Time averaging for the strongly confined nonlinear Schrödinger equation, using almost periodicity.

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Abstract

We study the limiting behavior of a nonlinear Schrödinger equation describing a 3 dimensional gas that is strongly confined along the vertical, z direction. The confinement induces fast oscillations in time, that need to be averaged out. Since the Hamiltonian in the z direction is merely assumed confining, without any further specification, the associated spectrum is discrete but arbitrary, and the fast oscillations induced by the nonlinear equation entail countably many frequencies that are arbitrarily distributed. For that reason, averaging can not rely on small denominator estimates or like.

To overcome these difficulties, we prove that the fast oscillations are *almost periodic* in time, with values in a *Sobolev-like* space that we completely identify. We then exploit the existence of *long time averages* for almost periodic function to perform the necessary averaging procedure in our nonlinear problem.

Key words : Adiabatic approximation, Sobolev scale associated with a self-adjoint operator, error estimates, nonlinear analysis, two dimensional electron gas, almost periodic functions.

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1 Introduction

In this paper, we study the asymptotic behavior of a *nonlinear* gas of quantum particles, evolving in the three dimensional space $(x,z) \in \mathbb{R}^3$ ($x \in \mathbb{R}^2$, $z \in \mathbb{R}$), yet *strongly confined* along the vertical z direction. The dynamics of the gas essentially occurs along the remaining, horizontal x plane, and our goal is to recover the limiting dynamics along x, by performing the relevant averaging procedure.

Such nonlinear and strongly confined gases are typically encountered in the study of Bose condensation, which is the example we have in mind throughout this paper. In this context, an atomic gas is confined in a given region of space, and an appropriate cooling procedure makes it possible to set all atoms in the *same* quantum state, described by the *same* wave function Ψ. This somehow "macroscopic" wave function Ψ satisfies a Schrödinger equation. The fact that the underlying gas is made up of many atoms which interact pairwise is usually taken into account using a mean-field model, and the appropriate Schrödinger equation is nonlinear.

Mathematically speaking, the present text is devoted to the study of a nonlinear Schrödinger equation in the presence of a small parameter. The mathematical context is similar in spirit to the so-called Born-Oppenheimer approximation: the confining Hamiltonian in the z direction, called H_z in the sequel, carries a weight $1/\varepsilon$ which, as $\varepsilon \to 0$, enhances the time oscillations of Ψ, of the form $\exp(-itH_z/\varepsilon)$ (roughly), and the difficulty is to average out these oscillations.

In this text, we show that the strong confinement allows to develop an averaged model over the discrete eigenspaces of H_z. This model describes the limiting dynamics along the x plane. The point is, we are able to completely develop the averaging procedure over *all* the eigenspaces *at once*. The limiting model is an infinite system of coupled, nonlinear, Schrödinger equations, describing the averaged evolution of Ψ over each eigenspace. In particular, all energy levels are coupled through the averaged nonlinearity. This contrasts with the previous study performed in [BMSW] where only the ground state, *i.e.* the eigenspace associated with the lowest eigenenergy of H_z, is treated, and the limiting model is a single, scalar, nonlinear Schrödinger equation, describing the averaged evolution of Ψ over this single eigenspace. This also contrasts with the Born-Oppenheimer situation (see [ST], [T], [HJ]), where the emphasis is more on the separation between two distinguished eigenspaces, but the spectrum is not necessarily discrete.

The key observation in the present study, that makes it possible to perform a clean averaging procedure, relies on the fact that the operator $\exp(-itH_z/\varepsilon)$ is *almost periodic* in time. In other words, it carries a discrete, possibly infinite, number of independent time-oscillations. This observation allows to average $\exp(-itH_z/\varepsilon)$ in time without having to deal with the difficulty of *small denominators* (see [BCD] in the context of laser-matter interaction). It also allows to formulate our limiting model in a "good" functional framework, without having to project it over all the eigenspace of H_z, a difficult if not impossible task, that is the very reason why the text [BMSW] restricts to a situation where only the ground state is occupied. Obviously, the counterpart is that our error terms are bounded by nothing better than $o(1)$: a simpler, periodic framework (i.e. only one time-oscillation, as in [BMSW]) certainly allows to obtain improved convergence rates, yet such a simplified framework is definitely *not* generic. Incidentally, in the course of the analysis, we are also led to identifying the Sobolev scale associated with the operator $H_z = -\partial^2/\partial z^2 + V_c(z)$ (see below for the notation), *i.e.* the domain of the successive powers $(-\partial^2/\partial z^2 + V_c(z))^m$ $(m \geq 0)$. This turns out to be an important and delicate step of our analysis, which leads us to use an appropriate pseudodifferential calculus, based on the Weyl-Hörmander calculus and on the associated Sobolev spaces developed by Bony and Chemin in [BC].

1.1 The model

Let (x, z) be the variable in $\mathbb{R}^3 = \mathbb{R}^2 \times \mathbb{R}$, where $z \in \mathbb{R}$ lies in the vertical direction (say), and $x \in \mathbb{R}^2$ belongs to the horizontal plane. It is important to stress that, though the present text presents a three-dimensional framework, our techniques are immediately adapted in any dimension $\mathbb{R}^d = \mathbb{R}^{d-p} \times \mathbb{R}^p$. In particular, the physically important case $p = 2$, $d = 3$ may be treated along the present lines.

According to the splitting $\mathbb{R}^3 = \mathbb{R}^2 \times \mathbb{R}$, take two Hamiltonians

$$H_x = -\Delta_x + V(x), \qquad \text{and} \quad H_z = -\frac{\partial^2}{\partial z^2} + V_c(z), \tag{1.1}$$

where both potentials $V(x)$ and $V_c(z)$ are assumed C^∞, real valued, and bounded from below. Without loss of generality, we may assume, by using the standard shift in time, that both potentials are bounded away from zero, i.e. we may assume

$$V(x) \geq 1 \text{ and } V_c(z) \geq 1.$$

Other, more specific, assumptions on the potentials $V_c(z)$ and $V(x)$ are needed in the present text, which are detailed now.

A key assumption of this paper is that V_c is *confining*, i.e.

$$V_c(z) \underset{|z|\to\infty}{\longrightarrow} +\infty. \tag{1.2}$$

As is well known [RS], this ensures that the spectrum of $H_z = -\partial^2/\partial z^2 + V_c(z)$ is discrete, when considered as a linear, unbounded operator over $L^2(\mathbb{R})$, with domain

$$D(H_z) = \{\Psi(z) \in L^2(\mathbb{R}) \text{ s.t. } \partial_z^2 \Psi \in L^2(\mathbb{R}) \text{ and } V_c(z)\,\Psi \in L^2(\mathbb{R})\}.$$

Throughout this paper, the eigenelements of H_z will be denoted by the collection of eigenenergies $E_p \geq 0$ and eigenfunctions $\chi_p(z)$, as p runs in $\mathbb{N}$. They satisfy, for any index p,

$$H_z \chi_p(z) = E_p\, \chi_p(z). \tag{1.3}$$

Also, it is well known that $E_p \to +\infty$ as $p \to \infty$, that the E_p's may be chosen to be non-decreasing, while the χ_p's may be chosen so as to form an orthonormal basis of

$L^2(\mathbb{R})$. We will assume these monotonicity and orthonormality properties hold true from now on.

For later functional analytic purposes, we shall actually assume a *reinforced version of confinement* in the z direction. This is a more technical point. Indeed, our study requires the following three conditions

$$\forall \alpha \in \mathbb{N}, \qquad \frac{\partial^\alpha V_c}{\partial z^\alpha}(z) = O\Big(V_c(z)\Big) \quad \text{as} \quad |z| \to \infty, \tag{1.4}$$

$$\exists M_z > 0, \qquad V_c(z) = O\left(|z|^{M_z}\right) \quad \text{as} \quad |z| \to \infty, \tag{1.5}$$

$$\exists M_z' > 0, \qquad \frac{|\partial_z V_c(z)|}{V_c(z)} = O\left(|z|^{-M_z'}\right) \quad \text{as} \quad |z| \to \infty. \tag{1.6}$$

In other words, $V_c(z)$ should roughly behave like a symbol at infinity in z (this is the meaning of assumptions (1.4) and (1.6)), and V should have at most polynomial growth at infinity in z (this is assumption (1.5)). These assumptions typically exclude potentials behaving like $\exp(|z|)$ at infinity or so, or potentials which oscillate too fast at infinity like $|z|^2 \sin(|z|^2)$ or so, for which the analysis we present in this text probably becomes false anyhow. However, assumptions (1.4) through (1.6) typically allow polynomial behavior of arbitrary degree. It even allows potentials that behave like $|z|^a$ at infinity in one direction, and $|z|^b$ for some $b \neq a$ at infinity in the other direction. Obviously, assumptions (1.2) and (1.4) are met in the case where H_z simply is the harmonic oscillator $-\partial^2/\partial z^2 + |z|^2$, which is the example we keep in mind throughout the paper, relevant in the context of Bose condensation.

Concerning the potential $V(x)$ in the x direction, the present study may be carried either when $V(x)$ is confining or when it is uniformly bounded. For definiteness, and because the physical situation we have in mind is again Bose condensation, we shall assume $V(x)$ is confining as is $V_c(z)$, namely

$$V(x) \underset{|x|\to\infty}{\longrightarrow} +\infty, \tag{1.7}$$

while we also assume a reinforced version of confinement in the x direction as we did in the z direction, namely

$$\forall \alpha \in \mathbb{N}^2, \quad \frac{\partial^\alpha V}{\partial x^\alpha}(x) = O\Big(V(x)\Big) \quad \text{as} \quad |x| \to \infty, \tag{1.8}$$

$$\exists M_x > 0, V(x) = O\left(|x|^{M_x}\right) \quad \text{as} \quad |x| \to \infty, \tag{1.9}$$

$$\exists M_x' > 0, \frac{|\nabla_x V(x)|}{V(x)} = O\left(|x|^{-M_x'}\right) \quad \text{as} \quad |x| \to \infty. \tag{1.10}$$

We stress that these assumptions are *not* essential in our analysis, and the alternative situation where $V(x) \in C_b^\infty(\mathbb{R}^2)$ (C^∞ and bounded functions) could be handled as well. Again, the typical potential we have in mind is the harmonic oscillator $-\Delta_x + |x|^2$.

Now, let $\varepsilon > 0$ be the small parameter that measures the strength of the confinement in the z direction, relative to that in the x plane. Take a nonlinearity

$$F : \mathbb{R} \mapsto \mathbb{R}, \quad F \in C^\infty(\mathbb{R}).$$

Our goal is to study the following nonlinear Schrödinger equation, written in dimensionless form, along the limit $\varepsilon \to 0$:

$$i\partial_t \Psi^\varepsilon(t, x, z) = H_x \, \Psi^\varepsilon + \frac{1}{\varepsilon} H_z \, \Psi^\varepsilon + F(|\Psi^\varepsilon|^2) \, \Psi^\varepsilon. \tag{1.11}$$

Here $H_x = -\Delta_x + V(x)$, and $H_z = -\partial^2/\partial z^2 + V_c(z)$, as before (see (1.1)). In other words, we study the idealized limit where confinement in z is infinite, and the quantum particles are essentially confined in the horizontal plane $\mathbb{R}^2$. The definite example we have in mind in the context of Bose condensation is $F(u) = \pm u$.

An initial datum is also prescribed for (1.11), namely

$$\Psi^\varepsilon(0, x, z) = \Psi_0(x, z) \in L^2(\mathbb{R}^2 \times \mathbb{R}). \tag{1.12}$$

In order to have "good" uniform bounds on Ψ^ε, and on the nonlinear term $F(|\Psi^\varepsilon|^2)$, we shall additionally assume that Ψ_0 possesses a "good" regularity in the Sobolev scale induced by the nonnegative, self-adjoint operators H_x and H_z. This is a delicate point of our analysis, which we now briefly discuss.

Namely, we shall suppose the following:

$$\begin{aligned} &\text{There exists an } m > 3/2 \text{ such that} \\ &\Psi_0 \in B_m := \left\{ u \in L^2(\mathbb{R}^3) \text{ s.t. } H_x^{m/2} u \in L^2(\mathbb{R}^3), \text{ and } H_z^{m/2} u \in L^2(\mathbb{R}^3) \right\}. \end{aligned} \tag{1.13}$$

As we show later, it turns out the spaces B_ℓ ($\ell \geq 0$) form a scale of Hilbert spaces, and they may be endowed with either the norm

$$\|u\|_{B_\ell}^2 := \|u\|_{L^2(\mathbb{R}^3)}^2 + \|H_x^{\ell/2} \, u\|_{L^2(\mathbb{R}^3)}^2 + \|H_z^{\ell/2} \, u\|_{L^2(\mathbb{R}^3)}^2, \tag{1.14}$$

or the equivalent norm (we use the same notation for simplicity)

$$\|u\|_{B_\ell}^2 := \|u\|_{H^\ell(\mathbb{R}^3)}^2 + \|V(x)^{\ell/2} \, u\|_{L^2(\mathbb{R}^3)}^2 + \|V_c(z)^{\ell/2} \, u\|_{L^2(\mathbb{R}^3)}^2, \tag{1.15}$$

where $H^\ell(\mathbb{R}^3)$ denotes the usual Sobolev space.

The reason for the present assumption is the following. First, the condition $m > 3/2$ in (1.13) makes B_m an algebra, as we show in Proposition 2.5, and the nonlinear application $\Psi^\varepsilon \mapsto F(|\Psi^\varepsilon|^2) \, \Psi^\varepsilon$ is seen to be locally Lipschitz in B_m. Second, and more importantly, the fact that the operators $H_x^{m/2}$ and $H_z^{m/2}$ commute with $H_x + H_z/\varepsilon$ in (1.11), allows to prove that Ψ^ε is uniformly bounded in B_m, despite the singular term H_z/ε. This observation is very reminiscent of the use of Heisenberg derivatives in the analysis of semi-classical Schrödinger equations.

Now, the crucial fact that both norms (1.14) and (1.15) are equivalent is *not* an obvious point, and the proof of this actually is an important and delicate step of our analysis (see Theorem 2.1, whose proof occupies the whole section 2). We refer to [He] for a similar equivalence of norms, in the particular case where $H_x \equiv -\Delta_x + |x|^2$ and $H_z \equiv -\partial^2/\partial z^2 + |z|^2$ are harmonic oscillators: even in this particular case, we stress that the proof of the equivalence is not obvious. Specifically, it turns out an appropriate pseudodifferential calculus needs to be used in order to prove both norms (1.14) and (1.15) are equivalent, even when $H_x = -\Delta_x + |x|^2$ and $H_z = -\partial^2/\partial z^2 + |z|^2$ (see [He] in this case), and our proof uses in a crucial way the Weyl-Hörmander calculus, following ideas by Bony and Chemin [BC], and the more recent work by Helffer and Nier [HN]. We stress that a "pedestrian" proof of the desired equivalence probably is out of reach, see below for further comments.

At this point of the discussion, we are in position to try to characterize the limit of Ψ^ε in B_m. This is where *almost-periodicity* enters, which is the key observation of the present text.

1.2 Heuristic approach to the strong confinement limit

Let us now give a flavor of the limiting behavior of $\Psi^\varepsilon(t,x,z)$ in the Schrödinger equation (1.11), and of the difficulties encountered in this text.

The probably most natural approach is to first *project* the Schrödinger equation (1.11) over the orthonormal basis $(\chi_p)_{p\in\mathbb{N}}$. Admitting for the moment there exists a time $T_0 > 0$ such that Ψ^ε is bounded in $C^0([0,T_0];B_m)$, uniformly with respect to ε, we may write the orthogonal decomposition

$$\Psi^\varepsilon(t,x,z) = \sum_{p\geq 0} \psi_p^\varepsilon(t,x)\;\chi_p(z) \qquad \text{with} \quad \psi_p^\varepsilon(t,x) := \langle \Psi^\varepsilon(t,x,z), \chi_p(z)\rangle,$$

and it may be assumed that the ψ_p^ε's possess nice uniform bounds in the space $C^0([0,T_0];l^2(\mathbb{N};L^2(\mathbb{R}^2)))$ (the l^2 norm may be improved into a weighted l^2 norm, using the E_p's). Here and throughout the paper, we use the notation

$$\langle u,v\rangle := \int_{\mathbb{R}} u\,\overline{v}\,dz. \tag{1.16}$$

Using this, the Schrödinger equation (1.11) may be decomposed into

$$\begin{aligned} i\partial_t\psi_p^\varepsilon(t,x) = H_x\,\psi_p^\varepsilon + \frac{E_p}{\varepsilon}\,\psi_p^\varepsilon \\ + \sum_{r\geq 0}\left\langle F\left(\Big|\sum_{q\geq 0}\psi_q^\varepsilon(t,x)\,\chi_q(z)\Big|^2\right), \overline{\chi_r(z)}\,\chi_p(z)\right\rangle \psi_r^\varepsilon, \end{aligned} \tag{1.17}$$

an infinite system of coupled, nonlinear, Schrödinger equations, on the $\psi_p^\varepsilon(t,x)$'s ($p\in\mathbb{N}$, $x\in\mathbb{R}^2$).

In view of (1.17), $\partial_t\psi_p^\varepsilon$ clearly has size $O(1/\varepsilon)$. For this reason, it is now natural to *filter out* the oscillations $\exp(-itE_p/\varepsilon)$ of ψ_p^ε induced by H_z, in the spirit of Schochet and Grenier's works [Sc], [Gr]. Hence, we define, for each $p\geq 0$, the new unknown

$$\phi_p^\varepsilon(t,x) := \psi_p^\varepsilon(t,x)\;\exp\left(+itE_p/\varepsilon\right). \tag{1.18}$$

The ϕ_p^ε's naturally satisfy the *filtered system*

$$i\partial_t \phi_p^\varepsilon(t,x) = H_x\, \phi_p^\varepsilon \\ + \sum_{r\geq 0} \mathrm{e}^{-it\frac{E_r - E_p}{\varepsilon}} \left\langle F\left(\left|\sum_{q\geq 0} \phi_q^\varepsilon(t,x)\, \chi_q(z)\, \mathrm{e}^{-it\frac{E_q}{\varepsilon}}\right|^2\right), \overline{\chi_r}\, \chi_p \right\rangle \phi_r^\varepsilon. \quad (1.19)$$

Clearly, $\partial_t \phi_p^\varepsilon$ is an $O(1)$ quantity. Even more, the system (1.19) is an infinite dimensional, nonlinear and coupled differential system on the ϕ_p^ε's ($p \in \mathbb{N}$), of the form

$$\partial_t u^\varepsilon = A u^\varepsilon + B(t/\varepsilon, u^\varepsilon), \quad (1.20)$$

and the nonlinearity B showing up on the right-hand-side of (1.19) clearly possesses some "periodicity" in time, due to the oscillatory factors $\exp(itE_p/\varepsilon)$ and like (to be more precise, the time dependence of the nonlinearity at hand turns out to be *almost-periodic*, as we discuss later in the text, see also section 3).

At this level, it now becomes quite tempting to *average* in time the system (1.19), or, equivalently, the toy model (1.20). This is actually the key ingredient in Schochet's work [Sc]. Indeed, it is well known that, provided the function $B(\tau, u)$ entering (1.20) possesses some *ergodicity* property in time, the reference system (1.20) converges towards

$$\partial_t u = Au + B_{\mathrm{av}}(u), \qquad \text{where } B_{\mathrm{av}}(u) := \lim_{T\to\infty} \frac{1}{T}\int_0^T B(\tau, u)\, d\tau. \quad (1.21)$$

We refer to [SV] and [LM] for statements of this form in the context of ODE's. We also refer to [BCD], [BCDG], or more recently [CDG1], [CDG2] for this kind of averaging procedure in the context of laser-matter interaction, yet for infinite dimensional systems. We also refer to the deep paper [MS] in the context of fluid mechanics, for the use of similar averaging tools in infinite dimensional systems (here, very fine *resonance* questions are considered). We last refer to the deep paper [L] for similar averaging techniques, yet in a context where *continuously many* frequencies are involved, a situation in which, as in the present paper, an appropriate non-standard analytic framework needs to be set up to deal with the rapid oscilations. In any circumstance, we mention that a typical "ergodicity" assumption on the time-behavior of $B(\tau, u)$ is that B is periodic in time. A more general assumption is that $B(\tau, u)$ is quasi-periodic in time, which means $B(\tau, u) \equiv \mathcal{B}(\omega_1\tau, \dots, \omega_N\tau, u)$, where $\mathcal{B}$ is 1-periodic in its first N arguments, and the ω_i's are rationally independent frequencies. An even more general assumption is that $B(\tau, u)$ is almost-periodic in time, which somehow corresponds to the quasi-periodic framework with $N = +\infty$ independent frequencies. We refer to the sequel on that important situation, which turns out to provide the natural framework in the present context.

For this reason, and despite the differential system satisfied by the ϕ_p^ε's is infinite dimensional, it is reasonable to expect that the ϕ_p^ε's in (1.19) converge *at least* *formally* towards the ϕ_p's, solution to the *averaged system*

$$i\partial_t \phi_p(t,x) = H_x\, \phi_p(t,x) + \sum_{r\geq 0} \phi_r(t,x) \tag{1.22}$$
$$\times \lim_{T\to\infty} \frac{1}{T} \int_0^T \left[\left\langle F\left(\Big| \sum_{q\geq 0} \phi_q(t,x)\, \chi_q(z)\, \mathrm{e}^{-i\tau E_q} \Big|^2 \right) \,,\, \overline{\chi_r(z)}\, \chi_p(z) \right\rangle \mathrm{e}^{-i\tau(E_r - E_p)} \right] d\tau.$$

All these steps require some care yet, before becoming rigorous statements. In some sense, the goal of this paper is to rigorously prove the convergence towards (1.22), and even more to exhibit a functional framework that is well adapted to this infinite dimensional problem.

