C. R. Acad. Sci. Paris, t. 332, Série I, p. 1–??, 2001 - PXMA????.TEX -
Equations différentielles/*Differential equations*

Analysis of a Poisson system with boundary conditions

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(Reçu le jour mois année, accepté après révision le jour mois année)

Abstract. We consider a class of problems originating from a Raman laser amplification model, for which the equations can be written as a Poisson system with boundary conditions. Once reformulated, this system becomes an integro-differential equation that we study here into some details. In particular, we show the existence of a smooth solution under general assumptions, and prove its uniqueness for boundary values that are not too far apart. Eventually, we completely solve the question of uniqueness for systems of dimensions one and two. © 2001 Académie des sciences/Éditions scientifiques et médicales Elsevier SAS

Analyse d'un système de Poisson avec conditions aux deux bouts

Résumé. *Nous étudions une classe de problèmes dont l'origine provient d'un modèle décrivant l'effet d'amplification Raman dans une fibre optique. Les équations s'écrivent sous la forme d'un système de Poisson avec conditions aux deux bouts. Après réduction, ce système s'écrit sous la forme d'une équation intégro-différentielle. Nous étudions ici cette classe de systèmes. Nous montrons l'existence d'une solution dans le cas général et l'unicité pour des données petites du problème. Pour des données quelconques, nous prouvons l'unicité en dimensions un et deux.* © 2001 Académie des sciences/Éditions scientifiques et médicales Elsevier SAS

Version française abrégée

Le problème ´etudi´e dans cette note tire son origine d'un modèle d´ecrivant le ph´enomène d'amplifi cation Raman dans une fi bre optique dop´ee. Dans le cas d'une fi bre id´ealis´ee, nous montrons dans [3] que les ´equations à r´esoudre peuvent être ´ecrites sous la forme (2.1), où G une matrice antisym´etrique, et où le hamiltonien est de la forme $H(u,d) = \sum_{i=1}^{N} d_i \sinh u_i$. Ici, u est un vecteur de dimension $N \geq 2$ de fonctions d´efi nies sur $[0,1]$, $d \in \mathbb{R}^N$ est un vecteur *inconnu* et les conditions aux limites sont de la forme

Note présentée par Prénom NOM

S0764-4442(00)0????-?/FLA

$u(0) = u_0$ et $u(1) = u_1$ avec u_0 et u_1 dans $\mathbb{R}^N$. Les coefficients d_i sont des invariants de Casimir de la structure de Poisson sous-jacente (voir [4]).

Plus généralement, nous considérons des problèmes de la forme suivante : soit f une fonction continuement différentiable de $\mathbb{R}$ dans $\mathbb{R}$ telle que $\forall v \in \mathbb{R}$, on ait $f(v) \geq 1$. Pour une matrice quelconque G d'ordre $N \geq 1$ et pour u_0 et u_1 donnés dans $\mathbb{R}^N$, nous considérons les équations (2.3-2.4-2.5) où $u \in C^1([0,1];\mathbb{R}^N)$, $f(u(x))$ est le vecteur de dimension N de composantes $f(u_i(x))$, $i = 1,\dots,N$, et où D est une matrice diagonale de coefficients inconnus d_i.

En intégrant et en utilisant les conditions aux limites, on peut "éliminer" les paramètres d_i, $i = 1,\dots,N$: si u est solution, on trouve $\delta u := u(1) - u(0) = GD\int_0^1 f(u(x))dx$. Si G est inversible, on pose $q = G^{-1}(\delta u)$, et on trouve l'équation $D\|f(u)\|_1 = q$ où $\|f(u)\|_1$ désigne le vecteur formé des normes L^1 des $f(u_i)$ sur $[0,1]$. Dans le cas général, soit q une solution de $Gq = \delta u$, et soit Q la matrice diagonale avec les q_i comme éléments diagonaux. Toute solution du système (3.7-3.8) où $\frac{f(u(x))}{\|f(u)\|_1}$ désigne le vecteur de composantes $\frac{f(u_i(x))}{\|f(u_i)\|_1}$, est alors solution du problème d'origine (2.3-2.4-2.5). Dans cette note, on prouve les résultats suivants : soit G une matrice (éventuellement singulière et non antisymétrique) de taille $N \geq 1$, et soit Q une matrice diagonale. Alors pour tout $u_0 \in \mathbb{R}^N$ le problème (3.7-3.8) possède au moins une solution régulière u^*. De plus, il existe un $\varepsilon > 0$ tel que pour tout $\|Q\|_\infty \leq \varepsilon$, la solution u^* est unique dans $\mathrm{L}^\infty([0,1];\mathbb{R}^N)$.

