ON THE WEAK-COUPLING LIMIT FOR BOSONS AND FERMIONS

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Abstract. In this paper we consider a large system of Bosons or Fermions. We start with an initial datum which is compatible with the Bose-Einstein, respectively Fermi-Dirac, statistics. We let the system of interacting particles evolve in a weak-coupling regime. We show that, in the limit, and up to the second order in the potential, the perturbative expansion expressing the value of the one-particle Wigner function at time t, agrees with the analogous expansion for the solution to the Uehling-Uhlenbeck equation.

This paper follows in spirit the companion work [2], where the authors investigated the weak-coupling limit for particles obeying the Maxwell-Boltzmann statistics: here, they proved a (much stronger) convergence result towards the solution of the Boltzmann equation.

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1. Introduction

In 1933 Uehling and Uhlenbeck in Ref. [17] proposed the following kinetic equation, called U-U in the sequel, for the time evolution of the one-particle Wigner function $f(x, v; t)$ associated with a large system of weakly interacting Bosons or Fermions (see Ref. [18] for the definition of the Wigner function). The U-U equation is

$$\begin{aligned}\partial_t f(x,v;t)+v\cdot\nabla_x f(x,v;t) = \int dv_* \int dv'_* \int dv'\; W(v,v_*|v',v'_*) \\ \Big\{ f'f'_*(1+8\pi^3\theta f)(1+8\pi^3\theta f_*) - ff_*(1+8\pi^3\theta f')(1+8\pi^3\theta f'_*)\Big\},\end{aligned} \tag{1.1}$$

where we use the standard short-hand notation

$$f = f(x,v;t), \quad f_* = f(x,v_*;t), \quad f' = f(x,v';t), \quad f'_* = f(x,v'_*;t).$$

Here, W denotes the transition kernel

$$\begin{aligned}W(v,v_*|v',v'_*) = \frac{1}{8\pi^2}\left[\widehat{\phi}(v'-v)+\theta\,\widehat{\phi}(v'-v_*)\right]^2 \\ \delta(v+v_*-v'-v'_*)\;\delta\Big(\frac{1}{2}(v^2+v_*^2-v'^2-v_*'^2)\Big).\end{aligned} \tag{1.2}$$

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Finally,

$$\widehat{\phi}(k) = \int dx \, \mathrm{e}^{-ik\cdot x}\phi(x) \tag{1.3}$$

is the Fourier transform of the two-body interaction potential ϕ, and $\theta = \pm 1$ for Bosons and Fermions respectively.

Note that the factors $8\pi^3$ do not appear in the original U-U equation in Ref. [17], because there, the distribution function is normalized in such a way that its integral on the velocity variable equals the space density times $8\pi^3$. At variance, in (1.1), f is just the standard Wigner function, whose integral on the velocities equals the space density. Let us mention that equation (1.1) actually is cubic (and not quartic) in the unknown f: apparent quartic terms have vanishing contribution, as shown by direct inspection.

Eq. (1.1) constitutes a natural modification of the usual quantum Boltzmann equation, in order to take into account statistics. In particular, there is a H-functional

$$\mathcal{H}(f) = \int dx \int dv \left\{ f \log f - \theta(1 + 8\pi^3\theta f) \log(1 + 8\pi^3\theta f) \right\} \tag{1.4}$$

driving the system to the Bose-Einstein and Fermi-Dirac equilibrium distribution:

$$M(v) = \frac{1}{\mathrm{e}^{\beta\mu + \beta v^2/2} - 8\pi^3\theta} \tag{1.5}$$

outside the Bose condensation region. Here β and μ denote the inverse temperature and the chemical potential respectively.

Eq. (1.1) is largely studied (see for istance [1] and [12] for physical consideration, and [15], [7], [13], [14] ... for a more mathematically oriented analysis concerning the existence of solutions and asymptotic behavior), so that it is certainly of great relevance to derive this equation from the first principles, namely from the Schrödinger equation.

As clearly explained by H. Spohn in [16], Eq (1.1) is indeed expected to hold in the so called weak-coupling limit, which consists in scaling space, time and the potential according to

$$x \to \varepsilon x, \quad t \to \varepsilon t, \quad \phi \to \sqrt{\varepsilon}\phi, \tag{1.6}$$

where ε is a small positive parameter.

A slightly different limit, usually called van Hove limit, scales t and ϕ as in (1.6) but leaves the microscopic space scale unchanged. Eq. (1.1) cannot be derived in the van Hove limit in general but, in case of translationally invariant states, we expect to achieve the homogeneous version of the U-U equation (for a large system). In fact Hugenholtz [11] proved formally that this happens. Later on Ho and Landau [10] proved that the homogeneous U-U equation holds rigorously up to the second order expansion in the potential. These approaches, as well as the recent contribution by Erdös et al. [8] (where the quantum analog of the Boltzmann's *Stosszahlansatz* is formulated), are based on the commutator expansion of the time evolution of the observables of the CCR and CAR algebras.

In the present paper we approach the problem from a different viewpoint. We start from the time evolution of a N particle quantum system in terms of the Wigner formalism. Here the statistics enters only through the choice of the admissible states we take as initial conditions. Such states, called quasi-free, must describe free Bosons and Fermions, so that they cannot have any other correlations but those arising from the statistics. Therefore the first step is to characterize quasi-free states (see for example Ref. [4]) in terms of the Wigner functions. Then we apply the dynamics (in terms of the usual hierarchy) and represent the solution as a perturbative expansion. The truncation of this expansion up to the second order in the potential is shown to converge to the expansion associated to the U-U equation, up to the first order in the scattering cross section.

In other words we recover the result in Ref. [10] with the following main differences. First we exploit the weak-coupling limit, so that we can deal with states which are not necessarily translationally invariant. Second, we work directly in terms of the Wigner formalism, in the same spirit of the Balescu book (see Ref. [1]). In doing so, we also follow a previous work [2] by the authors for the Maxwell-Boltzmann statistics. Hence the present work shows how the statistics can be handled in this formalism. Note in passing that the case of the Maxwell-Boltzmann statistics allows for a much stronger (but still partial) convergence result than the one presented here, see [2]. Note finally that the present formalism also allows to handle the low-density limit, see [3], see also the last section of this text.

It is also important to mention that a full rigorous derivation of the U-U equation (but also of the usual Boltzmann equation arising for the Maxwell-Boltzmann statistics) is still far beyond the present techniques and those of the previous references.

The plan of the paper is the following. In the next section we describe the particle system. In Section 3 we establish the result. The rest of the paper is devoted to the proofs.

2. The particle system.

We consider a Quantum particle system in $\mathbb{R}^3$. Let

$$\mathfrak{H} = \bigoplus_{n\geq 0} L_2(\mathbb{R}^3)^n := \bigoplus_{n\geq 0} \mathfrak{H}_n, \tag{2.1}$$

be the Fock space. A state of the system is a self-adjoint, positive trace class operator acting on $\mathfrak{H}$:

$$\sigma = \bigoplus_{n\geq 0} \sigma_n. \tag{2.2}$$

We assume

$$\operatorname{Tr}\sigma = 1. \tag{2.3}$$

The operator N, number of particles, is the multiplication by n on $\mathfrak{H}_n$ and hence

$$\langle N\rangle = \sum_{n\geq 0} n \operatorname{Tr}\sigma_n, \tag{2.4}$$

where the left hand side is the average number of particles in the state σ. If $\sigma_n(X_n;Y_n)$ is the kernel of σ_n, the Reduced Density Matrices (RDM) are defined by:

$$\rho_n(X_n;Y_n) = \sum_{m\geq 0} \frac{(n+m)!}{m!} \int \sigma_{n+m}(X_n,Z_m;Y_n,Z_m)\, dZ_m. \tag{2.5}$$

Here $X_n = (x_1, \dots, x_n)$, $x_i \in \mathbb{R}^3$ denotes the n-particle configuration. Note that

$$\begin{aligned}\operatorname{Tr}\rho_n &= \int dZ_n\ \rho_n(Z_n; Z_n) \\ &= \sum_{m\geq n} m(m-1)\dots(m-n+1)\operatorname{Tr}\sigma_m = \langle N(N-1)\dots(N-n+1)\rangle,\end{aligned} \tag{2.6}$$

and hence the RDM are equivalent to the classical correlation functions.

The Hamiltonian of the system is the self-adjoint operator acting on $\mathfrak{H}$ given by

$$H = \bigoplus_{n=1}^{\infty} H_n, \tag{2.7}$$

where

$$H_n = -\frac{1}{2}\sum_{j=1}^{n}\Delta_{x_j} + \sum_{1\leq i<j\leq n}\phi(x_i - x_j), \tag{2.8}$$

and the potential ϕ is a smooth two-body interaction. Here, $\hbar$ as well as the mass of the particles are normalized to unity.

Under these circumstances, the time evolved state is given by the usual

$$\sigma(t) = \mathrm{e}^{-iHt}\sigma\mathrm{e}^{iHt}. \tag{2.9}$$

Now, quantum statistics is taken into account by suitable properties of the physically relevant states. Namely, for the Maxwell-Boltzmann (M-B) statistics we require symmetry of $\rho_n(x_1, \dots, x_n; y_1 \dots, y_n)$ in the exchange of particle names. For the Bose-Einstein (B-E) and Fermi-Dirac (F-D) statistics we require additionally

$$\rho_n(x_1, \dots, x_n; y_1 \dots, y_n) = \theta^{s(\pi)}\rho_n(x_1, \dots, x_n; y_{\pi(1)} \dots, y_{\pi(n)}), \tag{2.10}$$

where $\pi \in \mathcal{P}_n$ is a permutation of n elements, and $s(\pi) = 0$ if the permutation is even, $s(\pi) = 1$ if it is odd.

Alternatively, the quantum statistics is automatically taken into account by considering states on the algebra generated by the annihilation and creation operators $a(x)$ and $a^\dagger(x)$ (with the commutation and anti-commutation relations according to the B-E and F-D statistics respectively). Then the RDM are defined as

$$\operatorname{Tr}\left[\sigma a^\dagger(x_n)\dots a^\dagger(x_1)a(y_1)\dots a(y_n)\right] = \rho_n(x_1, \dots, x_n; y_1 \dots, y_n). \tag{2.11}$$

However we do not use here the second quantization formalism.

Given a state σ, we define the Wigner transform [18] by

$$W_n(X_n; V_n) := \frac{1}{(2\pi)^{3n}}\int dY_n\ \mathrm{e}^{iY_n\cdot V_n}\sigma_n\Big(X_n - \frac{1}{2}Y_n; X_n + \frac{1}{2}Y_n\Big). \tag{2.12}$$

Therefore the analogous of the classical correlation functions are the j-particle Wigner functions defined through

$$F_j(X_j; V_j) = \sum_{n\geq 0}\frac{(n+j)!}{n!}\int dX_n\int dV_n\ W_{j+n}(X_j, X_n; V_j, V_n). \tag{2.13}$$

Note that the F_j's are the Wigner transforms of the RDM ρ_j, as one can easily check.

Due to the dynamics imposed by (2.9), it is a standard computation to check that the Wigner function W_n evolves according to the Wigner-Liouville equation

$$\partial_t W_n + \sum_{i=1}^{n} v_i \cdot \nabla_{x_i} W_n = T_n W_n. \tag{2.14}$$

As a consequence, the j-particle Wigner functions F_j's satisfy the associated hierarchy

$$\partial_t F_j + \sum_{i=1}^{j} v_i \cdot \nabla_{x_i} F_j = T_j F_j + C_{j+1} F_{j+1}, \tag{2.15}$$

where T_j and C_{j+1} will be defined below after Eq.(2.22), and the index j takes any value between 1 and N. Equations (2.15) are analogous to the usual BBGKY hierarchy for the classical systems and are derived in a similar manner. Note that by the definition of the RDM the coefficient in front of C_{j+1} is one instead of $N-j$.

We now want to analyze (2.15) in the weak-coupling regime (1.6). Therefore, we set

$$f_j^\varepsilon(X_j; V_j; t) := F_j(\varepsilon^{-1} X_j; V_j; \varepsilon^{-1} t), \tag{2.16}$$

where $\varepsilon > 0$ is a small parameter, and we scale the potential as well, by setting

$$\phi \to \sqrt{\varepsilon}\phi. \tag{2.17}$$

The resulting, scaled, equation is

$$\partial_t f_j^\varepsilon + \sum_{i=1}^{j} v_i \cdot \nabla_{x_i} f_j^\varepsilon = \frac{1}{\sqrt{\varepsilon}} T_j^\varepsilon f_j^\varepsilon + \frac{1}{\sqrt{\varepsilon}} \varepsilon^{-3} C_{j+1}^\varepsilon f_{j+1}^\varepsilon, \tag{2.18}$$

where

$$(T_j^\varepsilon f_j^\varepsilon)(X_j; V_j) = \sum_{0<k<\ell\leq j} (T_{k,l}^\varepsilon f_j^\varepsilon)(X_j; V_j), \tag{2.19}$$

and the $T_{k,l}^\varepsilon$'s are defined as follows: if $j=1$, we simply have $T_1^\varepsilon = 0$; otherwise,

$$\begin{aligned}(T_{k,l}^\varepsilon f_j^\varepsilon)(X_j; V_j) = &- i \sum_{\sigma=\pm 1} \sigma \int \frac{dh}{(2\pi)^3}\, \widehat{\phi}(h)\, \mathrm{e}^{i \frac{h}{\varepsilon} \cdot (x_k - x_\ell)} \\ & f_j^\varepsilon\Big(x_1, \dots, x_j; v_1, \dots, v_k - \sigma \frac{h}{2}, \dots, v_\ell + \sigma \frac{h}{2}, \dots, v_j\Big).\end{aligned} \tag{2.20}$$

On the other hand, the C_{j+1}^ε in (2.18) is computed as:

$$(C_{j+1}^\varepsilon f_{j+1}^\varepsilon)(X_j; V_j) = \sum_{k=1}^{j} (C_{k,j+1}^\varepsilon f_{j+1}^\varepsilon)(X_j; V_j), \tag{2.21}$$

where

$$(C^{\varepsilon}_{k,j+1}f^{\varepsilon}_{j+1})(X_j;V_j) = -\,i\sum_{\sigma=\pm1}\sigma\int\frac{dh}{(2\pi)^3}\int dx_{j+1}\int dv_{j+1}\,\widehat{\phi}(h)\,\mathrm{e}^{i\frac{h}{\varepsilon}\cdot(x_k-x_{j+1})}$$
$$f^{\varepsilon}_{j+1}\Big(x_1,\dots,x_{j+1};v_1,\dots,v_k-\sigma\frac{h}{2},\dots,v_{j+1}+\sigma\frac{h}{2}\Big). \tag{2.22}$$

Note that T_j and C_{j+1} are T^{ε}_j and C^{ε}_{j+1} for $\varepsilon = 1$. Last, we fix an initial condition sequence

$$\{f^{\varepsilon}_{0,j}\}_{j=1}^{\infty} \tag{2.23}$$

according to the quantum statistics, and perform the limit $\varepsilon \to 0$ in the resulting system.

Remark: Since

$$\int f^{\varepsilon}_{0,1}(x,v)dxdv = \varepsilon^3\langle N\rangle, \tag{2.24}$$

requiring $\|f^{\varepsilon}_{0,1}\|_{L_1} = O(1)$, implies $\langle N\rangle = O(\varepsilon^{-3})$. In other words we are working in the Grand-canonical formalism and the density is automatically rescaled. □

In the following we shall fix $f^{\varepsilon}_{0,1}$ to be a given (independent of ε) probability density f_0. This means that its inverse Wigner transform

$$\rho^{\varepsilon}(x,y) = \int dv\,\mathrm{e}^{i\frac{x-y}{\varepsilon}\cdot v}\,f_0\Big(\frac{x+y}{2},v\Big), \tag{2.25}$$

namely the one-particle rescaled RDM, is a superposition of WKB states.