1.3 Rigorous results, and statement of our main Theorem

The difficulty in making the above statements correct is twofold. Firstly, the above procedure requires to decompose Ψ^ε over the χ_p's, hence to write down series expansions of the form $\sum_{r\geq 0} \cdots$ as in (1.22). However, it turns out to be extremely difficult to control the *convergence* of these series expansions, despite the fact that we have nice $l^2(L^2)$ bounds on the ϕ_p^ε's. This is essentially due to the lack of information on the behavior of the coefficient $\langle F(|\cdots|^2)\,,\, \overline{\chi_r}\, \chi_p \rangle$ appearing above, for large values of r and p. Indeed, no orthogonality property is at hand to estimate this coefficient, except in the very special case where $\chi_p(z) = \exp(ipz)$, corresponding to periodic boundary conditions on z (we may yet refer to W.-M. Wang's delicate analysis [W1], [W2], of factors of the form $\displaystyle\int_{\mathbb{R}} \chi_p(x)\, \chi_q(x)\, \chi_r(x)\, \chi_s(x)\, dx$ - $p, q, r, s \in \mathbb{N}$ - in the case when the χ_p's are the eigenfunctions of the harmonic oscillator). Secondly, there is in fact a deeper difficulty. Indeed, in order to quantitatively prove the convergence of systems of the form (1.20) towards (1.21), one usually needs *small denominator estimates*. They turn out to be extremely difficult to recover in the present context, and in truth they are very probably false. For instance, in the reference situation where $F(u) = u$, equation (1.19) takes the simpler form

$$i\partial_t \phi_p^\varepsilon(t,x) = H_x\, \phi_p^\varepsilon + \sum_{r,s,q\geq 0} \phi_r^\varepsilon(t,x)\, \phi_q^\varepsilon(t,x)\, \overline{\phi_s^\varepsilon}(t,x)\, \mathrm{e}^{-it(E_q - E_s + E_r - E_p)/\varepsilon} \, \langle \chi_q\, \chi_r \,,\, \chi_s\, \chi_p \rangle.$$

As a consequence, the averaged system on the ϕ_p's is the same, up to the fact that the sum $\sum_{r,s,q\geq 0} \cdots$ eventually needs to be replaced by $\sum_{r,s,q\geq 0} \mathbf{1}[E_q - E_s + E_r - E_p = 0]$. Yet rigorously proving the associated convergence result requires to have has some lower bound on

$$\frac{\mathbf{1}[E_q - E_s + E_r - E_p \neq 0]}{E_q - E_s + E_r - E_p},$$

usually Diophantine estimates or like. However, except in the very special case where H_z is the harmonic oscillator for which the E_p's are known and have the value $E_p = 2p + 1$, such estimates are generally not at hand.

These two difficulties make it necessary to find an alternative route.

One such alternative way exists in the simplified situation where the initial datum lies in a *definite* energy level, or, more precisely, the case when the initial datum that lies in the fundamental energy level,

$$\Psi^\varepsilon(0,x,z) = \Psi_0(x,z) = \psi_0(x)\chi_0(z).$$

In this simpler case, it has been proved in [BMSW] that, for later times, the solution $\Psi^\varepsilon(t,x,z)$ to (1.11) remains of the form

$$\Psi^\varepsilon(t,x,z) = \psi_0^\varepsilon(t,x)\chi_0(z) + \text{small remainder},$$

thanks to an energy estimate. As a consequence, the sums entering (1.17), (1.19), and (1.22) turn out to actually contain *one single term* in that case. This is the key point. It obviously allows to circumvent all the above mentioned difficulties, and the limiting model is, in that case, a single, nonlinear, Schrödinger equation, of the form

$$i\partial_t\phi_0(t,x) = H_x\phi_0 + F_{\text{av}}(|\phi_0|^2)\,\phi_0.$$

Here, the new, averaged nonlinearity F_{av} is given, after the averaging procedure, by

$$F_{\text{av}}(u) := \left\langle F\left(u\,|\chi_0(z)|^2\right)\,,\,|\chi_0(z)|^2\right\rangle.$$

This gives a rigorous statement that fully justifies the heuristic limit (1.22) *in that particular case.*

Here, we definitely want to place ourselves in a situation where $\Psi^\varepsilon(t,x,z)$ contains many energy levels, a generic situation. As we said, the procedure of explicitly decomposing Ψ^ε over the χ_p's leads to hard small denominators difficulties, and the convergence of the sums entering the expected limiting system (1.22) is far from obvious. For this reason, we adopt the following completely different point of view.

Instead of filtering out the oscillations in (1.11) *after* the projection over the χ_p's, which leads to (1.19), we rather do it *without* projecting. For that reason, we define the new unknown

$$\Phi^\varepsilon(t,x,z) := \exp(+itH_z/\varepsilon)\,\Psi^\varepsilon(t,x,z), \tag{1.23}$$

in analogy with (1.18). It satisfies

$$i\partial_t\Phi^\varepsilon(t,x,z) = H_x\Phi^\varepsilon + \mathrm{e}^{+itH_z/\varepsilon}\,F\left(\left|\mathrm{e}^{-itH_z/\varepsilon}\,\Phi^\varepsilon\right|^2\right)\,\mathrm{e}^{-itH_z/\varepsilon}\,\Phi^\varepsilon. \tag{1.24}$$

In other words, introducing the function

$$\tau \mapsto G(\tau,u) := \mathrm{e}^{+i\tau H_z}\,F\left(\left|\mathrm{e}^{-i\tau H_z}\,u\right|^2\right)\mathrm{e}^{-i\tau H_z}\,u, \tag{1.25}$$

equation (1.24) reads

$$i\partial_t \Phi^\varepsilon(t,x,z) = H_x \Phi^\varepsilon + G\left(\frac{t}{\varepsilon}, \Phi^\varepsilon(t)\right). \tag{1.26}$$

This is an infinite dimensional ODE, which is still of the form (1.20).

The key point lies in the observation that, for any given function $u(x,z)$ having reasonable Sobolev-like regularity (namely $u \in B_m$ for some $m > 3/2$, see (1.13)), the to-be-averaged function $G(\tau, u)$ is *almost-periodic in time, with values in the Sobolev space* B_m.

The proof of these two facts is *not* obvious, and we refer to section 2 for the analysis and identification of the Sobolev spaces B_m, as well as to section 3 for the definition and functional analytic properties of almost periodic functions. The almost-periodicity of $G(\tau,u)$ roughly means that $G(\tau,u)$ has *countably* many frequencies in τ, which in turn translates the fact that the spectrum of H_z is discrete as well: in view of definition (1.25) indeed, the oscillations of $G(\tau,u)$ are only created by those of the propagator $\mathrm{e}^{\pm i\tau H_z}$ (the latter are discrete), appropriately combined with the nonlinearity $F(|u|^2)\,u$ (and almost periodicity usually is stable upon composition with nonlinearities).

The interesting fact about almost-periodic functions is, they do possess a well defined *long time average*, and the formula

$$G_{\mathrm{av}}(u) := \lim_{T\to\infty} \frac{1}{T}\int_0^T G(\tau,u)\,d\tau \tag{1.27}$$

makes sense in B_m. Of course, the convergence rate in (1.27) is $o(1)$ only, contrary to *periodic* functions, for which the convergence rate is $O(1/T)$: the point is, the long time average *exists*, *beyond* any “small denominator” consideration or like.

In any circumstance, the limiting equation for $\Phi = \lim \Phi^\varepsilon$ now naturally reads

$$i\partial_t \Phi(t,x,z) = H_x \Phi + G_{\mathrm{av}}(\Phi). \tag{1.28}$$

The present paper is devoted to rigorously proving the convergence of (1.11), or equivalently (1.26), towards (1.28). Note that equation (1.28) gives a rigorous statement corresponding to the heuristic limit (1.22) discussed before. Note also that the observation according to which we are here dealing with almost-periodic functions (hence the possibility to average in time), with values in a good Sobolev space (hence the possibility to do nonlinear analysis), are the two crucial ingredients in the present paper. They are rigorously stated in Proposition 3.3, resp. Proposition 2.1, and the associated proofs are given all through section 3, resp. section 2.

To summarize, in this paper, we prove the following

Main Theorem
Take $m > 3/2$. Take a function $\Psi_0(x,z)$ having the Sobolev-like regularity,

$$\Psi_0(x,z) \in B_m := \Big\{ u \in L^2(\mathbb{R}^3), \text{ s.t. } H_x^{m/2} u \in L^2(\mathbb{R}^3) \text{ and } H_z^{m/2} u \in L^2(\mathbb{R}^3) \Big\}.$$

Define $\Psi^\varepsilon(t,x,z)$ *as the solution to*

$$i\partial_t \Psi^\varepsilon = H_x \Psi^\varepsilon + \frac{1}{\varepsilon} H_z \Psi^\varepsilon + F\left(|\Psi^\varepsilon|^2\right)\,\Psi^\varepsilon, \quad \Psi^\varepsilon(0,x,z) = \Psi_0(x,z).$$

Equivalently, define the filtered function $\Phi^\varepsilon(t,x,z) = \exp(+itH_z/\varepsilon)\,\Psi^\varepsilon$ *as the solution to*

$$i\partial_t \Phi^\varepsilon = H_x \Phi^\varepsilon + G\left(\frac{t}{\varepsilon}, \Phi^\varepsilon\right), \quad \Phi^\varepsilon(0,x,z) = \Psi_0(x,z),$$

where $G(\tau,u) = \mathrm{e}^{+i\tau H_z}\, F\left(\left|\mathrm{e}^{-i\tau H_z}\, u\right|^2\right)\,\mathrm{e}^{-i\tau H_z} u$*. Lastly, define* $\Phi(t,x,z)$ *as the solution to the averaged equation*

$$i\partial_t \Phi = H_x \Phi + G_{\mathrm{av}}(\Phi), \quad \Phi(0,x,z) = \Psi_0(x,z),$$

where $G_{\mathrm{av}}(u) = \lim\limits_{T\to\infty} (1/T) \displaystyle\int_0^T G(\tau,u)d\tau$ *in* B_m*. Then, the following holds*

(i) There is a $T_0 > 0$*, depending only on* $\|\Psi_0\|_{B_m}$ *and on the nonlinear function* F*, such that* $\Psi^\varepsilon(t)$*,* $\Phi^\varepsilon(t)$*, and* $\Phi(t)$ *exist and possess the smoothness* $C^0([0,T_0]; B_m)$*, independently of* ε*. Besides,* B_m *is a Hilbert space and an algebra, when endowed with either of the norms (1.14) or (1.15).*

(ii) The following convergence holds

$$\|\Phi^\varepsilon - \Phi\|_{C^0([0,T_0];B_m)} \underset{\varepsilon\to 0}{\longrightarrow} 0.$$

(iii) The solution $\Phi(t)$ *to the averaged system has the following conserved quantities*

$$\|\Phi(t)\|_{L^2(\mathbb{R}^3)} = \mathrm{const}, \quad \langle \Phi(t), H_z \Phi(t)\rangle_{L^2(\mathbb{R}^3)} = \mathrm{const},$$
$$\left\langle H_x^{1/2}\,\Phi(t)\,,\, H_x^{1/2}\Phi(t)\right\rangle^2_{L^2(\mathbb{R}^3)} + \int_{\mathbb{R}^3} \mathcal{G}_{\mathrm{av}}(\Phi(t))\,dx\,dz = \mathrm{const},$$

where $\mathcal{G}_{\mathrm{av}}(\Psi)$ *is defined, for any* $\Psi \in B_m$*, as*

$$\mathcal{G}_{\mathrm{av}}(\Psi) := \lim_{T\to\infty} \frac{1}{T}\int_0^T \mathcal{G}\left(\left|\mathrm{e}^{-i\tau H_z}\,\Psi\right|^2\right) d\tau, \qquad \text{and, } \mathcal{G}(u) := \int_0^u F(v)\,dv.$$

Remarks on the Main Theorem:

• Obviously, after projecting Φ on the χ_p's, system (1.26) may be seen as an infinite system of coupled nonlinear Schrödinger equations, involving the quantities $\phi_p(t,x) := \langle \Phi, \chi_p\rangle$. The underlying system coincides with the formally expected system (1.22). This point is discussed further in the last section 5. We actually give at the end of this paper examples for which G_{av} is an explicitly computable nonlinearity.

• Needless to say, our main Theorem gives, as a particular case, the results obtained in [BMSW] when Ψ_0 is parallel with χ_0. Yet the (not to be improved) $o(1)$ convergence rate of our Theorem does not allow to recover the better convergence rates obtained in [BMSW] in this special situation.
• Note also that the above Theorem completely describes the asymptotic behavior of Ψ^ε, namely $\Psi^\varepsilon(t,x,z) \sim \exp(-itH_z/\varepsilon)\,\Phi(t,x,z)$ as $\varepsilon \to 0$.
• The reader's attention is drawn to the fact that the averaged system $i\partial_t\Phi = H_x\Phi + G_{\rm av}(\Phi)$ still is posed in the three dimensional space $\mathbb{R}^3$. It however entails a trivial dynamics in the vertical, z direction, which only plays the role of a parameter. Technically, factorizing out this z dependence is done by projecting the averaged system over the basis of the χ_p's.
• Point (iii) of the Theorem gives conservation of mass and energy H_z in z. The latter is natural since the dynamics of Φ eventually is trivial in the z direction. Point (iii) also gives the conservation of *total* energy in x. This piece of information may be useful when the nonlinearity F has definite *sign* properties, and the above local-in-time convergence results, may be turned into global ones. $\square$

The present paper is organized as follows.

In section 2, we identify the Sobolev scale associated with the non-negative, self-adjoint operators H_x and H_z. In particular, we establish the equivalence of both norms (1.14) and (1.15) for B_ℓ. We deduce the fact that Ψ^ε is uniformly bounded in $C^0([0,T_0];B_m)$. The main result of this section are Theorem 2.1, Proposition 2.5 and Corollary 2.6.

In section 3, we recall some known facts about almost-periodic functions, and properly define the space of almost-periodic functions with values in B_ℓ. This allows to prove that the averaged quantity $G_{\rm av}(\Psi)$ in the Main Theorem does exist, and enjoys nice functional properties. We also deduce that the solution Φ to the averaged system $i\partial_t\Phi = H_x\Phi + G_{\rm av}(\Phi)$ exists, possesses the claimed smoothness, and satisfies the conservation laws of point (iii). The main results of this section are Theorem 3.3 and Proposition 3.4.

In section 4, using the results of sections 2 and 3, we completely prove the convergence $\Phi^\varepsilon \to \Phi$ announced in the Main Theorem. Our proof relies on an adaptation of the averaging procedure for ODE's, for which we refer to [SV]. We do have a convergence rate that is slightly more precise than $o(1)$. Such an adaptation has been previously exploited in [BCD], [BCDG] in the context of laser-matter interaction, for which the natural model is an infinite dimensional system (a PDE).

Last, section 5 is devoted to the application of our Main Theorem in the case of the simplest model of Bose condensation, namely the cubic Schrödinger equation with harmonic confinement, for which we have $F(u) = u$, $H_x = -\Delta_x + |x|^2$, $H_z = -\partial^2/\partial z^2 + |z|^2$.

2 Sobolev scale adapted to H_x and H_z

In this section, we identify the Sobolev scale adapted to H_x and H_z. Specifically, given any real number $\ell \geq 0$, we completely identify the norm

$$\begin{aligned}\|u\|_{B_\ell}^2 &:= \|u\|_{L^2(\mathbb{R}^3)}^2 + \|H_x^{\ell/2}u\|_{L^2(\mathbb{R}^3)}^2 + \|H_z^{\ell/2}u\|_{L^2(\mathbb{R}^3)}^2, \\ &:= \|u\|_{L^2(\mathbb{R}^3)}^2 + \left\|(-\Delta_x + V(x))^{\ell/2}\ u\right\|_{L^2(\mathbb{R}^3)}^2 + \left\|\left(-\partial^2/\partial z^2 + V_{\rm c}(z)\right)^{\ell/2}\ u\right\|_{L^2(\mathbb{R}^3)}^2,\end{aligned}$$

whenever u is smooth enough. Our main result asserts the following equivalence between norms, valid for any real number $\ell \geq 0$,

$$\begin{aligned}\|u\|_{B_\ell}^2 \sim \|u\|_{L^2(\mathbb{R}^3)}^2 &+ \|\,(-\Delta_x)^{\ell/2}\ u\|_{L^2(\mathbb{R}^3)}^2 + \|\,\left(-\partial^2/\partial z^2\right)^{\ell/2}\ u\|_{L^2(\mathbb{R}^3)}^2 \\ &+ \|V(x)^{\ell/2}\,u\|_{L^2(\mathbb{R}^3)}^2 + \|V_{\rm c}(z)^{\ell/2}\,u\|_{L^2(\mathbb{R}^3)}^2, \end{aligned} \tag{2.1}$$

where, the symbol $\sim$ means that there are constants $c_0 > 0$ and $c_1 > 0$ such that $c_0 \times$ (r.h.s. of (2.1)) $\leq$ (l.h.s. of (2.1)) $\leq c_1 \times$ (r.h.s. of (2.1)), independently of u.

The identification of $\|u\|_{B_\ell}$ is a technically delicate, yet absolutely crucial step in the present paper. Indeed, the only uniform bound at hand on Ψ^ε, solution to (1.11), reads

$$\|\Psi^\varepsilon(t,x,z)\|_{L^2(\mathbb{R}^3)} + \|H_x^{m/2}\Psi^\varepsilon(t,x,z)\|_{L^2(\mathbb{R}^3)} + \|H_z^{m/2}\Psi^\varepsilon(t,x,z)\|_{L^2(\mathbb{R}^3)} = O(1),$$

on some non-trivial time interval $t \in [0, T_0]$, whenever the initial datum Ψ_0 belongs to B_m ($m > 3/2$). All other energy estimates (typically obtained by applying the operators ∂_x^α, ∂_z^α, $|x|^\alpha$, or $|z|^\alpha$ to the equation (1.11), and performing the natural integration by parts which lead to an L^2 bound on quantities like $\partial_x^{\alpha/2}\Psi^\varepsilon$ or so), give rise to commutators, hence diverging factors of the order $O(1/\varepsilon)$, due to the fast factor H_z/ε in (1.11). Hence they only give access to bounds of the size $O(1/\varepsilon)$ as well, a useless information.

The key tool we use to prove the equivalence (2.1) is the Weyl-Hörmander calculus, see *e.g.* [BC]. Let us comment on that point, keeping the discussion at a rather informal level for the time being.

In terms of *symbols* (in the sense of pseudodifferential calculus, for *some* pseudodifferential calculus to be precised below), assertion (2.1) is fairly natural. Indeed, the principal symbol of $1 + H_x^\ell + H_z^\ell$ is[1]

$$\sigma\left(1 + H_x^\ell + H_z^\ell\right)(x,z,\xi,\zeta) \equiv 1 + \left[\xi^2 + V(x)\right]^\ell + \left[\zeta^2 + V_{\rm c}(z)\right]^\ell,$$

where ξ and ζ are the Fourier variables associated with x resp. z, while the principal symbol of $1 + (-\Delta_x)^\ell + (-\partial_z^2)^\ell + V(x)^\ell + V_{\rm c}(z)^\ell$ is[2]

$$\sigma\left(1 + D_x^{2\ell} + D_z^{2\ell} + V(x)^\ell + V_{\rm c}(z)^\ell\right)(x,z,\xi,\zeta) \equiv 1 + \xi^{2\ell} + \zeta^{2\ell} + V(x)^\ell + V_{\rm c}(z)^\ell.$$

[1] From now on, given any N, and given any vector $y \in \mathbb{R}^N$, we use the notation $y^2 \equiv |y|^2$. Similarly we set $y^{2\ell} \equiv |y|^{2\ell}$ whenever $\ell \in \mathbb{R}$.

[2] where as usual $D_x \equiv -i\partial_x$ and $D_z = -i\partial_z$.

Using the identification of the operators with their associated principal symbols, the whole equivalence (2.1) eventually (and informally) reduces to the existence of positive, universal constants c_0 and c_1 such that

$$c_0 \leq \frac{1+\left[\xi^2+V(x)\right]^{\ell}+\left[\zeta^2+V_{\rm c}(z)\right]^{\ell}}{1+\xi^{2\ell}+\zeta^{2\ell}+V(x)^{\ell}+V_{\rm c}(z)^{\ell}} \leq c_1, \tag{2.2}$$

independently of $(x, z, \xi, \zeta) \in \mathbb{R}^3 \times \mathbb{R}^3$. The point is, passing from the equivalence between *symbols* (2.2) to the equivalence between *norms* (2.1), one needs to have a proper quantization of symbols, hence a proper pseudodifferential calculus. In other words, one needs appropriate *weights* together with appropriate *metrics* to deduce (2.1) from (2.2) using a pseudodifferential machinery.

