Large time dynamics of a classical system subject to a fast varying force

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Abstract

We investigate the asymptotic behavior of solutions to a kinetic equation describing the evolution of particles subject to the sum of a fixed, confining, Hamiltonian, and a small, time-oscillating, perturbation. The equation also involves an interaction operator which acts as a relaxation in the energy variable. This paper aims at providing a classical counterpart to the derivation of rate equations from the atomic Bloch equations. In the present classical setting, the homogenization procedure leads to a diffusion equation in the energy variable, rather than a rate equation, and the presence of the relaxation operator regularizes the limit process, leading to finite diffusion coefficients. The key assumption is that the time-oscillatory perturbation should have well-defined long time averages: our procedure includes general "ergodic" behaviors, amongst which periodic, or quasi-periodic potentials only are a particular case.

Key words: Kinetic equation, Homogenization, Diffusion limit.

AMS Subject classification: 74Q10, 35Q99, 35B25, 82C70

Comm. Math. Phys., Vol. 276, N. 1, pp. 23-49 (2007).

1 Setting of the Problem

We consider the asymptotic behavior as ε goes to 0 of the solutions $f^\varepsilon(t,x,v) \geq 0$ to the following kinetic equation with relaxation term

$$\varepsilon^2\, \partial_t f^\varepsilon(t,x,v) + \left\{H_0(x,v), f^\varepsilon\right\} + \varepsilon \left\{V\left(\frac{t}{\varepsilon^2}, x\right)\, ,\, f^\varepsilon\right\} = \gamma\, Q(f^\varepsilon)(t,x,v), \tag{1.1}$$

$$\text{where} \quad Q(f^\varepsilon)(t,x,v) := P(f^\varepsilon)(t,x,v) - f^\varepsilon. \tag{1.2}$$

Here, the Poisson bracket $\{\cdot,\cdot\}$ stands as usual for $\{f,g\} = \nabla_v f \cdot \nabla_x g - \nabla_x f \cdot \nabla_v g$. The position resp. velocity variables x resp. v both belong to the whole space $\mathbb{R}^d$ ($d \geq 1$), and we shall often make use of the phase-space variable $X = (x,v) \in \mathbb{R}^{2d}$. Throughout this text, the Hamiltonian $H_0(X) \in C^\infty(\mathbb{R}^{2d})$ is assumed given, and confining, *i.e.*

$$\lim_{|X|\to\infty} H_0(X) = +\infty. \tag{1.3}$$

The right-hand-side of (1.1)-(1.2) involves a projection operator P, whose value we define as

$$Pf^\varepsilon(t,X) := [\Pi f^\varepsilon](t, H_0(X)), \tag{1.4}$$

where the quantity $\Pi f^\varepsilon(t,E)$ is the mean value of f^ε over the energy shell

$$S_E := \{X \in \mathbb{R}^{2d} \ \text{ s.t. } \ H_0(X) = E\},$$

namely,

$$\Pi f^\varepsilon(t,E) := \frac{1}{h_0(E)} \int_{S_E} f^\varepsilon(t,X)\, \delta(H_0(X) - E), \tag{1.5}$$

$$\text{where } h_0(E) := \int_{S_E} \delta(H_0(X) - E) \quad \left(S_E = {H_0}^{-1}(E)\right). \tag{1.6}$$

Here, the measure $\delta(H_0(X) - E)$ over S_E is defined as

$$\delta(H_0(X) - E) := \frac{d\sigma_E(X)}{\left|\nabla_X H_0(X)\right|}, \tag{1.7}$$

where $d\sigma_E(X)$ denotes the induced euclidean surface measure over the energy shell S_E. The measure $\delta(H_0(X) - E)$ is the standard micro-canonical (or Liouville) measure of statistical physics. The fact that the above objects $Pf^\varepsilon(t,X)$ and $\Pi f^\varepsilon(t,E)$ are well defined is proved later, under the main assumption that the measure $\delta(H_0(X) - E)$ satisfies $h_0(E) = \displaystyle\int_{S_E} \delta(H_0(X) - E) < +\infty$ for almost every $E \in H_0(\mathbb{R}^{2d})$ (see Hypothesis 1), a requirement which somehow reinforces the fact that H_0 is assumed confining. Note that throughout this text, the prototype where H_0 is the harmonic oscillator $H_0 = H_{\text{harm}} = (x^2 + v^2)/2$ is relevant.

The kinetic equation (1.1) is written in dimensionless form. The important dimensionless parameters are $\varepsilon > 0$, which goes to zero, and the relaxation parameter $\gamma > 0$, considered fixed. We refer to [CDG] for a thorough discussion and motivation of the scaling.

