Moments' analysis in homogeneous Markov reward models

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Abstract

We analyze the moments of the accumulated reward over the interval $(0, t)$ in a continuous-time Markov chain. We develop a numerical procedure to compute efficiently the normalized moments using the uniformization technique. Our algorithm involves auxiliary quantities whose convergence is analyzed, and for which we provide a probabilistic interpretation.

Keywords: Markov models, accumulated reward, performability, uniformization.

1 Introduction

In the dependability analysis of repairable computing systems, there is an increasing interest in evaluating transient measures, and in particular, the accumulated reward over a given period. This measure is also known as a performability measure that takes into account both the performance of a system and its reliability. The state of the system is modeled by an irreducible continuous-time homogeneous Markov chain $X = \{X_t, t \geq 0\}$, over a finite state space S. To each state $i \in S$, is associated a reward rate $f(i)$, which is assumed to be a nonnegative real number, without any loss of generality. The accumulated reward $Y(t)$ over interval $(0, t)$ averaged over t and normalized by the maximum reward rate $f = \max\{f(j), j \in S\}$ is defined, for every $t > 0$, by

$$Y(t) = \frac{1}{ft} \int_0^t f(X_s)ds.$$

The distribution of accumulated reward has been considered in several papers ([9], [6], [7], [2], [14], [10], [4], [8], [5], [3]). This paper deals with the study and the evaluation of the moments of accumulated reward. As we shall see there is no need to evaluate the distribution if only some of the moments are required. Their evaluation has been studied in [11] where an algorithm has been proposed to compute them. In paper [11] the moments considered are not averaged over $(0, t)$ so that they increase to infinity when t goes to infinity. The first moment of $Y(t)$ has been analyzed in [13], where a stationary regime detection has been developed to improve its computational time.

In this paper, we consider the moments averaged over $(0,t)$. They are studied in Section 2 where we propose a first computational algorithm. Our approach, based on the uniformization technique, only involves numbers in $[0,1]$, which provides numerical stability. Besides, our method involves auxiliary quantities, defined as $U(n,r)$ in the sequel, whose convergence is studied in Section 3 which allows us to develop a new procedure to compute the moments more efficiently. The $U(n,r)$'s are considered in Section 4 and they are characterized using a probabilistic interpretation. Section 5 is devoted to a numerical example.

2 Moments analysis

We denote respectively by α and Q the initial distribution and the infinitesimal generator of the irreducible continuous-time homogeneous Markov process $X = \{X_t, t \geq 0\}$ over the finite state space S. For $i \in S$ and $r \geq 0$, we denote by $m_i^{(r)}(t)$ the r-th moment of the accumulated reward $Y(t)$ over $(0,t)$, given that the initial state of X is equal to i, i.e.

$$m_i^{(r)}(t) = E(Y(t)^r | X_0 = i).$$

We denote by $m^{(r)}(t)$ the column vector with the i-th element equal to $m_i^{(r)}(t)$. Clearly, we have $m^{(0)}(t) = \mathbb{1}$, where $\mathbb{1}$ is the column vector with all entries equal to 1. The r-th moment of the accumulated reward over interval $(0,t)$ is thus given, for $r \geq 0$, by

$$E(Y(t)^r) = \alpha m^{(r)}(t).$$

We denote by D the diagonal matrix defined by $D = \mathrm{diag}(d(i))$, where $d(i) = f(i)/f$. The column vectors $m^{(r)}(t)$ can be obtained recursively from the following result.

Theorem 1 *For every $r \geq 1$ and $t > 0$, we have*

$$m^{(r)}(t) = \frac{r}{t^r} \int_0^t u^{r-1} e^{Q(t-u)} D m^{(r-1)}(u) du. \tag{1}$$

Proof. Let $g_r(s)$ be the random variable defined, for $r \geq 0$ and $0 \leq s \leq t$, by

$$g_r(s) = \left(\int_s^t f(X_u) du \right)^r.$$

Differentiating with respect to s, we get

$$g_r'(s) = -r f(X_s) \left(\int_s^t f(X_u) du \right)^{r-1} = -r f(X_s) g_{r-1}(s),$$

which is equivalent to

$$g_r(s) = r \int_s^t f(X_u) g_{r-1}(u) du. \tag{2}$$

Note that from the homogeneity of the Markov chain X, we have, for $r \geq 0$ and $0 \leq s \leq t$,

$$\begin{aligned}
E(g_r(s)|X_s = j) &= E\left(\left(\int_s^t f(X_u)du\right)^r \Big| X_s = j\right) \\
&= E\left(\left(\int_0^{t-s} f(X_u)du\right)^r \Big| X_0 = j\right) \\
&= f^r(t-s)^r m_j^{(r)}(t-s). \qquad (3)
\end{aligned}$$

Taking the expectation in (2), we get, using the Fubini theorem

$$\begin{aligned}
E(g_r(s)|X_0 = i) &= r\int_s^t E(f(X_u)g_{r-1}(u)|X_0 = i)du \\
&= r\int_s^t \sum_{j\in S} E(f(X_u)g_{r-1}(u)1_{\{X_u=j\}}|X_0 = i)du \\
&= r\int_s^t \sum_{j\in S} f(j)E(g_{r-1}(u)|X_u = j, X_0 = i)\Pr\{X_u = j|X_0 = i\}du \\
&= r\int_s^t \sum_{j\in S} f(j)E(g_{r-1}(u)|X_u = j)\Pr\{X_u = j|X_0 = i\}du \\
&= rf^{r-1}\int_s^t \sum_{j\in S} f(j)(t-u)^{r-1}m_j^{(r-1)}(t-u)e_{i,j}^{Qu}du,
\end{aligned}$$

where the fourth equality follows from the Markov property and the fifth is due to homogeneity property (3). Taking $s = 0$, we obtain

$$m_i^{(r)}(t) = \frac{r}{t^r}\int_0^t \sum_{j\in S} d(j)(t-u)^{r-1}m_j^{(r-1)}(t-u)e_{i,j}^{Qu}du,$$

which can be written, in matrix notation

$$m^{(r)}(t) = \frac{r}{t^r}\int_0^t (t-u)^{r-1}e^{Qu}Dm^{(r-1)}(t-u)du,$$

or, by a variable change,

$$m^{(r)}(t) = \frac{r}{t^r}\int_0^t u^{r-1}e^{Q(t-u)}Dm^{(r-1)}(u)du,$$

which completes the proof. ∎

Differentiating relation (1) with respect to t, we get

$$rm^{(r)}(t) + t\frac{dm^{(r)}(t)}{dt} = tQm^{(r)}(t) + rDm^{(r-1)}(t). \qquad (4)$$