Now, the whole difficulty lies in the fact that the standard pseudodifferential calculus, based on the standard metrics

$$dx^2+dz^2+\frac{d\xi^2+d\zeta^2}{1+\xi^2+\zeta^2}$$

can only give access to usual Sobolev-like norms, where only powers of $-\Delta_x$, $-\partial_z^2$ are kept track of, or equivalently, one only takes into account powers of ξ^2 and ζ^2 as $|\xi|$ and/or $|\zeta|$ go to infinity. However, going from (2.2) to (2.1) requires not only counting powers of $-\Delta_x$, $-\partial_z^2$ (*i.e.* powers of ξ^2 and ζ^2), but also powers of $V(x)$ and $V_{\rm c}(z)$ as $|x|$ and $|z|$ go to infinity. Recall indeed that $V_{\rm c}$ and V are assumed confining, a key difficulty in the present perspective.

This is the reason why we need to consider an appropriate metric that keeps track of both aspects, and eventually develop the associated pseudodifferential machinery, based on the Weyl-Hörmander calculus.

Our main result in this section is the following

Theorem 2.1 [equivalence of norms].
Let $\ell \geq 0$ be a real number. Recall $H_x = -\Delta_x + V(x)$ and $H_z = -\partial^2/\partial z^2 + V_{\rm c}(z)$. The following two norms[3] are equivalent,

$$\begin{aligned} N_1(u) &:= \|u\|_{L^2(\mathbb{R}^3)} + \|H_x^{\ell/2}\, u\|_{L^2(\mathbb{R}^3)}, + \|H_z^{\ell/2}\, u\|_{L^2(\mathbb{R}^3)}, \\ N_2(u) &:= \|u\|_{H^{\ell}(\mathbb{R}^3)} + \|V(x)^{\ell/2}\, u\|_{L^2(\mathbb{R}^3)} + \|V_{\rm c}(z)^{\ell/2}\, u\|_{L^2(\mathbb{R}^3)}. \end{aligned}$$

Remark. As we already stressed, the proof of Theorem 2.1 is *not* direct, and our proof uses an appropriate pseudodifferential calculus adapted to the symbol $\xi^2+\zeta^2+V(x)+V_{\rm c}(z)$, see Bony and Chemin's work [BC]. This is also the route chosen by B. Helffer in the earlier work [He]: in this paper, B. Helffer completely identifies the Sobolev scale associated with the harmonic oscillator $-\Delta_x+|x|^2$, and the analogue of Theorem 2.1 is proved there in this very case. We stress that even the identification

[3]Note that the norms N_1 and N_2 are not labeled by ℓ, though they obviously depend on this parameter. This is done not to overweight notation.

of the norm $\|(1-\Delta_x+|x|^2)^\ell\, u\|_{L^2}$ with the obvious $\|u\|_{L^2}+\left\|(-\Delta_x)^\ell\ u\right\|_{L^2}+\|x^{2\ell}\, u\|_{L^2}$ is *not* an easy result: it readily requires developping a pseudodifferential calculus that is adapted to the symbol $1+\xi^2+x^2$.

In particular, a pedestrian proof of Theorem 2.1, directly using commutators of both operators $-\partial_z^2$ and $V_{\rm c}(z)$, respectively $-\Delta_x$ and $V(x)$, probably is *out of reach*, even for integer values of ℓ. Indeed, such an analysis anyhow fails when dealing with factors of the form

$$\left\|\left(-\partial_z^2\right)^{(\ell-k)/2}\ V_{\rm c}(z)^{k/2}\, u\right\|_{L^2(\mathbb{R}^3)}\ \text{ or }\ \left\|V_{\rm c}(z)^{k/2}\ \left(-\partial_z^2\right)^{(\ell-k)/2}\ u\right\|_{L^2(\mathbb{R}^3)},$$

whenever $0\leq k\leq \ell$, and when it comes to trying to control such terms with the help of the mere term

$$\|u\|_{L^2(\mathbb{R}^3)}+\left\|V_{\rm c}(z)^{\ell/2}\right\|_{L^2(\mathbb{R}^3)}+\left\|\left(-\partial_z^2\right)^{\ell/2}\ u\right\|_{L^2(\mathbb{R}^3)}.$$

$\square$

Remark. Our identification of $\|u\|_{B_\ell}$ uses the fact that $V_{\rm c}(z)$ (and $V(x)$) is confining, see (1.2). Even more, a crucial role is played by the reinforced assumptions (1.4) through (1.6), according to which $V_{\rm c}(z)$ behaves like a symbol at infinity in z, whose growth is at most polynomial (and similarly for $V(x)$). Note however that, would $V_{\rm c}(z)$ (and/or $V(x)$) be uniformly bounded together with all its derivatives (instead of being confining), the results below would hold just the same, the proofs being actually simpler. $\square$

2.1 Some basic facts about Weyl-Hörmander calculus

Our proof of Theorem 2.1 closely follows ideas developed by Bony and Chemin in [BC], and more recently by Helffer and Nier [HN]. We first recall here some basic facts about Weyl-Hörmander calculus.

Weyl-Hörmander calculus first requires a metric, and an appropriate weight function, both being required to satisfy some mild assumptions (slowness, temperance, uncertainty principle, and admissibility - see below, see also [BC]). For instance, the standard calculus, which is a particular case of Weyl-Hörmander calculus, is based on the metric $dx^2+dz^2+(d\xi^2+d\zeta^2)/(1+\xi^2+\zeta^2)$, and on the associated weights $(1+\xi^2+\zeta^2)^\ell$ ($\ell\in\mathbb{R}$). This is the calculus which is adapted when dealing with standard Sobolev spaces $H^\ell(\mathbb{R}^3)$.

In the present text, we define the weight

$$M(x,z,\xi,\zeta):=\sqrt{1+\xi^2+\zeta^2+V(x)+V_{\rm c}(z)}. \tag{2.3}$$

We also define the metric

$$g(x,z,\xi,\zeta):=dx^2+dz^2+\frac{d\xi^2+d\zeta^2}{M^2(x,z,\xi,\zeta)}, \tag{2.4}$$

meaning that for any $(x', z', \xi', \zeta') \in \mathbb{R}^3 \times \mathbb{R}^3$, we set $g(x, z, \xi, \zeta)(x', z', \xi', \zeta') = (x')^2 + (z')^2 + [(\xi')^2 + (\zeta')^2]/M^2(x, z, \xi, \zeta)$. Choosing to work within the metric g equivalently means that for any given $\ell \in \mathbb{R}$, we shall deal with the class $S(M^\ell, g)$ of symbols $a(x, z, \xi, \zeta) \in C^\infty(\mathbb{R}^3 \times \mathbb{R}^3)$ such that

$$\forall \alpha, \beta \in \mathbb{N}^3, \quad \exists C_{\alpha,\beta} > 0, \quad \forall (x, z, \xi, \zeta) \in \mathbb{R}^3 \times \mathbb{R}^3,$$
$$\left| \partial^\alpha_{x,z}\, \partial^\beta_{\xi,\zeta} a(x, z, \xi, \zeta) \right| \leq C_{\alpha,\beta}\, M(x, z, \xi, \zeta)^{\ell - |\beta|}. \tag{2.5}$$

The idea of using this class of symbols, *i.e.* this weight function and this metric, is actually borrowed from [HN]. The class $S(M^\ell, g)$ is a Fréchet space when endowed with the semi-norms $\left\| M^{-\ell+|\beta|} \partial^\alpha_{x,z}\, \partial^\beta_{\xi,\zeta} a \right\|_{L^\infty(\mathbb{R}^3)}$.

Following the usual terminology (see *e.g.* [BC]), we first claim that the metric g is *slow*, *temperate*, and it satisfies the *uncertainty principle*:

- The fact that g satisfies the uncertainty principle comes from the following easy computation. We first define the metric g^σ which is dual to g with respect to the symplectic form $\sigma = d(x, z) \wedge d(\xi, \zeta)$, see [BC], *i.e.* we set

  $$g^\sigma(x, z, \xi, \zeta)(\cdot) := \sup_{(x', z', \xi', \zeta') \neq 0} \frac{[\cdot, (x', z', \xi', \zeta')]^2}{g(x, z, \xi, \zeta)(x', z', \xi', \zeta')},$$

  where the Poisson bracket $[(x, z, \xi, \zeta), (x', z', \xi', \zeta')]$ equals $(\xi, \zeta) \cdot (x', z') - (x, z) \cdot (\xi', \zeta')$ as usual. In the present case, g^σ is easily computed, namely

  $$g^\sigma(x, z, \xi, \zeta) := M^2(x, z, \xi, \zeta)\, (dx^2 + dz^2) + d\xi^2 + d\zeta^2.$$

  Now, the uncertainty principle requires (see [BC])

  $$g \leq g^\sigma.$$

  In the present case, this assertion reduces to observing

  $$M \geq 1.$$

- The slowness of g comes from the fact that there exists $c > 0$ such that

  $$\left(\frac{M(x, z, \xi, \zeta)}{M(x', z', \xi', \zeta')} \right)^{\pm 1} \leq c,$$
  $$\text{whenever } |(x, z) - (x', z')| \leq c^{-1} \text{ and } |(\xi, \zeta) - (\xi', \zeta')| \leq c^{-1}\, M(x, z, \xi, \zeta).$$

  The proof of the latter assertion, which uses the reinforced assumptions (1.4) through (1.6), as well as (1.8) through (1.10), is left to the reader, see also [HN]. It implies that g is slow, *i.e.* there exists a $c > 0$ such that for any (x, z, ξ, ζ) and (x', z', ξ', ζ'), we have

  $$\sup_{(x'', z'', \xi'', \zeta'') \in \mathbb{R}^6} \left(\frac{g(x, z, \xi, \zeta)(x'', z'', \xi'', \zeta'')}{g(x', z', \xi', \zeta')(x'', z'', \xi'', \zeta'')} \right)^{\pm 1} \leq c$$
  $$\text{whenever } g(x, z, \xi, \zeta)(x - x', z - z', \xi - \xi', \zeta - \zeta') \leq c^{-1}.$$

- The fact that g is temperate comes from the existence of $c > 0$, $\nu > 0$, such that

$$\left(\frac{M(x,z,\xi,\zeta)}{M(x',z',\xi',\zeta')}\right)^{\pm 1} \leq c\,\left(1 + M(x,z,\xi,\zeta)^2\,\left[(x-x')^2 + (z-z')^2\right] + (\xi-\xi')^2 + (\zeta-\zeta')^2\right)^{\nu},$$

independently of $(x,z,\xi,\zeta) \in \mathbb{R}^3 \times \mathbb{R}^3$ and $(x',z',\xi',\zeta') \in \mathbb{R}^3 \times \mathbb{R}^3$. The proof of the latter assertion, which uses the reinforced assumptions (1.4) through (1.6), as well as (1.8) through (1.10), is left to the reader, see also [HN]. It implies g is temperate, namely there exists a c and a ν such that for any (x,z,ξ,ζ) and (x',z',ξ',ζ'), we have

$$\sup_{(x'',z'',\xi'',\zeta'')\in\mathbb{R}^6} \left(\frac{g(x,z,\xi,\zeta)(x'',z'',\xi'',\zeta'')}{g(x',z',\xi',\zeta')(x'',z'',\xi'',\zeta'')}\right)^{\pm 1} \leq c\,\left(1 + g^{\sigma}(x,z,\xi,\zeta)(x-x',z-z',\xi-\xi',\zeta-\zeta')\right)^{\nu}.$$

This being settled, we now assert that for any $\ell \in \mathbb{R}$, the weight M^ℓ is *admissible* for the metric g. This is our second claim. It comes from the following two assertions:

- for any $\ell \in \mathbb{R}$, there exists $c_\ell > 0$ (which depends on ℓ) such that

$$\left(\frac{M^\ell(x,z,\xi,\zeta)}{M^\ell(x',z',\xi',\zeta')}\right)^{\pm 1} \leq c_\ell,$$

$$\text{whenever } |(x,z)-(x',z')| \leq c_\ell^{-1} \text{ and } |(\xi,\zeta)-(\xi',\zeta')| \leq c_\ell^{-1}\,M(x,z,\xi,\zeta).$$

- for any $\ell \in \mathbb{R}$, there exists $c_\ell > 0$ and $\nu_\ell > 0$ (which depend on ℓ) such that

$$\left(\frac{M^\ell(x,z,\xi,\zeta)}{M^\ell(x',z',\xi',\zeta')}\right)^{\pm 1} \leq c_\ell\,\left(1 + M(x,z,\xi,\zeta)^2\,\left[(x-x')^2 + (z-z')^2\right] + (\xi-\xi')^2 + (\zeta-\zeta')^2\right)^{\nu_\ell},$$

independently of $(x,z,\xi,\zeta) \in \mathbb{R}^3 \times \mathbb{R}^3$ and $(x',z',\xi',\zeta') \in \mathbb{R}^3 \times \mathbb{R}^3$.

Lastly, there remains to observe that the value of the *gain* in the present calculus is, following Hörmander [Ho],

$$\lambda(x,z,\xi,\zeta) \equiv \left(\min_{(x',z',\xi',\zeta')\neq 0} \frac{g^{\sigma}(x,z,\xi,\zeta)(x',z',\xi',\zeta')}{g(x,z,\xi,\zeta)(x',z',\xi',\zeta')}\right)^{1/2} = M(x,z,\xi,\zeta). \quad (2.6)$$

Now, given the metric g (which is slow, temperate, and satisfies the uncertainty principle), and given the weight M^ℓ (which is admissible), to any symbol a in the class $S(M^\ell, g)$, Weyl-Hörmander calculus associates the *operator*

$u \in \mathcal{S}(\mathbb{R}^3) \mapsto a^{\mathrm{w}}\,u \in \mathcal{S}(\mathbb{R}^3)$ defined as

$$\left(a^{\mathrm{w}}\,u\right)(x) = \int_{\mathbb{R}^6} \mathrm{e}^{i(x-x')\cdot\xi + i(z-z')\cdot\zeta}\; a\left(\frac{x+x'}{2}, \frac{z+z'}{2}, \xi, \zeta\right)\,u(x',z')\,dx'\,dz'. \quad (2.7)$$

Operator a^{w} acts continuously on $\mathcal{S}(\mathbb{R}^3)$. Besides, the Weyl quantization has the following specific feature, which is implied by the particular symmetric arguments $(x+x')/2$ and $(z+z')/2$ in (2.7) :

the operator a^{w} is symmetric on $\mathcal{S}(\mathbb{R}^3)$ whenever a is real-valued.

In any circumstance, we shall make use of the following standard notation: when an operator A coincides with a^{w} for some $a \in S(M^\ell, g)$, we shall write $A \in \operatorname{Op} S(M^\ell, g)$.

One of the key result of Weyl-Hörmander's calculus is the following L^2 continuity statement, for which we refer to *e.g.* [BC]:

$$a \in S(1,g) \Rightarrow a^{\mathrm{w}} \in \mathcal{L}(L^2(\mathbb{R}^3)). \tag{2.8}$$

This statement extends the celebrated Calderon-Vaillancourt Theorem of standard pseudodifferential calculus. Note in passing that the above assertions prove that a real-valued symbol $a \in S(1,g)$ provides an essentially self-adjoint operator a^{w} in $\mathcal{L}(L^2(\mathbb{R}^3))$.

Naturally, the whole computational machinery of standard pseudodifferential calculus also extends to the present context. For instance, the composition rule of two operators a^{w} and b^{w} asserts that for $a \in S(M^m, g)$ and $b \in S(M^{m'}, g)$, there exists a $c \in S(M^{m+m'}, g)$ such that $a^{\mathrm{w}} \circ b^{\mathrm{w}} = c^{\mathrm{w}}$, and, for any $J \in \mathbb{N}$, we have the asymptotic expansion

$$\begin{aligned}
&c(x,z,\xi,\zeta) \equiv (a \sharp^{\mathrm{w}} b)\,(x,z,\xi,\zeta) \\
&= \sum_{j=0}^{J-1} \frac{\left(\frac{i}{2}\,[D_{x_1,z_1,\xi_1,\zeta_1}\ ,\ D_{x_2,z_2,\xi_2,\zeta_2}]\right)^j}{j!}\, a(x_1,z_1,\xi_1,\zeta_1)\, b(x_2,z_2,\xi_2,\zeta_2)\Big|_{\substack{(x,z,\xi,\zeta)=(x_1,z_1,\xi_1,\zeta_1)\\ =(x_2,z_2,\xi_2,\zeta_2)}} \\
&\qquad + R_J(a,b)(x,z,\xi,\zeta),
\end{aligned} \tag{2.9}$$

where $R_J \in S(M^{m+m'-J}, g)$, and $[.,.]$ again denotes the Poisson bracket. Naturally, the fact that R_J belongs to $S(M^{m+m'-J}, g)$, hence a gain of J factors M, comes from the relation (2.6). We again refer to [BC]. A consequence of the above expansion is that a similar result holds for the commutator $a^{\mathrm{w}} \circ b^{\mathrm{w}} - b^{\mathrm{w}} \circ a^{\mathrm{w}} = c^{\mathrm{w}}$ for some $c \in S(M^{m+m'-1}, g)$.

We last mention the following obvious fact, which actually is the whole motivation for introducing the weight M and the associated metric g. In the above defined language, we have

$$1 - \Delta_x - \partial_z^2 + V(x) + V_{\mathrm{c}}(z) = \left(1 + \xi^2 + \zeta^2 + V(x) + V_{\mathrm{c}}(z)\right)^{\mathrm{w}} \in \operatorname{Op} S(M^2, g).$$

2.2 Sobolev spaces associated with an admissible weight

Using the above defined language, proving the equivalence between norms (2.1) roughly reduces to proving the equivalence

$$\left\| \left((M^2)^{\mathrm{w}} \right)^{\ell/2} u \right\|_{L^2(\mathbb{R}^3)} \sim \left\| (M^\ell)^{\mathrm{w}}\, u \right\|_{L^2(\mathbb{R}^3)}$$

whenever $\ell \in \mathbb{R}$. This task, which is essentially performed in the next paragraph, requires some preliminary statements. We collect the necessary properties below.

Let now $\mathcal{M}(x, z, \xi, \zeta)$ be *any* weight which is admissible for the metric g. The text [BC] allows to define a Sobolev space associated with $\mathcal{M}$. The construction of Bony and Chemin is as follows. From now on, let us denote by

$$X = (x, z, \xi, \zeta)$$

a generic point in $\mathbb{R}^3 \times \mathbb{R}^3$. First, Bony and Chemin start with a g-partition of unity, namely a family of non-negative functions $\phi_X \in S(1, g)$, indexed by X, such that each ϕ_X has its support in the ball $B_g(X, r) := \{Y \in \mathbb{R}^6 \text{ s.t. } g(X)(Y - X) \leq r^2\}$, where the small parameter $r > 0$ is fixed at once (the very value of r depends on the constants appearing in the definition of the fact that g is slow and temperate), and one has the identity

$$\int_{\mathbb{R}^6} \phi_X(.)\ |\det(g(X))|^{1/2}\ dX = 1.$$

Here, $\det(g(X))$ denotes the determinant of the quadratic form $g(X)$. In this context, Bony and Chemin define the Sobolev space associated with the weight $\mathcal{M}(x, z, \xi, \zeta) \equiv \mathcal{M}(X)$, and denoted by $H(\mathcal{M}, g)$, as the set of functions $u = u(x, z)$ such that

$$\|u\|^2_{H(\mathcal{M},g)} \equiv \int_{\mathbb{R}^6} \mathcal{M}^2(X)\ \|\phi_X^{\mathrm{w}}\, u\|^2_{L^2(\mathbb{R}^3)}\ |\det(g(X))|^{1/2}\ dX < \infty. \tag{2.10}$$

Since the operator ϕ_X^{w} localizes u around the point X of phase-space, the set $H(\mathcal{M}, g)$ clearly extends the usual definition of the standard Sobolev spaces $H^s(\mathbb{R}^3)$ ($s \in \mathbb{R}$), in which case the weight $\mathcal{M}(X) = (1 + |\xi| + |\zeta|)^s$ is prescribed. We draw the reader's attention to the following point: our definition of $H(\mathcal{M}, g)$ uses the fact that the chosen metric g obviously is *strongly temperate* in the language of Bony and Chemin, (see [BC], Definition 4.1, Definition 7.1, and Theorem 7.8). The definition of $H(\mathcal{M}, g)$ would be slightly more involved without this property. Note that the natural orthogonality property ensures that definition (2.10) does not depend on the chosen partition of unity.

The above general definition allows to define the Sobolev scale associated with the weights M^ℓ ($\ell \in \mathbb{R}$). With the above notation, we clearly have

$$\forall \ell \leq \ell', \quad \mathcal{S}(\mathbb{R}^3) \subset H(M^\ell, g) \subset H(M^{\ell'}, g) \subset \mathcal{S}'(\mathbb{R}^3).$$

It is also proved in [BC] that

$$H(1, g) = L^2(\mathbb{R}^3),$$

while

$$\forall \ell, \ell', \quad \forall a \in S(M^\ell, g), \quad a^{\mathrm{w}} \in \mathcal{L}\left(H(M^{\ell'}, g); H(M^{\ell'-\ell}, g)\right).$$

Now, using this language, the following technical statements turn out to be a very important preliminary result in the sequel.