Pour des valeurs quelconques de Q, nous montrons les résultats suivants : si $N = 1$, alors pour tous réels G, Q et u_0, le système (3.7-3.8) a une unique solution u^* in $\mathrm{L}^\infty([0,1];\mathbb{R})$.

Si $N = 2$ et si G est une matrice antisymétrique, alors pour toute matrice diagonale Q et pour tout $u_0 \in \mathbb{R}^2$, le système (3.7-3.8) a une unique solution u^* in $\mathrm{L}^\infty([0,1];\mathbb{R}^2)$.

Pour $N = 1$, on montre l'unicité en se ramenant au cas d'une équation différentielle. Pour $N = 2$ et G antisymétrique, on utilise le fait qu'il existe un invariant (le hamiltonien du problème d'origine) pour obtenir une relation entre les normes L^1 et se ramener au cas d'un système différentiel.

1. Introduction

The problem described in this note originates from a model of Raman laser amplification in an optic fiber [5]. Standard discrete models of this phenomenon (see [1] or [6]) lead to a system of differential equations of Lotka-Volterra form with boundary conditions (see for instance [4]). These equations describe how high-frequency waves traveling forward and backward in the fiber disseminate part of their energy to low-frequency waves through a prey-predator process (Raman effect).

The numerical approximation of this problem seems difficult to obtain: for instance, the *shooting* method (see [2]) is to be banned due to the presence of nonlinearities (most initial values would lead to blow-up in finite time); more elaborated methods, such as finite differences, collocation, or multiple shooting, are possible alternatives, but might become prohibitively costly in large dimension.

In this note, we formulate the problem as a *Cauchy problem for a system of integro-differential equations* with mild nonlinearity. Using standard techniques (Schauder's theorem), the existence of solutions can thus be easily proved (see Section 4.). Uniqueness for boundary values that are not too far apart and an arbitrary dimension is shown in Section 5., while ad-hoc techniques allow for the treatment of the one and two-dimensional cases for arbitrary boundary values.

2. Setting of the problem

In an idealized situation, the equations governing the phenomenon of Raman laser amplification in an optical device are of Lotka-Volterra form (see [1] or [6]). In [3], we proved that they may be written as the following Poisson system

$$u' = G\nabla_u H(u,d) \quad \text{with} \quad H(u,d) = \sum_{i=1}^{N} d_i \sinh u_i \tag{2.1}$$

where u is an unknown vector of dimension $N \geq 2$ of functions defined on the interval $[0,1]$, d an unknown element of $\mathbb{R}^N$, G a skew symmetric matrix and $H(u,d)$ the hamiltonian of the problem[1]. Note that the d_i are Casimir invariants of the underlying Poisson structure (see [4]). The boundary conditions are of the form $u(0) = u_0$ and $u(1) = u_1$ where u_0 and u_1 are fixed in $\mathbb{R}^N$.

We generalize this problem in the following way: Let f be continuously differentiable function from $\mathbb{R}$ to $\mathbb{R}$, satisfying

$$\forall\, v \in \mathbb{R}, \quad f(v) \geq 1. \tag{2.2}$$

For a given (possibly non skew symmetric) matrix G of order $N \geq 1$, given u_0 and u_1 in $\mathbb{R}^N$, we consider the system of ordinary differential equations

$$u'(x) = GDf(u(x)), \quad \text{for} \quad x \in [0,1], \tag{2.3}$$

$$u(0) = u_0, \tag{2.4}$$

$$u(1) = u_1, \tag{2.5}$$

where $u \in C^1([0,1];\mathbb{R}^N)$, $f(u(x))$ is the N-dimensional vector with components $f(u_i(x))$, $i = 1,\ldots,N$, and where D is a diagonal matrix of free parameters d_i that have to be determined.