We now make use of the uniformization technique [12]. We introduce the uniformized Markov chain $Z = \{Z_n, n \geq 0\}$ associated to the Markov chain X, which is characterized by its uniformization rate ν and by its transition probability matrix P_ν. The uniformization rate ν verifies $\nu \geq \max\{-Q(i,i), i \in S\}$ and matrix P_ν is related to the infinitesimal generator Q by $P_\nu = I + Q/\nu$, where I denotes the identity matrix. To simplify notation, we write in the sequel P for P_ν. The number of transitions during the interval $(0,t)$, which we denote by N_t, is a Poisson process with rate ν. Since $Q = -\nu(I - P)$, relation (4) can be written as

$$rm^{(r)}(t) + t\frac{dm^{(r)}(t)}{dt} = -\nu t m^{(r)}(t) + \nu t P m^{(r)}(t) + rDm^{(r-1)}(t). \tag{5}$$

In the following theorem, we determine the sequence of column vectors $U(n,r)$ so that the solution to (5) has the form of the series (6).

Theorem 2 *For every $t \geq 0$, we have*

$$m^{(r)}(t) = \sum_{n=0}^{\infty} e^{-\nu t}\frac{(\nu t)^n}{n!}U(n,r), \tag{6}$$

where the column vectors $U(n,r) = (U_i(n,r), i \in S)$ are given by

$$U(0,r) = D^r \mathbb{1} \text{ for } r \geq 0, \tag{7}$$

$$U(n,0) = \mathbb{1} \text{ for } n \geq 0, \tag{8}$$

and, for $n, r \geq 1$,

$$U(n,r) = \frac{n}{n+r}PU(n-1,r) + \frac{r}{n+r}DU(n,r-1). \tag{9}$$

Proof. Differentiating expression (6) with respect to t, we get

$$t\frac{dm^{(r)}(t)}{dt} = -\nu t m^{(r)}(t) + \nu t \sum_{n=0}^{\infty} e^{-\nu t}\frac{(\nu t)^n}{n!}U(n+1,r) = -\nu t m^{(r)}(t) + \sum_{n=0}^{\infty} e^{-\nu t}\frac{(\nu t)^n}{n!}nU(n,r).$$

Replacing this expression together with expression (6) in equation (5), we obtain, for every $t \geq 0$,

$$\begin{aligned}\sum_{n=0}^{\infty} e^{-\nu t}\frac{(\nu t)^n}{n!}(n+r)U(n,r) &= \sum_{n=0}^{\infty} e^{-\nu t}\frac{(\nu t)^{n+1}}{n!}PU(n,r) + \sum_{n=0}^{\infty} e^{-\nu t}\frac{(\nu t)^n}{n!}rDU(n,r-1) \\ &= \sum_{n=0}^{\infty} e^{-\nu t}\frac{(\nu t)^n}{n!}(nPU(n-1,r) + rDU(n,r-1)).\end{aligned}$$

Note that for $n = 0$, it is not necessary to define $U(n-1, r)$ since, in that case, we have $nPU(n-1, r) = 0$. The equality holds for every $t \geq 0$, so we get, for every $n \geq 0$,

$$U(n,r) = \frac{n}{n+r} PU(n-1,r) + \frac{r}{n+r} DU(n,r-1).$$

The initial condition easily follows from the fact that the paths of the Markov chain X are supposed right continuous at $t = 0$. We thus have, for every $r \geq 0$

$$\lim_{t \longrightarrow 0} m^{(r)}(t) = D^r \mathbb{1}.$$

Taking $t = 0$ in relation (6), we get

$$U(0,r) = D^r \mathbb{1}.$$

Since $m^{(0)}(t) = \mathbb{1}$, we obtain

$$U(n,0) = \mathbb{1}, \text{ for every } n \geq 0,$$

which completes the proof. ■

2.1 The uniformization method

The computation of the moments of the normalized accumulated reward $E(Y(t)^r)$ is based on the following relation, which can be easily obtained from relation (6),

$$E(Y(t)^r) = \sum_{n=0}^{\infty} e^{-\nu t} \frac{(\nu t)^n}{n!} u(n,r),$$

where $u(n, r) = \alpha U(n, r)$. Note that since matrix P is stochastic and since diagonal matrix D has its entries in the interval $[0, 1]$, we easily obtain from relations (7), (8) and (9) that

$$0 \leq U_i(n,r) \leq 1, \text{ and thus } 0 \leq u(n,r) \leq 1.$$

Let $\varepsilon > 0$ be a given specified error tolerance and N be defined as

$$N = \min \left\{ n \in \mathbb{N} \;\middle|\; \sum_{j=0}^{n} e^{-\nu t} \frac{(\nu t)^j}{j!} \geq 1 - \varepsilon \right\}. \tag{10}$$

Then we obtain

$$E(Y(t)^r) = \sum_{n=0}^{N} e^{-\nu t} \frac{(\nu t)^n}{n!} u(n,r) + e(N),$$

where the remainder of the series $e(N)$ verifies

$$0 \leq e(N) = \sum_{n=N+1}^{\infty} e^{-\nu t} \frac{(\nu t)^n}{n!} u(n,r) \leq \sum_{n=N+1}^{\infty} e^{-\nu t} \frac{(\nu t)^n}{n!} = 1 - \sum_{n=0}^{N} e^{-\nu t} \frac{(\nu t)^n}{n!} \leq \varepsilon.$$

The computation of integer N can be made without any numerical problems even for large values of νt by using the method described in [1].