Proposition 2.2 **[self-adjointness of the operators $\left(M^\ell\right)^{\mathrm{w}}$, $\ell \in \mathbb{R}$].**
Given $\ell \in \mathbb{R}$, the operator $\left(M^\ell\right)^{\mathrm{w}}$ with domain $D\left(\left(M^\ell\right)^{\mathrm{w}}\right) = H(M^\ell, g)$ is self-adjoint on $L^2(\mathbb{R}^3)$. Besides, the norms $\|u\|_{H(M^\ell,g)}$ and $\| \left(M^\ell\right)^{\mathrm{w}} u\|_{L^2(\mathbb{R}^3)}$ are equivalent.

Remark. The reader's attention is drawn to the fact that operator $\left(M^\ell\right)^{\mathrm{w}}$ does not coincide with $(M^{\mathrm{w}})^\ell = (1 - \Delta_x - \partial_z^2 + V(x) + V_{\mathrm{c}}(z))^{\ell/2}$ (the latter being obviously self-adjoint thanks to the usual functional calculus for self-adjoint operators). $\square$

Proof of Proposition 2.2.
The proof relies on elliptic regularity, and on the existence of a left and right parametrix for $\left(M^\ell\right)^{\mathrm{w}}$, see *e.g.* (2.11) below. We refer to [HN], Chapter 4, Proposition 4.5 for a proof. $\square$

Proposition 2.3 **[resolvent of $(M^2)^{\mathrm{w}}$].**
The operator $(M^2)^{\mathrm{w}} = 1 - \Delta_x - \partial_z^2 + V(x) + V_{\mathrm{c}}(z)$ is such that for any λ in the resolvent set of $(M^2)^{\mathrm{w}}$, the operator $\left[(M^2)^{\mathrm{w}} + \lambda\right]^{-1}$ belongs to $\mathrm{Op}\, S(M^{-2}, g)$. Besides, whenever $\lambda \geq 0$, the semi-norms of the symbol of $\left[(M^2)^{\mathrm{w}} + \lambda\right]^{-1}$ in the class $S(M^{-2}, g)$ are bounded independently of $\lambda \geq 0$.

Proof of Proposition 2.3.
The proof of this Proposition is an easy application of the Beals criterion. We refer to [HN], Chapter 4 for a proof. $\square$

2.3 Weyl-Hörmander calculus for fully elliptic operators: functional calculus

In this paragraph, and with the help of the previously stated results and notation, we complete the proof of the equivalence (whenever $\ell \in \mathbb{R}$)

$$\left\| \left((M^2)^{\mathrm{w}}\right)^\ell u \right\|_{L^2(\mathbb{R}^3)} \sim \left\| \left(M^\ell\right)^{\mathrm{w}} u \right\|_{L^2(\mathbb{R}^3)}$$

A symbol $a \in S(M^\ell, g)$ is said to be fully elliptic whenever there is a $c > 0$ such that the reverse bound

$$\left|a(x, z, \xi, \zeta)\right| \geq c\, M^\ell(x, z, \xi, \zeta)$$

holds true, independently of $(x, z, \xi, \zeta) \in \mathbb{R}^3 \times \mathbb{R}^3$. Typically, the symbol $1 + \xi^2 + \zeta^2 + V(x) + V_{\mathrm{c}}(z) \in S(M^2, g)$ is fully elliptic. In the context of standard pseudodifferential calculus, it is well-known that a fully elliptic symbol a is such that the operator a^{w} admits an inverse, the principal symbol of which is $1/a$. In the present context (and because our metric is strongly temperate - see [BC] Theorem 7.6), this result extends to symbols in the class $S(M^\ell, g)$: whenever $a \in S(M^\ell, g)$ is fully elliptic, there exists a $b \in S(M^{-\ell}, g)$ and a $c \in S(M^{-\ell}, g)$ such that

$$a^{\mathrm{w}} \circ b^{\mathrm{w}} = c^{\mathrm{w}} \circ a^{\mathrm{w}} = \mathrm{Id}. \tag{2.11}$$

Now, one of the important successes of standard pseudodifferential calculus is the so-called functional calculus of Helffer and Robert [HR] (see also [Ma] or [DmS] for a modern presentation). It typically asserts that, for any function $f \in C^\infty(\mathbb{R};\mathbb{R})$ and under mild assumptions on the *real-valued* symbol a, the operator[4] $f(a^{\mathrm{w}})$ still is a pseudodifferential operator. Besides, the principal symbol of $f(a^{\mathrm{w}})$ coincides with $f(a)$. The key ingredient to the proof of this fact is the so-called Helffer-Sjöstrand formula, which asserts that, for any self-adjoint operator A, we have

$$f(A) = \frac{1}{2\pi} \int_{\mathbb{C}} \frac{\partial \widetilde{f}}{\partial \overline{\lambda}}(\lambda)\ [A - \lambda]^{-1}\ d\lambda \wedge d\overline{\lambda},$$

where $\lambda \in \mathbb{C}$, the measure $d\lambda \wedge d\overline{\lambda}$ is the standard 2-dimensional volume in $\mathbb{C}$, and $\widetilde{f}(\lambda)$ denotes an almost-analytic extension of f over $\mathbb{C}$. We refer, *e.g.* to [DmS] on that point. Typically, the Helffer-Sjöstrand formula establishes that computing $f(A)$ roughly reduces to computing the resolvent $[A-\lambda]^{-1}$ for any $\lambda \in \mathbb{C}$. In turn, this observation is the key to establish that $f(a^{\mathrm{w}})$ is a pseudodifferential operator as is a^{w}, and that the complete symbol of $f(a^{\mathrm{w}})$ may be computed as a full asymptotic expansion, using the asymptotic expansion of the symbol of $[a^{\mathrm{w}} - \lambda]^{-1}$.

As we now show, the similar results hold and can be proved along the same lines in the context of Weyl-Hörmander calculus: we now identify $f\left((M^2(x,z,\xi,\zeta))^{\mathrm{w}}\right)$ whenever $f(x) \equiv x^\ell$.

Proposition 2.4 *Let $\ell \in \mathbb{R}$. Then $\left[(M^2)^{\mathrm{w}}\right]^\ell \in \mathrm{Op}\, S(M^{2\ell}, g)$. Besides, the following assertion holds true*

$$\left[\left(M^2\right)^{\mathrm{w}}\right]^\ell - \left(M^{2\ell}\right)^{\mathrm{w}} \in \mathrm{Op}\, S(M^{2\ell-1}, g)$$

Remark. Note that the information $\left[(M^2)^{\mathrm{w}}\right]^\ell \in \mathrm{Op}\, S(M^{2\ell}, g)$ is *not* obvious. Note also that the second assertion of the Proposition does not give the complete asymptotic expansion of $\left[(M^2)^{\mathrm{w}}\right]^\ell$, but only its principal symbol. This turns out to be enough for our purposes. Note finally that our proof does not use the Helffer-Sjöstrand formula, but a simpler, particular, version of it, borrowed from [Yo]: this simplified approach is borrowed from [HN]. □

Proof of Proposition 2.4.

Our proof follows [HN], Chapter 4, proof of Theorem 4.8.

Whenever $\lambda \geq 0$, we know from Weyl-Hörmander calculus (see formula (2.9)) that

$$\left[\left(M^2\right)^{\mathrm{w}} + \lambda\right] \circ \left[\left(M^2 + \lambda\right)^{-1}\right]^{\mathrm{w}} = \mathrm{Id} + R(\lambda),$$

[4]which is well-defined thanks to the usual functional calculus for self-adjoint operators.

where $R(\lambda) \in \operatorname{Op} S(M^{-1}, g)$ has a symbol whose semi-norms all have size $O((1+\lambda)^{-1})$. Besides, the semi-norms of $\left((M^2)^{\mathrm{w}} + \lambda\right)^{-1}$ have size $O(1)$, uniformly with respect to $\lambda \geq 0$, thanks to Proposition 2.3. As a consequence, we recover for any $\lambda \geq 0$,

$$\left[(M^2)^{\mathrm{w}} + \lambda\right]^{-1} - \left[\left(M^2 + \lambda\right)^{-1}\right]^{\mathrm{w}} = R'(\lambda) \in \operatorname{Op} S(M^{-3}, g),$$

where all seminorms of $R'(\lambda)$ have size $O((1+\lambda)^{-1})$.

This first observation readily allows to deduce the result of the Proposition for *integer* values of the parameter ℓ. Indeed, taking $\lambda = 0$ in the above formula gives

$$\left[(M^2)^{\mathrm{w}}\right]^{-1} = \left[M^{-2}\right]^{\mathrm{w}} + R'(0)$$

and, since $\left[(M^2)^{\mathrm{w}}\right]^{-1} \in \operatorname{Op} S(M^{-2}, g)$ while $R'(0) \in \operatorname{Op} S(M^{-3}, g)$, iterating the above formula $|\ell|$ times (when $\ell \leq 0$), or simply iterating $(M^2)^{\mathrm{w}}$ ℓ times (when $\ell \geq 0$) provides, in conjunction with standard Weyl-Hörmander calculus (see (2.9), the identity

$$\forall \ell \in \mathbb{Z}, \quad \left[(M^2)^{\mathrm{w}}\right]^{\ell} - \left[M^{2\ell}\right]^{\mathrm{w}} \in \operatorname{Op} S(M^{2\ell-1}, g).$$

Let us now come to the case of real, non-integer values of ℓ.

The previous result, when combined with standard Weyl-Hörmander calculus (formula (2.9)), allows to reduce the proof to the mere case when $-1 < \ell < 0$ (actually any non-empty interval of $\mathbb{R}$ would do as well). This being observed, we use the following formula, valid for $-1 < \ell < 0$

$$\left[(M^2)^{\mathrm{w}}\right]^{\ell} = -\frac{\sin(\ell\pi)}{\pi} \int_0^{+\infty} \lambda^{\ell} \, \left[(M^2)^{\mathrm{w}} + \lambda\right]^{-1} \, d\lambda. \tag{2.12}$$

This formula actually holds true when M^{w} is replaced by *any* self-adjoint operator, and we use here the result of Proposition 2.2. Now, we may write

$$\left[(M^2)^{\mathrm{w}} + \lambda\right]^{-1} = \left[\left(M^2 + \lambda\right)^{-1}\right]^{\mathrm{w}} + R'(\lambda),$$

and the semi-norms of the involved pseudodifferential operators on the right-hand-side have size $O((1+\lambda)^{-1})$. As a consequence, the integral in (2.12) does converge, and we have

$$\begin{aligned}
\left[(M^2)^{\mathrm{w}}\right]^{\ell} &= -\frac{\sin(\ell\pi)}{\pi} \int_0^{+\infty} \lambda^{\ell} \, \left[(M^2)^{\mathrm{w}} + \lambda\right]^{-1} \, d\lambda \\
&= -\frac{\sin(\ell\pi)}{\pi} \int_0^{+\infty} \lambda^{\ell} \left(\left[\left(M^2 + \lambda\right)^{-1}\right]^{\mathrm{w}} + R'(\lambda)\right) \, d\lambda \\
&= \left(M^{2\ell}\right)^{\mathrm{w}} - \frac{\sin(\ell\pi)}{\pi} \int_0^{+\infty} \lambda^{\ell} \underbrace{R'(\lambda)}_{\in S(M^{-3}, g), \text{ with semi-norms } = O((1+\lambda)^{-1})} d\lambda,
\end{aligned}$$

where we again used formula (2.12) with $(M^2)^{\mathrm{w}}$ replaced by its symbol $M^{2\ell}$ to identify the term $\big(M^{2\ell}\big)^{\mathrm{w}}$. As a consequence, we have proved

$$\big[\big(M^2\big)^{\mathrm{w}}\big]^{\ell} - \big(M^{2\ell}\big)^{\mathrm{w}} \in S(M^{-3}, g) \text{ whenever } -1 < \ell < 0.$$

This finishes the proof of the Proposition. $\square$

2.4 Proof of Theorem 2.1

Take any $\ell \geq 0$ and $u \in B_\ell$. The proof is decomposed into two steps.

First step

We have,

$$\begin{aligned}
&\Big\| \big(1 - \Delta_x - \partial_z^2 + V(x) + V_{\mathrm{c}}(z)\big)^{\ell/2}\ u \Big\|^2_{L^2(\mathbb{R}^3)} \\
&\quad = \langle \big(1 - \Delta_x - \partial_z^2 + V(x) + V_{\mathrm{c}}(z)\big)^{\ell}\ u\,,\ u \rangle_{L^2(\mathbb{R}^3)} \\
&\quad = \langle \big(\big(M^2\big)^{\mathrm{w}}\big)^{\ell}\ u\,,\ u \rangle^2_{L^2(\mathbb{R}^3)} = \Big\| \big(\big(M^2\big)^{\mathrm{w}}\big)^{\ell/2}\ u \Big\|^2_{L^2(\mathbb{R}^3)}
\end{aligned}$$

hence, according to Proposition 2.4, we recover

$$\Big\| \big(1 - \Delta_x - \partial_z^2 + V(x) + V_{\mathrm{c}}(z)\big)^{\ell/2}\ u \Big\|^2_{L^2(\mathbb{R}^3)} = \big\| \big(M^{\ell}\big)^{\mathrm{w}}\ u + R\,u \big\|^2_{L^2(\mathbb{R}^3)}\,,$$

where $R \in \mathrm{Op}\, S(M^{\ell-1}, g)$. We now claim that for any $\varepsilon > 0$, there is a $C(\varepsilon) > 0$ such that

$$\|R\,u\|_{L^2(\mathbb{R}^3)} \leq C(\varepsilon)\ \|u\|_{L^2(\mathbb{R}^3)} + \varepsilon\ \big\| \big(M^{\ell}\big)^{\mathrm{w}}\ u \big\|_{L^2(\mathbb{R}^3)}\,. \tag{2.13}$$

Assuming for a while that (2.13) has been proved, we easily recover the equivalence

$$\Big\| \big(1 - \Delta_x - \partial_z^2 + V(x) + V_{\mathrm{c}}(z)\big)^{\ell/2}\ u \Big\|^2_{L^2(\mathbb{R}^3)} \sim \big\| \big(M^{\ell}\big)^{\mathrm{w}}\ u \big\|^2_{L^2(\mathbb{R}^3)}\,. \tag{2.14}$$

Let us now prove (2.13). The fact that $R \in \mathrm{Op}\, S(M^{\ell-1}, g)$ gives

$$\|R\,u\|_{L^2(\mathbb{R}^3)} \leq \|u\|_{H(M^{\ell-1}, g)}\,.$$

The definition of the space $H(M^{\ell-1}, g)$ (see (2.10)) provides

$$\begin{aligned}
\|u\|^2_{H(M^{\ell-1}, g)} &= \int_{\mathbb{R}^6} M^{2(\ell-1)}(X)\ \|\phi_X^{\mathrm{w}}\, u\|^2_{L^2(\mathbb{R}^3)}\ |\det(g(X))|^{1/2}\ dX \\
&= \int_{\mathbb{R}^6} \frac{M^{2\ell}(X)}{M^2(X)}\ \|\phi_X^{\mathrm{w}}\, u\|^2_{L^2(\mathbb{R}^3)}\ |\det(g(X))|^{1/2}\ dX.
\end{aligned}$$

Hence, decomposing $\mathbb{R}^6$ into $\{|X| \leq R\} \cup \{|X| > R\}$ for some large R, and using the fact that $1/M(X)$ goes to zero as X goes to infinity, we eventually recover

$$\|Ru\|^2_{L^2(\mathbb{R}^3)} \leq C(\varepsilon)\, \|u\|^2_{L^2(\mathbb{R}^3)} + \varepsilon\, \|u\|^2_{H(M^\ell,g)}.$$

There remains to observe that Proposition 2.2 asserts the equivalence of $\|u\|_{H(M^\ell,g)}$ with $\left\|\left(M^\ell\right)^{\mathrm{w}} u\right\|_{L^2(\mathbb{R}^3)}$, and we are in position to deduce (2.13).

Second step

From formula (2.2) we know there exist two positive constants c_0 and c_1 such that

$$c_0 \leq \frac{M^{2\ell}(x,z,\xi,\zeta)}{1+|\xi|^{2\ell}+|\zeta|^{2\ell}+V(x)^\ell+V_{\mathrm{c}}(z)^\ell} \leq c_1.$$

Hence, using the definition of $H(M^\ell, g)$, we recover

$$\begin{aligned}
&\|u\|^2_{H(M^\ell,g)} = \int_{\mathbb{R}^6} M^{2\ell}(X)\ \|\phi_X^{\mathrm{w}}\, u\|^2_{L^2(\mathbb{R}^3)}\ |\det(g(X))|^{1/2}\ dX \\
&\leq c_1 \int_{\mathbb{R}^6} \left(1+|\xi|^{2\ell}+|\zeta|^{2\ell}+V(x)^\ell+V_{\mathrm{c}}(z)^\ell\right)\ \|\phi_X^{\mathrm{w}}\, u\|^2_{L^2(\mathbb{R}^3)}\ |\det(g(X))|^{1/2}\ dX \\
&= c_1 \Big(\|u\|^2_{L^2(\mathbb{R}^3)} + \|\,(-\Delta_x)^{\ell/2}\ u\|^2_{L^2(\mathbb{R}^3)} + \|\,\left(-\partial_z^2\right)^{\ell/2}\ u\|^2_{L^2(\mathbb{R}^3)} \\
&\qquad + \|V(x)^{\ell/2}\, u\|^2_{L^2(\mathbb{R}^3)} + \|V_{\mathrm{c}}(z)^{\ell/2}\, u\|^2_{L^2(\mathbb{R}^3)}\Big).
\end{aligned}$$

The reverse inequality is obtained similarly. As a consequence, we recover the equivalence between norms

$$\begin{aligned}
&\left\|\left(M^\ell\right)^{\mathrm{w}}\ u\right\|^2_{L^2(\mathbb{R}^3)} \sim \|u\|^2_{L^2(\mathbb{R}^3)} + \left\|(-\Delta_x)^{\ell/2}\ u\right\|^2_{L^2(\mathbb{R}^3)} \\
&\quad + \left\|\left(-\partial_z^2\right)^{\ell/2}\ u\right\|^2_{L^2(\mathbb{R}^3)} + \left\|V(x)^{\ell/2}\, u\right\|^2_{L^2(\mathbb{R}^3)} + \left\|V_{\mathrm{c}}(z)^{\ell/2}\, u\right\|^2_{L^2(\mathbb{R}^3)}.
\end{aligned}$$

This, in combination with (2.14), ends the proof of Theorem 2.10. □

2.5 Various useful consequences of Theorem 2.1

This paragraph is devoted to the proof of the following

Proposition 2.5 [Properties of the Sobolev scale B_ℓ].
Take a real number $\ell > 3/2$. Define the Sobolev space B_ℓ as the completion of the set of smooth functions $u(x,z)$ under the norm

$$\|u\|^2_{B_\ell} := \|u\|^2_{H^\ell(\mathbb{R}^3)} + \|V(x)^{\ell/2}\, u\|^2_{L^2(\mathbb{R}^3)} + \|V_{\mathrm{c}}(z)^{\ell/2}\, u\|^2_{L^2(\mathbb{R}^3)}.$$

Then, B_ℓ is a Hilbert space and $B_\ell \subset L^\infty(\mathbb{R}^3)$ continuously. Moreover, the following properties hold true:

(i) **[algebra property]**.
Take any nonlinear function $f \in C^\infty(\mathbb{R})$ (with a possibly unbounded support), and satisfying $f(0) = 0$. Then, the mapping

$$u \in B_\ell \mapsto f(u) \in B_\ell$$

is well-defined and locally Lipschitz. It also satisfies the tame estimate

$$\|f(u)\|_{B_\ell} \leq C_f\Big(\|u\|_{L^\infty(\mathbb{R}^3)}\Big)\,\|u\|_{B_\ell}.$$

Here, $C_f(s) > 0$ depends on f and $s \geq 0$. It is a locally bounded function of s.

(ii) **[compact embeddings]**.
Take two real numbers $\ell' > \ell'' \geq 0$. Then, the embedding $B_{\ell'} \subset B_{\ell''}$ is compact.

As an immediate corollary of this result, we also have the following non-trivial uniform existence result.

Corollary 2.6 *Take a real number $m > 3/2$. Take an initial datum $\Psi_0(x, z)$ in (1.12) such that*

$$\Psi_0(x, z) \in B_m.$$

Then, there is a $T_0 > 0$, independent of ε, which only depends on $\|\Psi_0\|_{B_m}$ and on the nonlinear function F, such that the nonlinear Schrödinger equation (1.11) with initial datum Ψ_0 possesses a unique solution $\Psi^\varepsilon(t, x, z)$ with the smoothness

$$\Psi^\varepsilon(t, x, z) \in C^0([0, T_0], B_m).$$

Remark. Proposition 2.5 is essentially an immediate corollary of Theorem 2.1. Indeed, part (i) of the Proposition relies on the fact that the usual Sobolev space $H^\ell(\mathbb{R}^d)$ is an algebra whenever $\ell > d/2$, and that the tame estimate of point (i) then holds true when B_ℓ is replaced by $H^\ell(\mathbb{R}^d)$ - see *e.g.* [AG] - (here, $d = 3$), while part (ii) uses the fact that the embedding $H^{\ell'} \subset H^{\ell''}$ is locally compact, while $V_c(z)^{\ell'}$ resp. $V(x)^{\ell'}$ obviously dominate $V_c(z)^{\ell''}$ resp. $V(x)^{\ell''}$ at infinity, due to confinement. Similarly, Corollary 2.6 is essentially a consequence of Theorem 2.1, Proposition 2.5, and Gronwall's Lemma. The noticeable fact here is the independence of T_0 on ε, which is a direct consequence of the fact that H_x and H_z obviously commute with the fast oscillatory term H_z/ε on the right-hand-side of equation (1.11).
Lastly, note that Corollary 2.6 asserts that there is a common, non-trivial time-interval, such that all solutions to $i\partial_t \Psi^\varepsilon = H_x \Psi^\varepsilon + \varepsilon^{-1} H_z \Psi^\varepsilon + F(|\Psi^\varepsilon|^2)\,\Psi^\varepsilon$ exist on the *same* time interval $[0, T_0]$. Of course each maximal existence time $T_\varepsilon \leq +\infty$ associated with each solution Ψ^ε *a priori* depends on $\varepsilon > 0$. The point is, $\inf_\varepsilon T_\varepsilon \geq T_0$. $\square$

Proof of Proposition 2.5.