We now make assumptions on the initial state to take into account the statistics. For the M-B statistics a suitable initial sequence can be chosen completely factorized, e.g.

$$f^{\varepsilon}_{0,j} = f_0^{\otimes j}. \tag{2.26}$$

Such a notion of statistical independence, which corresponds to a complete factorization of the RDM's, is not compatible (but for the condensed Bose state) with the B-E and F-D statistics which exhibit intrinsic correlations even for non interacting particle systems. States describing free Bosons or Fermions are usually called quasi-free and are defined in terms of the RDM's by the following formula:

$$\rho_j(x_1,\dots,x_j;y_1,\dots,y_j) = \sum_{\pi\in\mathcal{P}_j}\theta^{s(\pi)}\prod_{i=1}^{j}\rho(x_i,y_{\pi(i)}), \tag{2.27}$$

for some positive definite operator ρ on $L_2(\mathbb{R}^3)$ with kernel $\rho(x,y)$. We show in Appendix how to construct explicitly quasi-free states for Bosons.

From now on we assume that the initial sequence (2.23) for the rescaled problem (2.18) is given by the Wigner transform of a quasi-free state (2.27) generated by $\rho(x,y) = \rho^{\varepsilon}(x,y)$ given by (2.25). As a consequence the initial sequence $\{f^0_j\}_{j=1}^{\infty}$ for the hierarchy (2.18) is of the form

$$f^0_j(X_j,V_j) = \sum_{\pi\in\mathcal{P}_j}\theta^{s(\pi)}f^{\pi}_j(X_j,V_j), \tag{2.28}$$

with

$$
\begin{aligned}
f_j^{\pi}(x_1,\dots,x_j,v_1,\dots,v_j) &= \frac{1}{(2\pi)^{3j}} \int dy_1 \cdots \int dy_j \int dw_1 \cdots \int dw_j \\
\prod_{k=1}^{j} \Big(\mathrm{e}^{iy_k\cdot v_k + iw_k\cdot \frac{x_k - x_{\pi(k)}}{\varepsilon} - iw_k \cdot \frac{y_k + y_{\pi(k)}}{2}} & f_0\Big(\frac{x_k + x_{\pi(k)}}{2} - \frac{\varepsilon}{4}(y_k - y_{\pi(k)}), w_k\Big)\Big) .
\end{aligned}
\tag{2.29}
$$

Eq. (2.29) follows from (2.27) and (2.25).

We underline once more that, in the present approach, the dynamics is given by the hierarchy of equations (2.18) which are completely equivalent to the Schrödinger equation, while the statistics enters only in the structure of the initial state.

In the weak-coupling limit $\varepsilon \to 0$, we expect that $f_j^{\varepsilon}(t)$ converges to a factorized state (because the effects of statistics disappear in the macroscopic limit). On the more each factor should be solution to the U-U equation (the collisions being affected by the statistics because they involve microscopic scales).

3. The main result.

Let $f = f(x,v,t)$ be a solution to the U-U equation and set $f_j(\,\cdot\,,\,\cdot\,,t) = f^{\otimes j}(\,\cdot\,,\,\cdot\,,t)$. Then the sequence $\{f_j\}_{j=1}^{\infty}$ satisfies the following hierachy of equations:

$$
(\partial_t + \sum_{i=1}^{j} v_i \cdot \nabla_{x_i}) f_j = Q_{j,j+1} f_{j+1} + Q_{j,j+2} f_{j+2}. \tag{3.1}
$$

Here the $Q_{j,j+1}$ contribution, a "two particles term" in the terminology used below, is

$$
\begin{aligned}
(Q_{j,j+1}f_{j+1})(X_j,V_j) = \sum_{k=1}^{j} \int dv'_k \int dv_{j+1} \int dv'_{j+1}\, W(v_k, v_{j+1}|v'_k, v'_{j+1}) \\
\Big\{ f_{j+1}(X_j, x_k; v_1,\dots,v'_k,\dots v'_{j+1}) - f_{j+1}(X_j, x_k; v_1,\dots,v_{j+1}) \Big\},
\end{aligned}
\tag{3.2}
$$

and the $Q_{j,j+2}$ contribution, a "three particles term", is

$$
\begin{aligned}
(Q_{j,j+2}f_{j+2})(X_j,V_j) &= 8\pi^3\theta \sum_{k=1}^{j} \int dv'_k \int dv_{j+1} \int dv'_{j+1} W(v_k, v_{j+1}|v'_k, v'_{j+1}) \\
\Big\{ f_{j+2}(X_j, x_k, x_k; v_1,\dots,v'_k,\dots v'_{j+1}, v_k) &+ f_{j+2}(X_j, x_k, x_k; v_1,\dots,v'_k,\dots v'_{j+1}, v_{j+1}) \\
- f_{j+2}(X_j, x_k\, x_k; v_1,\dots,v_{j+1}, v'_k) &- f_{j+2}(X_j\, x_k, x_k; v_1,\dots,v_{j+1}, v'_{j+1}) \Big\}.
\end{aligned}
\tag{3.3}
$$

Also, (X_n, y) denotes the $(n+1)$-sequence $(x_1,\dots,x_n,y)$.

A formal solution to the hierarchy (3.1) is given by the following series expansion:

$$
\begin{aligned}
f_j(t) = & \sum_{n\geq 0} \sum_{\substack{\alpha_1\cdots\alpha_n \\ \alpha_i=1,2}} \int_0^t dt_1 \int_0^{t_1} dt_2 \cdots \int_0^{t_{n-1}} dt_n \\
& S(t-t_1) Q_{j,j+\alpha_1} S(t_1-t_2) Q_{j+\alpha_1, j+\alpha_1+\alpha_2} S(t_2-t_3) \dots \\
& Q_{j+\alpha_1+\cdots+\alpha_{n-1}, j+\alpha_1+\cdots+\alpha_n} S(t_n) f_0^{\otimes(j+\alpha_1+\cdots+\alpha_n)},
\end{aligned}
\tag{3.4}
$$

where $S(t)$ denotes the fream stream operator, namely,

$$
(S(t)f_j)(X_j, V_j) = f_j(X_j - V_j t, V_j). \tag{3.5}
$$

As for the solution to the ε-dependent hierarchy (2.18), we can also expand f_ε^j in the similar way, at least at the formal level. This gives

$$
\begin{aligned}
f_j^\varepsilon(t) = & \sum_{n\geq 0} \sum_{\substack{\gamma_1\cdots\gamma_n \\ \gamma_i=0,1}} \int_0^t dt_1 \int_0^{t_1} dt_2 \cdots \int_0^{t_{n-1}} dt_n \\
& S(t-t_1) P^\varepsilon_{j,j+\gamma_1} S(t_1-t_2) P^\varepsilon_{j+\gamma_1, j+\gamma_1+\gamma_2} S(t_2-t_3) \dots \\
& P^\varepsilon_{j+\gamma_1+\cdots+\gamma_{n-1}, j+\gamma_1+\cdots+\gamma_n} S(t_n) f^0_{j+\gamma_1+\cdots+\gamma_n},
\end{aligned}
\tag{3.6}
$$

where f_j^0 is an initial quasi-free state given by (2.29), and

$$
P^\varepsilon_{j,j+1} = \varepsilon^{-\frac{7}{2}} C^\varepsilon_{j+1}, \quad P^\varepsilon_{j,j} = \varepsilon^{-\frac{1}{2}} T^\varepsilon_j. \tag{3.7}
$$

We are not able to show the convergence of $f_j^\varepsilon(t)$ to $f_j(t)$ in the limit $\varepsilon \to 0$ even for short times. However we are going to show that the two series agree up to the second order in the potential. Namely, defining the second order contributions

$$
g(t) := S(t)f_0 + \int_0^t dt_1\, S(t-t_1) Q_{1,2} S(t_1) f_0^{\otimes 2} + \int_0^t dt_1\, S(t-t_1) Q_{1,3} S(t_1) f_0^{\otimes 3}, \tag{3.8}
$$

associated with $f_j(t)$, and

$$
\begin{aligned}
g^\varepsilon(t) := & S(t) f_0 + \varepsilon^{-\frac{7}{2}} \int_0^t dt_1\, S(t-t_1) C_2^\varepsilon S(t_1) f_2^0 \\
& + \varepsilon^{-4} \int_0^t dt_1 \int_0^{t_1} dt_2\, S(t-t_1) C_2^\varepsilon S(t_1-t_2) T_2^\varepsilon S(t_2) f_2^0 \\
& + \varepsilon^{-7} \int_0^t dt_1 \int_0^{t_1} dt_2\, S(t-t_1) C_2^\varepsilon S(t_1-t_2) C_3^\varepsilon S(t_2) f_3^0,
\end{aligned}
\tag{3.9}
$$

associated with $f_j^\varepsilon(t)$, we rigorously prove below the convergence of $g^\varepsilon(t)$ to $g(t)$, under suitable assumptions on the data of the problem.

Assumptions: We require ϕ to be real and even, so that $\widehat{\phi}$ is real. In particular

$$
\widehat{\phi}(k) = \overline{\widehat{\phi}(-k)} = \widehat{\phi}(-k).
$$

This the most important assumption we need in the analysis. Besides, we shall need to deal with "smooth" data. Quantitatively, we assume the following regularity:

$$(1+|\xi|)^\alpha \sum_{|\beta|\le 2} |D_\xi^\beta \widehat{\phi}(\xi)| \in L_1,$$

for a sufficiently large α, and

$$\begin{aligned} &f_0(x,v) \in L_1, \\ &(1+|\xi|+|\eta|)^\alpha \sum_{0\le|\beta|\le 2}\sum_{0\le|\gamma|\le 2} |(D_\xi^\beta + D_\eta^\gamma)\widehat{f}_0(\xi,\eta)| \in L_1, \end{aligned} \tag{3.10}$$

for a sufficiently large α as well. In (3.10), β and γ denote multi-indices, and D_ξ^β, D_η^γ denote derivatives with respect to the variables ξ and η. Note that throughout this paper we use the following normalization for the Fourier transform:

$$\begin{aligned} \widehat{f}(h) &= (\mathcal{F}_x f)(h) = \int_{\mathbb{R}^n} dx\ f(x)\, \mathrm{e}^{-ih\cdot x}, \\ f(x) &= (\mathcal{F}_h^{-1}\widehat{f})(h) = \frac{1}{(2\pi)^n}\int_{\mathbb{R}^n} dh\ \widehat{f}(h)\, \mathrm{e}^{ih\cdot x}. \end{aligned} \tag{3.11}$$

□

Our main result is the

Theorem. *Under the above assumptions, we have*

$$\lim_{\varepsilon\to 0} \widehat{g}^\varepsilon(\xi,\eta,t) = \widehat{g}(\xi,\eta,t), \quad \textit{for any } t>0 \textit{ and any } (\xi,\eta)\in\mathbb{R}^6.$$

Remark: In the above statement (and the proofs given below), we found convenient to treat the terms in (3.8) and (3.9) in terms of their Fourier transforms, for which the convergence arises more naturally. However, we would like to stress that in the companion paper [2], a stronger, but analogous, result is formulated in terms of the pointwise convergence in the $x-v$ space, hence without going to the Fourier space. □

Before entering the details of the proof we first analyze all the contributions in the right hand side of (3.9).

The two-particle terms are (we skip the unessential operator $\int_0^t dt_1\ S(t-t_1)$)

$$\mathcal{S}_{2,\varepsilon}^\pi := \varepsilon^{-\frac{7}{2}} C_2^\varepsilon S(t_1) f_2^\pi, \tag{3.12}$$

where the permutation π may take the two values $\pi=(1,2)$ or $\pi=(2,1)$, together with

$$\mathcal{T}_{2,\varepsilon}^\pi := \varepsilon^{-4}\int_0^{t_1} dt_2\ C_2^\varepsilon S(t_1-t_2) T_2^\varepsilon S(t_2) f_2^\pi, \tag{3.13}$$

with π taking the values $\pi=(1,2)$ or $\pi=(2,1)$. There are four such terms.

The three-particle terms are twelve, namely:

$$\mathcal{W}^{\pi}_{3,\varepsilon} := \varepsilon^{-7} \int_0^{t_1} dt_2 \; C^{\varepsilon}_{1,2} S(t_1 - t_2) C^{\varepsilon}_{1,3} S(t_2) f^{\pi}_3, \tag{3.14}$$

and

$$\mathcal{V}^{\pi}_{3,\varepsilon} := \varepsilon^{-7} \int_0^{t_1} dt_2 \; C^{\varepsilon}_{1,2} S(t_1 - t_2) C^{\varepsilon}_{2,3} S(t_2) f^{\pi}_3, \tag{3.15}$$

with $\pi \in \mathcal{P}_3$, the set of the permutations of three objects, whose cardinality is six.

Note that all the above terms are funtions of (x, v) (and t_1 of course).

For further convenience, and in view of the proof of our main result, we readily express all these contributions in terms of their Fourier transforms.

We start with the following obvious three formulae for the basic operators $S(t)$, T_2, and C_2 (see (3.5), (2.19)-(2.20), and (2.21)-(2.22), respectively):

$$\widehat{T^{\varepsilon}_2} \widehat{f}(\xi_1, \xi_2; \eta_1, \eta_2) = -i \sum_{\sigma=\pm 1} \sigma \int dh \; \frac{\widehat{\phi}(h)}{(2\pi)^3} \; \mathrm{e}^{i\sigma \frac{h}{2}\cdot(\eta_2 - \eta_1)} \widehat{f}\left(\xi_1 - \frac{h}{\varepsilon}, \xi_2 + \frac{h}{\varepsilon}; \eta_1, \eta_2\right),$$

$$\widehat{C^{\varepsilon}_2} \widehat{f}(\xi; \eta) = \widehat{T^{\varepsilon}_2} \widehat{f}(\xi, 0; \eta, 0) = -i \sum_{\sigma=\pm 1} \sigma \int dh \; \frac{\widehat{\phi}(h)}{(2\pi)^3} \; \mathrm{e}^{-i\sigma \frac{h}{2}\cdot\eta} \widehat{f}\left(\xi - \frac{h}{\varepsilon}, \frac{h}{\varepsilon}; \eta, 0\right),$$

$$\widehat{S}(t) \widehat{f}(\xi; \eta) = \widehat{f}(\xi, \eta + \xi t).$$

These relations give in (3.12) through (3.15):

$$\widehat{\mathcal{S}}^{\pi}_{2,\varepsilon}(\xi, \eta) = -i \frac{\varepsilon^{-\frac{7}{2}}}{(2\pi)^3} \sum_{\sigma=\pm 1} \sigma \int dh \; \widehat{\phi}(h) \; \mathrm{e}^{-i\frac{\sigma}{2} h\cdot\eta} \; \widehat{f^{\pi}_2}\left(\xi - \frac{h}{\varepsilon}, \frac{h}{\varepsilon}; \eta + t_1(\xi - \frac{h}{\varepsilon}), t_1 \frac{h}{\varepsilon}\right). \tag{3.16}$$

$$\begin{aligned}\widehat{\mathcal{T}}^{\pi}_{2,\varepsilon}(\xi, \eta) = &- \frac{\varepsilon^{-4}}{(2\pi)^6} \sum_{\sigma_1,\sigma_2=\pm 1} \sigma_1\sigma_2 \int_0^{t_1} dt_2 \int dh_1 \int dh_2 \; \widehat{\phi}(h_1)\, \widehat{\phi}(h_2) \\ &\mathrm{e}^{-i\frac{\sigma_1}{2} h_1\cdot\eta} \, \mathrm{e}^{-i\frac{\sigma_2}{2} h_2\cdot(\eta + \xi(t_1 - t_2) - \frac{2}{\varepsilon}(t_1 - t_2) h_1)} \\ &\widehat{f^{\pi}_2}\left(\xi - \frac{1}{\varepsilon}(h_1 + h_2), \frac{1}{\varepsilon}(h_1 + h_2); \eta + t_1\xi - \frac{t_1 h_1 + t_2 h_2}{\varepsilon}, \frac{t_1 h_1 + t_2 h_2}{\varepsilon}\right),\end{aligned} \tag{3.17}$$