3. Reformulation of the problem

Writing the boundary conditions, we can immediately obtain an equation for the parameters d_i, $i = 1,\ldots,N$, in terms of the L^1-norms that we denote for short

$$\|f(u_i)\|_1 = \int_0^1 |f(u_i(x))| dx = \int_0^1 f(u_i(x)) dx, \quad \text{for} \quad i = 1,\ldots,N.$$

If a solution u exists, we have indeed

$$\delta u := u_1 - u_0 = u(1) - u(0) = GD \int_0^1 f(u(x)) dx. \tag{3.6}$$

If G is assumed to be invertible, then we can define $q = G^{-1}(\delta u)$ and get $D\|f(u)\|_1 = q$ where $\|f(u)\|_1$ denotes the vector of L^1-norms of $f(u_i)$. This can be written as

$$d_i = \frac{1}{\|f(u_i)\|_1} \sum_{j=1}^{N} \omega_{ij} (\delta u)_j \quad \text{for} \quad i = 1,\ldots,N,$$

where $\Omega = (\omega_{ij}) = G^{-1}$. More generally, let q be a solution of $Gq = \delta u$, and let Q be the diagonal matrix with diagonal elements q_i. Then any solution of the system

$$u'(x) = GQ \frac{f(u(x))}{\|f(u)\|_1}, \quad \text{for} \quad x \in [0,1], \tag{3.7}$$

$$u(0) = u_0 \in \mathbb{R}^N, \tag{3.8}$$

[1] At this stage, getting a canonical Poisson system requires only to bring the *constant skew-symmetric* matrix G to canonical form.

is a solution of the system (2.3-2.4-2.5). Here, we have used the consistent notation

$$\frac{f(u(x))}{\|f(u)\|_1} = \left(\frac{f(u_1(x))}{\|f(u_1)\|_1}, \ldots, \frac{f(u_N(x))}{\|f(u_N)\|_1}\right)^T.$$

4. An existence result

We now state the following theorem :

THEOREM 1. – *Let G be a (possibly singular and non skew-symmetric) matrix of size $N \geq 1$, and let Q be a diagonal matrix. Then for all $u_0 \in \mathbb{R}^N$, the initial-value integro-differential problem* (3.7-3.8) *has at least one infinitely differentiable solution u^*.*

Proof : Let $L^\infty([0,1];\mathbb{R}^N)$ be the space of bounded functions from $[0,1]$ to $\mathbb{R}^N$, equipped with the standard L^∞-norm, and consider the functional Φ from $L^\infty([0,1];\mathbb{R}^N)$ to itself, defined by

$$\Phi(u)(x) = u_0 + GQ\int_0^x \frac{f(u(y))}{\|f(u)\|_1}\,dy, \quad \text{for} \quad x \in [0,1].$$

We first notice that

$$\|\Phi(u)\|_\infty \leq \|u_0\|_\infty + \|GQ\|_\infty,$$

where $\|\cdot\|_\infty$ denote either the L^∞-norm on $[0,1]$ or the standard infinity norm for vector and matrix in $\mathbb{R}^N$. This means that $\Phi(L^\infty([0,1];\mathbb{R}^N))$ is bounded by $\bar{M} := \|u_0\|_\infty + \|GQ\|_\infty$. Then, we thus consider for a real $M > 0$, the image by Φ of the L^∞-ball $\mathcal{B}_M$ of radius M. For any $w = \Phi(u)$ in $\Phi(\mathcal{B}_M)$, and for any x and y in $[0,1]$, we have

$$\|w(x) - w(y)\|_\infty \quad \leq \quad 2\|GQ\|_\infty (\sup_{|x|\leq M} f(x))|x-y|,$$