The truncation level N is in fact a function of t, say $N(t)$. For a fixed value of ε, $N(t)$ is an increasing function of t. It follows that if we want to compute the R first moments $E(Y(t)^r)$ for J distinct values of t, denoted by $t_1 < \cdots < t_J$, we only need to compute $u(n, r)$ for $n = 1, \ldots, N(t_J)$ since the values of $u(n, r)$ are independent of the parameter t.

The pseudo code of the uniformization method can then be written as follows.

input : ε, $t_1 < \cdots < t_J$, R
output : $E(Y(t_j)^r)$ for $j = 1, \ldots, J$ and $r = 1, \ldots, R$
Compute N from Relation (10) with $t = t_J$
for $r = 0$ **to** R **do**
    $U(0, r) = D^r \mathbb{1}$
    $u(0, r) = \alpha U(0, r)$
endfor
for $n = 1$ **to** N **do**
    $U(n, 0) = \mathbb{1}$
    for $r = 1$ **to** R **do**
        $U(n, r) = \dfrac{n}{n + r} PU(n - 1, r) + \dfrac{r}{n + r} DU(n, r - 1)$
        $u(n, r) = \alpha U(n, r)$
    endfor
endfor
for $j = 1$ **to** J **do**
    for $r = 1$ **to** R **do**
        $E(Y(t_j)^r) = \displaystyle\sum_{n=0}^{N} e^{-\nu t_j} \frac{(\nu t_j)^n}{n!} u(n, r)$
    endfor
endfor

Table 1: Algorithm for the computation of $E(Y(t)^r)$.

3 Stationarity detection

We consider in this section the sequence of column vectors $U(n, r)$ and we show that it converges when n tends to infinity. This allows us to stop the computation of the $U(n, r)$ as soon as they are close enough to their limit. In the following theorem, we express the vectors $U(n, r)$ recursively over index r. This recursive expression will be used in Theorem 4 to prove the convergence of the sequence $U(n, r)$.

Theorem 3 *For every $n \geq 0$ and $r \geq 1$,*

$$U(n,r) = \frac{1}{\binom{n+r}{r}} \sum_{\ell=0}^{n} \binom{\ell+r-1}{r-1} P^{n-\ell} D U(\ell, r-1). \tag{11}$$

Proof. We prove this relation by recurrence over index n. For $n = 0$, this relation gives $U(0,r) = DU(0,r-1)$. This leads to $U(0,r) = D^r U(0,0) = D^r \mathbb{1}$, which is relation (7).

Suppose that relation (11) is true for integer n. We have to show that

$$U(n+1,r) = \frac{1}{\binom{n+1+r}{r}} \sum_{\ell=0}^{n+1} \binom{\ell+r-1}{r-1} P^{n+1-\ell} D U(\ell, r-1).$$

From relation (9) and using the recurrence hypothesis, we have

$$\begin{aligned}
U(n+1,r) &= \frac{n+1}{n+1+r} PU(n,r) + \frac{r}{n+1+r} DU(n+1, r-1) \\
&= \frac{n+1}{(n+1+r)\binom{n+r}{r}} \sum_{\ell=0}^{n} \binom{\ell+r-1}{r-1} P^{n+1-\ell} DU(\ell, r-1) \\
&\quad + \frac{r}{n+1+r} DU(n+1, r-1) \\
&= \frac{1}{\binom{n+1+r}{r}} \sum_{\ell=0}^{n} \binom{\ell+r-1}{r-1} P^{n+1-\ell} DU(\ell, r-1) \\
&\quad + \frac{r}{n+1+r} DU(n+1, r-1) \\
&= \frac{1}{\binom{n+1+r}{r}} \sum_{\ell=0}^{n+1} \binom{\ell+r-1}{r-1} P^{n+1-\ell} DU(\ell, r-1),
\end{aligned}$$

which is the desired result. ∎

We denote by π the stationary probability distribution of the Markov chain X. This row vector satisfies $\pi A = 0$ or equivalently $\pi = \pi P$.

Theorem 4 *For every $r \geq 0$, we have*

$$\lim_{n \longrightarrow \infty} U(n,r) = (\pi D \mathbb{1})^r \mathbb{1}. \tag{12}$$

Proof. We proceed by recurrence over integer r. The result is true for $r = 0$, since we have $U(n,0) = \mathbb{1}$.

Suppose the result is true for integer $r-1$, i.e. suppose that $\lim_{n\longrightarrow\infty} U(n, r-1) = (\pi D\mathbb{1})^{r-1}\mathbb{1}$.

Let us define

$$\beta(n,\ell) = \frac{\binom{\ell+r-1}{r-1}}{\binom{n+r}{r}}, \quad H(\ell) = P^\ell D, \quad U(\ell) = U(\ell, r-1), \text{ and } V(n) = U(n,r).$$

We have $\beta(n,\ell) \geq 0$ and

$$\sum_{\ell=0}^{n} \beta(n,\ell) = 1,$$

and, since the sequence $\beta(n,\ell)$ is increasing with ℓ,

$$\lim_{n\longrightarrow\infty} \sup_{0\leq\ell\leq n} \beta(n,\ell) = \lim_{n\longrightarrow\infty} \beta(n,n) = \lim_{n\longrightarrow\infty} \frac{r+1}{n+r+1} = 0.$$

To ensure the convergence of the sequence of matrices P^n, we require either that the output rates $-Q(i,i)$, $i \in S$, of the Markov chain X are not all equal or that the uniformization rate ν is such that $\nu > \max\{-Q(i,i), i \in S\}$. This guarantees that the transition probability matrix P is aperiodic which means that it is ergodic and thus, we have

$$H = \lim_{\ell\longrightarrow\infty} H(\ell) = \Pi D,$$

where Π is the matrix with all its lines equal to π. By using the recurrence hypothesis, we get

$$U = \lim_{\ell\longrightarrow\infty} U(\ell) = (\pi D\mathbb{1})^{r-1}\mathbb{1} = (\Pi D)^{r-1}\mathbb{1}.$$