The dynamics induced by equation (1.1) may be described as follows:

(a) at leading order, the evolution is driven by transport along the Hamiltonian flow of H_0, *i.e.* along the solutions to the Hamiltonian ODE

$$\begin{aligned} \partial_t \overline{x}(t,x,v) &= \nabla_v H_0\left(t, \overline{x}(t,x,v), \overline{v}(t,x,v)\right), & \overline{x}(0,x,v) &= x, \\ \partial_t \overline{v}(t,x,v) &= -\nabla_x H_0\left(t, \overline{x}(t,x,v), \overline{v}(t,x,v)\right), & \overline{v}(0,x,v) &= v. \end{aligned} \tag{1.8}$$

We shall often use the phase-space notation $\overline{X}(t,X) = (\overline{x}(t,x,v), \overline{v}(t,x,v))$. Due to the scaling in (1.1), transport occurs at the fast time scale t/ε^2 and, since H_0 is confining, transport roughly induces "oscillatory" trajectories at the fast scale t/ε^2.

(b) the reference Hamiltonian H_0 is perturbed by the small and oscillatory potential $\varepsilon\, V(t/\varepsilon^2)$.

On the one hand, this perturbation is of size $\varepsilon > 0$ when compared to H_0, and ε perturbations of a Hamiltonian flow are known to modify the dynamics by an $O(\varepsilon^2)$ quantity, on time scales of the order 1. Time being rescaled by a factor $1/\varepsilon^2$ in our case, the perturbing term εV is expected to modify the dynamics by a quantity of the order 1. This is usually called a weak-coupling regime.

On the other hand, the oscillations carried by the potential $V(t/\varepsilon^2)$ at the fast scale t/ε^2 may interact with those induced by the transport term at the same scale. Hence only the average effect – in time – of these combined oscillations is expected to influence the dynamics at dominant order.

(c) the relaxation term $Q(f^\varepsilon) = P(f^\varepsilon) - f^\varepsilon$, discussed below, models complex interaction phenomena. Since the projection operator $P(f^\varepsilon)$ projects f^ε onto functions of the energy $H_0(X)$ only, it is clear that $Q(f^\varepsilon)$ redistributes the energy uniformly on each energy shell S_E, or, in other words, it relaxes f^ε to a solution of $Pf^\varepsilon = f^\varepsilon$. Yet the fluctuations $Pf^\varepsilon - f^\varepsilon$, small but definitely non-zero in (1.1), are transported along the Hamiltonian flow, which eventually give rise to diffusion in the energy variable. This is typically what already happens for standard diffusion limits in kinetic theory.

In a previous text [CDG], we studied the asymptotic behavior of (1.1) under the main assumption that the perturbing potential $V(t/\varepsilon^2, x)$ is periodic or quasi-periodic in the fast time variable. Assuming also that the Hamiltonian flow of H_0 has some stability property (an assumption that we shall need here as well, see Hypothesis 2 below), we proved that f^ε goes, in some weak topology, to a function of the energy only, say $F(t, H_0(X))$, and that the limiting profile $F(t, E)$ ($E \in \mathbb{R}$) satisfies a diffusion equation in energy, of the form

$$\partial_t \big[h_0(E)\, F(t,E)\big] - \partial_E \big[b(E)\, h_0(E)\, \partial_E F(t,E)\big] = 0, \tag{1.9}$$

where $h_0(E)$ is defined in (1.5). The effective coefficient $b(E)$ obtained in [CDG] is non-negative. It takes into account the average effect, in time, of the resonant interactions between $\overline{X}(t/\varepsilon^2)$ and the perturbation $V(t/\varepsilon^2)$. It also is given by an almost explicit formula involving auxiliary profile equations.

For several reasons, the periodic or quasi-periodic case is unsatisfactory.

Technically speaking, the limit equation is easily guessed in the (quasi-)periodic case, upon simply performing a double scale expansion in the spirit of classical homogenization theory [BLP]. On top of that, the actual proofs given in [CDG] rely on the user-friendly framework of double-scale convergence introduced in [Ng] and [A]. In essence, (quasi-)periodicity turns out to be a strong, and quantitative version of ergodicity, for which the analysis eventually reduces to conveniently adapting the tools of double-scale convergence.

(Quasi-)periodicity is also restrictive from the physical viewpoint. As discussed below, equation (1.1) may describe the classical evolution, in phase space, of a collection of atoms undergoing the influence of the atomic Hamiltonian H_0, and weakly coupled to an external laser field through $\varepsilon V(t/\varepsilon^2)$. The potential V then is roughly the electric field. Hence restricting to (quasi-)periodic V's means restricting to (quasi-)periodic fields.