Take u and f as in the statement of the Proposition. We know from Proposition 2.1 that the B_ℓ norm of u is equivalent with

$$\|u\|_{H^\ell(\mathbb{R}^3)} + \|V_c(z)^{\ell/2}\,u\|_{L^2(\mathbb{R}^3)} + \|V(x)^{\ell/2}\,u\|_{L^2(\mathbb{R}^3)},$$

where $H^\ell(\mathbb{R}^3)$ is the usual Sobolev space. Since $\ell > 3/2$, we readily know have the usual tame estimate for $f(u)$,

$$\|f(u)\|_{H^\ell(\mathbb{R}^3)} \leq C_f\Big(\|u\|_{L^\infty(\mathbb{R}^3)}\Big)\,\|u\|_{H^\ell(\mathbb{R}^3)}.$$

(We have used the notation of the Proposition). On the other hand, we have

$$\|V_c(z)^{\ell/2}\,f(u)\|_{L^2(\mathbb{R}^3)} \leq C_f\Big(\|u\|_{L^\infty(\mathbb{R}^3)}\Big)\,\|V_c(z)^{\ell/2}\,u\|_{L^2(\mathbb{R}^3)},$$

where $C_f(.)$ is as in the Proposition. Obviously, the similar inequalities hold when $V_c(z)$ is replaced by $V(x)$. This ends the proof of the tame estimate. The fact that $u \mapsto f(u)$ is locally Lipschitz in B_ℓ is proved along the same lines. This ends the proof of part (i) of the Proposition. Similarly, part (ii) is the consequence of the locally compact embedding $H^{\ell'}(\mathbb{R}^3) \subset H^{\ell''}(\mathbb{R}^3)$, together with the fact that V and V_c go to infinity at infinity, so that $\|V(x)^{\ell'/2}\,u\|_{L^2}$ strictly dominates $\|V(x)^{\ell''/2}\,u\|_{L^2}$ and the same for V_c. We skip the easy details. $\square$

Proof of Corollary 2.6.

Take $\Psi_0 \in B_m$. For any given $\varepsilon > 0$, we look for a solution $\psi^\varepsilon(t,x,z)$ to (1.11) as a solution of the fixed point equation

$$\begin{aligned}\Psi^\varepsilon(t,x,z) &= \exp\left(-it\left[H_x + \frac{H_z}{\varepsilon}\right]\right)\,\Psi_0(x,z)\\ &\quad - i\int_0^t \exp\left(-i(t-s)\left[H_x + \frac{H_z}{\varepsilon}\right]\right)\,F\left(|\Psi^\varepsilon(s,x,z)|^2\right)\,\Psi^\varepsilon(s,x,z)\,ds. \qquad (2.15)\end{aligned}$$

Thanks to point (i) of Proposition 2.5, the mapping $u \in B_m \mapsto F(|u|^2)\,u \in B_m$ is locally Lipschitz. Even more, due to the fact that both operators H_x and H_z commute with the propagator $\exp\left(-it\left[H_x + \frac{H_z}{\varepsilon}\right]\right)$, it is readily seen that the following Lipschitz estimate holds true whenever $t \geq 0$, namely

$$\begin{aligned}&\Big\|\int_0^t \exp\left(-i(t-s)\left[H_x + \frac{H_z}{\varepsilon}\right]\right)\\ &\qquad \left[F\left(|u(s,x,z)|^2\right)\,u(s,x,z) - F\left(|v(s,x,z)|^2\right)\,v(s,x,z)\right]\,ds\Big\|_{B_m}\\ &\leq t \times \sup_{s\in[0,t]}\Big\|F\left(|u(s,x,z)|^2\right)\,u(s,x,z) - F\left(|v(s,x,z)|^2\right)\,v(s,x,z)\Big\|_{B_m}\\ &\leq t \times C_F\left(\|u\|_{C^0([0,t];B_m)};\|v\|_{C^0([0,t];B_m)}\right) \times \|u - v\|_{C^0([0,t];B_m)},\end{aligned}$$

where u and v belong to the space $C^0([0,t];B_m)$. Here, the function $C_F(.,.)$ is independent of ε, it is a nondecreasing function of its arguments, and it does depend on the nonlinearity F. With these ingredients at hand, it is now an easy task to deduce, see e.g. [C], that for any given $\varepsilon > 0$, there exists a (possibly small) time $T_\varepsilon > 0$, and a unique solution $\Psi^\varepsilon(t,x,z) \in C^0([0,T_\varepsilon];B_m)$ to the integral equation (2.15). This provides incidentally the unique solution to (1.11) with initial datum Ψ_0.

Let us now prove that there is a common lower bound T_0 such that $T_\varepsilon \geq T_0$ for any $\varepsilon > 0$. Taking the scalar product of the equation with Ψ^ε first gives the conservation of mass

$$\partial_t\Big(\|\Psi^\varepsilon(t,x,z)\|^2_{L^2(\mathbb{R}^3)}\Big) = 0.$$

Next, multiplying the equation by $H_x^m\overline{\Psi^\varepsilon} + H_z^m\overline{\Psi^\varepsilon}$ and integrating by parts gives, using the crucial fact that H_x and H_z commute with $H_x + H_z/\varepsilon$,

$$\begin{aligned}
&\partial_t\Big(\|H_x^{m/2}\Psi^\varepsilon(t,x,z)\|^2_{L^2(\mathbb{R}^3)} + \|H_z^{m/2}\Psi^\varepsilon(t,x,z)\|^2_{L^2(\mathbb{R}^3)}\Big)\\
&\quad = \Big\langle \left[H_x^{m/2} + H_z^{m/2}\right]\left[F(|\Psi^\varepsilon|^2)\,\Psi^\varepsilon\right],\left[H_x^{m/2} + H_z^{m/2}\right]\Psi^\varepsilon\Big\rangle\\
&\quad \leq \Big\|F(|\Psi^\varepsilon|^2)\,\Psi^\varepsilon\Big\|_{B_m}\|\Psi^\varepsilon\|_{B_m} \leq C_F\Big(\big\|\Psi^\varepsilon\big\|^2_{L^\infty}\Big)\,\big\|\Psi^\varepsilon\big\|^2_{B_m},
\end{aligned}$$

where $C_F(u)$ is a locally bounded, non-decreasing function of $u \geq 0$, and we have used Proposition 2.5. Eventually, we have obtained

$$\partial_t\Big(\|\Psi^\varepsilon(t,x,z)\|^2_{B_m}\Big) \leq C_F\Big(\big\|\Psi^\varepsilon\big\|^2_{L^\infty}\Big)\,\big\|\Psi^\varepsilon\big\|^2_{B_m},$$

hence, using the Sobolev embedding $B_m \subset L^\infty$

$$\partial_t\Big(\|\Psi^\varepsilon(t,x,z)\|^2_{B_m}\Big) \leq C_F\Big(\big\|\Psi^\varepsilon\big\|^2_{B_m}\Big)\,\big\|\Psi^\varepsilon\big\|^2_{B_m},$$

for a possibly larger value of $C_F(.)$. Gronwall's Lemma then proves that there is a common $T_0 > 0$ such that $\|\Psi^\varepsilon(t,x,z)\|_{B_m}$ remains bounded on $[0,T_0]$ whenever $\varepsilon > 0$.

This completes the proof. $\square$

3 Almost periodic functions (in time) with values in B_m (in space)

In this section, we first collect various known facts about Hilbert valued almost-periodic functions $\Theta(\tau)$. The corresponding results are Propositions 3.1 and 3.2 below. For obvious reasons, our focus is on almost-periodic functions with values in the Sobolev spaces B_ℓ. Next, we deduce from these known facts the properties that will be useful for our asymptotic analysis. Our main result is Proposition 3.3. Needless to say, in our perspective, the key fact about almost-periodic functions $\Theta(t)$ is the existence of their long time average $\Theta_{\rm av} := \lim_{T\to\infty} \frac{1}{T}\int_0^T \Theta(\tau)\,d\tau$, and the point is, no small-divisors estimate (or like) is needed to define such averages. In some sense, the small divisor estimates are encoded in the very definition of almost-periodic functions.

We begin with the

Definition and Proposition 3.1 (Borrowed from [LZ])
Let $\ell \geq 0$. A function $\Theta : \tau \in \mathbb{R} \mapsto \Theta(\tau) \in B_\ell$, with $\Theta \in C^0(\mathbb{R}; B_\ell)$, is called almost-periodic*, and we note $\Theta \in \mathrm{AP}(\mathbb{R}, B_\ell)$, whenever the set of translates*

$$\{\tau \mapsto \Theta(\tau + h),\ h \in \mathbb{R}\}$$

has compact closure in the norm $L^\infty(\mathbb{R}, B_\ell)$.

Equivalently, $\Theta \in \mathrm{AP}(\mathbb{R}, B_\ell)$ if and only if *$\Theta(\tau)$ is the strong limit of trigonometric polynomials, i.e. for any $\delta > 0$, there exists a trigonometric polynomial*

$$\Theta^\delta(\tau) = \sum_{n=1}^{N_\delta} \theta_{n,\delta} \exp(i\lambda_{n,\delta}\tau), \quad \text{such that} \quad \sup_{\tau\in\mathbb{R}} \|\Theta(\tau) - \Theta^\delta(\tau)\|_{B_\ell} \leq \delta,$$

where the $\theta_{n,\delta}$'s belong to B_ℓ, the $\lambda_{n,\delta}$'s belong to $\mathbb{R}$, and N_δ is some finite integer.

Remark. The above definition, namely the precompactness criterion, is usually called Bochner's criterion for almost-periodicity. The equivalence with being the uniform limit of trigonometric polynomials is a standard (and crucial) fact about almost-periodic functions. It is proved, e.g., in [LZ], and in any textbook about almost-periodic functions. □

Remark. Note that a function $\Theta \in \mathrm{AP}(\mathbb{R}, B_\ell)$ necessarily satisfies $\Theta \in L^\infty(\mathbb{R}, B_\ell)$. □

With this definition, it turns out that one may do *Fourier analysis* on almost periodic functions. In particular, the long-time average of $\Theta(\tau)$ (which plays the role of the mean mode in standard Fourier analysis), is well defined, as shown by the

Proposition 3.2 (Borrowed from [LZ])
Let $\ell \geq 0$ and take $\Theta \in \mathrm{AP}(\mathbb{R}, B_\ell)$. Then, the following strong limit exists in B_ℓ,

$$\Theta_{\rm av} := \lim_{T\to\infty} \frac{1}{T}\int_0^T \Theta(\tau)\,d\tau.$$

More generally, for any $\lambda \in \mathbb{R}$, the Fourier-like coefficient

$$\widehat{\Theta}(\lambda) := \lim_{T\to\infty} \frac{1}{T} \int_0^T \Theta(\tau)\, \exp(-i\lambda\tau)\, d\tau,$$

is well-defined as a limit in B_ℓ. Last, Bessel's inequality holds, i.e. for any sequence $\{\lambda_n\}_{n\in\mathbb{N}} \in \mathbb{R}^{\mathbb{N}}$, we have

$$\sum_{n\in\mathbb{N}} \left\|\widehat{\Theta}(\lambda_n)\right\|_{B_\ell}^2 \leq \lim_{T\to\infty} \frac{1}{T} \int_0^T \left\|\Theta(\tau)\right\|_{B_\ell}^2 d\tau \qquad \left(\leq \|\Theta\|_{L^\infty(\mathbb{R},B_\ell)}\right).$$

Remark. As an immediate consequence of Bessel's inequality, the coefficients $\widehat{\Theta}(\lambda)$ are non-zero for *countably many* values of λ only. The λ's such that $\widehat{\Theta}(\lambda)$ is non-zero are called the *almost-periods* of Ψ. Actually, Parseval's equality holds as well, namely

$$\sum_{\lambda\in\Lambda} \left\|\widehat{\Theta}(\lambda)\right\|_{B_\ell}^2 = \lim_{T\to\infty} \frac{1}{T} \int_0^T \left\|\Theta(\tau)\right\|_{B_\ell}^2 d\tau,$$

where the sum runs over $\Lambda := \{\lambda \text{ such that } \widehat{\Theta}(\lambda) \neq 0\}$. Again, we refer to [LZ]. □

Remark. A particular case of almost-periodic functions is formed by *quasi-periodic* functions. Given any finite-dimensional vector $\omega = (\omega_1, \ldots, \omega_N) \in \mathbb{R}^N$, called frequency-vector, whose components are assumed pairwise rationally independent for simplicity, and given a set of Fourier coefficients $\widehat{\Theta}(\alpha) \in B_\ell$, indexed by $\alpha \in \mathbb{Z}^N$, a typical quasi-periodic (hence almost periodic) function is

$$\Theta(\tau) := \sum_{\alpha\in\mathbb{Z}^N} \widehat{\Theta}(\alpha)\, \exp(i\,\omega\cdot\alpha\tau)$$

provided the sum converges. In that direction, one may assume that the frequency vector ω satisfies the Diophantine estimate

$$\exists C > 0, \quad \exists\gamma > 0, \quad \forall\alpha \in \mathbb{Z}^N \setminus \{0\}, \quad |\omega\cdot\alpha| \geq \frac{C}{|\alpha|^\gamma},$$

(a generic estimate whenever $\gamma > N-1$, see [A]), while the $\widehat{\Theta}(\alpha)$'s decay fast enough,

$$\|\widehat{\Theta}(\alpha)\|_{B_\ell} \leq C|\alpha|^{-\gamma-N-1},$$

for some C independent of α. In that case, the mean value of Θ, namely $\Theta_{\rm av} = \lim_{T\to\infty}(1/T)\int_0^T \Theta(\tau)\, d\tau$, coincides with the coefficient $\widehat{\Theta}(0)$ associated with $\alpha = 0$ in the sum that defines $\Theta(\tau)$. Even more, one has the convergence rate

$$\left\|\Theta_{\rm av} - \frac{1}{T}\int_0^T \Theta(\tau)\, d\tau\right\|_{B_\ell} \leq \frac{C}{T},$$

for some C independent of T. In this picture, the almost-periodic framework corresponds to the case of infinitely many independent frequencies ($N = \infty$), and when no Diophantine estimate is at hand. $\square$

We now turn to drawing the consequences of Proposition 3.2 that are of interest in our study of the equation $i\partial_t \Psi^\varepsilon = H_x \Psi^\varepsilon + \varepsilon^{-1} H_z \Psi^\varepsilon + F(|\Psi^\varepsilon|^2)\Psi^\varepsilon$. Our first result in that direction is the

Proposition 3.3 *Take $\ell > 3/2$ and take $\Theta(x,z) \in B_\ell$. Under these circumstances, the following holds:*
(i) the function

$$\tau \mapsto \mathrm{e}^{+i\tau H_z}\, F\left(\left|\mathrm{e}^{-i\tau H_z}\,\Theta\right|^2\right) \mathrm{e}^{-i\tau H_z}\,\Theta =: G(\tau, \Theta)$$

belongs to $\mathrm{AP}(\mathbb{R}; B_\ell)$. *Hence, one may define the long time average as the limit in B_ℓ,*

$$G_{\mathrm{av}}(\Theta) := \lim_{T\to\infty} \frac{1}{T} \int_0^T G(\tau, \Theta)\, d\tau.$$

(ii) the function $\Theta \mapsto G_{\mathrm{av}}(\Theta)$ is locally Lipschitz in B_ℓ. Moreover, for any ℓ' such that $3/2 < \ell' \leq \ell$, it satisfies the tame estimate

$$\|G_{\mathrm{av}}(\Theta)\|_{B_\ell} \leq C_{F,\ell'}\left(\|\Theta\|_{B_{\ell'}}\right)\, \|\Theta\|_{B_\ell},$$

where $C_{F,\ell'}(s)$ only depends on F, ℓ' and $s \geq 0$, and is locally bounded in s.

Next, we have all the necessary tools that allow to perform the natural nonlinear analysis of the averaged model $i\partial_t \Phi = H_x \Phi + G_{\mathrm{av}}(\Phi)$, obtained from the oscillatory equation $i\partial_t \Psi^\varepsilon = H_x \Psi^\varepsilon + \varepsilon^{-1} H_z \Psi^\varepsilon + F(|\Psi^\varepsilon|^2)\, \Psi^\varepsilon$.

Proposition 3.4 *Take $m > 3/2$ and $\Psi_0 \in B_m$. Then, there is a $T_0 > 0$, only depending on $\|\Psi_0\|_{B_m}$ and the nonlinear function F, such that the solution Φ to the averaged equation*

$$i\partial_t \Phi = H_x \Phi + G_{\mathrm{av}}(\Phi), \qquad \Phi(0,x,z) = \Psi_0(x,z),$$

exists and is unique in $C^0([0,T_0]; B_m)$. Even more, it satisfies the conservation laws of the Main Theorem, namely

$$\|\Phi(t)\|_{L^2(\mathbb{R}^3)} = \mathrm{const}, \quad \left\langle H_z^{1/2}\,\Phi(t)\,,\, H_z^{1/2}\Phi(t)\right\rangle_{L^2(\mathbb{R}^3)} = \mathrm{const},$$
$$\left\langle H_x^{1/2}\,\Phi(t)\,,\, H_x^{1/2}\Phi(t)\right\rangle^2_{L^2(\mathbb{R}^3)} + \int_{\mathbb{R}^3} \mathcal{G}_{\mathrm{av}}(\Phi(t))\, dx\, dz = \mathrm{const}$$

on the time interval $t \in [0, T_0]$. Here $\mathcal{G}_{\mathrm{av}}(\Psi) := \lim_{T\to\infty} \frac{1}{T}\int_0^T \mathcal{G}\left(\left|\mathrm{e}^{-i\tau H_z}\,\Psi\right|^2\right) d\tau$ for any $\Psi \in B_m$, and $\mathcal{G}(u) := \int_0^u F(v)\, dv$ ($u \in \mathbb{R}$).

The remaining part of this section is essentially devoted to the proof of Proposition 3.3, for which parts (i) and (ii) are established separately. Then, the proof of Proposition 3.4 comes as an easy corollary.

Proof of Proposition 3.3 - Part (i).
The proof is performed in three steps.

First step: $\mathrm{AP}(\mathbb{R}; B_\ell)$ **is stable upon multiplication by** $\exp(i\tau H_z)$
We first claim that, given *any* function $\Theta \in \mathrm{AP}(\mathbb{R}, B_\ell)$, the new function $\tau \mapsto \exp(i\tau H_z)\Theta(\tau)$ belongs to $\mathrm{AP}(\mathbb{R}, B_\ell)$ as well.

In order to prove this, we use the characterization of almost-periodic functions as strong limits of trigonometric polynomials (Proposition 3.1). The Bessel inequality gives us the necessary uniform bound needed to pass to the limit in the approximation process.

Let us come to the details. Take a small $\delta > 0$. We wish to approximate $\exp(i\tau H_z)\Theta(\tau)$ by a trigonometric polynomial, to within δ.

Firstly, since $\Theta \in \mathrm{AP}(\mathbb{R}; B_\ell)$, we may find a trigonometric polynomial

$$\Theta^\delta(\tau) := \sum_{n=1}^{N_\delta} \theta_{n,\delta} \exp(i\,\lambda_{n,\delta}\tau), \quad \text{such that} \quad \left\|\Theta(\tau) - \Theta^\delta(\tau)\right\|_{C^0(\mathbb{R};B_\ell)} \le \delta,$$

where the $\theta_{n,\delta}$'s belong to B_ℓ and the $\lambda_{n,\delta}$'s belong to $\mathbb{R}$. We clearly have the uniform bound $\|\Theta^\delta(\tau)\|_{C^0(\mathbb{R};B_\ell)} \le C$, for some $C > 0$ independent of δ. On top of that, we obviously have

$$\left\| \exp(i\tau H_z)\,\Theta(\tau) - \exp(i\tau H_z)\,\Theta^\delta(\tau)\right\|_{C^0(\mathbb{R};B_\ell)} \le \delta, \tag{3.1}$$

since $\exp(i\,\tau H_z)$ preserves the B_ℓ norm.