From relation (11) and by using Lemma 10, we get

$$\lim_{n\longrightarrow\infty} U(n,r) = \lim_{n\longrightarrow\infty} V(n) = \Pi D(\Pi D)^{r-1}\mathbb{1} = (\Pi D)^r\mathbb{1} = (\pi D\mathbb{1})^r\mathbb{1},$$

which completes the proof. ∎

Using this result, the algorithm of Table 1 can be modified as follows to take into account the convergence of the sequence $U(n,r)$ and thus of the sequence $u(n,r)$. In order to do that we consider that we make the following assumption, which is satisfied in practice.

$$\text{if } \exists\, K < N \text{ s. t. } \max_{r\leq R} |u(K,r) - (\pi D\mathbb{1})^r| \leq \varepsilon \text{ then } \forall n \geq K,\ \max_{r\leq R} |u(n,r) - (\pi D\mathbb{1})^r| \leq \varepsilon.$$

If such a K does not exist, there is no stationarity detection and we come back to the previous algorithm. Such a situation means that the value of N is not large and neither is the value of t_J. If such a K exists, we have

$$E(Y(t)^r) = \sum_{n=0}^{K} e^{-\nu t}\frac{(\nu t)^n}{n!} u(n,r) + (\pi D\mathbb{1})^r \left(1 - \sum_{n=0}^{K} e^{-\nu t}\frac{(\nu t)^n}{n!}\right) + e_1(K),$$

where the remainder of the series $e_1(K)$ verifies

$$|e_1(K)| = \left| \sum_{n=K+1}^{\infty} e^{-\nu t} \frac{(\nu t)^n}{n!} \left(u(n,r) - (\pi D \mathbb{1})^r\right) \right| \leq \sum_{n=K+1}^{\infty} e^{-\nu t} \frac{(\nu t)^n}{n!} \left|u(n,r) - (\pi D \mathbb{1})^r\right| \leq \varepsilon.$$

We obtain the following algorithm, in which we suppose that $\pi D \mathbb{1}$ has been computed beforehand.

input : ε, $t_1 < \cdots < t_J$, R
output : $E(Y(t_j)^r)$ for $j = 1, \ldots, J$ and $r = 1, \ldots, R$
Compute N from Relation (10) with $t = t_J$
for $r = 0$ **to** R **do**
  $U(0,r) = D^r \mathbb{1}$
  $u(0,r) = \alpha U(0,r)$
endfor
$K = N$
for $n = 1$ **to** N **do**
  $U(n,0) = \mathbb{1}$
  for $r = 1$ **to** R **do**
    $U(n,r) = \dfrac{n}{n+r} P U(n-1,r) + \dfrac{r}{n+r} D U(n,r-1)$
    $u(n,r) = \alpha U(n,r)$
  endfor
  if $|u(n,r) - (\pi D \mathbb{1})^r| \leq \varepsilon$ **for every** $r = 1, \ldots, R$ **then**
    K = n
    break
  endif
endfor
for $j = 1$ **to** J **do**
  if $K = N$ **then**
    for $r = 1$ **to** R **do**

$$E(Y(t_j)^r) = \sum_{n=0}^{N} e^{-\nu t_j} \frac{(\nu t_j)^n}{n!} u(n,r)$$

    endfor
  endif
  if $K < N$ **then**
    for $r = 1$ **to** R **do**

$$E(Y(t_j)^r) = \sum_{n=0}^{K} e^{-\nu t_j} \frac{(\nu t_j)^n}{n!} u(n,r) + (\pi D \mathbb{1})^r \left(1 - \sum_{n=0}^{K} e^{-\nu t_j} \frac{(\nu t_j)^n}{n!}\right)$$

    endfor
  endif
endfor

Table 2: Algorithm for the computation of $E(Y(t)^r)$ using stationarity detection.

The following section is devoted to probabilistic interpretations of the sequence $U(n,r)$.

4 Probabilistic interpretation

In the following theorem, we obtain an explicit expression of the column vectors $U(n,r)$ using the matrices P and D.

Theorem 5 *For every $n \geq 0$ and $r \geq 1$, we have*

$$U(n,r) = \frac{1}{\binom{n+r}{r}} \sum_{\ell_1=0}^{n} P^{n-\ell_1} D \sum_{\ell_2=0}^{\ell_1} P^{\ell_1-\ell_2} D \cdots \sum_{\ell_{r-1}=0}^{\ell_{r-2}} P^{\ell_{r-2}-\ell_{r-1}} D \sum_{\ell_r=0}^{\ell_{r-1}} P^{\ell_r} D \mathbb{1}. \tag{13}$$

Proof. For $r = 1$, the expression (13) reads

$$U(n,1) = \frac{1}{n+1} \sum_{\ell_1=0}^{n} P^{\ell_1} D \mathbb{1},$$

which is obtained by (11) and (8).

Suppose that relation (13) is true for integer $r-1$, i.e. suppose that, for every $\ell_1 \geq 0$,

$$U(\ell_1, r-1) = \frac{1}{\binom{\ell_1+r-1}{\ell_1-1}} \sum_{\ell_2=0}^{\ell_1} P^{\ell_1-\ell_2} D \sum_{\ell_3=0}^{\ell_2} P^{\ell_2-\ell_3} D \cdots \sum_{\ell_{r-1}=0}^{\ell_{r-2}} P^{\ell_{r-2}-\ell_{r-1}} D \sum_{\ell_r=0}^{\ell_{r-1}} P^{\ell_r} D \mathbb{1}.$$

Using relation (11), we obtain directly (13). ■

The first interpretation of the $U(n,r)$ we obtain is given by the following theorem.