so that $\Phi(\mathcal{B}_M)$ is a uniformly equicontinuous bounded set. By Ascoli's theorem, $\overline{\Phi(\mathcal{B}_M)}$ is compact and thus Φ is a compact function from $L^\infty([0,1];\mathbb{R}^N)$ to itself. Now, for u and v in $L^\infty([0,1];\mathbb{R}^N)$ such that $\|u-v\|_\infty \leq \delta$, we have

$$\|\Phi(u) - \Phi(v)\|_\infty \quad \leq \quad 2\|GQ\|_\infty \left(\sup_{|\theta|\leq\|u\|_\infty+\delta} |f'(\theta)|\right)\|u-v\|_\infty, \tag{4.9}$$

implying that Φ is continuous. Eventually, Schauder's theorem implies that there exists a fixed point u^* of Φ in $\mathcal{B}_{\bar{M}}$. An easy induction shows that u^* is infinitely differentiable. $\square$

5. Uniqueness for small values of Q

As announced in the introduction section, showing uniqueness of the solution for any diagonal matrix Q has proven difficult. However, a weaker result exists:

THEOREM 2. – *Let G be a matrix of size $N \geq 1$ and $u_0 \in \mathbb{R}^N$. There exists $\varepsilon > 0$ such that for all diagonal matrix Q satisfying $\|Q\|_\infty \leq \varepsilon$, the initial-value integro-differential problem* (3.7-3.8) *has a unique solution u^* in $L^\infty([0,1];\mathbb{R}^N)$.*

Proof : All we have to show here is that the fixed-point u^* of Φ found in Theorem 1 is unique. To see this, we consider another fixed-point v^* of Φ. As noticed at the end of Theorem 1, although u^* and v^* are only assumed at first place to belong to $\mathrm{L}^\infty([0,1];\mathbb{R}^N)$, they are in fact bounded by $\bar{M} := \|u_0\|_\infty + \|GQ\|_\infty$. Hence, estimate (4.9) with $u = u^*$ and $v = v^*$ can be refined as follows

$$\|\Phi(u^*) - \Phi(v^*)\|_\infty \leq 2\|GQ\|_\infty \left(\sup_{|\theta| \leq \|u_0\|_\infty + \|GQ\|_\infty} |f'(\theta)| \right) \|u^* - v^*\|_\infty .$$

Now, let $k = 2\|GQ\|_\infty \left(\sup_{|\theta|\leq\|u_0\|_\infty+\|GQ\|_\infty} |f'(\theta)|\right)$. We have $k \to 0$ as $\|Q\|_\infty \to 0$. Thus there exists $\varepsilon > 0$ such that for $\|Q\|_\infty \leq \varepsilon$, we have $k < 1$ and the application Φ is contractant in $\mathcal{B}_{\bar{M}}$. In this case, we thus have $v^* = u^*$. $\square$

It must be emphasized that for a singular[2] matrix G, equations (2.3-2.4-2.5) have several distinct solutions.

Note also that under the hypothesis of Theorem 2, the standard Picard-like scheme $u^{k+1} = \Phi(u^k)$ converges toward u^* linearly.

6. Uniqueness for large values of Q

When the norm of Q is not small, we are left with the open question of uniqueness of the solution of (3.7-3.8).

For $N = 1$, the uniqueness of the solution can be proved as follows: equations (3.7-3.8) read here

$$u'(x) = \mu \frac{f(u(x))}{\|f(u)\|_1}, \quad u(0) = u_0 \in \mathbb{R}, \tag{6.10}$$

where $\mu \in \mathbb{R}$. Note that we have by integration $u(1) = u_0 + \mu$. If $\mu = 0$, the solution $u(x) \equiv u_0$ is unique. If $\mu \neq 0$, we define the function $I(u) = \int_0^u (1/f(v))dv$. This function is strictly increasing, and it can be checked that (6.10) is equivalent to the differential equation

$$u'(x) = (I(u_0 + \mu) - I(u_0))f(u(x)), \quad u(0) = u_0 \in \mathbb{R},$$

whose solution is unique. We thus get

PROPOSITION 1. – *If $N = 1$, for any real numbers G, Q and u_0, the initial-value integro-differential problem* (3.7-3.8) *has a unique solution u^* in* $\mathrm{L}^\infty([0,1];\mathbb{R})$.