$$\begin{aligned}\widehat{\mathcal{W}}^{\pi}_{3,\varepsilon}(\xi, \eta) = &- \frac{\varepsilon^{-7}}{(2\pi)^6} \sum_{\sigma_1,\sigma_2=\pm 1} \sigma_1\sigma_2 \int_0^{t_1} dt_2 \int dh_1 \int dh_2 \; \widehat{\phi}(h_1)\, \widehat{\phi}(h_2) \\ &\mathrm{e}^{-i\frac{\sigma_1}{2} h_1\cdot\eta} \, \mathrm{e}^{-i\frac{\sigma_2}{2} h_2\cdot(\eta + (\xi - \frac{h_1}{\varepsilon})(t_1 - t_2))} \\ &\widehat{f^{\pi}_3}\left(\xi - \frac{1}{\varepsilon}(h_1 + h_2), \frac{h_1}{\varepsilon}, \frac{h_2}{\varepsilon}; \eta + t_1\xi - \frac{t_1 h_1 + t_2 h_2}{\varepsilon}, \frac{t_1 h_1}{\varepsilon}, \frac{t_2 h_2}{\varepsilon},\right),\end{aligned} \tag{3.18}$$

$$\begin{aligned}\widehat{\mathcal{V}}^{\pi}_{3,\varepsilon}(\xi, \eta) = &- \frac{\varepsilon^{-7}}{(2\pi)^6} \sum_{\sigma_1,\sigma_2=\pm 1} \sigma_1\sigma_2 \int_0^{t_1} dt_2 \int dh_1 \int dh_2 \; \widehat{\phi}(h_1)\, \widehat{\phi}(h_2) \\ &\mathrm{e}^{-i\frac{\sigma_1}{2} h_1\cdot\eta} \, \mathrm{e}^{-i\frac{\sigma_2}{2} h_2\cdot\frac{h_1}{\varepsilon}(t_1 - t_2)} \\ &\widehat{f^{\pi}_3}\left(\xi - \frac{h_1}{\varepsilon}, \frac{h_1 - h_2}{\varepsilon}, \frac{h_2}{\varepsilon}; \eta + t_1\xi - \frac{t_1 h_1}{\varepsilon}, \frac{t_1 h_1 - t_2 h_2}{\varepsilon}, \frac{t_2 h_2}{\varepsilon}\right).\end{aligned} \tag{3.19}$$

Starting form those expressions, the plan of the proof is the following. In Section 4 we evaluate the two particle terms $\mathcal{S}^{\pi}_{2,\varepsilon}$ and $\mathcal{T}^{\pi}_{2,\varepsilon}$. We prove that they converge towards the associated two particles terms in the U-U equation. In Section 5 we deal with the three-particle terms associated to the permutations π with a fixed element. Those are shown to converge towards the associated three particles terms in the U-U equation, while contributing by the quantity $\widehat{\phi}(v'-v)^2+\widehat{\phi}(v'-v_*)^2$ to the transition kernel W (see (1.2)). Finally in Section 6 we treat the three particle terms relative to cyclic permutations. We recover in this way the missing contribution to the transition kernel, namely the cross term $\theta\widehat{\phi}(v'-v)\,\widehat{\phi}(v'-v_*)$.

For sake of simplicity we shall carry out the computations for Bosons ($\theta=1$), being clear that the Fermionic case is just the same with suitable changes of sign.

4. Two-particle terms.

We introduce the partial Fourier transform

$$\widetilde{f}^{\pi}_j(x_1,\dots,x_j;\eta_1,\dots,\eta_j):=\int dv_1\cdots\int dv_j\ \mathrm{e}^{-i\sum_{k=1}^{j}v_k\cdot\eta_k}\,f^{\pi}_j(x_1,\dots,x_j;v_1,\dots,v_j). \tag{4.1}$$

As a consequence of (2.29) we have

$$\begin{aligned}\widetilde{f}^{\pi}_j&(x_1,\dots,x_j;\eta_1,\dots,\eta_j)=\\&\prod_{k=1}^{j}\widetilde{f}_0\left(\frac{x_k+x_{\pi(k)}}{2}-\varepsilon\frac{\eta_k-\eta_{\pi(k)}}{4},-\frac{x_k-x_{\pi(k)}}{\varepsilon}+\frac{\eta_k+\eta_{\pi(k)}}{2}\right).\end{aligned} \tag{4.2}$$

In particular, we have the obvious

$$\widetilde{f}^{(1,2)}_2(x_1,x_2;\eta_1,\eta_2)=\widetilde{f}_0(x_1,\eta_1)\widetilde{f}_0(x_2,\eta_2), \tag{4.3}$$

together with

$$\begin{aligned}\widetilde{f}^{(2,1)}_2(x_1,x_2;\eta_1,\eta_2)=&\widetilde{f}_0\left(\frac{x_1+x_2}{2}-\frac{\varepsilon}{4}(\eta_1-\eta_2);-\frac{x_1-x_2}{\varepsilon}+\frac{\eta_1+\eta_2}{2}\right)\\&\widetilde{f}_0\left(\frac{x_1+x_2}{2}+\frac{\varepsilon}{4}(\eta_1-\eta_2);+\frac{x_1-x_2}{\varepsilon}+\frac{\eta_1+\eta_2}{2}\right).\end{aligned} \tag{4.4}$$

Hence, upon now performing the complete Fourier transform, we obtain,

$$\widehat{f}^{(1,2)}_2(\xi_1,\xi_2;\eta_1,\eta_2)=\widehat{f}_0(\xi_1,\eta_1)\widehat{f}_0(\xi_2,\eta_2), \tag{4.5}$$

together with

$$\begin{aligned}\widehat{f}^{(2,1)}_2(\xi_1,\xi_2;\eta_1,\eta_2)=&\varepsilon^3\int dy_1\int dy_2\ \mathrm{e}^{-i\xi_1\cdot(y_1+\frac{\varepsilon}{2}y_2)}\,\mathrm{e}^{-i\xi_2\cdot(y_1-\frac{\varepsilon}{2}y_2)}\\&\widetilde{f}_0\left(y_1-\frac{\varepsilon}{4}(\eta_1-\eta_2);-y_2+\frac{\eta_1+\eta_2}{2}\right)\\&\widetilde{f}_0\left(y_1+\frac{\varepsilon}{4}(\eta_1-\eta_2);+y_2+\frac{\eta_1+\eta_2}{2}\right).\end{aligned} \tag{4.6}$$

We are now in position to analyse the term $\mathcal{S}^{\pi}_{2,\varepsilon}$ for $\pi = (1,2)$ and $\pi = (2,1)$. First, using the identity

$$\sum_{\sigma=\pm 1} \sigma \mathrm{e}^{-i\sigma \frac{h}{2}\cdot\eta} = -2i \sin \frac{h\cdot\eta}{2},$$

we get the the explicit expression:

$$\widehat{\mathcal{S}}^{\pi}_{2,\varepsilon} = -\frac{2\varepsilon^{-\frac{7}{2}}}{(2\pi)^3} \int dh\, \widehat{\phi}(h) \sin\left(\frac{h\cdot\eta}{2}\right) \widehat{f}^{\pi}_2 \left(\xi - \frac{h}{\varepsilon}, \frac{h}{\varepsilon}; \eta + (\xi - \frac{h}{\varepsilon})t_1, \frac{h}{\varepsilon}t_1\right). \tag{4.7}$$

In the case of $\widehat{\mathcal{S}}^{(1,2)}_{2,\varepsilon}$, a change of variable $h \to \varepsilon h$ then gives, using (4.5), the value

$$\widehat{\mathcal{S}}^{(1,2)}_{2,\varepsilon}(\xi,\eta) = -\frac{2\varepsilon^{-\frac{1}{2}}}{(2\pi)^3} \int dh\, \widehat{\phi}(\varepsilon h) \sin\left(\varepsilon\frac{h\cdot\eta}{2}\right) \widehat{f}_0(\xi - h; \eta + (\xi - h)t_1)\widehat{f}_0(h, ht_1). \tag{4.8}$$

Therefore, we may estimate

$$\begin{aligned}
&|\widehat{\mathcal{S}}^{(1,2)}_{2,\varepsilon}| \le \\
&\le C\sqrt{\varepsilon}\, \|\widehat{\phi}\|_{L_\infty} \int dh\, |h|\ (|\eta + (\xi - h)t_1| + |\xi - h|t_1)\, |\widehat{f}_0(h; ht_1)|\, |\widehat{f}_0|(\xi - h; \eta + (\xi - h)t_1) \\
&\le C\sqrt{\varepsilon} \left(\sup_{\xi,\eta} |\eta||\widehat{f}_0|(\xi;\eta) \int d\xi \sup_{\eta} |\xi||\widehat{f}_0|(\xi;\eta) + t_1 \sup_{\xi,\eta} |\xi||\widehat{f}_0|(\xi;\eta) \int d\xi |\xi| \sup_{\eta} |\widehat{f}_0(\xi;\eta)| \right) \\
&\le C\sqrt{\varepsilon} \sup_{\xi,\eta} \Big((|\xi| + |\eta|)\, |\widehat{f}_0|(\xi;\eta) \Big) \int d\xi\, \sup_{\eta} \Big((|\xi| + |\eta|)\, |\widehat{f}_0|(\xi;\eta) \Big),
\end{aligned} \tag{4.9}$$

and the corresponding contribution vanishes with ε.

In the case of $\widehat{\mathcal{S}}^{(2,1)}_{2,\varepsilon}$ on the other hand, equations (4.7) and (4.6) give

$$\begin{aligned}
\widehat{\mathcal{S}}^{(2,1)}_{2,\varepsilon}(\xi,\eta) = &-\frac{2\varepsilon^{-\frac{1}{2}}}{(2\pi)^3} \int dh \int dy_1 \int dy_2\, \widehat{\phi}(h)\, \sin\left(\frac{h\cdot\eta}{2}\right) \mathrm{e}^{-i(y_1 + \frac{\varepsilon}{2}y_2)\cdot\xi}\, \mathrm{e}^{ih\cdot y_2} \\
&\widetilde{f}_0 \left(y_1 - \frac{\varepsilon}{4}\left[\eta + \xi t_1 - \frac{2h}{\varepsilon t_1}\right]; -y_2 + \frac{\eta + \xi t_1}{2}\right) \\
&\widetilde{f}_0 \left(y_1 + \frac{\varepsilon}{4}\left[\eta + \xi t_1 - \frac{2h}{\varepsilon t_1}\right]; y_2 + \frac{\eta + \xi t_1}{2}\right) \\
= &-\frac{2\varepsilon^{-\frac{1}{2}}}{(2\pi)^3} \int dh \int dy_1 \int dy_2\, \widehat{\phi}(h)\, \sin\left(\frac{h\cdot\eta}{2}\right) \mathrm{e}^{-i(y_1 + \varepsilon\frac{y_2}{2})\cdot\xi} \mathrm{e}^{ih\cdot y_2} \\
&\widetilde{f}_0 \left(y_1 + \frac{h}{2}t_1; -y_2 + \frac{\eta + \xi t_1}{2}\right) \widetilde{f}_0 \left(y_1 - \frac{h}{2}t_1; y_2 + \frac{\eta + \xi t_1}{2}\right) + O(\sqrt{\varepsilon}).
\end{aligned} \tag{4.10}$$

By the parity of ϕ, the first term in the right hand side is vanishing: Indeed, it is antisymmetric in the exchange $h \to -h$ and $y_2 \to -y_2$. Note that the mechanism that makes the dominant, $O(\varepsilon^{-1/2})$, contribution of $\widehat{\mathcal{S}}^{(2,1)}_{2,\varepsilon}$, vanish in the limit, is very different from the one involved in the vanishing of $\widehat{\mathcal{S}}^{(1,2)}_{2,\varepsilon}$: here, antisymmetry plays a crucial role. This aspect will play an even more important, and more intricate, role in the next two sections.

There remains to prove that the $O(\sqrt{\varepsilon})$ term in (4.10) indeed has the claimed size. It can be written as

$$-\frac{2\varepsilon^{\frac{1}{2}}}{(2\pi)^6}\int dh\,dy_2\,d\xi_1\,\widehat{\phi}(h)\,\sin\left(\frac{h\cdot\eta}{2}\right)\widehat{f}_0\left(\xi_1;\frac{\eta+\xi t_1}{2}-y_2\right)\widehat{f}_0\left(\xi-\xi_1;\frac{\eta+\xi t_1}{2}+y_2\right)$$
$$\mathrm{e}^{ih\cdot y_2+i\frac{t_1}{2}h\cdot(2\xi_1-\xi)}\left(1-\mathrm{e}^{i\frac{\varepsilon}{2}\left(-y_2\cdot\xi+(\xi-2\xi_1)\cdot\left(\frac{\eta+\xi t_1}{2}\right)\right)}\right).$$

It may be estimated by

$$C\varepsilon^{\frac{1}{2}}\int dh\,|\widehat{\phi}|(h)\int dy_2\,d\xi_1\,|\widehat{f}_0|\left(\xi_1;\frac{\eta+\xi t_1}{2}+y_2\right)|\widehat{f}_0|\left(\xi-\xi_1;\frac{\eta+\xi t_1}{2}-y_2\right)$$
$$\left(|\xi_1|\left|\frac{\eta+\xi t_1}{2}+y_2\right|+|\xi-\xi_1|\left|\frac{\eta+\xi t_1}{2}-y_2\right|\right).$$

Therefore the term $\mathcal{S}^{(2,1)}_{2,\varepsilon}$ vanishes as well.

As a conclusion, all terms $\mathcal{S}^{\pi}_{2,\varepsilon}$, which are the ones that are linear in ϕ, vanish in the limit $\varepsilon\to 0$.

We now pass to the evaluation of the terms $\mathcal{T}^{\pi}_{2,\varepsilon}$.