Theorem 6 *For every $n \geq 0$, $r \geq 1$ and $i \in S$, we have*

$$U_i(n,r) = \frac{1}{\binom{n+r}{r}} E\left(\sum_{\ell_1=0}^{n} d(Z_{\ell_1}) \sum_{\ell_2=\ell_1}^{n} d(Z_{\ell_2}) \cdots \sum_{\ell_{r-1}=\ell_{r-2}}^{n} d(Z_{\ell_{r-1}}) \sum_{\ell_r=\ell_{r-1}}^{n} d(Z_{\ell_r}) \middle| Z_0 = i \right). \tag{14}$$

Proof. The relation is true for $r=1$ since the right hand side becomes

$$\begin{aligned}
\frac{1}{n+1}E\left(\sum_{\ell_1=0}^{n} d(Z_{\ell_1})\Big| Z_0=i\right) &= \frac{1}{n+1}\sum_{\ell_1=0}^{n} E\left(d(Z_{\ell_1})|Z_0=i\right)\\
&= \frac{1}{n+1}\sum_{\ell_1=0}^{n}\sum_{j\in S} P_{i,j}^{\ell_1} d(j)\\
&= \frac{1}{n+1}\sum_{\ell_1=0}^{n}(P^{\ell_1}D\mathbb{1})_i\\
&= U_i(n,1).
\end{aligned}$$

Suppose that relation (14) is true for integer $r-1$, i.e.

$$U_i(n,r-1) = \frac{E\left(\sum_{\ell_1=0}^{n} d(Z_{\ell_1})\sum_{\ell_2=\ell_1}^{n} d(Z_{\ell_2})\cdots\sum_{\ell_{r-2}=\ell_{r-3}}^{n} d(Z_{\ell_{r-2}})\sum_{\ell_{r-1}=\ell_{r-2}}^{n} d(Z_{\ell_{r-1}})\Big| Z_0=i\right)}{\binom{n+r-1}{r-1}}.$$

We then have

$$E\left(\sum_{\ell_1=0}^{n} d(Z_{\ell_1})\sum_{\ell_2=\ell_1}^{n} d(Z_{\ell_2})\cdots\sum_{\ell_{r-1}=\ell_{r-2}}^{n} d(Z_{\ell_{r-1}})\sum_{\ell_r=\ell_{r-1}}^{n} d(Z_{\ell_r})\Big| Z_0=i\right)$$

$$
\begin{aligned}
&= \sum_{j\in S} E\left(\sum_{\ell_1=0}^{n} d(Z_{\ell_1}) \sum_{\ell_2=\ell_1}^{n} d(Z_{\ell_2}) \cdots \sum_{\ell_{r-1}=\ell_{r-2}}^{n} d(Z_{\ell_{r-1}}) \sum_{\ell_r=\ell_{r-1}}^{n} d(Z_{\ell_r}) \Big| Z_{\ell_1}=j, Z_0=i\right) P_{i,j}^{\ell_1} \\
&= \sum_{\ell_1=0}^{n} \sum_{j\in S} P_{i,j}^{\ell_1} d(j) E\left(\sum_{\ell_2=\ell_1}^{n} d(Z_{\ell_2}) \cdots \sum_{\ell_{r-1}=\ell_{r-2}}^{n} d(Z_{\ell_{r-1}}) \sum_{\ell_r=\ell_{r-1}}^{n} d(Z_{\ell_r}) \Big| Z_{\ell_1}=j, Z_0=i\right) \\
&= \sum_{\ell_1=0}^{n} \sum_{j\in S} P_{i,j}^{\ell_1} d(j) E\left(\sum_{\ell_2=\ell_1}^{n} d(Z_{\ell_2}) \cdots \sum_{\ell_{r-1}=\ell_{r-2}}^{n} d(Z_{\ell_{r-1}}) \sum_{\ell_r=\ell_{r-1}}^{n} d(Z_{\ell_r}) \Big| Z_{\ell_1}=j\right) \\
&= \sum_{\ell_1=0}^{n} \sum_{j\in S} P_{i,j}^{\ell_1} d(j) E\left(\sum_{\ell_2=0}^{n-\ell_1} d(Z_{\ell_2}) \cdots \sum_{\ell_{r-1}=\ell_{r-2}}^{n-\ell_1} d(Z_{\ell_{r-1}}) \sum_{\ell_r=\ell_{r-1}}^{n-\ell_1} d(Z_{\ell_r}) \Big| Z_0=j\right) \\
&= \sum_{\ell_1=0}^{n} \sum_{j\in S} P_{i,j}^{\ell_1} d(j) \binom{n-\ell_1+r-1}{r-1} U_j(n-\ell_1, r-1) \\
&= \sum_{\ell_1=0}^{n} \binom{\ell_1+r-1}{r-1} \sum_{j\in S} P_{i,j}^{n-\ell_1} d(j) U_j(\ell_1, r-1) \\
&= \sum_{\ell_1=0}^{n} \binom{\ell_1+r-1}{r-1} (P^{n-\ell_1} D U(\ell_1, r-1))_i \\
&= \binom{n+r}{r} U_i(n,r),
\end{aligned}
$$

which completes the proof. Note that the third equality follows from the Markov property, the fourth equality follows from the homogeneity of the Markov chain Z, the fifth one follows from the recurrence hypothesis and the last one is relation (11). ■

Clearly, for $r=1$, $U_i(n,1)$ is the expected reward accumulated, during the n first transitions of Z, starting from state i and averaged by $n+1$. For $r \geq 2$, such a simple interpretation of $U_i(n,r)$ is not at hand. The remainder part of this section is devoted to establishing another way to obtain a probabilistic interpretation of $U_i(n,r)$. The two following lemmas will be used to prove Theorem 9.

Lemma 7 *For every $t \geq 0$, we have*

$$
\int_0^t (t-u)^\ell u^n du = \frac{t^{n+\ell+1}}{(n+\ell+1)\binom{n+\ell}{\ell}}.
$$

Proof. Let $I(n,\ell)$ denote the integral in the right hand side. Using an integration by parts, we obtain

$$
I(n,\ell) = \frac{n}{\ell+1} I(n-1, \ell+1).
$$

This leads to

$$I(n,\ell)=\binom{n+\ell}{\ell}I(0,n+\ell)=\binom{n+\ell}{\ell}\frac{t^{n+\ell+1}}{n+\ell+1},$$

which completes the proof. ■

Let us define $G_i(t,n,r)=E(Y(t)^r|N_t=n,X_0=i)$.