Secondly, the Bessel inequality, when applied to Θ^δ, gives the crucial uniform bound

$$\sum_{n=1}^{N_\delta} \|\theta_{n,\delta}\|_{B_\ell}^2 \le \lim_{T\to\infty} \frac{1}{T}\int_0^T \|\Theta^\delta(\tau)\|_{B_\ell}^2\,d\tau \le \sup_{\tau\in\mathbb{R}} \|\Theta^\delta(\tau)\|_{B_\ell}^2 \le C,$$

for some C independent of δ. This is due to the fact that the $\theta_{n,\delta}$'s coincide with the Fourier-like coefficients $\lim\limits_{T\to\infty}(1/T)\displaystyle\int_0^T \Theta^\delta(\tau)\,\exp(-i\lambda_{n,\delta}\tau)\,d\tau$. For this reason, we recover the uniform bound (by definition of the B_ℓ norm)

$$\sum_{n=1}^{N_\delta}\sum_{p\ge 0}(1+E_p^\ell)\left\|\langle\theta_{n,\delta}\,,\,\chi_p\rangle\right\|_{L^2(\mathbb{R}^2)}^2 + \sum_{n=1}^{N_\delta}\sum_{p\ge 0}\left\|\langle H_x^{\ell/2}\,\theta_{n,\delta}\,,\,\chi_p\rangle\right\|_{L^2(\mathbb{R}^2)}^2 \le C.$$

Hence, we may approximate $\Theta^\delta(\tau) = \sum_{p\geq 0} \langle \Theta^\delta(\tau)\,,\,\chi_p\rangle\,\chi_p$ by a *finite* sum. More precisely, given δ, one may find an integer P_δ such that

$$\sum_{n=1}^{N_\delta} \sum_{p>P_\delta} (1+E_p^\ell)\,\left\|\langle \theta_{n,\delta}\,,\,\chi_p\rangle\right\|^2_{L^2(\mathbb{R}^2)} + \sum_{n=1}^{N_\delta} \sum_{p>P_\delta} \left\|\langle H_x^{\ell/2}\,\theta_{n,\delta}\,,\,\chi_p\rangle\right\|^2_{L^2(\mathbb{R}^2)} \leq \delta.$$

In particular, we recover the estimate

$$\sup_{\tau\in\mathbb{R}} \left\| \Theta^\delta(\tau) - \sum_{p=0}^{P_\delta} \langle \Theta^\delta(\tau)\,,\,\chi_p\rangle\,\chi_p \right\|_{B_\ell} \leq \delta.$$

Note that the existence of such a truncation parameter P_δ is intimately related with the Bessel inequality. As a consequence,

$$\sup_{\tau\in\mathbb{R}} \left\| \exp(i\,\tau H_z)\Theta^\delta(\tau) - \sum_{p=0}^{P_\delta} \mathrm{e}^{i\,\tau E_p}\,\langle \Theta^\delta(\tau)\,,\,\chi_p\rangle\,\chi_p \right\|_{B_\ell} \leq \delta. \tag{3.2}$$

Third, it turns out that the function

$$\sum_{p=0}^{P_\delta} \mathrm{e}^{i\,\tau E_p}\,\langle \Theta^\delta(\tau)\,,\,\chi_p\rangle\,\chi_p = \sum_{n=0}^{N_\delta} \sum_{p=0}^{P_\delta} \mathrm{e}^{i\,\tau\,[E_p+\lambda_{n,\delta}]}\,\langle \theta_{n,\delta}\,,\,\chi_p\rangle\,\chi_p$$

obviously is a trigonometric polynomial, with coefficients in B_ℓ. Even more, one deduces from (3.1) and (3.2) the estimate

$$\sup_{\tau\in\mathbb{R}} \left\| \exp(i\,\tau H_z)\,\Theta(\tau) - \sum_{p=0}^{P_\delta} \mathrm{e}^{i\,\tau E_p}\,\langle \Theta^\delta(\tau)\,,\,\chi_p\rangle\,\chi_p \right\|_{B_\ell} \leq 2\delta.$$

This establishes the fact that the function $\tau \mapsto \exp(i\,\tau H_z)\,\Theta(\tau)$ belongs to $\mathrm{AP}(\mathbb{R};B_\ell)$.

Second step: $\mathrm{AP}(\mathbb{R};B_\ell)$ is an algebra

We now prove that, given any $\Theta \in \mathrm{AP}(\mathbb{R};B_\ell)$, the function $\tau \mapsto F(|\Theta(\tau)|^2)\,\Theta(\tau)$ belongs to $\mathrm{AP}(\mathbb{R};B_\ell)$ as well. In other words, $\mathrm{AP}(\mathbb{R};B_\ell)$ is an algebra. This obviously uses the fact that B_ℓ is an algebra whenever $\ell > 3/2$ (Proposition 2.5).

Since $\Theta \in \mathrm{AP}(\mathbb{R};B_\ell)$, Θ belongs in particular to $L^\infty(\mathbb{R};B_\ell)$. Since $\ell > 3/2$ and $B_\ell \subset H^\ell(\mathbb{R}^3) \subset L^\infty(\mathbb{R}^3)$, the function $\Theta(\tau)$ belongs to $L^\infty(\mathbb{R}\times\mathbb{R}^3)$. As a consequence, there exists a C_c^∞ function $F_M(x)$ $(x \in \mathbb{R})$ such that

$$F(|\Theta(\tau)|^2) \equiv F_M(|\Theta(\tau)|^2).$$

(In essence, one may take F_M as any regularization of the function $x \mapsto F(x)\times\mathbf{1}[|x| \leq M]$, where M is an upper bound of $\|\Theta(\tau)\|_{L^\infty(\mathbb{R}\times\mathbb{R}^3)}$).

Now, since $\Theta \in \mathrm{AP}(\mathbb{R};B_\ell)$, $\Theta(\tau)$ is also the limit in $L^\infty(\mathbb{R},B_\ell)$ of trigonometric polynomials $\Theta^\delta(\tau)$. On top of that, since $F_M(x)$ belongs to C_c^∞, it also is the limit in $C_c^\infty(\mathbb{R})$ of polynomials, denoted by, say, $F_{M,\delta}$. Hence, using the fact that B_ℓ is an algebra, for each $\delta > 0$, the function $F_{M,\delta}(|\Theta^\delta(\tau)|^2)\,\Theta^\delta(\tau)$ is a trigonometric polynomial (in time τ), with coefficients in B_ℓ (in space). Even more, the sequence $F_{M,\delta}(|\Theta^\delta(\tau)|^2)\,\Theta^\delta(\tau)$ approaches $F_M(|\Theta(\tau)|^2)\,\Theta(\tau) \equiv F(|\Theta(\tau)|^2)\,\Theta(\tau)$ as $\delta \to 0$, in the space $L^\infty(\mathbb{R}; B_\ell)$.

This ends the proof that the mapping $\tau \mapsto F(|\Theta(\tau)|^2)\,\Theta(\tau)$ belongs to $\mathrm{AP}(\mathbb{R}; B_\ell)$.

<u>Third step: conclusion</u>
The proof of Proposition 3.3, part (i), is now immediate.

Given $\Theta \in B_\ell$, the first step ensures $\tau \mapsto \exp(-i\,\tau H_z)\,\Theta(\tau)$ belongs to $\mathrm{AP}(\mathbb{R}; B_\ell)$. Then, the second step ensures that $\tau \mapsto F(|\exp(-i\,\tau H_z)\,\Theta(\tau)|^2)\,\exp(-i\,\tau H_z)\,\Theta(\tau)$ as well belongs to $\mathrm{AP}(\mathbb{R}; B_\ell)$. The first step in turn ensures that $\tau \mapsto \exp(+i\,\tau H_z)$ $F(|\exp(-i\,\tau H_z)\,\Theta(\tau)|^2)\ \ \exp(-i\,\tau H_z)\,\Theta(\tau)$ belongs to $\mathrm{AP}(\mathbb{R}; B_\ell)$.

This proves Proposition 3.3 - Part (i). $\square$

Proof of Proposition 3.3 - Part (ii).
We only prove the tame estimate, since the locally Lipschitz property is established along the same lines. We take a function Θ in B_ℓ, and estimate

$$\begin{aligned}
&\Big\|G_{\mathrm{av}}(\Theta)\Big\|_{B_\ell} = \Big\| \lim_{T\to\infty} \frac{1}{T} \int_0^T G(\tau,\Theta)\,d\tau \Big\|_{B_\ell} \\
&\quad \leq \lim_{T\to\infty} \frac{1}{T} \int_0^T \Big\|G(\tau,\Theta)\Big\|_{L^\infty(\mathbb{R};B_\ell)}\,d\tau \\
&\quad \text{(this uses the fact that the } \lim_{T\to\infty} \ldots \text{ holds in } B_\ell) \\
&\quad \leq \sup_{\tau\in\mathbb{R}} \Big\|G(\tau,\Theta)\Big\|_{B_\ell} = \sup_{\tau\in\mathbb{R}} \Big\|\mathrm{e}^{+i\tau H_z}\,F\left(\left|\mathrm{e}^{-i\tau H_z}\,\Theta\right|^2\right)\,\mathrm{e}^{-i\tau H_z}\,\Theta\Big\|_{B_\ell} \\
&\quad = \sup_{\tau\in\mathbb{R}} \Big\|F\left(\left|\mathrm{e}^{-i\tau H_z}\,\Theta\right|^2\right)\,\mathrm{e}^{-i\tau H_z}\,\Theta\Big\|_{B_\ell} \\
&\quad \text{(we use the fact that } \mathrm{e}^{i\tau H_z} \text{ is unitary in the } B_\ell \text{ norm).}
\end{aligned}$$

Then, the Proposition becomes essentially clear. Indeed, we may further estimate, for any $3/2 < \ell' \leq \ell$,

$$\begin{aligned}
\Big\|G_{\mathrm{av}}(\Theta)\Big\|_{B_\ell} &\leq C_F\left(\sup_{\tau\in\mathbb{R}} \Big\|\exp(-i\,\tau H_z)\,\Theta\Big\|_{L^\infty(\mathbb{R}^3)}\right)\,\sup_{\tau\in\mathbb{R}} \Big\|\exp(-i\,\tau H_z)\,\Theta\Big\|_{B_\ell} \\
&\leq C_{F,\ell'}\left(\sup_{\tau\in\mathbb{R}} \Big\|\exp(-i\,\tau H_z)\,\Theta\Big\|_{B_{\ell'}}\right)\,\sup_{\tau\in\mathbb{R}} \Big\|\exp(-i\,\tau H_z)\,\Theta\Big\|_{B_\ell} \\
&= C_{F,\ell'}\left(\big\|\Theta\big\|_{B_{\ell'}}\right)\,\big\|\Theta\big\|_{B_\ell},
\end{aligned}$$

where $C_F(s)$ resp. $C_{F,\ell'}(s)$ depend on F resp. F and ℓ', they are locally bounded function of $s \geq 0$, and we have used the tame estimate of Proposition 2.5-(i) together with the Sobolev embedding $B_{\ell'} \subset L^\infty(\mathbb{R}^3)$. □

Proof of Proposition 3.4.
The existence, regularity, and uniqueness part of Proposition 3.4 is an immediate consequence of Proposition 3.3. Indeed, as already underlined in the proof of Corollary 2.6, which provides a similar statement, the key ingredient to proving existence and uniqueness of a local-in-time solution to $i\partial_t \Phi = H_x \Phi + G_{\rm av}\Phi$, with prescribed initial datum in B_m, is the fact that the mapping $\Phi \in B_m \mapsto G_{\rm av}(\Phi) \in B_m$ is locally Lipschitz, combined with the fact that the propagator $\exp(itH_x)$ is unitary in B_m. We again refer to [C] on these matters.

There only remains to prove the claimed conservation laws.

Conservation of mass is easy. Indeed, we write

$$\begin{aligned}\frac{d}{dt}\left(\|\Phi(t)\|^2_{L^2(\mathbb{R}^3)}\right) &= 2\,\mathrm{Re}\left\langle \frac{1}{i}\left[H_x\Phi + G_{\rm av}(\Phi)\right]\,,\,\Phi\right\rangle_{L^2(\mathbb{R}^3)}\\ &= 2\,\mathrm{Im}\,\langle G_{\rm av}(\Phi)\,,\,\Phi\rangle_{L^2(\mathbb{R}^3)}\,,\end{aligned}$$

due to the fact that H_x is self-adjoint. Now, recalling that

$$G_{\rm av}(\Phi) = \lim_{T\to\infty}\frac{1}{T}\int_0^T \mathrm{e}^{+i\tau H_z}\, F\left(\left|\mathrm{e}^{-i\tau H_z}\,\Phi\right|^2\right)\,\mathrm{e}^{-i\tau H_z}\,\Phi\,d\tau,$$

we may write, for any *given* $T>0$,

$$\begin{aligned}&\mathrm{Im}\left\langle \frac{1}{T}\int_0^T \mathrm{e}^{+i\tau H_z}\, F\left(\left|\mathrm{e}^{-i\tau H_z}\,\Phi\right|^2\right)\,\mathrm{e}^{-i\tau H_z}\,\Phi\,d\tau\,,\,\Phi\right\rangle_{L^2(\mathbb{R}^3)}\\ &\quad= \frac{1}{T}\int_0^T \mathrm{Im}\left\langle F\left(\left|\mathrm{e}^{-i\tau H_z}\,\Phi\right|^2\right)\,\mathrm{e}^{-i\tau H_z}\,\Phi\,,\,\mathrm{e}^{-i\tau H_z}\,\Phi\right\rangle_{L^2(\mathbb{R}^3)} d\tau = 0.\end{aligned}$$

Hence, passing to the limit $T\to\infty$ in B_m, which is licit thanks to Proposition 3.2, we recover

$$\mathrm{Im}\,\langle G_{\rm av}(\Phi)\,,\,\Phi\rangle_{L^2(\mathbb{R}^3)} = 0.$$

The conclusion is $\|\Phi(t)\|_{L^2(\mathbb{R}^3)} = \text{const}$.

We stress in passing that all integrations by parts we perform here and below are perfectly licit when $m > 3/2$ is sufficiently large (to ensure, say, $H_x\Phi \in L^2(\mathbb{R}^3)$, $G_{\rm av}(\Phi) \in L^2(\mathbb{R}^3)$, or so). Hence the relation $\|\Phi(t)\|_{L^2(\mathbb{R}^3)} = $ const. is true for *any* $m > 3/2$, as shown by a standard regularization procedure (take a sequence of initial data $\Psi_0^n \in B_{m'}$ for some large m', which converges towards the given initial data Ψ_0 in B_m).

Conservation of H_z (energy along the z-axis) is proved in the same spirit, *i.e.*

$$\begin{aligned}\frac{d}{dt}\left(\langle \Phi(t)\,,\,H_z\,\Phi(t)\rangle_{L^2(\mathbb{R}^3)}\right) &= 2\,\mathrm{Re}\left\langle \frac{1}{i}\left[H_x\Phi + G_{\rm av}(\Phi)\right]\,,\,H_z\,\Phi\right\rangle_{L^2(\mathbb{R}^3)}\\ &= 2\,\mathrm{Im}\,\langle G_{\rm av}(\Phi)\,,\,H_z\,\Phi\rangle_{L^2(\mathbb{R}^3)}\,,\end{aligned}$$

due to the fact that H_x and H_z are self-adjoint. Besides, recalling the definition of $G_{\rm av}$, we may write, for any *given* $T>0$,

$$\begin{aligned}
&\operatorname{Im}\left\langle \frac{1}{T}\int_0^T \mathrm{e}^{+i\tau H_z}\, F\left(\left|\mathrm{e}^{-i\tau H_z}\,\Phi\right|^2\right)\, \mathrm{e}^{-i\tau H_z}\,\Phi\, d\tau\,,\, H_z\,\Phi\right\rangle_{L^2(\mathbb{R}^3)}\\
&\quad=\frac{1}{T}\int_0^T \operatorname{Im}\left\langle F\left(\left|\mathrm{e}^{-i\tau H_z}\,\Phi\right|^2\right)\, \mathrm{e}^{-i\tau H_z}\,\Phi\,,\, H_z\,\mathrm{e}^{-i\tau H_z}\,\Phi\right\rangle_{L^2(\mathbb{R}^3)} d\tau = 0.
\end{aligned}$$

where we used the fact that H_z is self-adjoint and it commutes with $\mathrm{e}^{-i\tau H_z}$ (a crucial fact throughout our analysis). Passing to the limit $T\to\infty$ thanks to Proposition 3.2, we recover

$$\operatorname{Im}\left\langle G_{\rm av}(\Phi)\,,\, H_z\,\Phi\right\rangle_{L^2(\mathbb{R}^3)} = 0.$$

The conclusion is

$$\langle\Phi(t)\,,\, H_z\,\Phi(t)\rangle_{L^2(\mathbb{R}^3)} = \langle H_z^{1/2}\Phi(t)\,,\, H_z^{1/2}\,\Phi(t)\rangle_{L^2(\mathbb{R}^3)} = \text{const.}$$

We end with the proof of the conservation of total energy along x. We write

$$\begin{aligned}
\frac{d}{dt}\left(\langle\Phi(t)\,,\, H_x\,\Phi(t)\rangle_{L^2(\mathbb{R}^3)}\right) &= 2\operatorname{Re}\left\langle \frac{1}{i}\left[H_x\Phi + G_{\rm av}(\Phi)\right]\,,\, H_x\,\Phi\right\rangle_{L^2(\mathbb{R}^3)}\\
&= 2\operatorname{Im}\left\langle G_{\rm av}(\Phi)\,,\, H_x\,\Phi\right\rangle_{L^2(\mathbb{R}^3)} = 2\operatorname{Im}\left\langle G_{\rm av}(\Phi)\,,\, i\partial_t\Phi - G_{\rm av}(\Phi)\right\rangle_{L^2(\mathbb{R}^3)}\\
&= -2\operatorname{Re}\left\langle G_{\rm av}(\Phi)\,,\, \partial_t\Phi\right\rangle_{L^2(\mathbb{R}^3)}.
\end{aligned}$$

We may further observe that, for any *given* $T>0$, we have

$$\begin{aligned}
&\operatorname{Re}\left\langle \frac{1}{T}\int_0^T \mathrm{e}^{+i\tau H_z}\, F\left(\left|\mathrm{e}^{-i\tau H_z}\,\Phi\right|^2\right)\, \mathrm{e}^{-i\tau H_z}\,\Phi\, d\tau\ ,\ \partial_t\Phi\right\rangle_{L^2(\mathbb{R}^3)}\\
&=\frac{1}{T}\int_0^T \operatorname{Re}\left\langle F\left(\left|\mathrm{e}^{-i\tau H_z}\,\Phi(t)\right|^2\right)\, \mathrm{e}^{-i\tau H_z}\,\Phi(t)\,,\, \partial_t\left[\mathrm{e}^{-i\tau H_z}\,\Phi(t)\right]\right\rangle_{L^2(\mathbb{R}^3)} d\tau\\
&=\frac{1}{T}\int_0^T \operatorname{Re}\int_{\mathbb{R}^3} F\left(\left|\mathrm{e}^{-i\tau H_z}\,\Phi\right|^2\right)\times \mathrm{e}^{-i\tau H_z}\,\Phi\times\partial_t\overline{\left[\mathrm{e}^{-i\tau H_z}\,\Phi\right]}\,dx\,dz\,d\tau\\
&=\frac{1}{2T}\int_0^T\int_{\mathbb{R}^3} F\left(\left|\mathrm{e}^{-i\tau H_z}\,\Phi\right|^2\right)\times\partial_t\left[\left|\mathrm{e}^{-i\tau H_z}\,\Phi\right|^2\right]\,dx\,dz\,d\tau\\
&=\frac{1}{2}\frac{d}{dt}\left(\frac{1}{T}\int_0^T\int_{\mathbb{R}^3}\mathcal{G}\left(\left|\mathrm{e}^{-i\tau H_z}\,\Phi\right|^2\right)\right)\,dx\,dz\,d\tau,
\end{aligned}$$

where we used the notation $\mathcal{G}(u)=\displaystyle\int_0^u F(v)\,dv$ whenever $u\in\mathbb{R}$. Passing to the limit $T\to\infty$ thanks to Proposition 3.2 eventually produces the relation

$$\begin{aligned}
2\operatorname{Re}\left\langle G_{\rm av}(\Phi)\,,\, \partial_t\Phi\right\rangle_{L^2(\mathbb{R}^3)} &= \lim_{T\to\infty}\frac{d}{dt}\left(\frac{1}{T}\int_0^T\int_{\mathbb{R}^3}\mathcal{G}\left(\left|\mathrm{e}^{-i\tau H_z}\,\Phi\right|^2\right)\right)\,dx\,dz\,d\tau\\
&=\frac{d}{dt}\int_{\mathbb{R}^3}\mathcal{G}_{\rm av}(\Phi)\,dx\,dz.
\end{aligned}$$

This ends the proof of the conservation of energy along x.

Note that in these estimates, all limits are justified by using repeatedly Proposition 3.2 (which allows to perform long-time averages), Proposition 3.3 (which allows to do nonlinear analysis), and the regularity at hand for Φ, while all integrations by parts are justified as well, at least when m is large enough. The obtained conservation laws then hold for *any* $m > 3/2$, using the standard regularization argument.

Proposition 3.4 is now proved. $\square$

4 Proof of the Main Theorem

Points (i) and (iii) of the Main Theorem are already proved. There remains to perform the averaging procedure in time, i.e. to prove point (ii).

In order to do so, we first prove a reduced version of the result, for initial data Ψ_0 that possess the improved regularity $\Psi_0 \in B_{m+2}$ (instead of B_m as in our main Theorem). It turns out that part (ii) of our main Theorem then comes as an easy corollary.