The contribution $\mathcal{T}^{(1,2)}_{2,\varepsilon}$ has been already considered in Ref. [2]. However, for sake of completeness, we analyze this term in the present context as well. Using (4.5) in (3.17), and performing the change of variables:

$$\frac{h_1+h_2}{\varepsilon}=k,\quad h_2=h,\quad \frac{t_1-t_2}{\varepsilon}=s, \tag{4.11}$$

we arrive at

$$\begin{aligned}\widehat{\mathcal{T}}^{(1,2)}_{2,\varepsilon}(\xi,\eta)=&-\frac{1}{(2\pi)^6}\sum_{\sigma_1,\sigma_2=\pm1}\sigma_1\sigma_2\int_0^{\frac{t_1}{\varepsilon}}ds\int dh\int dk\,\widehat{\phi}(h)\widehat{\phi}(-h+\varepsilon k)\\&\mathrm{e}^{-i[\eta\cdot h\left(\frac{\sigma_2-\sigma_1}{2}\right)+\sigma_2h^2s]}\,\mathrm{e}^{-i\varepsilon\frac{\sigma_1}{2}k\cdot\eta}\,\mathrm{e}^{-i\varepsilon\sigma_2\frac{h}{2}\cdot[\xi s-2sk]}\\&\widehat{f}_0(\xi-k;\eta+t_1\xi+hs-kt_1)\,\widehat{f}_0(k;kt_1-hs).\end{aligned}\tag{4.12}$$

This term converges formally to

$$\begin{aligned}\widehat{\mathcal{T}}^{(1,2)}_{2}(\xi,\eta)=&-\frac{1}{(2\pi)^6}\sum_{\sigma_1,\sigma_2=\pm1}\sigma_1\sigma_2\int_0^{+\infty}ds\int dh\int dk\,|\widehat{\phi}(h)|^2\\&\mathrm{e}^{-i[\eta\cdot h\left(\frac{\sigma_2-\sigma_1}{2}\right)+\sigma_2h^2s]}\,\widehat{f}_0(\xi-k;\eta+t_1\xi+hs-kt_1)\,\widehat{f}_0(k;kt_1-hs).\end{aligned}\tag{4.13}$$

To justify the limit we split the integration in ds over the two intervals $[0,1]$, $[1,+\infty]$. In the first interval we bound the integrand by

$$\|\widehat{\phi}\|_{L_\infty}\,\|\widehat{f}_0\|_{L_\infty}\,|\widehat{\phi}(h)|\sup_\eta|\widehat{f}_0(k,\eta)|, \tag{4.14}$$

which is a $L_1(dk\,dh)$ function for any $s \in [0,1]$. In the second part of the integration domain, after the change of variables $h \to (kt_1 - hs)$, we bound the integrand by

$$\|\widehat{\phi}\|^2_{L_\infty} \|\widehat{f}_0\|_{L_\infty} |\widehat{f}_0(k,\eta)| \frac{1}{s^3}, \tag{4.15}$$

which is a $L_1(dk\,dh\,ds)$ function on $\mathbb{R}^3 \times \mathbb{R}^3 \times (1,+\infty)$. The claimed convergence in (4.13) is then consequence of the Dominated Convergence Theorem. It holds uniformly in ξ, η.

We now evaluate $\widehat{\mathcal{T}}^{(2,1)}_{2,\varepsilon}$. Inserting (4.6) in (3.17), and rescaling time $t_1 - t_2 = \varepsilon s$, the resulting expression is:

$$\begin{aligned}\widehat{\mathcal{T}}^{(2,1)}_{2,\varepsilon}(\xi,\eta) = &-\frac{1}{(2\pi)^6} \sum_{\sigma_1,\sigma_2=\pm 1} \sigma_1\sigma_2 \int_0^{\frac{t_1}{\varepsilon}} ds \int dh_1 \int dh_2 \int dy_1 \int dy_2\, \widehat{\phi}(h_1)\,\widehat{\phi}(h_2) \\ &\mathrm{e}^{-i\frac{\sigma_1}{2}h_1\cdot\eta}\,\mathrm{e}^{-i\frac{\sigma_2}{2}h_2\cdot(\eta+\varepsilon s\xi-2sh_1)}\,\mathrm{e}^{-i\xi\cdot(y_1+\frac{\varepsilon}{2}y_2)}\,\mathrm{e}^{i(h_1+h_2)\cdot y_2} \\ &\widetilde{f}_0\left(y_1 - \frac{\varepsilon}{4}\left[\eta+\xi t_1 - 2\frac{h_1t_1+h_2(t_1-\varepsilon s)}{\varepsilon}\right]; -y_2 + \frac{\eta+\xi t_1}{2}\right) \\ &\widetilde{f}_0\left(y_1 + \frac{\varepsilon}{4}\left[\eta+\xi t_1 - 2\frac{h_1t_1+h_2(t_1-\varepsilon s)}{\varepsilon}\right]; y_2 + \frac{\eta+\xi t_1}{2}\right).\end{aligned} \tag{4.16}$$

Now the formal limit is:

$$\begin{aligned}\widehat{\mathcal{T}}^{(2,1)}_{2}(\xi,\eta) = &-\frac{1}{(2\pi)^6} \sum_{\sigma_1,\sigma_2=\pm 1} \sigma_1\sigma_2 \int_0^{+\infty} ds \int dh_1 \int dh_2 \int dy_1 \int dy_2\, \widehat{\phi}(h_1)\,\widehat{\phi}(h_2) \\ &\mathrm{e}^{-i\frac{\sigma_1}{2}h_1\cdot\eta}\,\mathrm{e}^{-i\frac{\sigma_2}{2}h_2\cdot(\eta-2sh_1)}\,\mathrm{e}^{-i\xi\cdot y_1}\,\mathrm{e}^{i(h_1+h_2)\cdot y_2} \\ &\widetilde{f}_0\left(y_1 + t_1\frac{h_1+h_2}{2}; -y_2 + \frac{\eta+\xi t_1}{2}\right)\widetilde{f}_0\left(y_1 - t_1\frac{h_1+h_2}{2}; y_2 + \frac{\eta+\xi t_1}{2}\right).\end{aligned} \tag{4.17}$$

To justify the limit we have to show the uniform (with respect to ε) integrability of the integrand in the right hand side of (4.16). To outline the decay with respect to the s variable we observe

$$\mathrm{e}^{i\sigma_2 h_1\cdot h_2 s} = -\frac{1}{s^2|h_2|^4}(h_2\cdot\nabla_{h_1})^2\mathrm{e}^{i\sigma_2 h_1\cdot h_2 s}, \tag{4.18}$$

and then proceed with the natural integration by parts with respect to h_1 in (4.16) (Recall that $1/|h_2|^2$ is integrable close to the origin in dimension $d=3$). Splitting the integral in ds as before, we may apply the Dominated Convergence Theorem, upon using the smoothness of $\widehat{\phi}$ and $\widetilde{f}_0$, thus justifying the above formal limit.

Our last task is to interpret the result we have obtained, in terms of the U-U equation. To do so, we first go back to the original variables, expressing $\mathcal{T}^{(1,2)}_2$ and $\mathcal{T}^{(2,1)}_2$ as functions of (x,v). A straightforward computation yields, on the one hand,

$$\begin{aligned}\mathcal{T}^{(1,2)}_2(x,v) = &-\frac{1}{(2\pi)^3} \sum_{\sigma_1,\sigma_2=\pm 1} \sigma_1\sigma_2 \int dh \int dv_1 \int dv_2\, \delta\left(v - v_1 - h\frac{\sigma_2-\sigma_1}{2}\right) \\ &\Delta^+\left(-h\cdot(\sigma_2 h + (v_1 - v_2))\right)|\widehat{\phi}(h)|^2 f_0(x - v_1t_1, v_1)\, f_0(x - v_2 t_1, v_2),\end{aligned} \tag{4.19}$$

and, on the other hand (with $h_2 = h$),

$$\begin{aligned}\mathcal{T}_2^{(2,1)}(x,v) = -\frac{1}{(2\pi)^3} \sum_{\sigma_1,\sigma_2=\pm 1} \sigma_1\sigma_2 \int dh \int dv_1 \int dv_2 \\ \delta\left(v - v_1\frac{1-\sigma_1}{2} - v_2\frac{1+\sigma_1}{2} - h\frac{\sigma_2-\sigma_1}{2}\right) \\ \Delta^+(\sigma_2 h\cdot(-h - v_1 + v_2))\,\widehat{\phi}(h)\,\widehat{\phi}(-h - v_1 + v_2) \\ f_0(x - v_1 t_1, v_1)\, f_0(x - v_2 t_1, v_2).\end{aligned} \tag{4.20}$$

Here we define the distribution

$$\Delta^+(\alpha) := \int_0^\infty ds\, \mathrm{e}^{i\alpha s} = \pi\delta(\alpha) + i\text{p.v.}\left(\frac{1}{\alpha}\right). \tag{4.21}$$

Now, on both preceding formulae, we readily observe the following important fact. The parity of $\widehat{\phi}$, and the symmetries $h \to -h$, $\sigma_1 \to -\sigma_1$, $\sigma_2 \to -\sigma_2$ in (4.19), and $h \to -h$, $\sigma_1 \to -\sigma_1$, $\sigma_2 \to -\sigma_2$, $v_1 \leftrightarrow v_2$ in (4.20), show that Δ^+ may be replaced by $\pi\delta$ everywhere. This will eventually give, as shown next, the desired conservation of energy in the limiting U-U equation.

There remains to actually perform the sum $\sum \sigma_1\sigma_2$ in (4.19) and (4.20), in order to identify the very value of $\mathcal{T}_2^{(1,2)}$ and $\mathcal{T}_2^{(2,1)}$. For $\mathcal{T}_2^{(1,2)}$ we make the following choice:

σ_1	σ_2	h	v_1	v_2
1	1	$v' - v$	v	v_*
-1	-1	$v - v'$	v	v_*
1	-1	$v' - v$	v'	v'_*
-1	1	$v - v'$	v'	v'_*

This results in the final expression:

$$\begin{aligned}\mathcal{T}_2^{(1,2)}(x,v) =& \frac{1}{4\pi^2} \int dv_* \int dv' \int dv'_*\; \delta(v_* + v - v'_* - v') \\ & \delta\left(\frac{1}{2}({v_*}^2 + v^2 - {v'_*}^2 - v'^2)\right)\, |\widehat{\phi}(v' - v)|^2\, (f' f'_* - f f_*),\end{aligned} \tag{4.22}$$

where, with abuse of notation, we set the "transported quantities"

$$f = f_0(x - vt_1, v), \quad f_* = f_0(x - v_* t_1, v_*), \quad f' = f_0(x - v' t_1, v'), \quad f'_* = f_0(x - v'_* t_1, v'_*).$$

Notice that, by changing $v' \leftrightarrow v'_*$, and using the conservation of momentum, we may replace $\frac{1}{2}|\widehat{\phi}(v' - v)|^2$ by $\frac{1}{2}|\widehat{\phi}(v' - v_*)|^2$ in (4.22).

Besides, for $\mathcal{T}_2^{(2,1)}$ we make the following changes of variables:

σ_1	σ_2	h	v_1	v_2
1	1	$v - v'$	v_*	v
-1	-1	$v' - v$	v	v_*
1	-1	$v' - v$	v'_*	v'
-1	1	$v - v'$	v'	v'_*

This results in the final expression:

$$\begin{aligned} \mathcal{T}_2^{(2,1)}(x,v) =& \frac{1}{4\pi^2} \int dv_* \int dv' \int dv'_*\, \delta(v_* + v - v'_* - v') \\ & \delta\left(\frac{1}{2}({v_*}^2 + v^2 - {v'_*}^2 - v'^2)\right) \widehat{\phi}(v'-v)\,\widehat{\phi}(v'-v_*)\,(f'f'_* - ff_*). \end{aligned} \tag{4.23}$$

As a conclusion for the $\mathcal{T}_2$ terms, we have eventually established the (desired) equality

$$\mathcal{T}_2^{(1,2)} + \mathcal{T}_2^{(2,1)} = \int dv_* \int dv' \int dv_* W(v, v_*|v', v'_*)(f'f'_* - ff_*). \tag{4.24}$$

This ends up the analysis of the two-particle terms.

5. Three-particle terms: permutations with a fixed element

In this section we analyze $\mathcal{W}^\pi_{3,\varepsilon}$ and $\mathcal{V}^\pi_{3,\varepsilon}$ for the permutations π with a fixed element. To simplify the notation we set

$$\mathcal{W}^0_{3,\varepsilon} \quad \text{and } \mathcal{V}^0_{3,\varepsilon} \qquad \text{for } \pi = (1,2,3),$$

and

$$\mathcal{W}^i_{3,\varepsilon} \quad \text{and } \mathcal{V}^i_{3,\varepsilon}, \quad i = 1,2,3,$$

for the three permutations leaving i fixed. To state the result briefly, let us readily say that the factors

$$\mathcal{W}^0_{3,\varepsilon}, \quad \mathcal{V}^0_{3,\varepsilon}, \qquad \text{together with } \mathcal{W}^2_{3,\varepsilon}, \quad \mathcal{V}^1_{3,\varepsilon},$$

give a vanishing contribution. Also, the sum

$$\mathcal{W}^3_{3,\varepsilon} + \mathcal{V}^3_{3,\varepsilon}$$

is shown to vanish asymptotically, while each of these two terms is $O(\varepsilon^{-1})$ separately. Here, anti-symmetry will play a central role. Finally, the two terms

$$\mathcal{W}^1_{3,\varepsilon}, \quad \mathcal{V}^2_{3,\varepsilon},$$

do contribute to the limiting U-U equation through the cubic term. They build up the transition kernel $\widehat{\phi}(v'-v)^2 + \widehat{\phi}(v'-v_*)^2$. The missing cross term $2\widehat{\phi}(v'-v)\,\widehat{\phi}(v'-v_*)$ in $W(v, v_*|v', v'_*)$ will come up in the next section.

Let us show first that $\mathcal{W}^0_{3,\varepsilon}$ and $\mathcal{V}^0_{3,\varepsilon}$ are vanishing. From (3.18), scaling h_1 and h_2 and summing on σ_1, σ_2, we have

$$\begin{aligned} \widehat{\mathcal{W}^0_{3,\varepsilon}} &= +4\frac{\varepsilon^{-1}}{(2\pi)^6} \int_0^{t_1} dt_2 \int dh_1 \int dh_2\, \widehat{\phi}(\varepsilon h_1)\,\widehat{\phi}(\varepsilon h_2) \\ &\quad \sin\left(\varepsilon \frac{h_1\cdot\eta}{2}\right) \sin\left(\varepsilon \frac{h_2}{2}\cdot(\eta + (t_1 - t_2)(\xi - h_1)\right) \\ &\quad \widehat{f_0}(\xi - (h_1 + h_2), \eta + t_1\xi - (t_1 h_1 + t_2 h_2))\,\widehat{f_0}(h_1, t_1 h_1)\,\widehat{f_0}(h_2, t_2 h_2) \\ &= O(\varepsilon), \end{aligned} \tag{5.1}$$

due to the decay properties of $\widehat{f}_0$. The same argument easily leads to $\widehat{\mathcal{V}}^0_{3,\varepsilon} = O(\varepsilon)$.