Lemma 8 *For every $t\geq 0$, $n\geq 1$, $r\geq 0$ and $i\in S$, we have*

$$G_i(t,n,r)=\frac{n}{t^{n+r}}\sum_{\ell=0}^{r}\binom{r}{\ell}d(i)^\ell\sum_{j\in S}p_{i,j}\int_0^t (t-u)^\ell u^{n+r-\ell-1}G_j(u,n-1,r-\ell)du.$$

Proof. Let us define the quantity $V_i(t,n,r)=E(Y(t)^r 1_{\{N_t=n\}}|X_0=i)$. Let T_1 be the sojourn time in the initial state. Using a renewal argument, we have

$$\begin{aligned}
&V_i(t,n,r)\\
&= E(Y(t)^r 1_{\{N_t=n\}}|X_0=i)\\
&= \int_0^t\sum_{j\in S}p_{i,j}E(Y(t)^r 1_{\{N_t=n\}}|X_u=j,T_1=u,X_0=i)\nu e^{-\nu u}du\\
&= \int_0^t\sum_{j\in S}p_{i,j}E\left(\left[f_i u+\int_u^t f(X_s)ds\right]^r 1_{\{N_t-N_u=n-1\}}|X_u=j,T_1=u,X_0=i\right)\nu e^{-\nu u}du\\
&= \int_0^t\sum_{j\in S}p_{i,j}E\left(\left[f_i u+\int_u^t f(X_s)ds\right]^r 1_{\{N_t-N_u=n-1\}}\Big|X_u=j\right)\nu e^{-\nu u}du\\
&= \int_0^t\sum_{j\in S}p_{i,j}E\left([f_i u+Y(t-u)]^r 1_{\{N_t-N_u=n-1\}}|X_u=j\right)\nu e^{-\nu u}du\\
&= \int_0^t\sum_{j\in S}p_{i,j}E\left([f_i u+Y(t-u)]^r 1_{\{N_{t-u}=n-1\}}|X_0=j\right)\nu e^{-\nu u}du\\
&= \sum_{\ell=0}^{r}\binom{r}{\ell}f(i)^\ell\sum_{j\in S}p_{i,j}\int_0^t u^\ell E\left(Y(t-u)^{r-\ell}1_{\{N_{t-u}=n-1\}}|X_0=j\right)\nu e^{-\nu u}du\\
&= \sum_{\ell=0}^{r}\binom{r}{\ell}f(i)^\ell\sum_{j\in S}p_{i,j}\int_0^t u^\ell V_j(t-u,n-1,r-\ell)\nu e^{-\nu u}du.
\end{aligned}$$

Unconditioning on the number of transitions during $(0,t)$, we get

$$G_i(t,n,r)=\frac{n!V_i(t,n,r)}{f^r t^r e^{-\nu t}(\nu t)^n},$$

and thus

$$\begin{aligned} G_i(t,n,r) &= \frac{n}{t^{n+r}} \sum_{\ell=0}^{r} \binom{r}{\ell} d(i)^\ell \sum_{j\in S} p_{i,j} \int_0^t u^\ell (t-u)^{n+r-\ell-1} G_j(t-u, n-1, r-\ell) du \\ &= \frac{n}{t^{n+r}} \sum_{\ell=0}^{r} \binom{r}{\ell} d(i)^\ell \sum_{j\in S} p_{i,j} \int_0^t (t-u)^\ell u^{n+r-\ell-1} G_j(u, n-1, r-\ell) du, \end{aligned}$$

which completes the proof. ■

The following theorem gives a clearer interpretation of $U_i(n,r)$ in terms of the normalized accumulated reward $Y(t)$ and of the Poisson process N_t.

Theorem 9 *For every $n \geq 0$, $r \geq 0$, $i \in S$ and $t \geq 0$, we have*

$$U_i(n,r) = E(Y(t)^r | N_t = n, X_0 = i). \tag{15}$$

Proof. Conditioning on N_t, we have

$$E(Y(t)^r | X_0 = i) = \sum_{n=0}^{\infty} e^{-\nu t} \frac{(\nu t)^n}{n!} E(Y(t)^r | N_t = n, X_0 = i),$$

and from (6), we have

$$E(Y(t)^r | X_0 = i) = \sum_{n=0}^{\infty} e^{-\nu t} \frac{(\nu t)^n}{n!} U_i(n,r).$$

So, to prove the theorem, we just have to show that $E(Y(t)^r | N_t = n, X_0 = i)$ is independent of t. We proceed by recurrence over index n.

For $n = 0$, we have $E(Y(t)^r | N_t = 0, X_0 = i) = d(i)^r$ since $Y(t) = f(i)t$ when $N_t = 0$ and $X_0 = i$.

Suppose that $U_i(n-1,r) = E(Y(t)^r | N_t = n-1, X_0 = i)$. Using this hypothesis and Lemma 8, we obtain

$$G_i(t,n,r) = \frac{n}{t^{n+r}} \sum_{\ell=0}^{r} \binom{r}{\ell} d(i)^\ell \sum_{j\in S} p_{i,j} U_j(n-1, r-\ell) \int_0^t (t-u)^\ell u^{n+r-\ell-1} du.$$

Using now Lemma 7, we obtain

$$\begin{aligned} G_i(t,n,r) &= \frac{n}{t^{n+r}} \sum_{\ell=0}^{r} \binom{r}{\ell} d(i)^\ell \sum_{j\in S} p_{i,j} U_j(n-1, r-\ell) \int_0^t (t-u)^\ell u^{n+r-\ell-1} du \\ &= \frac{n}{n+r} \sum_{\ell=0}^{r} \frac{\binom{r}{\ell} d(i)^\ell}{\binom{n-1+r}{\ell}} \sum_{j\in S} p_{i,j} U_j(n-1, r-\ell), \end{aligned}$$

which is independent of t. ■

It is worthwhile noting that the moments of the normalized accumulated reward $Y(t)$, given the number of transitions over $(0, t)$ in the uniformized process, are independent of t.

