann case where the initial datum is uniformly smooth, and where the oscillatory factors simply come from the collision operators $T^\varepsilon_{k,\ell}$ and $C^\varepsilon_{k,j+1}$. These new oscillatory factors naturally play a crucial role. Indeed, as we saw in the previous section, oscillations dominate the asymptotic process, and they are the building blocks that allow to recover the relevant Boltzmann equation in the limit. This is the very reason why a cubic Boltzmann equation is obtained in the Fermi-Dirac or Bose-Einstein case, while the equation simply is quadratic in the Maxwell-Boltzmann situation.

Technically, we analyze in [BCEP2] the repeated application of the collision-recollision operators C^ε_{j+1}, T^ε_{j+1}, as we did in the previous section, when they act on initial states of the form (4.3). The analysis is similar in spirit to the one we used to study $\mathcal{I}_0$, …, $\mathcal{I}_4$ in the previous section. Our approach yields various terms: two of them are bilinear in the initial condition f_0, and twelve are trilinear in f_0. Some of these terms vanish in the limit due to a non-stationary phase argument. Others give rise to truly diverging contributions (negative powers of ε). However, when grouping the terms in the appropriate way, those terms are seen to cancel each other. Last, some terms give the collision operator (2.24). The computation is heavy and hence we address the reader to [BCEP2] for the details.

This ends the analysis of terms up to second order in the potential.

Obviously, and as in the Maxwell-Boltzmann case, we could extend the result in [BCEP2] and try to resum the dominant terms, as we did in the previous section when extending the analysis of (3.21) to that of (3.22). This would lead to analyzing a true subseries of the complete series expansion expressing $f_j^N(t)$. We do not see any conceptual difficulty. However, this resummation procedure has not been explicitely done in [BCEP2].

To end this paragraph, we mention that a similar analysis, using commutator expansions in the framework of the second quantization formalism, has been performed in [HL] (following [H]) in the case of the van Hove limit for lattice systems (that is the same as the weak-coupling limit, yet without rescaling the distances). For more recent formal results in this direction, but in the context of the weak-coupling limit, we also wish to quote [ESY].

5 The low density limit

Up to here, we only have investigated the weak-coupling regime. In this section, we tackle the low-density regime, a technically more difficult situation.

Before coming to the details, we first recall that in the low-density regime, the statistics is expected to play no role in the asymptotics, due to the fact that the gas is rarefied. For that reason, we limit ourselves to completely factorized initial states, corresponding to a Maxwell-Boltzmann statistics, as in (2.18).

In the low-density case, the number of particles N diverges moderately, namely as ε^{-2} (in three dimensions of space), while the potential ϕ is not scaled at all. As a result, when keeping the kinetic approach already described in section 3, the low-density regime gives rise to exactly the same collision operators $C^{e}_{k,j+1}$ and $T^{\varepsilon}_{k,j+1}$ than in the weak coupling regime (they are given in (2.17), and (2.13), respectively), and the underlying hierarchy is also similar to (2.15), with a different normalization in ε yet: the new point is that the prefactor $1/\sqrt{\varepsilon}$ that we have in the weak-coupling regime in front of T^{ε}_j, is now replaced by $1/\varepsilon$ (a stronger prefactor), and the prefactor $(N-j)/\sqrt{\varepsilon} \sim \varepsilon^{-7/2}$ in front of C^{ε}_{j+1} is now replaced by $(N-j)/\varepsilon \sim \varepsilon^{-3}$ (a weaker prefactor). Quantitatively, the hierarchy is, in the low-density case:

$$\left(\partial_t + \sum_{k=1}^{j} v_k \cdot \nabla_k \right) f_j^N(X_j, V_j) = \frac{1}{\varepsilon} T_j^\varepsilon f_j^N + \frac{(N-j)}{\varepsilon} C_{j+1}^\varepsilon f_{j+1}^N . \tag{5.1}$$

4.14

Starting from (5.1), we may now solve (5.2) iteratively, as we did in section 3. This gives rise to a huge series expansion. Similarly to what has been done in section 3, we only analyze the subseries of the true series expansion of $f_j^N(t)$, that is obtained upon retaining the dominant terms only (see (3.22) in the weak-coupling case).