We now pass to the computation of $\widehat{\mathcal{W}}^1_{3,\varepsilon}$. This term is associated with the permutation $\pi = (1,3,2)$. Upon Fourier transforming in x the relation (4.2) for $\widetilde{f}^{\pi}_j$ (with $j = 3$), and using the change of variables $y_1 = (x_2 + x_3)/2$, $y_2 = (x_2 - x_3)/\varepsilon$ in the corresponding formula, we recover

$$
\begin{aligned}
\widehat{f}^{(1,3,2)}_{3,\varepsilon}(\xi_1,\xi_2,\xi_3,\eta_1,\eta_2,\eta_3) = \varepsilon^3 \widehat{f}_0(\xi_1,\eta_1) \int dy_1 \int dy_2 \, \mathrm{e}^{-i\xi_2\cdot(y_1+\frac{\varepsilon}{2}y_2)} \, \mathrm{e}^{-i\xi_3\cdot(y_1-\frac{\varepsilon}{2}y_2)} \\
\widetilde{f}_0\left(y_1 - \varepsilon\frac{\eta_2-\eta_3}{4}, -y_2 + \frac{\eta_2+\eta_3}{2}\right) \widetilde{f}_0\left(y_1 + \varepsilon\frac{\eta_2-\eta_3}{4}, y_2 + \frac{\eta_2+\eta_3}{2}\right).
\end{aligned}
\tag{5.2}
$$

Then, inserting (5.2) in the formula (3.18) relating the value of $\mathcal{W}^{\pi}_{3,\varepsilon}$, we arrive at

$$
\begin{aligned}
\widehat{\mathcal{W}}^1_{3,\varepsilon}(\xi,\eta) = -\frac{\varepsilon^{-4}}{(2\pi)^6} \sum_{\sigma_1,\sigma_2=\pm1} \sigma_1\sigma_2 \int_0^{t_1} dt_2 \int dh_1 \int dh_2 \int dy_1 \int dy_2 \, \widehat{\phi}(h_1)\, \widehat{\phi}(h_2) \\
\mathrm{e}^{-i\left(\frac{h_1+h_2}{\varepsilon}\cdot y_1 + \frac{h_1-h_2}{2}\cdot y_2\right)} \, \mathrm{e}^{-i\frac{\sigma_1}{2}h_1\cdot\eta} \, \mathrm{e}^{-i\frac{\sigma_2}{2}h_2\cdot\left(\eta+(t_1-t_2)(\xi-\frac{h_1}{\varepsilon})\right)} \\
\widehat{f}_0\left(\xi - \frac{h_1+h_2}{\varepsilon}, \eta + t_1\xi - \frac{t_1h_1+t_2h_2}{\varepsilon}\right) \\
\widetilde{f}_0\left(y_1 - \frac{t_1h_1-t_2h_2}{4}, \frac{t_1h_1+t_2h_2}{2\varepsilon} - y_2\right) \\
\widetilde{f}_0\left(y_1 + \frac{t_1h_1-t_2h_2}{4}, \frac{t_1h_1+t_2h_2}{2\varepsilon} + y_2\right).
\end{aligned}
\tag{5.3}
$$

Changing variables

$$
k = \frac{h_1+h_2}{\varepsilon}, \quad h = h_2, \quad s = \frac{t_1-t_2}{\varepsilon},
$$

we obtain

$$
\begin{aligned}
\widehat{\mathcal{W}}^1_{3,\varepsilon}(\xi,\eta) = -\frac{1}{(2\pi)^6} \sum_{\sigma_1,\sigma_2=\pm1} \sigma_1\sigma_2 \int_0^{\frac{t_1}{\varepsilon}} ds \int dh \int dk \int dy_1 \int dy_2 \, \widehat{\phi}(-h+\varepsilon k)\, \widehat{\phi}(h) \\
\mathrm{e}^{-i\frac{\sigma_1}{2}\eta\cdot(-h+\varepsilon k)} \, \mathrm{e}^{-i\frac{\sigma_2}{2}h\cdot(\eta+sh+\varepsilon s(\xi-k))} \, \mathrm{e}^{-ik\cdot y_1 - iy_2\left(-h+\varepsilon\frac{k}{2}\right)} \, \widehat{f}_0(\xi-k, \eta+t_1(\xi-k)+sh) \\
\widetilde{f}_0\left(y_1 + \frac{t_1h}{2} - \varepsilon\frac{kt_1+sh}{4}, \frac{kt_1-sh}{2} - y_2\right) \widetilde{f}_0\left(y_1 - \frac{t_1h}{2} + \varepsilon\frac{kt_1+sh}{4}, \frac{kt_1-sh}{2} + y_2\right).
\end{aligned}
\tag{5.4}
$$

We are now in position to identify the rigorous limit of $\widehat{\mathcal{W}}^1_{3,\varepsilon}$, using the assumed decay of $\widehat{\phi}$ and $\widehat{f}_0$. The argument are those used in the previous section: we do not repeat them here and in the sequel. Passing to the limit we get, eventually,

$$
\begin{aligned}
\widehat{\mathcal{W}}^1_3(\xi,\eta) = -\frac{1}{(2\pi)^6} \sum_{\sigma_1,\sigma_2=\pm1} \sigma_1\sigma_2 \int_0^{+\infty} ds \int dh \int dk \int dy_1 \int dy_2 \, |\widehat{\phi}(h)|^2 \\
\mathrm{e}^{i\frac{\sigma_1-\sigma_2}{2}h\cdot\eta} \mathrm{e}^{-i\frac{\sigma_2}{2}h^2 s} \, \mathrm{e}^{-ik\cdot y_1 + ih\cdot y_2} \, \widehat{f}_0(\xi-k, \eta+t_1(\xi-k)+sh) \\
\widetilde{f}_0\left(y_1 + \frac{t_1h}{2}, \frac{kt_1-sh}{2} - y_2\right) \widetilde{f}_0\left(y_1 - \frac{t_1h}{2}, \frac{kt_1-sh}{2} + y_2\right).
\end{aligned}
\tag{5.5}
$$

Last, we go back to the (x, v) variables, computing the inverse Fourier transform of the above term. This gives

$$\begin{aligned} \mathcal{W}_3^1(x,v) = -\pi \sum_{\sigma_1,\sigma_2=\pm1} \sigma_1\sigma_2 \int dv_1 \int dv_2 \int dv_3 \, |\widehat{\phi}(v_2-v_3)|^2 \\ \delta\left([v_2-v_3]\cdot\left[-v+v_2\frac{1+\sigma_1}{2}+v_3\frac{1-\sigma_1}{2}\right]\right)\delta\left(v-v_1+\frac{\sigma_1-\sigma_2}{2}(v_3-v_2)\right) \\ f_0(x-v_1t_1,v_1)\, f_0(x-v_2t_1,v_2)\, f_0(x-v_3t_1,v_3). \end{aligned} \tag{5.6}$$

This is our final expression of $\mathcal{W}_3^1$. It will be interpreted later in terms of the v, v_*, v', v'_* variables of the U-U equation.

In a similar fashion we compute $\mathcal{V}_{3,\varepsilon}^2$ and its limit $\mathcal{V}_3^2$. We write

$$\begin{aligned} \widehat{f}_{3,\varepsilon}^{(3,2,1)}(\xi_1,\xi_2,\xi_3,\eta_1,\eta_2,\eta_3) = \varepsilon^3 \widehat{f}_0(\xi_2,\eta_2) \int dy_1 \int dy_2 \, \mathrm{e}^{-i\xi_1\cdot(y_1+\frac{\varepsilon}{2}y_2)}\, \mathrm{e}^{-i\xi_3\cdot(y_1-\frac{\varepsilon}{2}y_2)} \\ \widetilde{f}_0\left(y_1-\varepsilon\frac{\eta_1-\eta_3}{4},-y_2+\frac{\eta_2+\eta_1}{2}\right)\widetilde{f}_0\left(y_1+\varepsilon\frac{\eta_1-\eta_3}{4},y_2+\frac{\eta_1+\eta_3}{2}\right). \end{aligned} \tag{5.7}$$

We insert this expression into (3.19), and perform the change of variables $h = h_2$, $k = (h_1 - h_2)/\varepsilon$, and $s = (t_1 - t_2)/\varepsilon$. Passing to the limit $\varepsilon \to 0$ at once, gives the asymptotic value

$$\begin{aligned} \widehat{\mathcal{V}}_3^2(\xi,\eta) = -\frac{1}{(2\pi)^6} \sum_{\sigma_1,\sigma_2=\pm1} \sigma_1\sigma_2 \int_0^{+\infty} ds \int dh \int dk \int dy_1 \int dy_2 \, |\widehat{\phi}(h)|^2 \\ \widetilde{f}_0\left(y_1+\frac{t_1h}{2},\frac{\eta+\xi t_1-kt_1-sh}{2}-y_2\right)\widetilde{f}_0\left(y_1-\frac{t_1h}{2},\frac{\eta+\xi t_1-kt_1-sh}{2}+y_2\right) \\ \widehat{f}_0(k,t_1k+sh)\,\mathrm{e}^{-i(\xi-k)\cdot y_1}\,\mathrm{e}^{ih\cdot y_2}\mathrm{e}^{-i\frac{\sigma_1}{2}h\cdot\eta}\,\mathrm{e}^{-i\frac{\sigma_2}{2}h^2 s}, \end{aligned} \tag{5.8}$$

whose inverse Fourier transform is

$$\begin{aligned} \mathcal{V}_3^2(x,v) = -\pi \sum_{\sigma_1,\sigma_2=\pm1} \sigma_1\sigma_2 \int dv_1 \int dv_2 \int dv_3 \, |\widehat{\phi}(v_1-v_3)|^2 \\ \delta\left([v_1-v_3]\cdot\left[v_2-v_1\frac{1+\sigma_2}{2}-v_3\frac{1-\sigma_2}{2}\right]\right)\delta\left(v-v_1\frac{1-\sigma_1}{2}-v_3\frac{1+\sigma_1}{2}\right) \\ f_0(x-v_1t_1,v_1)f_0(x-v_2t_1,v_2)\, f_0(x-v_3t_1,v_3). \end{aligned} \tag{5.9}$$

Before coming to the computation of the other $\mathcal{W}_3^i$'s and $\mathcal{V}_3^i$'s, let us now identify the link between the obtained values of $\mathcal{W}_3^1$, $\mathcal{V}_3^2$, and the U-U equation. The following changes of variables in $\mathcal{W}_3^1$

σ_1	σ_2	v_1	v_2	v_3
1	1	v	v'_*	v_*
-1	-1	v	v_*	v'_*
1	-1	v'	v'_*	v_*
-1	1	v'	v_*	v'_*

yields the explicit value

$$\begin{aligned}\mathcal{W}_3^1(x,v) = 2\pi \int dv_* \int dv' \int dv'_*\, |\widehat{\phi}(v'-v)|^2 \\ \delta(v+v_*-v'-v'_*)\,\delta\left(\frac{1}{2}(v^2+v_*^2-v'^2-v_*'^2)\right)\,[f_*f'f'_*-ff_*f'_*].\end{aligned} \tag{5.10}$$

For $\mathcal{V}_3^2$ we set

σ_1	σ_2	v_1	v_2	v_3
1	1	v'_*	v_*	v
-1	-1	v	v_*	v'_*
1	-1	v'_*	v'	v
-1	1	v	v'	v'_*

and obtain the final

$$\begin{aligned}\mathcal{V}_3^2(x,v) = 2\pi \int dv_* \int dv' \int dv'_*\, |\widehat{\phi}(v-v'_*)|^2 \\ \delta(v+v_*-v'-v'_*)\,\delta\left(\frac{1}{2}(v^2+v_*^2-v'^2-v_*'^2)\right)\,[ff'f'_*-f'_*ff_*].\end{aligned} \tag{5.11}$$

Last, using the symmetry $v' \leftrightarrow v'_*$ (note that $v'-v=-(v'_*-v_*)$ and $v'-v_*=v-v'_*$) we finally conclude after some computations that:

$$\begin{aligned}(\mathcal{W}_3^1+\mathcal{V}_3^2)(x,v) = \pi \int dv_* \int dv' \int dv'_*\, [(f+f_*)f'f'_*-(f'+f'_*)ff_*)] \\ \delta(v+v_*-v'-v'_*)\,\delta\left(\frac{1}{2}(v^2+v_*^2-v'^2-v_*'^2)\right)\,(|\widehat{\phi}(v'-v)|^2+|\widehat{\phi}(v'-v_*)|^2).\end{aligned} \tag{5.12}$$

This is the desired cubic term in the U-U equation, up to the fact that we only recover here part of the total cross-section $W(v,v_*|v',v'_*) = [\widehat{\phi}(v-v')+\widehat{\phi}(v-v_*)]^2$. The missing cross term will come up in the next section.

We now show that all other terms associated to permutations with a fixed element, namely $\mathcal{W}_{3,\varepsilon}^2$, $\mathcal{V}_{3,\varepsilon}^1$, and $\mathcal{W}_{3,\varepsilon}^3+\mathcal{V}_{3,\varepsilon}^3$, give a vanishing contribution in the limit $\varepsilon \to 0$.

We begin with $\mathcal{W}_{3,\varepsilon}^2$. Inserting (5.7) into (3.18) and changing variables $k=h_1\varepsilon^{-1}$, $h_2=h$, we readily obtain:

$$\begin{aligned}\widehat{\mathcal{W}}_{3,\varepsilon}^2(\xi,\eta) = -\frac{\varepsilon^{-1}}{(2\pi)^6} \sum_{\sigma_1,\sigma_2=\pm1} \sigma_1\sigma_2 \int_0^{t_1} dt_2 \int dk \int dh \int dy_1 \int dy_2\, \widehat{\phi}(h)\,\widehat{\phi}(\varepsilon k) \\ e^{-\frac{i}{2}(\sigma_1\eta\cdot k\varepsilon+\sigma_2 h\cdot(\eta+(t_1-t_2)(\xi-k))}\, e^{iy_2\cdot h}\, e^{-i\left(y_1+\varepsilon\frac{y_2}{2}\right)\cdot(\xi-k)} \\ \widetilde{f}_0\left(y_1-\frac{\varepsilon}{4}\left[\eta+\xi t_1-t_1k-\frac{2}{\varepsilon}t_2h\right];\frac{1}{2}(\eta+\xi t_1-t_1k)-y_2\right) \\ \widetilde{f}_0\left(y_1+\frac{\varepsilon}{4}\left[\eta+\xi t_1-t_1k-\frac{2}{\varepsilon}t_2h\right];\frac{1}{2}(\eta+\xi t_1-t_1k)+y_2\right) \\ \widehat{f}_0(k,t_1k).\end{aligned} \tag{5.13}$$

Summing on σ_1 and σ_2 allows to compute the limit which is:

$$\begin{aligned}\widehat{\mathcal{W}}_3^2(\xi,\eta) &= \frac{4}{(2\pi)^6}\int_0^{t_1} dt_2 \int dk \int dh \int dy_1 \int dy_2\, \widehat{\phi}(h)\,\widehat{\phi}(0)\,\mathrm{e}^{iy_2\cdot h}\,\mathrm{e}^{-iy_1\cdot(\xi-k)}\\ &\frac{\eta\cdot k}{2}\sin\left(\frac{1}{2}h\cdot(\eta+(t_1-t_2)(\xi-k)\right)\widetilde{f}_0\left(y_1+\frac{1}{2}t_2h;\frac{1}{2}(\eta+\xi t_1-t_1k)-y_2\right)\\ &\widetilde{f}_0\left(y_1-\frac{1}{2}t_2h;\frac{1}{2}(\eta+\xi t_1-t_1k)+y_2\right)\widehat{f}_0(k,t_1k).\end{aligned}\tag{5.14}$$

Note that the product of the two $\widetilde{f}_0$'s is invariant under the change of variables $h\to -h$, $y_2\to -y_2$, as well as the oscillatory factor $\mathrm{e}^{iy_2\cdot h}$. Therefore, using the parity of $\widehat{\phi}$, it follows that $\widehat{\mathcal{W}}_3^2(\xi,\eta) = -\widehat{\mathcal{W}}_3^2(\xi,\eta)$. Hence

$$\widehat{\mathcal{W}}_3^2 = 0.$$

The term $\widehat{\mathcal{V}}_3^1$ is treated in the analogous way. We insert (5.2) into (3.19) and make the change of variables $h = h_2$, $k = \varepsilon^{-1}h_1$, obtaining after summation over σ_1 and σ_2:

$$\begin{aligned}\widehat{\mathcal{V}}_{3,\varepsilon}^1(\xi,\eta) &= \frac{4\varepsilon^{-1}}{(2\pi)^6}\int_0^{t_1} dt_2 \int dk \int dh \int dy_1 \int dy_2\, \widehat{\phi}(h)\,\widehat{\phi}(\varepsilon k)\\ &\mathrm{e}^{-iy_1\cdot k}\,\mathrm{e}^{iy_2\cdot\left(h-\varepsilon\frac{k}{2}\right)}\sin\left(\frac{\varepsilon}{2}\eta\cdot k\right)\sin\left(\frac{1}{2}h\cdot k(t_1-t_2)\right)\\ &\widetilde{f}_0\left(y_1-\frac{\varepsilon}{4}\left[t_1k-\frac{2}{\varepsilon}t_2h\right];\frac{1}{2}t_1k-y_2\right)\widetilde{f}_0\left(y_1+\frac{\varepsilon}{4}\left[t_1k-\frac{2}{\varepsilon}t_2h\right];\frac{1}{2}t_1k+y_2\right)\\ &\widehat{f}_0(\xi-k,\eta+t_1(\xi-k)).\end{aligned}\tag{5.15}$$

This term clearly goes to

$$\begin{aligned}\widehat{\mathcal{V}}_3^1(\xi,\eta) &= \frac{4}{(2\pi)^6}\int_0^{t_1} dt_2 \int dk \int dh \int dy_1 \int dy_2\, \widehat{\phi}(h)\widehat{\phi}(0)\\ &\mathrm{e}^{-iy_1\cdot k}\,\mathrm{e}^{iy_2\cdot h}\left(\frac{1}{2}\eta\cdot k\right)\sin\left(\frac{1}{2}h\cdot k(t_1-t_2)\right)\widetilde{f}_0\left(y_1+\frac{1}{2}t_2h;\frac{1}{2}t_1k-y_2\right)\\ &\widetilde{f}_0\left(y_1-\frac{1}{2}t_2h;\frac{1}{2}t_1k+y_2\right)\widehat{f}_0(\xi-k,\eta+t_1(\xi-k)).\end{aligned}\tag{5.16}$$

Hence $\widehat{\mathcal{V}}_3^1 = 0$ for the same reason as before.