In the next section, we present some numerical experiments to show the importance of the stationarity detection in the reduction of the computational time of the moments $E(Y(t)^r)$.

5 Numerical example

We consider a fault-tolerant multiprocessor system which consists of n identical processors and b buffer stages. Processors fail independently at rate λ and are repaired singly with rate μ. Buffers stages fail independently at rate γ and are repaired with rate τ. Processor failures cause a graceful degradation of the system and the number of operational processors is decreased by one. The system is in a failed state when all the processors have failed or any of the buffer stages has failed. No additional processor failures are assumed to occur when the system is in a failed state. The model is represented by a Markov process with state transition diagram shown in Figure 1. The state space of the system is $S = \{(i, j); 0 \le i \le n,\ j = 0, 1\}$. The component i of a state (i, j) means that there are i operational processors and the component j is zero if any of the buffer stages has failed, otherwise it is one. It follows that the set U of operational states is $U = \{(i, 1); 1 \le i \le n\}$. The reward structure we choose here is given by $f(i) = 1$ if $i \in U$ and $f(i) = 0$ otherwise. We suppose that the initial state of the system is state $(n, 1)$. The number of processors is fixed to 16, each with a failure rate $\lambda = 0.00006$ per hour and a repair rate $\mu = 0.1666$ per hour. The number of buffer stages is fixed to 1024, each with a failure rate $\gamma = 0.00131$ per hour and a repair rate $\tau = 0.1666$ per hour. The error tolerance is $\varepsilon = 0.00001$.

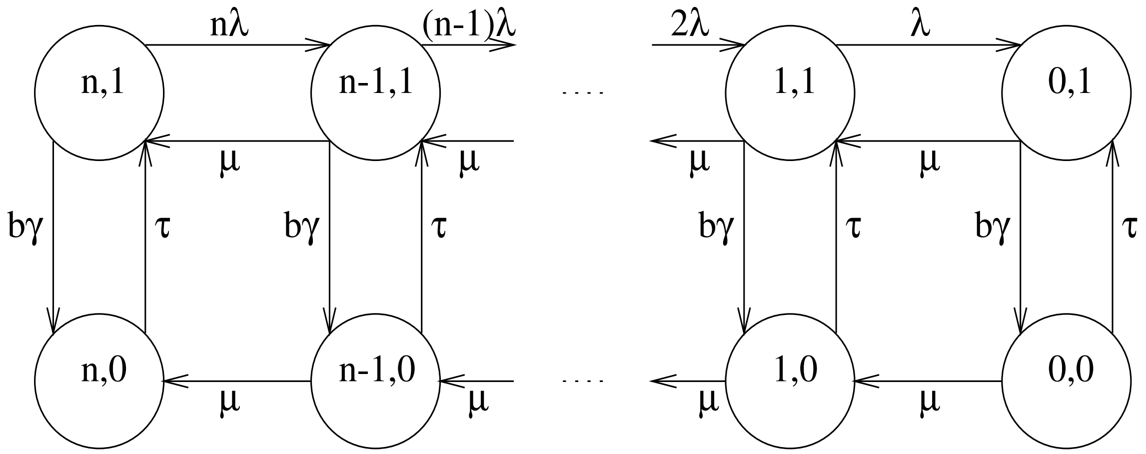

Figure 1: State-transition diagram for a n-processor system.

In Figure 2, we plot the first five moments $Y(t)$, as a function of t.

Let us now consider higher values of t. Solving the linear system $\pi A = 0$ with $\pi\mathbb{1} = 1$, we obtain $\pi D\mathbb{1} = 0.110475$. The complexities of the two algorithms in Table 1 and Table 2 are

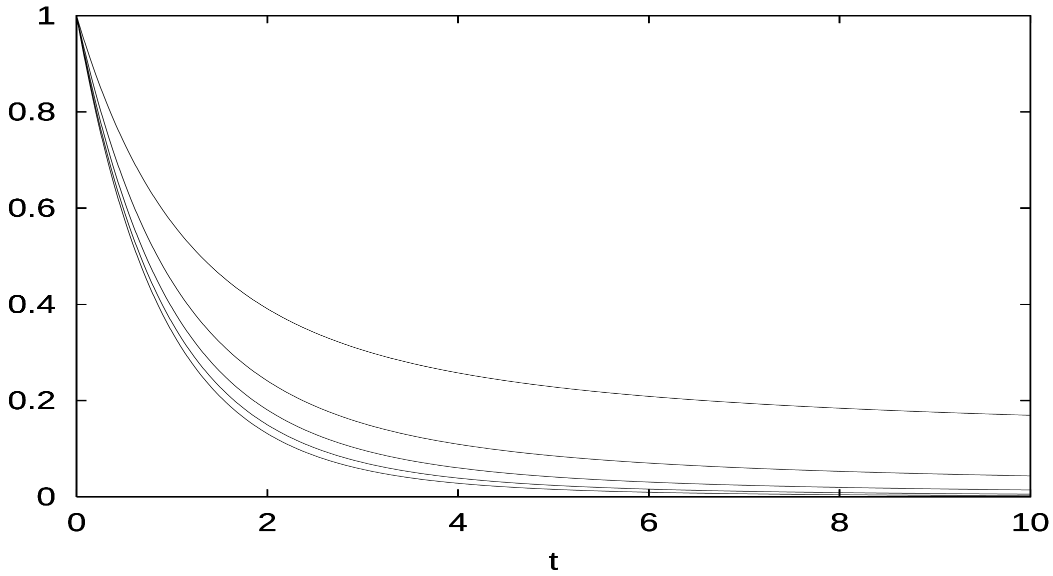

Figure 2: (Top to bottom) $E(Y(t)^r)$ for $r = 1, 2, 3, 4, 5$.

mainly due to the products of matrix P by vector $U(n-1,r)$. The number of such products is equal to NR for the algorithm of Table 1 and it is equal to KR, with $K \leq N$, for the algorithm of Table 2. We show in Figure 3, the values of K and N obtained when $R = 5$, for different values of $t = t_J$. As expected the value of $N = N(t_J)$ increases with t_J and the value of K constant after the first instant where $K < N$. This instant, which we denote by T_ε, does depend on ε, and may be called the time to stationarity. It is defined as

$$T_\varepsilon = \inf\{t \geq 0 | K < N(t)\}.$$

In our example, it is between 50000 and 60000. A more detailed analysis shows that $\lceil T_\varepsilon \rceil = 55482$.

t	50000	60000	70000	80000	90000	100000
N	76621	91823	107015	122200	137379	152154
K	76621	84955	84955	84955	84955	84955

Figure 3: Stationarity detection for different values of t, $R = 5$.