We state and prove the

Proposition 4.1 *Let $\Psi_0 \in B_{m+2}$ with $m > 3/2$. Let $\Phi^\varepsilon(t)$ and $\Phi(t)$ satisfy, as in the main theorem, $i\partial_t \Phi^\varepsilon = H_x \Phi^\varepsilon + G(t/\varepsilon, \Phi^\varepsilon)$, resp. $i\partial_t \Phi = H_x \Phi + F_{\rm av}(\Phi)$, with initial datum Ψ_0, where $G(\tau, u) = \mathrm{e}^{+i\tau H_z} F(|\mathrm{e}^{-i\tau H_z} u|^2)\, \mathrm{e}^{-i\tau H_z} u$. Then,*

$$\|\Phi^\varepsilon(t) - \Phi(t)\|_{C^0([0,T_0];B_m)} \underset{\varepsilon \to 0}{\longrightarrow} 0.$$

Corollary 4.2 *Let $\Psi_0 \in B_m$ with $m > 3/2$. Let $\Phi^\varepsilon(t)$ and $\Phi(t)$ satisfy, as in the main theorem, $i\partial_t \Phi^\varepsilon = H_x \Phi^\varepsilon + G(t/\varepsilon, \Phi^\varepsilon)$, resp. $i\partial_t \Phi = H_x \Phi + F_{\rm av}(\Phi)$, with initial datum Ψ_0. Then,*

$$\|\Phi^\varepsilon(t) - \Phi(t)\|_{C^0([0,T_0];B_m)} \underset{\varepsilon \to 0}{\longrightarrow} 0.$$

Remark. Note that, though the solutions Φ^ε and Φ in Proposition 4.1 both possess the improved regularity $C^0([0, T_0]; B_{m+2})$, the convergence $\Phi^\varepsilon \to \Phi$ only holds in the weaker space $C^0([0, T_0]; B_m)$. Technically speaking, this stems from the fact that the proof of Proposition 4.1 requires that $\partial_t \Phi^\varepsilon$ and $\partial_t \Phi$ belong to B_m, while the equations $i\partial_t \Phi^\varepsilon = H_x \Phi^\varepsilon + G(t/\varepsilon, \Phi^\varepsilon)$ and $i\partial_t \Phi^\varepsilon = H_x \Phi^\varepsilon + G_{\rm av}(\Phi)$ only provide $\partial_t \Phi^\varepsilon \in B_m$ under the condition $H_x \Phi^\varepsilon \in B_m$, which in turn requires $\Phi^\varepsilon \in B_{m+2}$, and similarly for Φ.

The Corollary 4.2 next ensures $\Phi^\varepsilon \to \Phi$ in $C^0([0, T_0]; B_m)$ provided Φ^ε and Φ belong to B_m only, a property that is deduced from Proposition 4.1 using a regularization procedure. $\square$

Proof of Proposition 4.1. .

First step: reduction of the proof

We follow the strategy developed in [SV] for finite-dimensional ODE's (see [BCD] for an adaptation in the infinite-dimensional situation). The filtered function Φ^ε satisfies

$$i\partial_t \Phi^\varepsilon = H_x\, \Phi^\varepsilon + G(t/\varepsilon, \Phi^\varepsilon), \qquad \Phi^\varepsilon(0) = \Psi_0,$$
$$\text{where} \quad G(t, \Psi) := \mathrm{e}^{+itH_z}\, F(|\mathrm{e}^{-itH_z}\, \Psi|^2)\, \mathrm{e}^{-itH_z}\, \Psi \text{ is almost periodic.} \tag{4.1}$$

We wish to estimate the difference with the averaged system

$$i\partial_t \Phi = H_x\, \Phi + G_{\mathrm{av}}(\Phi), \qquad \Phi(0) = \Psi_0,$$
$$\text{where} \quad G_{\mathrm{av}}(\Psi) := \lim_{T\to\infty} \frac{1}{T} \int_0^T G(\tau, \Psi)\, d\tau. \tag{4.2}$$

In order to do so, we choose a (large) time $T(\varepsilon)$ such that $T(\varepsilon) = o(1/\varepsilon)$ as $\varepsilon \to 0$. The "good" choice for $T(\varepsilon)$ is made precise below - see (4.14). Associated with $T(\varepsilon)$, we introduce the auxiliary solution $\widetilde{\Phi}^\varepsilon$ to

$$i\partial_t \widetilde{\Phi}^\varepsilon = H_x\, \widetilde{\Phi}^\varepsilon + \widetilde{G}_\varepsilon(t/\varepsilon, \widetilde{\Phi}^\varepsilon), \qquad \widetilde{\Phi}^\varepsilon(0) = \Psi_0,$$
$$\text{where} \quad \widetilde{G}_\varepsilon(t, \Psi) := \frac{1}{T(\varepsilon)} \int_t^{t+T(\varepsilon)} G(s, \Psi)\, ds. \tag{4.3}$$

Our strategy is to successively prove that the two terms $\Phi^\varepsilon - \widetilde{\Phi}^\varepsilon$ and $\widetilde{\Phi}^\varepsilon - \Phi$ go to zero in $C^0([0, T_0]; B_m)$. As we shall see, each term requires specific arguments.

Second step: some preliminary bounds

Take $\Psi_0 \in B_{m+2}$.

Before estimating $\Phi^\varepsilon - \widetilde{\Phi}^\varepsilon$ and $\widetilde{\Phi}^\varepsilon - \Phi$, a preliminary remark is in order.

Repeating the Proof of Proposition 3.3 - Part (ii) given before, the function $u \mapsto \widetilde{G}_\varepsilon(t, u)$ clearly is locally Lipschitz in all Sobolev spaces B_ℓ ($\ell > 3/2$). Even more, the following estimate holds true, independently of ε,

$$\left\| \widetilde{G}_\varepsilon(t, u) \right\|_{B_\ell} \leq C_F \left(\|u\|_{B_\ell} \right) \|u\|_{B_\ell} \qquad (\ell > 3/2),$$

for some $C_F(s)$ which is locally bounded in $s \geq 0$. As a consequence, there exists a T_0, independent of ε, such that the solution $\widetilde{\Phi}^\varepsilon$ to (4.3) exists, is unique, and has the regularity $C^0([0, T_0], B_{m+2})$. Even more, there exists a common upper-bound $M > 0$ such that

$$\sup_{0<\varepsilon<1} \left[\left\| \Phi^\varepsilon \right\|_{C^0([0,T_0];B_{m+2})} + \left\| \widetilde{\Phi}^\varepsilon \right\|_{C^0([0,T_0];B_{m+2})} + \left\| \Phi \right\|_{C^0([0,T_0];B_{m+2})} \right] \leq M. \tag{4.4}$$

Similarly, the following uniform Lipschitz property may be stated:

$$\sup_{0<\varepsilon<1} \; \sup_{0\leq\tau\leq T_0/\varepsilon} \; \sup_{\sup\left(\|u\|_{B_{m+2}}, \|v\|_{B_{m+2}}\right)\leq M} \Big[\|G(\tau, u) - G(\tau, v)\|_{B_{m+2}} \tag{4.5}$$
$$+ \|\widetilde{G}_\varepsilon(\tau, u) - \widetilde{G}_\varepsilon(\tau, v)\|_{B_{m+2}} + \|G_{\mathrm{av}}(u) - G_{\mathrm{av}}(v)\|_{B_{m+2}} \Big] \leq C(F, M)\, \|u - v\|_{B_{m+2}},$$

for some constant $C(F,M) > 0$ that depends on F and M only. The analogous Lipschitz estimate actually holds with B_{m+2} everywhere replaced by B_m as well.

Third step: estimating $\widetilde{G}_\varepsilon(t/\varepsilon, u) - G_{\rm av}(u)$ for each $u \in B_{m+2}$

Estimating the difference between $\widetilde{\Phi}^\varepsilon - \Phi$ in equations (4.2) and (4.3), clearly requires to estimate, for any given $u \in B_{m+2}$, the difference $\widetilde{G}_\varepsilon(\tau, u) - G_{\rm av}(u)$, for τ's belonging to the interval $[0, T_0/\varepsilon]$. This is what we do in the present step.

For any *given* $u \in B_{m+2}$, we introduce the two convergence rates

$$\delta^{(0)}(\varepsilon, u) := \sup_{0 \le \tau \le 2T_0/\varepsilon} \left\| \frac{\varepsilon}{2T_0} \int_0^\tau \Big[G(\sigma, u) - G_{\rm av}(u) \Big] \, d\sigma \right\|_{B_{m+2}},$$

$$\delta^{(2)}(\varepsilon, u) := \sup_{0 \le \tau \le 2T_0/\varepsilon} \left\| \frac{\varepsilon}{2T_0} \int_0^\tau \Big[G(\sigma, u) - G_{\rm av}(u) \Big] \, d\sigma \right\|_{B_m}. \tag{4.6}$$

Note that $\delta^{(2)}$ measures a convergence rate *with loss* of smoothness (loss of "two derivatives"). This explains the exponent "(2)". Note also the obvious relation

$$\delta^{(2)}(\varepsilon, u) \le \delta^{(0)}(\varepsilon, u).$$

The reason for introducing separately $\delta^{(2)}$ and $\delta^{(0)}$ becomes clear below: it is mainly due to the fact that $\delta^{(2)}(\varepsilon, u) \to 0$ *uniformly* with respect to u, while $\delta^{(0)}$ probably does not share this uniformity.

We are now in position to state the

Lemma 4.3
(i) For any given $u \in B_{m+2}$ $(m > 3/2)$, we have

$$\delta^{(0)}(\varepsilon, u) \underset{\varepsilon \to 0}{\longrightarrow} 0, \quad \text{and} \quad \delta^{(2)}(\varepsilon, u) \underset{\varepsilon \to 0}{\longrightarrow} 0. \tag{4.7}$$

(ii) Take M is as in (4.4), and introduce the uniform convergence rate

$$\delta_M^{(2)}(\varepsilon) := \sup_{\|u\|_{B_{m+2}} \le M} \delta^{(2)}(\varepsilon, v). \tag{4.8}$$

Then,

$$\delta_M^{(2)}(\varepsilon) \underset{\varepsilon \to 0}{\longrightarrow} 0. \tag{4.9}$$

(iii) For any $0 \le t \le T_0$, we have (here, M is as in (4.4)),

$$\sup_{\|u\|_{B_{m+2}} \le M} \left\| \widetilde{G}_\varepsilon(t/\varepsilon, u) - G_{\rm av}(u) \right\|_{B_m} \le 2 \, T_0 \, \frac{\delta_M^{(2)}(\varepsilon)}{\varepsilon \, T(\varepsilon)}. \tag{4.10}$$

Remark. Note that the right-hand-side of (4.10) does not necessarily go to zero with ε: an appropriate choice of $T(\varepsilon)$ has to be done, and only $\delta_M^{(2)}(\varepsilon)$ goes to zero at this stage. We recall in passing that $T(\varepsilon)$ will be chosen so that $\varepsilon T(\varepsilon) \to 0$ as $\varepsilon \to 0$ ($\varepsilon T(\varepsilon) = \sqrt{\delta_M^{(2)}(\varepsilon)}$ will do). $\square$

Proof of Lemma 4.3.
For any given $u \in B_{m+2}$, the quantity $(1/T)\displaystyle\int_0^T G(\tau, u)\, d\tau$ goes to $G_{\rm av}(u)$ in B_{m+2}, hence in B_m, as $T \to \infty$. This is the definition of $G_{\rm av}$. This proves part (i) of the Lemma. Here, we have used the information $m > 3/2$, together with Proposition 3.3.

To prove point (ii), we argue by contradiction. In the opposite case where $\delta_M^{(2)}(\varepsilon) \not\to 0$, we would be able to build up two sequences $\varepsilon_n \to 0$, and u_n such that $\|u_n\|_{B_m} \le M$, with $\delta^{(2)}(\varepsilon_n, u_n) \not\to 0$. Yet, since the embedding $B_{m+2} \subset B_m$ is locally compact (Proposition 2.5), one may build up a subsequence, still denoted by u_n, such that $u_n \to u$ in B_m, for some limit u. Even more, the following estimate is obvious

$$|\delta^{(2)}(\varepsilon_n, u_n) - \delta^{(2)}(\varepsilon_n, u)| \underset{n\to\infty}{\longrightarrow} 0,$$

while we clearly have $\delta^{(2)}(\varepsilon_n, u) \underset{n\to\infty}{\to} 0$. We deduce $\delta^{(2)}(\varepsilon_n, u_n) \to 0$. This establishes the contradiction. Point (ii) is proved.

Last, (iii) is easily established. Indeed, we may write

$$\begin{aligned} &\sup_{\|u\|_{B_{m+2}} \le M} \left\| \widetilde{G}_\varepsilon(t/\varepsilon, u) - G_{\rm av}(u) \right\|_{C^0([0,T_0];B_m)} \\ &\quad = \sup_{\|u\|_{B_m} \le M} \ \sup_{0 \le \tau \le T_0/\varepsilon} \left\| \frac{1}{T(\varepsilon)} \int_\tau^{\tau+T(\varepsilon)} \big[G(\sigma, u) - G_{\rm av}(u) \big]\, d\sigma \right\|_{B_m}, \end{aligned}$$

so that, writing $\displaystyle\int_\tau^{\tau+T(\varepsilon)} ... = \int_0^{\tau+T(\varepsilon)} ... - \int_0^\tau ...$, we eventually recover the estimate (iii). Here we use the fact that throughout the computations, we assume $\varepsilon\, T(\varepsilon) \le T_0$.

This ends the proof of the Lemma. $\square$

Fourth step: estimating $\widetilde{\Phi}^\varepsilon - \Phi$
This becomes an easy task once (4.10) is established.

Indeed, the difference $\Delta^\varepsilon(t) := \widetilde{\Phi}^\varepsilon(t) - \Phi(t)$ satisfies

$$i\partial_t \Delta^\varepsilon(t) = H_x\, \Delta^\varepsilon(t) + \widetilde{G}_\varepsilon(t/\varepsilon, \widetilde{\Phi}^\varepsilon(t)) - G_{\rm av}(\Phi(t)), \qquad \Delta^\varepsilon(0) = 0,$$

from which it follows that for any $t \in [0, T_0]$ we have

$$\begin{aligned}
\|\Delta^\varepsilon(t)\|_{B_m} &\leq \Big\| \int_0^t \exp(-i(t-s)H_x) \ \Big[\widetilde{G}_\varepsilon(s/\varepsilon, \widetilde{\Phi}^\varepsilon(s)) - G_{\rm av}(\Phi(s))\Big] \ ds \Big\|_{B_m} \\
&\leq \int_0^t \Big\| \widetilde{G}_\varepsilon(s/\varepsilon, \widetilde{\Phi}^\varepsilon(s)) - G_{\rm av}(\Phi(s)) \Big\|_{B_m} ds.
\end{aligned}$$

Hence, using the uniform Lipschitz property (4.5) (with B_{m+2} replaced by B_m), together with the estimate (4.10), we recover

$$\begin{aligned}
\|\Delta^\varepsilon(t)\|_{B_m} &\leq C(F,M) \ \int_0^t \|\Delta^\varepsilon(s)\|_{B_m} \ ds + \int_0^t \Big\| \widetilde{G}_\varepsilon(s/\varepsilon, \Phi(s)) - G_{\rm av}(\Phi(s)) \Big\|_{B_m} ds \\
&\leq C(F,M) \ \int_0^t \|\Delta^\varepsilon(s)\|_{B_m} \ ds + \int_0^t \sup_{\|u\|_{B_{m+2}} \leq M} \Big\| \widetilde{G}_\varepsilon(s/\varepsilon, u) - G_{\rm av}(u) \Big\|_{B_m} ds \\
&\leq C(F,M) \ \int_0^t \|\Delta^\varepsilon(s)\|_{B_m} \ ds + 2\, T_0 \, \frac{\delta_M^{(2)}(\varepsilon)}{\varepsilon\, T(\varepsilon)}.
\end{aligned}$$

Now, Gronwall's Lemma gives

$$\forall 0 \leq t \leq T_0, \qquad \|\widetilde{\Phi}^\varepsilon(t) - \Phi(t)\|_{B_m} \leq C \, \frac{\delta_M^{(2)}(\varepsilon)}{\varepsilon\, T(\varepsilon)}, \tag{4.11}$$

where $C > 0$ only depends on T_0, M, F.

Fifth step: estimating $\Phi^\varepsilon - \widetilde{\Phi}^\varepsilon$

This estimate is more delicate than the previous one. It relies on an appropriate "integration by parts", see below.

Introducing the difference $\Delta^\varepsilon(t) := \Phi^\varepsilon(t) - \widetilde{\Phi}^\varepsilon(t)$ as before (we use the same letter Δ^ε since no confusion is possible), we readily have

$$i\partial_t \Delta^\varepsilon(t) = H_x \, \Delta^\varepsilon(t) + G(t/\varepsilon, \Phi^\varepsilon(t)) - \widetilde{G}_\varepsilon(t/\varepsilon, \widetilde{\Phi}^\varepsilon(t)), \qquad \Delta^\varepsilon(0) = 0.$$

Hence, for $0 \leq t \leq T_0$, we recover, using (4.5) again,

$$\begin{aligned}
\|\Delta^\varepsilon(t)\|_{B_m} &\leq \Big\| \int_0^t \ {\rm e}^{i(t-s)H_x} \ \Big[G(s/\varepsilon, \Phi^\varepsilon(s)) - \widetilde{G}_\varepsilon(s/\varepsilon, \widetilde{\Phi}^\varepsilon(s)) \Big] \, ds \Big\|_{B_m} \\
&\leq C(F,M) \ \int_0^t \|\Delta^\varepsilon(s)\|_{B_m} \ ds \\
&\qquad + \Big\| \int_0^t \ {\rm e}^{i(t-s)H_x} \ \Big[G(s/\varepsilon, \Phi^\varepsilon(s)) - \widetilde{G}_\varepsilon(s/\varepsilon, \Phi^\varepsilon(s)) \Big] \, ds \Big\|_{B_m}.
\end{aligned} \tag{4.12}$$

We are thus led to estimating the second term on the right-hand-side of (4.12). To do so, we write, whenever $0 \le t \le T_0$,

$$\begin{aligned}
&\int_0^t \mathrm{e}^{i(t-s)H_x} \left[G(s/\varepsilon, \Phi^\varepsilon(s)) - \widetilde{G}_\varepsilon(s/\varepsilon, \Phi^\varepsilon(s)) \right] ds \\
&= \int_0^t \mathrm{e}^{i(t-s)H_x}\, G\left(\frac{s}{\varepsilon}, \Phi^\varepsilon(s)\right)\, ds - \int_0^1 \int_0^t \mathrm{e}^{i(t-s)H_x}\, G\left(\frac{s+\varepsilon\, T(\varepsilon)\, u}{\varepsilon}, \Phi^\varepsilon(s)\right)\, ds\, du \\
&= \int_0^t \mathrm{e}^{i(t-s)H_x}\, G\left(\frac{s}{\varepsilon}, \Phi^\varepsilon(s)\right)\, ds \\
&\qquad - \int_0^1 \int_0^t \mathrm{e}^{i(t-s)H_x}\, G\left(\frac{s+\varepsilon\, T(\varepsilon)\, u}{\varepsilon}, \Phi^\varepsilon(s+\varepsilon\, T(\varepsilon) u)\right)\, ds\, du\, + R_1^\varepsilon \\
&= \int_0^t \mathrm{e}^{i(t-s)H_x}\, G\left(\frac{s}{\varepsilon}, \Phi^\varepsilon(s)\right)\, ds - \int_0^1 \int_{\varepsilon\, T(\varepsilon)\, u}^{t+\varepsilon\, T(\varepsilon)\, u} \mathrm{e}^{i(t-s)H_x}\, G\left(\frac{s}{\varepsilon}, \Phi^\varepsilon(s)\right)\, ds\, du \\
&\qquad + R_1^\varepsilon + R_2^\varepsilon \\
&= \int_0^t \mathrm{e}^{i(t-s)H_x}\, G\left(\frac{s}{\varepsilon}, \Phi^\varepsilon(s)\right)\, ds - \int_0^1 \int_0^t \mathrm{e}^{i(t-s)H_x}\, G\left(\frac{s}{\varepsilon}, \Phi^\varepsilon(s)\right)\, ds\, du \\
&\qquad + R_1^\varepsilon + R_2^\varepsilon + R_3^\varepsilon.
\end{aligned}$$

Eventually, we have established

$$\int_0^t \mathrm{e}^{i(t-s)H_x} \left[G(s/\varepsilon, \Phi^\varepsilon(s)) - \widetilde{G}_\varepsilon(s/\varepsilon, \Phi^\varepsilon(s)) \right] ds = R_1^\varepsilon + R_2^\varepsilon + R_3^\varepsilon,$$

and we have postponed the task of estimating the (implicitely defined) remainders R_1^ε, R_2^ε, and R_3^ε, for the moment.