To end up this paragraph, let us last prove that the sum $\mathcal{W}_{3,\varepsilon}^3 + \mathcal{V}_{3,\varepsilon}^3$ vanishes asymptotically. First, we write

$$\begin{aligned}\widehat{f}_{3,\varepsilon}^3(\xi_1,\xi_2,\xi_3,\eta_1,\eta_2,\eta_3) &= \varepsilon^3\widehat{f}_0(\xi_3,\eta_3)\int dy_1\int dy_2\,\mathrm{e}^{-i\xi_1\cdot(y_1+\frac{\varepsilon}{2}y_2)}\,\mathrm{e}^{-i\xi_2\cdot(y_1-\frac{\varepsilon}{2}y_2)}\\ &\widetilde{f}_0\left(y_1-\varepsilon\frac{\eta_1-\eta_2}{4},-y_2+\frac{\eta_1+\eta_2}{2}\right)\widetilde{f}_0\left(y_1+\varepsilon\frac{\eta_1-\eta_2}{4},y_2+\frac{\eta_1+\eta_2}{2}\right).\end{aligned}\tag{5.17}$$

We insert this formmula in (3.18), and perform the change of variables $h_1 = h$, $h_2 = \varepsilon k$. In this way we recover, upon computing the sum $\sum \sigma_1\sigma_2$,

$$\begin{aligned}
\widehat{\mathcal{W}}^3_{3,\varepsilon}(\xi,\eta) = &\frac{4\varepsilon^{-1}}{(2\pi)^6}\int_0^{t_1} dt_2 \int dk \int dh \int dy_1 \int dy_2\, \widehat{\phi}(h)\, \widehat{\phi}(\varepsilon k) \\
&\mathrm{e}^{iy_2\cdot h}\, \mathrm{e}^{-i(y_1+\frac{\varepsilon}{2}y_2)\cdot(\xi-k)} \sin\left(\frac{1}{2}\eta\cdot h\right) \sin\left(\frac{\varepsilon k}{2}\cdot(\eta+(t_1-t_2)\xi) - \frac{1}{2}k\cdot h(t_1-t_2)\right) \\
&\widetilde{f}_0\left(y_1 - \frac{\varepsilon}{4}\left[\eta+\xi t_1 - kt_2 - \frac{2}{\varepsilon}t_1 h\right]; \frac{1}{2}(h+\xi t_1 - t_2 k) - y_2\right) \\
&\widetilde{f}_0\left(y_1 + \frac{\varepsilon}{4}\left[\eta+\xi t_1 - kt_2 - \frac{2}{\varepsilon}t_1 h\right]; \frac{1}{2}(\eta+\xi t_1 - t_2 k) + y_2\right) \\
&\widehat{f}_0(k, t_2 k).
\end{aligned} \tag{5.18}$$

Similarly, using (5.17), (3.19), and performing again the change of variables $h_1 = h$, $h_2 = \varepsilon k$, we obtain:

$$\begin{aligned}
\widehat{\mathcal{V}}^3_{3,\varepsilon}(\xi,\eta) = &\frac{4\varepsilon^{-1}}{(2\pi)^6}\int_0^{t_1} dt_2 \int dk \int dh \int dy_1 \int dy_2\, \widehat{\phi}(h)\, \widehat{\phi}(\varepsilon k) \\
&\mathrm{e}^{iy_2\cdot h}\, \mathrm{e}^{-iy_1\cdot(\xi-k)}\, \mathrm{e}^{-i\frac{\varepsilon}{2}y_2\cdot(\xi+k)} \\
&\sin\left(\frac{1}{2}\eta\cdot h\right) \sin\left(\frac{1}{2}k\cdot h(t_1-t_2)\right) \\
&\widetilde{f}_0\left(y_1 + \frac{\varepsilon}{4}\left[\eta+\xi t_1 + kt_2 - \frac{2}{\varepsilon}t_1 h\right]; \frac{1}{2}(\eta+\xi t_1 - t_2 k) + y_2\right) \\
&\widetilde{f}_0\left(y_1 - \frac{\varepsilon}{4}\left[\eta+\xi t_1 + kt_2 - \frac{2}{\varepsilon}t_1 h\right]; \frac{1}{2}(\eta+\xi t_1 - t_2 k) - y_2\right) \\
&\widehat{f}_0(k, t_2 k).
\end{aligned} \tag{5.19}$$

Hence, both terms $\widehat{\mathcal{W}}^3_{3,\varepsilon}$ and $\widehat{\mathcal{V}}^3_{3,\varepsilon}$ are $O(\varepsilon^{-1})$. However we have the following expansions:

$$\varepsilon\widehat{\mathcal{W}}^3_{3,\varepsilon} = A_0 + \varepsilon A_1 + O(\varepsilon^2), \quad \varepsilon\widehat{\mathcal{V}}^3_{3,\varepsilon} = B_0 + \varepsilon B_1 + O(\varepsilon^2),$$

and it is easy to realize that $A_0 = -B_0$. Moreover, after some straightforward calculation, we obtain at the next order:

$$\begin{aligned}
A_1 + B_1 = &\frac{4\varepsilon^{-1}}{(2\pi)^6}\int_0^{t_1} dt_2 \int dk \int dh \int dy_1 \int dy_2\, \widehat{\phi}(h)\, \widehat{\phi}(0)\mathrm{e}^{iy_2\cdot h}\, \mathrm{e}^{-iy_1\cdot(\xi-k)} \sin\left(\frac{1}{2}\eta\cdot h\right) \\
&\widehat{f}_0(k,t_2k)\widetilde{f}_0\left(y_1 - \frac{1}{2}t_1 h; \frac{1}{2}(\eta+\xi t_1 - t_2 k) + y_2\right) \widetilde{f}_0\left(y_1 + \frac{1}{2}t_1 h; \frac{1}{2}(\eta+\xi t_1 - t_2 k) - y_2\right) \\
&\left\{\frac{k}{2}\cdot(\eta+(t_1-t_2)\xi) \cos\left(\frac{1}{2}k\cdot h(t_1-t_2)\right) + \right. \\
&\left.\sin\left(\frac{1}{2}k\cdot h(t_1-t_2)\right)\left(\frac{1}{2}t_2 k\cdot\nabla_x \log\frac{\widetilde{f}_0((y_1-\frac{1}{2}t_1h;\frac{1}{2}(\eta+\xi t_1-t_2k)+y_2)}{\widetilde{f}_0((y_1+\frac{1}{2}t_1h;\frac{1}{2}(\eta+\xi t_1-t_2k)-y_2)} - iy_2\cdot k\right)\right\}.
\end{aligned} \tag{5.20}$$

Let us now exchange $h \to -h$ and $y_2 \to -y_2$. The term in braces is invariant because the log term changes its sign. All the other terms are invariant but $\sin(\eta \cdot h/2)$, which changes sign. Therefore $A_1 + B_1 = -(A_1 + B_1)$, hence $A_1 + B_1 = 0$. This shows that $\mathcal{W}^3_{3,\varepsilon} + \mathcal{V}^3_{3,\varepsilon}$ vanishes in the limit $\varepsilon \to 0$.

6. Three-particle terms: cyclic permutations

We still have to evaluate $\mathcal{W}^{\pi}_{3,\varepsilon}$, $\mathcal{W}^{\pi^{-1}}_{3,\varepsilon}$, $\mathcal{V}^{\pi}_{3,\varepsilon}$ and $\mathcal{V}^{\pi^{-1}}_{3,\varepsilon}$ with $\pi = (2,3,1)$ and $\pi^{-1} = (3,1,2)$.

We first observe, for later convenience, the two relations

$$
\begin{aligned}
\widehat{f}_3^{\pi^{-1}}(\xi_1,\xi_2,\xi_3;\eta_1,\eta_2,\eta_3) = & \int dx_1 \int dx_2 \int dx_3 \; \mathrm{e}^{-i\sum_{k=1}^3 x_k\cdot\xi_k} \\
& \widetilde{f}_0\left(\frac{x_1+x_2}{2} + \frac{\varepsilon}{4}(\eta_1-\eta_2); \frac{x_1-x_2}{\varepsilon} + \frac{\eta_1+\eta_2}{2}\right) \\
& \widetilde{f}_0\left(\frac{x_2+x_3}{2} + \frac{\varepsilon}{4}(\eta_2-\eta_3); \frac{x_2-x_3}{\varepsilon} + \frac{\eta_2+\eta_3}{2}\right) \\
& \widetilde{f}_0\left(\frac{x_3+x_1}{2} + \frac{\varepsilon}{4}(\eta_3-\eta_1); \frac{x_3-x_1}{\varepsilon} + \frac{\eta_3+\eta_1}{2}\right),
\end{aligned}
\tag{6.1}
$$

$$
\begin{aligned}
\widehat{f}_3^{\pi}(\xi_1,\xi_2,\xi_3;\eta_1,\eta_2,\eta_3) = & \int dx_1 \int dx_2 \int dx_3 \; \mathrm{e}^{-i\sum_{k=1}^3 x_k\cdot\xi_k} \\
& \widetilde{f}_0\left(\frac{x_2+x_1}{2} + \frac{\varepsilon}{4}(\eta_2-\eta_1); \frac{x_2-x_1}{\varepsilon} + \frac{\eta_2+\eta_1}{2}\right) \\
& \widetilde{f}_0\left(\frac{x_3+x_2}{2} + \frac{\varepsilon}{4}(\eta_3-\eta_2); \frac{x_3-x_2}{\varepsilon} + \frac{\eta_3+\eta_2}{2}\right) \\
& \widetilde{f}_0\left(\frac{x_1+x_3}{2} + \frac{\varepsilon}{4}(\eta_1-\eta_3); \frac{x_1-x_3}{\varepsilon} + \frac{\eta_1+\eta_3}{2}\right).
\end{aligned}
\tag{6.2}
$$

Armed with these expressions, we begin with the computation of $\widehat{\mathcal{W}}^{\pi^{-1}}_{3,\varepsilon}$ and $\widehat{\mathcal{W}}^{\pi}_{3,\varepsilon}$. To do so, we insert (6.1) and (6.2) in the general formula (3.18) relating the value of the $\widehat{\mathcal{W}}^{\pi}_{3,\varepsilon}$'s. In the so-obtained formulae, we also change variables, $h_1 \to -h_1$, $h_2 \to -h_2$ for π^{-1}, and $\sigma_1 \to -\sigma_1$, $\sigma_2 \to -\sigma_2$ for π. With this new set of variables, the ξ's and η's involved in (3.18) are

$$
\begin{aligned}
&\xi_1 = \xi \pm \frac{h_1+h_2}{\varepsilon}, && \xi_2 = \mp\frac{h_1}{\varepsilon}, && \xi_3 = \mp\frac{h_2}{\varepsilon}, \\
&\eta_1 = \eta + \xi t_1 \pm \frac{t_1h_1+t_2h_2}{\varepsilon}, && \eta_2 = \mp\frac{t_1h_1}{\varepsilon}, && \eta_3 = \mp\frac{t_2h_2}{\varepsilon},
\end{aligned}
\tag{6.3}
$$

for π^{-1} and π respectively. Also, the phases appearing in (3.18) are:

$$
\begin{aligned}
\mathcal{S}_{\pi^{-1}} &= \frac{\sigma_1}{2}h_1\cdot\eta + \frac{\sigma_2}{2}h_2\left(\eta + (t_1-t_2)\left[\xi + \frac{h_1}{\varepsilon}\right]\right), \\
\mathcal{S}_{\pi} &= \frac{\sigma_1}{2}h_1\cdot\eta + \frac{\sigma_2}{2}h_2\left(\eta + (t_1-t_2)\left[\xi - \frac{h_1}{\varepsilon}\right]\right).
\end{aligned}
\tag{6.4}
$$

All this gives in (3.18), the two values

$$
\begin{aligned}
\widehat{\mathcal{W}}^{\pi^{-1}}_{3,\varepsilon}(\xi,\eta) = &- \frac{\varepsilon^{-7}}{(2\pi)^6} \sum_{\sigma_1\sigma_2=\pm1} \sigma_1\sigma_2 \int_0^{t_1} dt_2 \int dh_1 \int dh_2 \int dx_1 \int dx_2 \int dx_3 \\
&\widetilde{f}_0\left(\frac{x_1+x_2}{2} + \frac{2t_1h_1+t_2h_2}{4} + \frac{\varepsilon}{4}\bar{\eta}; \frac{x_1-x_2}{\varepsilon} + \frac{t_2h_2}{2\varepsilon} + \frac{\bar{\eta}}{2}\right) \\
&\widetilde{f}_0\left(\frac{x_2+x_3}{2} - \frac{t_1h_1-t_2h_2}{4}; \frac{x_2-x_3}{\varepsilon} - \frac{t_1h_1+t_2h_2}{2\varepsilon}\right) \\
&\widetilde{f}_0\left(\frac{x_3+x_1}{2} - \frac{t_1h_1+2t_2h_2}{4} - \frac{\varepsilon}{4}\bar{\eta}; \frac{x_3-x_1}{\varepsilon} + \frac{t_1h_1}{2\varepsilon} + \frac{\bar{\eta}}{2}\right) \\
&\widehat{\phi}(h_1)\, \widehat{\phi}(h_2)\, \mathrm{e}^{-ix_1\cdot(\xi+\frac{h_1+h_2}{\varepsilon})}\, \mathrm{e}^{ix_2\cdot\frac{h_1}{\varepsilon}}\, \mathrm{e}^{ix_3\cdot\frac{h_2}{\varepsilon}}\, \mathrm{e}^{i\mathcal{S}_{\pi^{-1}}},
\end{aligned}
\tag{6.5}
$$

$$
\begin{aligned}
\widehat{\mathcal{W}}^{\pi}_{3,\varepsilon}(\xi,\eta) = &- \frac{\varepsilon^{-7}}{(2\pi)^6} \sum_{\sigma_1\sigma_2=\pm1} \sigma_1\sigma_2 \int_0^{t_1} dt_2 \int dh_1 \int dh_2 \int dx_1 \int dx_2 \int dx_3 \\
&\widetilde{f}_0\left(\frac{x_2+x_1}{2} + \frac{2t_1h_1+t_2h_2}{4} - \frac{\varepsilon}{4}\bar{\eta}; \frac{x_2-x_1}{\varepsilon} - \frac{t_2h_2}{2\varepsilon} + \frac{\bar{\eta}}{2}\right) \\
&\widetilde{f}_0\left(\frac{x_3+x_2}{2} - \frac{t_1h_1-t_2h_2}{4}; \frac{x_3-x_2}{\varepsilon} + \frac{t_1h_1+t_2h_2}{2\varepsilon}\right) \\
&\widetilde{f}_0\left(\frac{x_1+x_3}{2} - \frac{t_1h_1+2t_2h_2}{4} + \frac{\varepsilon}{4}\bar{\eta}; \frac{x_1-x_3}{\varepsilon} - \frac{t_1h_1}{2\varepsilon} + \frac{\bar{\eta}}{2}\right) \\
&\widehat{\phi}(h_1)\, \widehat{\phi}(h_2)\, \mathrm{e}^{-ix_1\cdot(\xi-\frac{h_1+h_2}{\varepsilon})}\, \mathrm{e}^{-ix_2\cdot\frac{h_1}{\varepsilon}}\, \mathrm{e}^{-ix_3\cdot\frac{h_2}{\varepsilon}}\, \mathrm{e}^{i\mathcal{S}_{\pi}},
\end{aligned}
\tag{6.6}
$$

where we use the notation $\bar{\eta} = \eta + \xi t_1$.