Now, due to the fact that the potential is stronger, the selection of dominant terms is somewhat different than in the weak-coupling situation. Actually, collision-recollision terms (one C operator followed by one T - see (3.21)) *do not* dominate the asymptotics, contrary to the weak-coupling case: one has to consider *all* terms obtained through a creation-recollision *sequence* with 1 operator C followed by n operators T, for any value of n. Namely, the dominant terms are all the

$$\begin{aligned} &\int_0^t dt_1 \int_0^{t_1} d\tau_1 \cdots \int_0^{\tau_{n-1}} d\tau_n \; S(t-t_1) C_{\alpha,\beta}^\varepsilon S(t_1-\tau_1) T_{\alpha,\beta}^\varepsilon S(\tau_1-\tau_2) T_{\alpha,\beta}^\varepsilon \\ &\qquad \cdots \; S(\tau_{n-1}-\tau_n) T_{\alpha,\beta}^\varepsilon S(\tau_n) f_{j+1}^0, \end{aligned} \tag{5.2}$$

4.1

for any values of the particles number α and β (compare with (3.21), for which $n = 1$). Such terms certainly behave in a different way for the weak-coupling and the low-density regimes. Indeed the coefficient in front of such sequences are:

$$\varepsilon^{-(\frac{7}{2}+\frac{n}{2})} \qquad \text{in the weak-coupling regime,} \tag{5.3}$$

$$\varepsilon^{-(3+n)} \qquad \text{in the low-density regime.} \tag{5.4}$$

Besides, as we have seen in the computations of section 3, each C operator gives a gain of ε^3 due to oscillations, and each T operator gives a gain of ε due to the associated time integration (see the change of variable (3.14)). As a result, the term involved in (5.2) has the effective size

$$\varepsilon^{+(\frac{n}{2}-\frac{1}{2})} \qquad \text{in the weak-coupling regime,} \tag{5.5}$$

$$O(1) \qquad \text{in the low-density regime.} \tag{5.6}$$

In conclusion, for the weak-coupling regime, only the term for $n = 1$ is $O(1)$, all the others being negligible - in agreement with what we assert in section

3. On the contrary, for the low-density regime, all the terms of the type (5.2) are $O(1)$ and, therefore, they have to be resummed.

Our contribution in [BCEP3] is the following. First, we analyze each term of the form (5.2), for each value of n. Using stationary phase methods in large dimensions and carefully analyzing the phase factors involved, we are able to pass to the limit in these terms. We refer to [C1] for a similar analysis in the context of the linear Boltzmann equation. Second, we resum these terms with respect to n. Using the very specific algebraic structure of the underlying series, and using a previous identity proved in [C2], we show the typical relation

$$\begin{aligned}
&\lim_{\varepsilon\to 0}\Bigg(\sum_{n\geq 1}\varepsilon^{-3+n}\int_0^t dt_1\int_0^{t_1}d\tau_1\ \cdots\int_0^{\tau_{n-1}}d\tau_n \\
&\qquad S(t-t_1)C^{\varepsilon}_{1,2}S(t_1-\tau_1)T^{\varepsilon}_{1,2}\cdots\ S(\tau_{n-1}-\tau_n)T^{\varepsilon}_{1,2}S(\tau_n)f_2^0\Bigg) \\
&= \int dv_1 d\omega B_\ell(\omega, v-v_1)\left[f_0(t,x,v')\,f_0(t,x,v_1')-f_0(t,x,v)\,f_0(t,x,v_1)\right],
\end{aligned} \tag{5.7}$$

bs

where B_ℓ is the full Born series expansion of quantum scattering (see (2.28)). The difficulty actually lies in *identifying* the coefficient B_ℓ at this step. Last, we resum *all term* of the form (5.2). We refer to (3.22) for the analogous approach in the weak-coupling regime. We do not write the corresponding formulae. We simply mention that the corresponding series is proved to converge for small times, uniformly in ε, towards the perturbative series expansion of the solution to the Boltzmann equation (2.26)-(2.27)-(2.28). We refer to [BCEP3] for the details. Note that our results need a smallness assumption on the potential, as does the Born series expansion of quantum scattering.