The third remainder R_3^ε is easily estimated by

$$\begin{aligned}
\|R_3^\varepsilon\|_{B_m} &\le \int_0^{\varepsilon\, T(\varepsilon)} \left\| G\left(\frac{s}{\varepsilon}, \Phi^\varepsilon(s)\right) \right\|_{B_m} ds + \int_t^{t+\varepsilon\, T(\varepsilon)} \left\| G\left(\frac{s}{\varepsilon}, \Phi^\varepsilon(s)\right) \right\|_{B_m} ds \\
&\le 2\, \varepsilon\, T(\varepsilon) \left\| G\left(\frac{s}{\varepsilon}, \Phi^\varepsilon(s)\right) \right\|_{C^0([0,T_0+\varepsilon\, T(\varepsilon)];B_m)} \\
&\le C(F,M)\, \varepsilon\, T(\varepsilon).
\end{aligned}$$

The last line uses the Lipschitz condition (4.5), together with the fact that $\Phi^\varepsilon(s)$ is uniformly bounded in B_m, whenever $0 \le s \le T_0 + \varepsilon\, T(\varepsilon)$. Note that, *stricto sensu*, $\Phi^\varepsilon(s)$ is only defined for $0 \le s \le T_0$. However, since $\varepsilon\, T(\varepsilon) \to 0$, the solution $\Phi^\varepsilon(s)$ is easily seen to exist up to slightly larger times $T_0 + \varepsilon\, T(\varepsilon)$ (T_0 is *not* the maximal existence time of $\Phi^\varepsilon(s)$).

There remains to estimate R_1^ε and R_2^ε.

Concerning R_1^ε, we write

$$\|R_1^\varepsilon\|_{B_m} \le C(F,M)\, \varepsilon\, T(\varepsilon)\, \|\partial_t \Phi^\varepsilon(s)\|_{C^0([0,T_0+\varepsilon\, T(\varepsilon)];B_m)}.$$

Yet, the equation

$$i\partial_t \Phi^\varepsilon = H_x \Phi^\varepsilon + G(s/\varepsilon, \Phi^\varepsilon),$$

together with the bounds at hand for Φ^ε in $C^0([0, T_0 + \varepsilon\, T(\varepsilon)]; B_{m+2})$ and the uniform Lipschitz property (4.5) satisfied by $G(s/\varepsilon, .)$, clearly imply

$$\|\partial_t \Phi^\varepsilon(s)\|_{C^0([0,T_0+\varepsilon\, T(\varepsilon)];B_m)} \leq C,$$

for some $C > 0$ independent of ε. Eventually, we have established

$$\|R_1^\varepsilon\|_{B_m} \leq C\, \varepsilon\, T(\varepsilon),$$

for some $C > 0$ independent of ε.

Concerning R_2^ε, we write in the similar spirit

$$\begin{aligned}
&\|R_2^\varepsilon\|_{B_m} \leq (T + \varepsilon\, T(\varepsilon)) \left\| \left[\mathrm{e}^{i\varepsilon T(\varepsilon) u H_x} - 1 \right]\; G\left(\frac{s}{\varepsilon}, \Phi^\varepsilon(s)\right) \right\|_{C^0([0,1]\times[0,T_0+\varepsilon\, T(\varepsilon)];B_m)} \\
&\leq \varepsilon T(\varepsilon)\; (T + \varepsilon\, T(\varepsilon)) \left\| H_x\, G\left(\frac{s}{\varepsilon}, \Phi^\varepsilon(s)\right) \right\|_{C^0([0,T_0+\varepsilon\, T(\varepsilon)];B_m)} \\
&\leq \varepsilon T(\varepsilon)\; (T + \varepsilon\, T(\varepsilon)) \left\| G\left(\frac{s}{\varepsilon}, \Phi^\varepsilon(s)\right) \right\|_{C^0([0,T_0+\varepsilon\, T(\varepsilon)];B_{m+2})} \\
&\qquad \text{(note the loss of "two derivatives")} \\
&\leq C(F, M)\; \varepsilon T(\varepsilon).
\end{aligned}$$

Summarizing, the Gronwall Lemma gives in (4.12),

$$\forall 0 \leq t \leq T_0, \quad \|\Phi^\varepsilon(t) - \widetilde{\Phi}^\varepsilon(t)\|_{B_m} \leq C\, \varepsilon\, T(\varepsilon), \tag{4.13}$$

for some $C > 0$ independent of ε.

Sixth and last step: conclusion

Gathering the results established in the fourth and fifth steps, we recover

$$\forall 0 \leq t \leq T_0, \quad \|\Phi^\varepsilon(t) - \Phi(t)\|_{B_m} \leq C\, \left(\varepsilon\, T(\varepsilon) + \frac{\delta_M^{(2)}(\varepsilon)}{\varepsilon\, T(\varepsilon)} \right).$$

For that reason, the optimal choice for $T(\varepsilon)$ is

$$T(\varepsilon) = \sqrt{\delta_M^{(2)}(\varepsilon)}/\varepsilon. \tag{4.14}$$

We check, *a posteriori*, that $\varepsilon\, T(\varepsilon) = \sqrt{\delta_M^{(2)}(\varepsilon)} \to 0$, a property that has been used many times in the above estimates. This choice of $T(\varepsilon)$ gives

$$\forall 0 \leq t \leq T_0, \quad \|\Phi^\varepsilon(t) - \Phi(t)\|_{B_m} \leq C\, \sqrt{\delta_M^{(2)}(\varepsilon)} \to 0. \tag{4.15}$$

This ends the proof of Proposition 4.1. $\square$

Proof of Corollary 4.2.
Take $m > 3/2$ and $\Psi_0 \in B_m$. Given a small $\delta > 0$, pick a $\Psi_{0,\delta} \in B_{m+2}$ such that

$$\|\Psi_0 - \Psi_{0,\delta}\|_{B_m} \leq \delta.$$

Associated with $\Psi_{0,\delta}$, define the solutions $\Psi^\varepsilon_\delta(t)$, $\Phi^\varepsilon_\delta(t)$, $\Phi_\delta(t)$ to $i\partial_t \Psi^\varepsilon_\delta = H_x \Psi^\varepsilon_\delta + \varepsilon^{-1} H_z \Psi^\varepsilon_\delta + F(|\Psi^\varepsilon_\delta|^2)\,\Psi^\varepsilon_\delta$, resp. $i\partial_t \Phi^\varepsilon_\delta = H_x \Phi^\varepsilon_\delta + G(t/\varepsilon, \Phi^\varepsilon_\delta)$, resp. $i\partial_t \Phi_\delta = H_x \Phi_\delta + G_{\mathrm{av}}(\Phi_\delta)$, with initial data $\Psi_{0,\delta}$.

We already know from our previous results that there is a $T_0 > 0$, independent of ε, such that $\Psi^\varepsilon(t)$, $\Phi^\varepsilon(t)$, $\Phi(t)$ belong to $C^0([0, T_0], B_m)$, uniformly in ε. This comes from the (uniform in ε) Gronwall estimate

$$\partial_t \|\Psi^\varepsilon(t)\|_{B_m} \leq C_F \left(\|\Psi^\varepsilon(t)\|_{B_m}\right) \|\Psi^\varepsilon(t)\|_{B_m},$$

and similarly for $\Phi^\varepsilon(t)$ and $\Phi(t)$ In particular, as a consequence of the above bound, T_0 may be estimated from below by a quantity that only depends on F and $\|\Psi_0\|_{B_m}$.

We also know that for each δ, there is a T_0^δ such that $\Psi^\varepsilon_\delta(t)$, $\Phi^\varepsilon_\delta(t)$, $\Phi_\delta(t)$ belong to $C^0([0, T_0^\delta], B_{m+2})$, uniformly in ε. More precisely, the tame estimates of Propositions 2.5-(i) and 3.3-(ii) establish the following Gronwall estimate

$$\partial_t \|\Psi^\varepsilon_\delta(t)\|_{B_{m+2}} \leq C_F \left(\|\Psi^\varepsilon_\delta(t)\|_{B_m}\right) \|\Psi^\varepsilon_\delta(t)\|_{B_{m+2}},$$

and similarly for $\Phi^\varepsilon_\delta(t)$ and $\Phi_\delta(t)$. As a consequence, the existence time T_0^δ in B_{m+2} of all these functions may be estimated from below by a quantity that only depends on F and $\|\Psi_{0,\delta}\|_{B_m}$. Since $\|\Psi_0 - \Psi_{0,\delta}\|_{B_m} \leq \delta$, we may ensure that the $\|\Psi_{0,\delta}\|_{B_m}$ is a close as we wish to $\|\Psi_0\|_{B_m}$, so that T_0^δ may be in turn assumed as close as needed to T_0. For this reason, and without loss of generality, we may safely assume in the remaining part of this argument that *all* functions $\Psi^\varepsilon(t)$, $\Phi^\varepsilon(t)$, $\Phi(t)$, and $\Psi^\varepsilon_\delta(t)$, $\Phi^\varepsilon_\delta(t)$, $\Phi_\delta(t)$ are defined on the *same* time interval $[0, T_0]$.

Now, Proposition 4.1 asserts

$$\|\Phi^\varepsilon_\delta(t) - \Phi_\delta(t)\|_{C^0([0,T_0];B_m)} \underset{\varepsilon \to 0}{\longrightarrow} 0.$$

On the other hand, the tame estimates of Propositions 2.5-(i) and 3.3-(ii) and Gronwall's Lemma ensure

$$\begin{aligned}\|\Phi^\varepsilon_\delta(t) - \Phi^\varepsilon(t)\|_{C^0([0,T_0];B_m)} &\leq C\left(\|\Psi_{0,\delta}\|_{B_m}\right) \|\Psi_{0,\delta} - \Psi_0\|_{B_m} \leq C\left(\|\Psi_{0,\delta}\|_{B_m}\right) \delta \\ &\leq C\left(\|\Psi_0\|_{B_m}\right) \delta,\end{aligned}$$

as well as

$$\begin{aligned}\|\Phi_\delta(t) - \Phi(t)\|_{C^0([0,T_0];B_m)} &\leq C\left(\|\Psi_{0,\delta}\|_{B_m}\right) \|\Psi_{0,\delta} - \Psi_0\|_{B_m} \leq C\left(\|\Psi_{0,\delta}\|_{B_m}\right) \delta \\ &\leq C\left(\|\Psi_0\|_{B_m}\right) \delta.\end{aligned}$$

We are now in position to conclude: first choosing a δ such that the various bounds $C\left(\|\Psi_0\|_{B_m}\right)\delta$ become small, then choosing ε such that $\|\Phi^\varepsilon_\delta(t)-\Phi_\delta(t)\|_{C^0([0,T_0];B_{m+2})}$ becomes small as well, the Corollary is proved.

We stress the importance of the tame estimate of Proposition 2.5-(i), as well as that of Proposition 3.3-(ii), throughout this proof: it is the key ingredient to have the necessary uniformity along the regularizing process $\delta \to 0$. □

5 Application: the cubic Schrödinger equation, with harmonic confinement

In this section, we apply our Main Theorem to the following simplest model of Bose condensation

$$i\partial_t \Psi^\varepsilon(t) = \left(-\Delta_x + x^2\right)\,\Psi^\varepsilon(t) + \frac{1}{\varepsilon}\,\left(-\partial^2/\partial z^2 + z^2\right)\,\Psi^\varepsilon(t) + \left|\Psi^\varepsilon(t)\right|^2\,\Psi^\varepsilon(t). \tag{5.1}$$

In other words, we specify our discussion to the case

$$H_x = -\Delta_x + |x|^2, \quad H_z = -\partial^2/\partial z^2 + |z|^2, \quad F(u) = +u.$$

We know from the Main Theorem that this model is asymptotically described by

$$\begin{aligned} &i\partial_t \Phi(t) = \left(-\Delta_x + x^2\right)\,\Phi(t) \\ &+ \lim_{T\to\infty}\frac{1}{T}\int_0^T \mathrm{e}^{+i\tau[-\partial^2/\partial z^2+|z|^2]}\,\left|\mathrm{e}^{-i\tau[-\partial^2/\partial z^2+|z|^2]}\,\Phi(t)\right|^2\,\mathrm{e}^{-i\tau[-\partial^2/\partial z^2+|z|^2]}\,\Phi(t)\,d\tau. \end{aligned} \tag{5.2}$$

The conservation of total energy along x, here takes the following form

$$\langle \Phi(t)\,,\,H_x\,\Phi(t)\rangle + \frac{1}{4}\lim_{T\to\infty}\frac{1}{T}\int_0^T\int_{\mathbb{R}^3}\left|\mathrm{e}^{-i\tau H_z}\,\Phi(t)\right|^4\,dx\,dz\,d\tau = \text{const}, \tag{5.3}$$

which involves the sum of two *non-negative* terms, hence each term is uniformly bounded, and the solution exists globally in time in B_1, *i.e.* $\Phi(t)\in C^0([0,+\infty[;B_1)$.

Let us now give a more explicit form to (5.2). We know that the eigenelements of the harmonic oscillator $-\partial^2/\partial z^2+|z|^2$ are

$$E_p = (2p+1), \quad \text{and} \quad \chi_p(z) = H_p(z)\,\exp(-|z|^2/2),$$

where H_p is the p-th Hermite polynomial. Hence, introducing the quantities

$$\phi_p(t,x) = \langle \Phi(t,x,z)\,,\,\chi_p(z)\rangle\,, \qquad (p\in\mathbb{N}),$$

equation (5.2) readily becomes

$$
\begin{aligned}
i\partial_t \phi_p &= \left(-\Delta_x + x^2\right)\ \phi_p \\
&\quad + \lim_{T\to\infty} \frac{1}{T} \int_0^T \sum_{r,s,q\in\mathbb{N}} \phi_r(t)\, \phi_q(t)\, \overline{\phi_s(t)}\, \mathrm{e}^{-i\tau[E_q - E_s + E_r - E_p]}\ \langle \chi_q\, \chi_r\ ,\ \chi_s\, \chi_p \rangle\ d\tau,
\end{aligned}
$$

Now, since the E_p's are integers, the $\lim_{T\to\infty} \frac{1}{T} \int_0^T \ldots$ simply becomes averaging over one period, namely $\frac{1}{2\pi}\int_0^{2\pi} \ldots$, and the latter integral transforms the sum $\sum_{q,r,s} \ldots$ into a sum over those integers such that $E_q + E_r = E_p + E_s$, or, in other words, $q + r = p + s$. We thus recover the averaged model

$$
\begin{aligned}
i\partial_t \phi_p(t,x) &= \left(-\Delta_x + x^2\right)\ \phi_p + \sum_{r,s,q/q+r=p+s} A_{p,q,r,s}\ \phi_r\, \phi_q\, \overline{\phi_s} \qquad (5.4) \\
&\qquad \text{where} \quad A_{p,q,r,s} := \langle \chi_q\, \chi_r\ ,\ \chi_s\, \chi_p \rangle\ .
\end{aligned}
$$

This is an infinite system of cubic Schrödinger equations along the x plane. Note that we do not have any simple information about the behavior of the given coefficients $A_{p,q,r,s}$ entering the system, despite the fact that the eigenfunctions χ_p are explicitly known. This makes it definitely easier to deal directly with the equation on Φ (without projecting).

As a special case, equation (5.4) allows to recover the one mode situation treated in [BMSW]. Indeed, when the initial datum satisfies

$$\Phi(0,x,z) = \phi_0(0,x)\, \chi_0(z),$$

i.e. when $\Phi(0)$ lies entirely in the eigenspace associated with the lowest energy $E_0 = 1$, it is easily seen that the function

$$\Phi(t,x,z) = \phi_0(t,x)\, \chi_0(z)$$

solves the averaged system (5.2), provided $\phi_0(t,x)$ solves the one-mode problem

$$i\partial_t \phi_0(t,x) = \left(-\Delta_x + x^2\right)\ \phi_0 + A_{0,0,0,0}\ |\phi_0(t)|^2\, \phi_0(t). \qquad (5.5)$$

This is due to the fact that, starting from the mode $p = 0$, equation (5.4) can only feed the mode $p = 0$ and no new mode is switched on. Uniqueness of the solutions to (5.2) then establishes that the above Φ is *the* relevant solution. As desired, we exactly recover the one-mode averaged model derived in [BMSW] (see introduction).

One can even go a bit further, namely, when the initial datum is *any* one-mode function

$$\Phi(0,x,z) = \phi_p(0,x)\, \chi_p(z),$$

for some given index p, i.e. when $\Phi(0)$ lies entirely in the eigenspace associated with the energy E_p, it is easily seen that the function

$$\Phi(t,x,z) = \phi_p(t,x)\,\chi_p(z)$$

solves the averaged system (5.2), provided $\phi_p(t,x)$ solves the one-mode problem

$$i\partial_t \phi_p(t,x) = \left(-\Delta_x + x^2\right)\,\phi_p + A_{p,p,p,p}\,|\phi_p(t)|^2\,\phi_p(t). \tag{5.6}$$

Again, starting from the mode p, equation (5.4) can only feed the same mode p and no new mode is switched on. Uniqueness again establishes that the above Φ is *the* relevant solution. This observation extends the results of [BMSW] to *any* one-mode solution. Note that uniqueness is *not* obvious when arguing directly on the projected system (5.4).

Now, in the opposite case where the initial datum contains at least *two* distinct modes, say p_0 and p_1, it is clear that equation (5.4) immediately allows to switch on the modes $2p_0 - p_1$, $2p_1 - p_0$, hence the modes $4p_0 - 3p_1$ and so on, so that eventually *an infinite number of modes* is switched on, and the need for a clean functional analytic framework to treat equation (5.4), namely the formulation (5.2), becomes transparent. This observation is the reason why we actually tackle the *generic* multi-mode case in this article.

We wish to end this text with a last, bibliographical comment.

In [BaMSW], the above problem (5.4) has been *formally* derived. The authors study the existence and uniqueness for a simplified problem by proceeding to a *truncation of the modes*. Namely they consider the problem

$$i\partial_t \Phi_p = H_x \phi_p + \sum_{r,s,q/q+r=p+s \text{ and } p,q,r,s\leq L} A_{p,q,r,s}\phi_r\,\phi_q\,\overline{\phi_s}. \tag{5.7}$$

Needless to say, the truncated problem (5.7) is considerably simpler than (5.4), in that all the convergence issues of the series expansion are then removed. To study the above truncated problem, they consider it as a cubic Schrödinger system in $\mathbb{R}^2$. The local existence in $H^1(\mathbb{R}^2)^L$ is proved by showing that the cubic term is locally a Lipschitz function in $H^1(\mathbb{R}^2)^L$. The global existence of solutions is shown by using the defocusing character of the cubic term, which is a version of the above energy conservation (5.3). Unfortunately, the Lipschitz constant tends to $+\infty$ as L tends to $+\infty$, so that the approach of [BaMSW] seemingly does not allow the construction of solutions to the whole limit problem. This again shows the *necessity* to avoid projections in the present context, be it at the level of the asymptotic process, or even at the level of the limiting model itself: the compact formulation (5.2) contains more information than its projection (5.4).

In that perspective, another aspect of our approach is that it eventually *justifies* the fact that a truncated system can be considered for numerical purposes, even though the convergence rate is not known. Indeed, our approach actually allows to construct the solutions of all *truncated* problems at once, and to show that they provide indeed a good approximation of the untruncated one in B_m ($m > 3/2$), as $L \to \infty$. Let us show this last statement. Let θ_L be a cutoff function in $C^\infty(\mathbb{R})$ such that

$$0 \le \theta_L \le 1; \quad \theta_L(u) = 1 \quad \text{for } u \le E_L; \quad \theta_L(u) = 0 \quad \text{for } u \ge E_{L+1}.$$

The truncated problem may be written

$$i\partial_t \Phi_L(t) = H_x \Phi + F_{\text{av}}^L(\Phi_L), \quad \Phi_L(0) = \theta_L(H_z)\Psi_0, \tag{5.8}$$

where

$$F_{\text{av}}^L(u) := \theta_L(H_z) F_{\text{av}} \left(\theta_L(H_z)\, u\right),$$

whenever $u \in B_m$. Since $\theta_L(H_z)$ is a bounded operator on B_m, with norm equal to 1, it is clear that F_{av}^L exists and has essentially the same properties as F_{av}. Since $\theta_L(H_z)$ converges strongly, as L tends to $+\infty$, to the identity in B_m, we have

$$\lim_{L\to+\infty} \sup_{u\in C_m} \|\theta_L(H_z)u - u\|_{B_m} = 0,$$

where C_m may be *any* given compact subset of B_m. Consequently, it is readily seen that we have

$$\lim_{L\to+\infty} \sup_{u\in C_m} \|F_{\text{av}}(u) - F_{\text{av}}^L(u)\|_{B_m} = 0.$$

We also have the uniform Lipschitz property

$$\sup_{\|u\|_{B_m}, \|v\|_{B_m} \le M} \|F_{\text{av}}^L(u) - F_{\text{av}}^L(v)\|_{B_m} \le C(M)\, \|u - v\|_{B_m},$$

with a constant $C(M)$ independent of L. These two properties are enough to show that if $[0, T)$ is the maximal existence interval for the untruncated problem in B_m, then for any $T_0 < T$, the truncated problem, with L large enough, has a unique solution Φ_L in $C^0([0, T_0], B_m)$ and we have

$$\lim_{L\to+\infty} \|\Phi - \Phi_L\|_{C^0([0,T_0],B_m)} = 0.$$

Let us finally notice that the energy estimate for the truncated problem is obviously obtained by replacing the function $G_{\text{av}}(\Psi)$ by $G_{\text{av}}^L(\Psi) := G_{\text{av}}(\theta_L(H_z)\Psi)$.

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