Now, we perform the following natural change of variables:

$$
\begin{aligned}
&\text{for } \pi^{-1}: \qquad x_2 = x_1 + \frac{1}{2}t_2h_2 - \varepsilon y_1, \quad x_3 = x_1 - \frac{1}{2}t_1h_1 + \varepsilon y_3, \\
&\text{for } \pi: \qquad x_2 = x_1 + \frac{1}{2}t_2h_2 + \varepsilon y_1, \quad x_3 = x_1 - \frac{1}{2}t_1h_1 - \varepsilon y_3.
\end{aligned}
\tag{6.7}
$$

In both cases x_1 is unchanged. This finally gives the two relations

$$
\begin{aligned}
\widehat{\mathcal{W}}^{\pi^{-1}}_{3,\varepsilon}(\xi,\eta) = &- \frac{\varepsilon^{-1}}{(2\pi)^6} \sum_{\sigma_1\sigma_2=\pm1} \sigma_1\sigma_2 \int_0^{t_1} dt_2 \int dh_1 \int dh_2 \int dx_1 \int dy_1 \int dy_3 \\
&\widetilde{f}_0\left(x_1 + \frac{t_1h_1+t_2h_2}{2} + \frac{\varepsilon}{2}\left[\frac{\bar{\eta}}{2} - y_1\right]; y_1 + \frac{\bar{\eta}}{2}\right) \\
&\widetilde{f}_0\left(x_1 - \frac{t_1h_1-t_2h_2}{2} - \frac{\varepsilon}{2}(y_1-y_3); -y_1-y_3\right) \\
&\widetilde{f}_0\left(x_1 - \frac{t_1h_1+t_2h_2}{2} - \frac{\varepsilon}{2}\left[\frac{\bar{\eta}}{2} - y_3\right]; y_3 + \frac{\bar{\eta}}{2}\right) \\
&\widehat{\phi}(h_1)\, \widehat{\phi}(h_2)\, \mathrm{e}^{-ix_1\cdot\xi}\, \mathrm{e}^{-iy_1\cdot h_1}\, \mathrm{e}^{iy_3\cdot h_2}\, \mathrm{e}^{-\frac{i}{2\varepsilon}h_1\cdot h_2(t_1-t_2)}\, \mathrm{e}^{i\mathcal{S}_{\pi^{-1}}},
\end{aligned}
\tag{6.8}
$$

$$
\begin{aligned}
\widehat{\mathcal{W}}^{\pi}_{3,\varepsilon}(\xi,\eta) = &- \frac{\varepsilon^{-1}}{(2\pi)^6} \sum_{\sigma_1\sigma_2=\pm 1} \sigma_1\sigma_2 \int_0^{t_1} dt_2 \int dh_1 \int dh_2 \int dx_1 \int dy_1 \int dy_3 \\
& \widetilde{f}_0\left(x_1 + \frac{t_1h_1+t_2h_2}{2} - \frac{\varepsilon}{2}\left[\frac{\bar\eta}{2} - y_1\right]; y_1 + \frac{\bar\eta}{2}\right) \\
& \widetilde{f}_0\left(x_1 - \frac{t_1h_1-t_2h_2}{2} + \frac{\varepsilon}{2}(y_1-y_3); -y_1-y_3\right) \\
& \widetilde{f}_0\left(x_1 - \frac{t_1h_1+t_2h_2}{2} + \frac{\varepsilon}{2}\left[\frac{\bar\eta}{2} - y_3\right]; y_3 + \frac{\bar\eta}{2}\right) \\
& \widehat{\phi}(h_1)\,\widehat{\phi}(h_2)\,\mathrm{e}^{-ix_1\cdot\xi}\,\mathrm{e}^{-iy_1\cdot h_1}\,\mathrm{e}^{iy_3\cdot h_2}\,\mathrm{e}^{\frac{i}{2\varepsilon}h_1\cdot h_2(t_1-t_2)}\,\mathrm{e}^{i\mathcal{S}_\pi}.
\end{aligned}
\tag{6.9}
$$

Next, we come to the computation of $\widehat{\mathcal{V}}^{\pi}_{3,\varepsilon}$ and $\widehat{\mathcal{V}}^{\pi^{-1}}_{3,\varepsilon}$. We insert (6.1) and (6.2) in (3.19). We also change $h_1 \to -h_1$, $\sigma_2 \to -\sigma_2$ for π^{-1} and $h_2 \to -h_2$, $\sigma_2 \to -\sigma_2$ for π. This gives the two identities

$$
\begin{aligned}
\widehat{\mathcal{V}}^{\pi^{-1}}_{3,\varepsilon}(\xi,\eta) = &- \frac{\varepsilon^{-7}}{(2\pi)^6} \sum_{\sigma_1\sigma_2=\pm 1} \sigma_1\sigma_2 \int_0^{t_1} dt_2 \int dh_1 \int dh_2 \int dx_1 \int dx_2 \int dx_3 \\
& \widetilde{f}_0\left(\frac{x_1+x_2}{2} + \frac{2t_1h_1+t_2h_2}{4} + \frac{\varepsilon}{4}\bar\eta; \frac{x_1-x_2}{\varepsilon} - \frac{t_2h_2}{2\varepsilon} + \frac{\bar\eta}{2}\right) \\
& \widetilde{f}_0\left(\frac{x_3+x_1}{2} - \frac{t_1h_1-t_2h_2}{4} - \frac{\varepsilon}{4}\bar\eta; \frac{x_3-x_1}{\varepsilon} + \frac{t_1h_1+t_2h_2}{2\varepsilon} + \frac{\bar\eta}{2}\right) \\
& \widetilde{f}_0\left(\frac{x_2+x_3}{2} - \frac{t_1h_1+2t_2h_2}{4}; \frac{x_2-x_3}{\varepsilon} - \frac{t_1h_1}{2\varepsilon}\right) \\
& \widehat{\phi}(h_1)\,\widehat{\phi}(h_2)\,\mathrm{e}^{-ix_1\cdot(\xi+\frac{h_1}{\varepsilon})}\,\mathrm{e}^{ix_2\cdot\frac{h_1+h_2}{\varepsilon}}\,\mathrm{e}^{-ix_3\cdot\frac{h_2}{\varepsilon}}\,\mathrm{e}^{i\widetilde{\mathcal{S}}_{\pi^{-1}}},
\end{aligned}
\tag{6.10}
$$

$$
\begin{aligned}
\widehat{\mathcal{V}}^{\pi}_{3,\varepsilon}(\xi,\eta) = &- \frac{\varepsilon^{-7}}{(2\pi)^6} \sum_{\sigma_1\sigma_2=\pm 1} \sigma_1\sigma_2 \int_0^{t_1} dt_2 \int dh_1 \int dh_2 \int dx_1 \int dx_2 \int dx_3 \\
& \widetilde{f}_0\left(\frac{x_2+x_1}{2} + \frac{2t_1h_1+t_2h_2}{4} - \frac{\varepsilon}{4}\bar\eta; \frac{x_2-x_1}{\varepsilon} + \frac{t_2h_2}{2\varepsilon} + \frac{\bar\eta}{2}\right) \\
& \widetilde{f}_0\left(\frac{x_1+x_3}{2} - \frac{t_1h_1-t_2h_2}{4} + \frac{\varepsilon}{4}\bar\eta; \frac{x_1-x_3}{\varepsilon} - \frac{t_1h_1+t_2h_2}{2\varepsilon} + \frac{\bar\eta}{2}\right) \\
& \widetilde{f}_0\left(\frac{x_3+x_2}{2} - \frac{t_1h_1+2t_2h_2}{4}; \frac{x_3-x_2}{\varepsilon} + \frac{t_1h_1}{2\varepsilon}\right) \\
& \widehat{\phi}(h_1)\,\widehat{\phi}(h_2)\,\mathrm{e}^{-ix_1\cdot(\xi-\frac{h_1}{\varepsilon})}\,\mathrm{e}^{-ix_2\cdot\frac{h_1+h_2}{\varepsilon}}\,\mathrm{e}^{ix_3\cdot\frac{h_2}{\varepsilon}}\,\mathrm{e}^{i\widetilde{\mathcal{S}}_{\pi}}.
\end{aligned}
\tag{6.11}
$$

Here the phases are:

$$
\begin{aligned}
\widetilde{\mathcal{S}}_{\pi^{-1}} &= \frac{\sigma_1}{2}h_1\cdot\eta + \frac{\sigma_2}{2}h_2\cdot\frac{h_1}{\varepsilon}(t_1-t_2), \\
\widetilde{\mathcal{S}}_{\pi} &= \frac{\sigma_1}{2}h_1\cdot\eta - \frac{\sigma_2}{2}h_2\cdot\frac{h_1}{\varepsilon}(t_1-t_2).
\end{aligned}
\tag{6.12}
$$

We make the following natural change of variable

$$
\begin{aligned}
&\text{for } \pi^{-1}: \qquad x_1 = x_1' + \frac{1}{2}t_2h_2, \quad x_2 = x_1' - \varepsilon y_1, \quad x_3 = x_1' - \frac{1}{2}t_1h_1 - \varepsilon\left(y_1 + y_3 + \frac{\bar\eta}{2}\right),\\
&\text{for } \pi: \qquad x_1 = x_1' + \frac{1}{2}t_2h_2, \quad x_2 = x_1' + \varepsilon y_1, \quad x_3 = x_1' - \frac{1}{2}t_1h_1 + \varepsilon\left(y_1 + y_3 + \frac{\bar\eta}{2}\right).
\end{aligned}
\tag{6.13}
$$

With this change of variables, we eventually obtain

$$
\begin{aligned}
\widehat{\mathcal{V}}^{\pi^{-1}}_{3,\varepsilon}(\xi,\eta) = -\frac{\varepsilon^{-1}}{(2\pi)^6}\sum_{\sigma_1\sigma_2=\pm1}\sigma_1\sigma_2\int_0^{t_1}dt_2\int dh_1\int dh_2\int dx_1'\int dy_1\int dy_3\\
\widetilde{f}_0\left(x_1' + \frac{t_1h_1+t_2h_2}{2} + \frac{\varepsilon}{2}\left[\frac{\bar\eta}{2} - y_1\right]; y_1 + \frac{\bar\eta}{2}\right)\\
\widetilde{f}_0\left(x_1' - \frac{t_1h_1-t_2h_2}{2} - \frac{\varepsilon}{2}(\bar\eta + y_1 + y_3); -y_1 - y_3\right)\\
\widetilde{f}_0\left(x_1' - \frac{t_1h_1+t_2h_2}{2} - \frac{\varepsilon}{2}\left[\frac{\bar\eta}{2} + 2y_1 + y_3\right]; y_3 + \frac{\bar\eta}{2}\right)\\
\widehat{\phi}(h_1)\,\widehat{\phi}(h_2)\,\mathrm{e}^{-ix_1'\cdot\xi}\,\mathrm{e}^{-iy_1\cdot h_1}\,\mathrm{e}^{iy_3\cdot h_2}\,\mathrm{e}^{\frac{i}{2\varepsilon}h_1\cdot h_2(t_1-t_2)}\,\mathrm{e}^{i\frac{h_2}{2}\cdot(\eta+\xi(t_1-t_2))}\,\mathrm{e}^{i\widetilde{\mathcal{S}}_{\pi^{-1}}},
\end{aligned}
\tag{6.14}
$$

as well as

$$
\begin{aligned}
\widehat{\mathcal{V}}^{\pi}_{3,\varepsilon}(\xi,\eta) = -\frac{\varepsilon^{-1}}{(2\pi)^6}\sum_{\sigma_1\sigma_2=\pm1}\sigma_1\sigma_2\int_0^{t_1}dt_2\int dh_1\int dh_2\int dx_1'\int dy_1\int dy_3\\
\widetilde{f}_0\left(x_1' + \frac{t_1h_1+t_2h_2}{2} - \frac{\varepsilon}{2}\left[\frac{\bar\eta}{2} - y_1\right]; y_1 + \frac{\bar\eta}{2}\right)\\
\widetilde{f}_0\left(x_1' - \frac{t_1h_1-t_2h_2}{2} + \frac{\varepsilon}{2}(\bar\eta + y_1 + y_3); -y_1 - y_3\right)\\
\widetilde{f}_0\left(x_1' - \frac{t_1h_1+t_2h_2}{2} + \frac{\varepsilon}{2}\left[\frac{\bar\eta}{2} + 2y_1 + y_3\right]; y_3 + \frac{\bar\eta}{2}\right)\\
\widehat{\phi}(h_1)\,\widehat{\phi}(h_2)\,\mathrm{e}^{-ix_1'\cdot\xi}\,\mathrm{e}^{-iy_1\cdot h_1}\,\mathrm{e}^{iy_3\cdot h_2}\,\mathrm{e}^{-\frac{i}{2\varepsilon}h_1\cdot h_2(t_1-t_2)}\,\mathrm{e}^{i\frac{h_2}{2}\cdot(\eta+\xi(t_1-t_2))}\,\mathrm{e}^{i\widetilde{\mathcal{S}}_{\pi}}.
\end{aligned}
\tag{6.15}
$$

Let us come to the computation of the limit of the above four terms $\widehat{\mathcal{W}}^{\pi^{-1}}_{3,\varepsilon}$, $\widehat{\mathcal{W}}^{\pi}_{3,\varepsilon}$, $\widehat{\mathcal{V}}^{\pi^{-1}}_{3,\varepsilon}$, $\widehat{\mathcal{V}}^{\pi}_{3,\varepsilon}$. The phases carried by these terms are respectively:

$$
\begin{aligned}
&\frac{\sigma_1h_1+\sigma_2h_2}{2}\cdot\eta + \frac{\sigma_2}{2}h_2\cdot\xi(t_1-t_2) - x_1\cdot\xi - y_1\cdot h_1 + y_3\cdot h_2 - \frac{1-\sigma_2}{2}\frac{t_1-t_2}{\varepsilon}h_1\cdot h_2,\\
&\frac{\sigma_1h_1+\sigma_2h_2}{2}\cdot\eta + \frac{\sigma_2}{2}h_2\cdot\xi(t_1-t_2) - x_1\cdot\xi - y_1\cdot h_1 + y_3\cdot h_2 + \frac{1-\sigma_2}{2}\frac{t_1-t_2}{\varepsilon}h_1\cdot h_2,\\
&\frac{\sigma_1h_1+h_2}{2}\cdot\eta + \frac{1}{2}h_2\cdot\xi(t_1-t_2) - x_1'\cdot\xi - y_1\cdot h_1 + y_3\cdot h_2 + \frac{1+\sigma_2}{2}\frac{t_1-t_2}{\varepsilon}h_1\cdot h_2,\\
&\frac{\sigma_1h_1+h_2}{2}\cdot\eta + \frac{1}{2}h_2\cdot\xi(t_1-t_2) - x_1'\cdot\xi - y_1\cdot h_1 + y_3\cdot h_2 - \frac{1+\sigma_2}{2}\frac{t_1-t_2}{\varepsilon}h_1\cdot h_2.
\end{aligned}
$$