The uniqueness result for $N = 2$ and G skew-symmetric is a direct by-product of the existence of an invariant. Indeed, we can easily show

LEMMA 1. – *Let $N \geq 2$, G a skew-symmetric $N \times N$ matrix, and Q a diagonal matrix. Let F be a primitive of f, u a solution of* (3.7-3.8) *and q the vector with elements the diagonal elements of Q. The quantity $H(u(x))$ defined by :*

$$H(u(x)) = \sum_{j=1}^{N} q_j \frac{F(u_j(x))}{\|f(u_j)\|_1},$$

is invariant along any solution u.

For $N = 2$, the matrix G is of the form $G = \begin{pmatrix} 0 & -\sigma \\ \sigma & 0 \end{pmatrix}$ with $\sigma \in \mathbb{R}$. Note that if $\sigma = 0$ the solution is obviously unique. If $\sigma \neq 0$, from $H(u(0)) = H(u(1))$, we get

$$\frac{q_1(\int_{u_1(0)}^{u_1(1)} f(u)du)}{\|f(u_1)\|_1} + \frac{q_2(\int_{u_2(0)}^{u_2(1)} f(u)du)}{\|f(u_2)\|_1} = 0.$$

[2] For odd values of N, this is necessarily the case if G is skew-symmetric.

If $q_1(\int_{u_1(0)}^{u_1(1)} f(u)du) = q_2(\int_{u_2(0)}^{u_2(1)} f(u)du) = 0$, it is easy to check that the equations degenerate into a system of ordinary differential equations with a unique solution. Whenever $q_1(\int_{u_1(0)}^{u_1(1)} f(u)du) \neq 0$, we get the equation

$$\frac{1}{\|f(u_1)\|_1} = \frac{\lambda}{\|f(u_2)\|_1},$$

where $\lambda = -\big(q_2 \int_{u_2(0)}^{u_2(1)} f(u)du\big)\big(q_1 \int_{u_1(0)}^{u_1(1)} f(u)du)\big)^{-1}$. As a consequence, we can express the L^1-norm of $f(u_1)$ in terms of the L^1-norm of $f(u_2)$. Now if u is a continuously differentiable solution of (3.7-3.8) defined on $[0,1]$, then v, defined on $[0,(\|f(u_2)\|_1)^{-1}]$ by $v(x) = u(\|f(u_2)\|_1 x)$ is a continuously differentiable solution of the system of ordinary differential equations

$$\left\{ \begin{array}{lcll} v_1'(x) & = & -\sigma q_2 f(v_2(x)), & v_1(0) = (u_0)_1, \\ v_2'(x) & = & \lambda\sigma q_1 f(v_1(x)), & v_2(0) = (u_0)_2, \end{array} \right. \tag{6.11}$$

and satisfies

$$\int_0^{(\|f(u_2)\|_1)^{-1}} f(v_2(x))dx = 1.$$

However, since $f \geq 1$, this defines $(\|f(u_2)\|_1)^{-1}$ uniquely, given that (6.11) has a unique solution on an interval $[0,X[$, $X > 0$. We thus have :

PROPOSITION 2. – *If G is a 2-by-2 skew-symmetric matrix, Q is a diagonal matrix, then for all $u_0 \in \mathbb{R}^2$, the initial-value integro-differential problem* (3.7-3.8) *for $N = 2$ has a unique solution u^* in* $\mathrm{L}^\infty([0,1];\mathbb{R}^2)$.

Acknowledgements. This work has been initiated during a collaboration with the Transmission Department of ALCATEL Research & Innovation (Marcoussis, France). In particular the authors are glad to thank Dominique Bayart, Florence Leplingard and Catherine Martinelli for fruitful discussions.
Besides the authors are grateful to Monique Dauge for her constructive criticisms.

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