As in section 3, this analysis only yields a partial result, stating that a subseries of the true series expansion of $f_j^N(t)$ converges to the appropriate Boltzmann equation: neither are we able to bound the true series, nor are we able to pass to the limit term-by-term.

References

Ba [Ba] R. Balescu, **Equilibrium and Nonequilibrium Statistical Mechanics**, John Wiley & Sons, New-York (1975).

BCEP1 [BCEP1] D. Benedetto, F. Castella, R. Esposito, M. Pulvirenti, *Some Considerations on the derivation of the nonlinear Quantum Boltzmann Equation*, J. Stat. Phys. Vol 116, Nos 114 pp 381-410 (2004).

BCEP2 [BCEP2] D. Benedetto, F. Castella, R. Esposito, M. Pulvirenti, *On The Weak-Coupling Limit for Bosons and Fermions* , Math. Mod. Meth. Appl. Sci., Vol. 15, N. 12, pp. 1811–1843 (2005).

BCEP3 [BCEP3] D. Benedetto, F. Castella, R. Esposito, M. Pulvirenti, *Some Considerations on the derivation of the nonlinear Quantum Boltzmann Equation II: the low-density regime*, To appear in J. Stat. Phys. (2006).

C1 [C1] F. Castella, *From the von Neumann equation to the Quantum Boltzmann equation in a deterministic framework*, J. Stat. Phys., Vol. 104, N. 1/2, pp. 387-447 (2001).

C2 [C2] F. Castella, *From the von Neumann equation to the Quantum Boltzmann equation II: identifying the Born series*, J. Stat. Phys., Vol. 106, N. 5/6, pp. 1197-1220 (2002).

CC [CC] S. Chapman, T. G. Cowling, **The Mathematical Theory of Non-uniform Gases**, Cambridge Univ. Press, Cambridge, England (1970).

CIP [CIP] C. Cercignani, R. Illner, M. Pulvirenti, **The mathematical theory of dilute gases**, Applied Mathematical Sciences, Vol. 106, Springer-Verlag, New York (1994).

ESY [ESY] L. Erdös, M. Salmhofer, H.-T. Yau, *On the quantum Boltzmann equation*, J. Stat. Phys., Vol. 116, N. 1-4, pp.367-380 (2004).

H [H] M.N. Hugenholtz, *Derivation of the Boltzmann equation for a Fermi gas*, J. Stat. Phys 32, 231-254, (1983).

HL [HL] N.T. Ho, L.J. Landau, *Fermi gas in a lattice in the van Hove limit*, J. Stat. Phys 87, 821-845, (1997).

IP [IP] R. Illner, M. Pulvirenti, *Global Validity of the Boltzmann equation for a two-dimensional rare gas in the vacuum* ,Comm. Math. Phys. Vol. 105 189-203 (1986), *Erratum and improved result*, Comm. Math. Phys. Vol 121 143-146 (1989).

L [L] O. Lanford III, *Time evolution of large classical systems*, Lecture Notes in Physics, Vol. 38, pp. 1-111, E.J. Moser ed., Springer-Verlag (1975).

LP [LP] P.L.Lions, T.Paul, *Sur les mesures de Wigner*, Revista Matematica ibero americana, Vol.9, n.3, 553-618 (1993).

RS [RS] M. Reed, B. Simon, **Methods of modern mathematical physics III. Scattering theory**, Academic Press, New York-London (1979).

Sp1 [Sp1] H. Spohn, **Large scale dynamics of interacting particles**, Springer (1991)

Sp2 [Sp2] H. Spohn, *Quantum kinetic equations*, in "On three levels: micro-meso and Macro Approaches in Physics" M.Fannes, C.Maes, A Verbeure eds, Nato series B: Physics Vol. 324, 1-10, (1994).

UU [UU] E.A. Uehling, G.E. Uhlenbeck, *Transport Phenomena in Einstein-Bose and Fermi-Dirac Gases. I*, Phys. Rev. Vol 43, 552-561 (1933)

W [W] E.P. Wigner, *On the quantum correction for thermodynamic equilibrium*, Phys. Rev., Vol. 40 (1932).