Appendix

Let $\beta(n,\ell)$, for $n \geq 0$ and $0 \leq \ell \leq n$, be real numbers, $H(\ell)$ be a sequence of matrices and $U(\ell)$ be a sequence of column vectors with the same finite dimension. We define the sequence of column vectors $V(n)$ by

$$V(n) = \sum_{\ell=0}^{n} \beta(n,\ell) H(n-\ell) U(\ell).$$

We have the following result.

Lemma 10 *Assume that*

$$\beta(n,\ell)\geq 0,\ \ \sum_{\ell=0}^{n}\beta(n,\ell)=1,\ \ \lim_{n\longrightarrow\infty}\sup_{0\leq\ell\leq n}\beta(n,\ell)=0,\ \ \lim_{\ell\longrightarrow\infty}H(\ell)=H\ \textit{and}\ \lim_{\ell\longrightarrow\infty}U(\ell)=U$$

then

$$\lim_{n\longrightarrow\infty}V(n)=HU.$$

Proof. The convergence of the sequences $H(\ell)$ and $U(\ell)$ implies that both sequences are uniformly bounded, i.e. there exists an integer M such that, for every $\ell\geq 0$,

$$\|H(\ell)\|\leq M \text{ and } \|U(\ell)\|\leq M.$$

Moreover, their convergence implies that for any $\varepsilon>0$, there exists L such that for any $\ell,m\geq L$, we have

$$\|H(m)-H\|+\|U(\ell)-U\|\leq\frac{\varepsilon}{2M}.$$

As a consequence, for $n\geq 2L$, we recover

$$\begin{aligned}
\|V(n)-HU\| &= \left\|\sum_{\ell=0}^{n}\beta(n,\ell)H(n-\ell)U(\ell)-HU\right\| \\
&= \left\|\sum_{\ell=0}^{n}\beta(n,\ell)[H(n-\ell)U(\ell)-HU]\right\| \\
&\leq \left\|\sum_{\ell=0}^{L-1}\beta(n,\ell)[H(n-\ell)U(\ell)-HU]\right\|+\left\|\sum_{\ell=L}^{n-L}\beta(n,\ell)[H(n-\ell)U(\ell)-HU]\right\| \\
&\quad +\left\|\sum_{\ell=n-L+1}^{n}\beta(n,\ell)[H(n-\ell)U(\ell)-HU]\right\|.
\end{aligned}$$

We start with the second term. Since $L\leq\ell\leq n-L$, we have $\ell\geq L$ and $n-\ell\geq L$, we can write

$$\begin{aligned}
\|H(n-\ell)U(\ell)-HU\| &= \|[H(n-\ell)-H]U(\ell)+H[U(\ell)-U]\| \\
&\leq \|H(n-\ell)-H\|\,\|U(\ell)\|+\|H\|\,\|U(\ell)-U\| \\
&\leq M\left(\|H(n-\ell)-H\|+\|U(\ell)-U\|\right)\leq\frac{\varepsilon}{2}.
\end{aligned}$$

We thus have

$$\begin{aligned}
\left\|\sum_{\ell=L}^{n-L}\beta(n,\ell)[H(n-\ell)U(\ell)-HU]\right\| &\leq \sum_{\ell=L}^{n-L}\beta(n,\ell)\,\|H(n-\ell)U(\ell)-HU\| \\
&\leq \frac{\varepsilon}{2}\sum_{\ell=L}^{n-L}\beta(n,\ell)\ \leq\ \frac{\varepsilon}{2}.
\end{aligned}$$

Concerning the first and the third terms, we use the fact that, for every $0 \leq \ell \leq n$, we have

$$\|H(n-\ell)U(\ell) - HU\| = \|H(n-\ell)U(\ell)\| + \|HU\| \leq \|H(n-\ell)\|\,\|U(\ell)\| + \|H\|\,\|U\| \leq 2M^2.$$

Let us define the sequence $\beta(n) = \sup_{0 \leq \ell \leq n} \beta(n,\ell)$. Since, by hypothesis, $\beta(n)$ converges to 0 when n goes towards infinity and since L is fixed, we can determine N such that for any $n \geq N$, we have $\beta(n) \leq \varepsilon/(8M^2 L)$.

We then have

$$\begin{aligned}
&\left\| \sum_{\ell=0}^{L-1} \beta(n,\ell)[H(n-\ell)U(\ell) - HU] \right\| + \left\| \sum_{\ell=n-L+1}^{n} \beta(n,\ell)[H(n-\ell)U(\ell) - HU] \right\| \\
&\quad \leq \sum_{\ell=0}^{L-1} \beta(n,\ell)\,\|H(n-\ell)U(\ell) - HU]\| + \sum_{\ell=n-L+1}^{n} \beta(n,\ell)\,\|H(n-\ell)U(\ell) - HU]\| \\
&\quad \leq 2M^2 \left(\sum_{\ell=0}^{L-1} \beta(n,\ell) + \sum_{\ell=n-L+1}^{n} \beta(n,\ell) \right) \leq 4M^2 L \beta(n) \leq \frac{\varepsilon}{2}.
\end{aligned}$$

Using the previous results, we obtain that for every $n \geq \max(2L, N)$,

$$\|V(n) - HU\| \leq \varepsilon,$$

which completes the proof. ■

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