Denoting by $\widehat{\mathcal{W}}^{\pi^{-1},\pm}_{3,\varepsilon}$, $\widehat{\mathcal{W}}^{\pi,\pm}_{3,\varepsilon}$, $\widehat{\mathcal{V}}^{\pi^{-1},\pm}_{3,\varepsilon}$ $\widehat{\mathcal{V}}^{\pi,\pm}_{3,\varepsilon}$, the eight terms relative to the values of $\sigma_2 = \pm 1$, we realize that $\widehat{\mathcal{W}}^{\pi^{-1},+}_{3,\varepsilon}$, $\widehat{\mathcal{W}}^{\pi,+}_{3,\varepsilon}$, $\widehat{\mathcal{V}}^{\pi^{-1},-}_{3,\varepsilon}$ $\widehat{\mathcal{V}}^{\pi,-}_{3,\varepsilon}$, have only slowly varying phases, so that they are individually $O(\varepsilon^{-1})$. However, setting

$$\begin{aligned} \varepsilon\widehat{\mathcal{W}}^{\pi^{-1},+}_{3,\varepsilon} &= A_0 + A_1\varepsilon + O(\varepsilon^2), \quad \varepsilon\widehat{\mathcal{W}}^{\pi,+}_{3,\varepsilon} = B_0 + B_1\varepsilon + O(\varepsilon^2), \\ \varepsilon\widehat{\mathcal{V}}^{\pi^{-1},-}_{3,\varepsilon} &= C_0 + C_1\varepsilon + O(\varepsilon^2), \qquad \varepsilon\widehat{\mathcal{V}}^{\pi,-}_{3,\varepsilon} = D_0 + D_1\varepsilon + O(\varepsilon^2), \end{aligned} \tag{6.16}$$

an easy first order Taylor expansion gives $A_0+C_0 = B_0+D_0 = 0$ and $A_1+B_1 = C_1+D_1 = 0$. Hence

$$\lim_{\varepsilon\to 0}\left(\widehat{\mathcal{W}}^{\pi^{-1},+}_{3,\varepsilon} + \widehat{\mathcal{W}}^{\pi,+}_{3,\varepsilon} + \widehat{\mathcal{V}}^{\pi^{-1},-}_{3,\varepsilon} + \widehat{\mathcal{V}}^{\pi,-}_{3,\varepsilon}\right) = 0. \tag{6.17}$$

For the other terms, which carry a rapidly oscillating phases, it is natural to rescale time, setting $s = \varepsilon^{-1}(t_1 - t_2)$. Then, an easy computation shows

$$\lim_{\varepsilon\to 0}\left(\widehat{\mathcal{W}}^{\pi,-}_{3,\varepsilon} + \widehat{\mathcal{W}}^{\pi^{-1},-}_{3,\varepsilon}\right) =: \widehat{\mathcal{W}}^{\pi}_{3}, \tag{6.18}$$

where

$$\begin{aligned} \widehat{\mathcal{W}}^{\pi}_{3}(\xi,\eta) =& \frac{1}{(2\pi)^6}\sum_{\sigma_1\equiv\pm 1}\sigma_1\int_0^\infty ds\int dh_1\int dh_2\,\widehat{\phi}(h_1)\,\widehat{\phi}(h_2)\left(\mathrm{e}^{-ih_1\cdot h_2 s} + \mathrm{e}^{ih_1\cdot h_2 s}\right) \\ & \int dx_1\int dy_1\int dy_3\; \mathrm{e}^{\frac{i}{2}(\sigma_1 h_1 - h_2)\cdot\eta}\,\mathrm{e}^{-ix_1\cdot\xi}\,\mathrm{e}^{-y_1\cdot h_1}\,\mathrm{e}^{iy_3\cdot h_2} \\ & \widetilde{f}_0\left(x_1 + t_1\frac{h_1+h_2}{2}; y_1 + \frac{\eta}{2}\right)\widetilde{f}_0\left(x_1 - t_1\frac{h_1-h_2}{2}; -y_1 - y_3\right) \\ & \widetilde{f}_0\left(x_1 - t_1\frac{h_1+h_2}{2}; y_3 + \frac{\eta}{2}\right). \end{aligned} \tag{6.19}$$

Finally, taking the inverse Fourier transform of this term, we obtain:

$$\begin{aligned} \mathcal{W}^{\pi}_{3}(x,v) =& 2\pi\int dv_1\int dv_2\int dv_3\;\widehat{\phi}(v_2 - v_1)\,\widehat{\phi}(v_3 - v_2) \\ & [\delta(v + v_2 - v_1 - v_3) - \delta(v - v_3)]\,\delta((v_2 - v_1)\cdot(v_3 - v_2)) \\ & f_0(x - v_1t_1, v_1)f_0(x - v_2t_1, v_2)f_0(x - v_3t_1, v_3) \\ =& 2\pi\int dv_*\int dv'_*\int dv'\;\widehat{\phi}(v' - v_*)\,\widehat{\phi}(v' - v)\left\{f_*f'f'_* - ff_*f'\right\} \\ & \delta(v + v_* - v' - v'_*)\,\delta\left(\frac{1}{2}(v^2 + v_*^2 - v'^2 - {v'_*}^2)\right). \end{aligned} \tag{6.20}$$

In the similar way we compute the limit

$$\lim_{\varepsilon\to 0}\left(\mathcal{V}^{\pi^{-1},+}_{3,\varepsilon} + \mathcal{V}^{\pi,+}_{3,\varepsilon}\right) = \mathcal{V}^{\pi}_{3},$$

whose inverse Fourier transform admits the value

$$\begin{aligned}\mathcal{V}_3^\pi(x,v) =&2\pi \int dv_1 \int dv_2 \int dv_3\, \widehat{\phi}(v_2-v_1)\, \widehat{\phi}(v_3-v_2)\\ &[\delta(v-v_2)-\delta(v-v_1)]\, \delta((v_2-v_1)\cdot(v_3-v_2))\\ &f_0(x-v_1t_1,v_1)f_0(x-v_2t_1,v_2)f_0(x-v_3t_1,v_3)\\ =&2\pi \int dv_* \int dv'_* \int dv'\, \widehat{\phi}(v'-v_*)\, \widehat{\phi}(v'-v)\{ff'f'_*-ff_*f'_*\}\\ &\delta(v+v_*-v'-v'_*)\, \delta\left(\frac{1}{2}(v^2+v_*^2-v'^2-{v'_*}^2)\right).\end{aligned} \tag{6.21}$$

There remains to sum up the contributions of the terms $\mathcal{W}_3^\pi$ and $\mathcal{V}_3^\pi$. It gives, after some computations using the exchange of variables $v' \leftrightarrow v'_*$, the missing cross term

$$\begin{aligned}(\mathcal{W}_3^\pi+\mathcal{V}_3^\pi)(x,v) = 2\pi \int dv_* \int dv' \int dv'_*\, [(f+f_*)f'f'_* - (f'+f'_*)ff_*]\\ \delta(v+v_*-v'-v'_*)\, \delta\left(\frac{1}{2}(v^2+v_*^2-v'^2-{v'_*}^2)\right)\, \widehat{\phi}(v'-v)\, \widehat{\phi}(v'-v_*).\end{aligned}$$

This completes the proof of the theorem.

7. Concluding Remarks

It is well known that other possible scaling lead to kinetic equations as well. The most important is the low-density limit (or Boltzmann-Grad limit): it is the regime in which classical rarefied gases are described by the usual Boltzmann equation.

In our grand-canonical formalism it can be introduced in the following way. We do not rescale ϕ which is $O(1)$. On the other hand the rarefaction hypothesis is given by the condition $\varepsilon^2\langle N\rangle = O(1)$. This means that (see (2.24))

$$f_0^\varepsilon(x,v) = O(\varepsilon). \tag{7.1}$$

Under this scaling, the hierarchy becomes (see (2.18))

$$\partial_t f_j^\varepsilon + \sum_{k=1}^{j} v_k\cdot\nabla_{x_k} f_j^\varepsilon = \frac{1}{\varepsilon}T_j^\varepsilon f_j^\varepsilon + \frac{1}{\varepsilon^4}C_{j+1}^\varepsilon f_{j+1}^\varepsilon. \tag{7.2}$$

Rescaling the correlation functions by defining:

$$\bar{f}_j^\varepsilon = \varepsilon^{-j} f_j^\varepsilon, \tag{7.3}$$

we arrive at the hierarchy

$$\partial_t \bar{f}_j^\varepsilon + \sum_{k=1}^{j} v_k\cdot\nabla_{x_k} \bar{f}_j^\varepsilon = \frac{1}{\varepsilon}T_j^\varepsilon \bar{f}_j^\varepsilon + \frac{1}{\varepsilon^3}C_{j+1}^\varepsilon \bar{f}_{j+1}^\varepsilon, \tag{7.4}$$

with a fixed initial datum of $O(1)$. It is now clear that the terms CC are vanishing in the limit $\varepsilon \to 0$ and the statistical correlations are lost. On the other hand many terms of the type $CT\dots T$ are finite in the limit. It turns out that the sum of these terms lead to the Born series expansion of the cross section. The underlying series actually converges provided the potential ϕ is small. This task is performed in the case of the Maxell-Boltzmann statistic in Ref. [3] by the authors. Here, a difficult point lies in the identification of the cross section as the Born series expansion of quantum scattering, a task which is achieved using an original identity derived in [6].

Another comment is in order. The U-U equation has been partially derived whenever f_0 is the Wigner transform of a one-particle quasi-free state. As shown in Appendix, a sufficient condition for the explicit construction of such a state is a small value of the activity z. On the other hand the U-U equation for Bosons makes sense also for more general initial conditions describing states with large activity. It seems very interesting to understand whether the U-U dynamics of such states make sense from a physical point of view and whether it can describe dynamical condensation phenomena.

Appendix: Quasi-free states for Bosons

Let r be a one-particle state i.e. a self-adjoint positive operator whose kernel is denoted by $r(x,y)$. We want to construct a state which is compatible with the B-E statistics and with a given average particle number.

Let σ_n be a n-particle completely symmetric state given by

$$\sigma_n(X_n, Y_n) = \sum_{\pi \in \mathcal{P}_n} r(x_1, y_{\pi(1)}) \dots r(x_n, y_{\pi(n)}). \tag{A.1}$$

The state

$$\sigma^z = \frac{1}{\Xi(z)} \bigoplus_{n=0}^{\infty} \frac{z^n}{n!} \sigma_n, \tag{A.2}$$

where

$$\Xi(z) = \sum_{n \geq 0} \frac{z^n}{n!} \operatorname{Tr} \sigma_n, \tag{A.3}$$

is a normalized state for Bosons and

$$\langle N \rangle = \operatorname{Tr}[\sigma^z N] = z \frac{d}{dz} \log \Xi. \tag{A.4}$$

We now compute the partition function $\Xi(z)$ (see [9]):

$$\Xi(z) = \sum_{n \geq 0} \frac{z^n}{n!} \int dx_1 \dots dx_n \sum_{\pi \in \mathcal{P}_n} r(x_1, x_{\pi(1)}) \dots r(x_n, x_{\pi(n)}). \tag{A.5}$$

Given π, let $\alpha_1, \dots, \alpha_n$, be non negative integers, α_j denoting the number of cycles of length j in π. Clearly

$$\sum_{j=1}^{n} j \alpha_j = n. \tag{A.6}$$

Given the sequence $\alpha_1, \dots, \alpha_n$,

$$\int dx_1 \dots dx_n \; r(x_1, x_{\pi(1)}) \dots r(x_n, x_{\pi(n)}) = \prod_{j=1}^{n} \left(\mathrm{Tr}\, r^j \right)^{\alpha_j}. \tag{A.7}$$

The number of permutations associated to a given sequence $\alpha_1, \dots, \alpha_n$ is

$$n! \prod_{j=1}^{n} \frac{1}{\alpha_j! j^{\alpha_j}}. \tag{A.8}$$

Hence

$$\begin{aligned}\Xi(z) &= \sum_{n\geq 0} \sum_{\substack{\alpha_1,\dots,\alpha_n \geq 0 \\ \sum j\alpha_j = n}} \prod_{j=1}^{n} \frac{\left(\mathrm{Tr}\, r^j\right)^{\alpha_j} (z^j)^{\alpha_j}}{\alpha_j! j^{\alpha_j}} \\ &= \sum_{n\geq 0}\sum_{s\geq 0} \frac{1}{s!} \sum_{\substack{j_1,\dots,j_s \geq 1 \\ \sum j_k = n}} \prod_{\ell=1}^{s} \frac{\mathrm{Tr}\, r^{j_\ell} z^{j_\ell}}{j_\ell} = \exp\left[\sum_{j\geq 1} \frac{\mathrm{Tr}\, r^j z^j}{j} \right].\end{aligned} \tag{A.9}$$

In the second equality s denotes the number of actual cycles in each permutation and $j_1, \dots, j_s$ are the lenghts of the cycles. The last sum is convergent for z sufficiently small (away from Bose condensation region). Then, by (A.4) and (A.9)

$$\langle N \rangle = \sum_{j\geq 1} \mathrm{Tr}\,(rz)^j = z + o(z), \tag{A.10}$$

for z small.

The RDM's according to (2.5) are:

$$\rho_j(X_j, Y_j) = \frac{1}{\Xi(z)} \sum_{n\geq 0} \frac{(n+j)!}{n!} \int dZ_n \; \frac{z^{n+j}}{(n+j)!} \sigma_{j+n}(X_j, Z_n; Y_j, Z_n). \tag{A.11}$$

Therefore we have:

$$\rho_j(X_j, Y_j) = \frac{1}{\Xi(z)} \sum_{n\geq j} \frac{z^n}{(n-j)!} \sum_{\pi\in\mathcal{P}_n} \int dz_{j+1} \cdots \int dz_n \; r(x_1, \xi_{\pi(1)}) \dots r(z_n, \xi_{\pi(n)}), \tag{A.12}$$

where $\xi = (Y_j, Z_{n-j})$. Hence

$$\begin{aligned}\rho_j(X_j, Y_j) = \frac{1}{\Xi(z)} \sum_{n\geq j} \frac{z^n}{(n-j)!} \sum_{s=0}^{n-j} \binom{n-j}{s} \sum_{\pi'\in\mathcal{P}_s} \int dZ_s \; r(z_1, z_{\pi'(1)}) \dots r(z_s, z_{\pi'(s)}) \\ (n-j-s)! \sum_{\pi\in\mathcal{P}_j} \sum_{\substack{k_1,\dots,\kappa_j \geq 1 \\ \sum k_j = n-s}} \prod_{i=1}^{j} r^{k_1}(x_i, y_{\pi(i)})\end{aligned} \tag{A.13}$$

with $r^k(x,y)$ the kernel of r^k.

Since

$$\binom{n-j}{s}\frac{(n-j-s)!}{(n-j)!}=\frac{1}{s!},\qquad z^n=z^s z^{n-s}=z^s z^{\Sigma k_\ell},\tag{A.14}$$

we obtain

$$\rho_j(X_j,Y_j)=\frac{1}{\Xi(z)}\sum_{s\geq 0}\frac{z^s}{s!}\mathrm{Tr}\,\sigma_s\sum_{\pi\in\mathcal{P}_j}\prod_{\ell=1}^{j}\sum_{k_\ell=1}^{\infty}z^{k_\ell}r^{k_\ell}(x_\ell,y_{\pi(\ell)}).\tag{A.15}$$

Defining the one-particle operator

$$r_z=\sum_{k\geq 1}z^k r^k=\frac{zr}{1-zr},\tag{A.16}$$

for z sufficiently small, we arrive at

$$\rho_j(X_j,Y_j)=\sum_{\pi\in\mathcal{P}_j}r_z(x_1,y_{\pi(1)})\ldots r_z(x_j,y_{\pi(j)}),\tag{A.17}$$

that is the characterization of the quasi-free state in terms of RDM.

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