Last, the quantum equation analogous to (1.1) has been previously studied in [BCD, BCDG], and other, more general situations than the mere (quasi-)periodic setting have been analyzed there. The analysis performed in [BCD, BCDG] actually shows the key point is that the perturbing potential V should possess well defined long time averages of the form $\lim_{T\to\infty} \dfrac{1}{T}\displaystyle\int_0^T V(s)\, ds$ or so. The existence of such long time averages is certainly implied by the much stronger (quasi-)periodicity assumption, but it definitely includes much more general "ergodic" behaviors.

Adopting this point of view, our goal in the present paper is to fill the gap, *i.e.* to describe the asymptotic $\varepsilon \to 0$ for more general oscillating potentials V that are not (quasi-)periodic in time. The potentials we are interested in are actually of KBM type (after: Krylov, Bogolioubov, Mitropolski): they are required to possess specific long-time averages (Hypothesis 3 below).

We stress that the present extension is by no means trivial.

First, the mere formulation of the assumption we need on the potential, though of KBM type, is not immediate. It is an original requirement. Second, the (quasi-)periodic setting essentially requires an adaptation of tools from double-scale analysis, and allows to pass to the limit in the standard two-scale topology directly on f^ε. This strategy cannot be extended in any way here. We need at variance to split f^ε as $Pf^\varepsilon + (\mathrm{Id} - P)f^\varepsilon$ and to pass to the limit in Pf^ε, exploiting a specific compactness property inherited from the structure of the equation. From this point of view, our proof shares a lot of features with the derivation of Kubo-like formula in [GP3]. Last, an important difficulty in the analysis of (1.1) is created by the fact that the operator $f \mapsto Pf$ does not preserve the smoothness of f, and Pf is not even once differentiable in general. This difficulty is linked with the general lack of smoothness of the energy levels S_E as E varies. In the (quasi-)periodic setting, the use of a double-scale analysis allows to circumvent this difficulty, with the mere drawback that our homogenization procedure does not provide a corrector. In the present framework, the lack of smoothness of Pf is more problematic, and we do need to deal with this difficulty in order to recover the necessary compactness.

Our main result is the following.

Theorem 1.1 *Let $f_0^\varepsilon \geq 0$ be the initial data for (1.1). We assume that f_0^ε is bounded in $L^2(\mathbb{R}^{2d})$. We also suppose the Hamiltonian has well-defined energy levels with finite measure (Hypothesis 1 below), the Hamiltonian flow of H_0 is polynomially stable (Hypothesis 2), and the potential V has some well-defined long-time averages (Hypothesis 3). Then, the following holds:*

(i) The solution $f^\varepsilon(t,X)$ to (1.1) admits the decomposition

$$f^\varepsilon(t,X) = Pf^\varepsilon(t,X) + \varepsilon g_\varepsilon(t,X),$$

where g_ε is bounded in $L^2((0,T)\times\mathbb{R}^{2d})$ and $Pf^\varepsilon(t,X)$ is bounded in $C^0([0,T];L^2(\mathbb{R}^{2d}))$.

(ii) up to subsequences, $Pf^\varepsilon(t,X)$ converges in $C^0([0,T];L^2(\mathbb{R}^{2d})-\text{weak})$ towards a function $F(t,H_0(X))$.

(iii) the limiting function $F:\mathbb{R}^+\times\mathbb{R}\to\mathbb{R}^+$ satisfies the following diffusion equation in $\mathcal{D}'([0,\infty)\times\mathbb{R})$:

$$\begin{aligned}
&\partial_t\big[h_0(E)\,F(t,E)\big] - \partial_E\big[h_0(E)\,b(E)\,\partial_E F(t,E)\big] = 0,\\
&F(0,E) = \lim_{\varepsilon\to 0}\Pi f_0^\varepsilon(E) \qquad \textit{(the limit is in } L^2(\mathbb{R})\textit{-weak).}
\end{aligned}$$

Equivalently, F satisfies

$$\begin{aligned}
&\partial_t\big[h_0(E)\,F(t,E)\big] - h_0(E)\,a(E)\,\partial_E F(t,E) - h_0(E)\,b(E)\,\partial^2_{E,E}F(t,E) = 0,\\
&F(0,E) = \lim_{\varepsilon\to 0}\Pi f_0^\varepsilon(E).
\end{aligned}$$

Here the coefficient a and b are defined from (1.20) *below through*

$$a(E) = \Pi\langle A\rangle(E), \quad b(E) = \Pi\langle B\rangle(E).$$

They satisfy

$$h_0(E)\,a(E) = \partial_E\big[h_0(E)\,b(E)\big], \quad b(E)\geq 0.$$

Naturally, the above theorem extends the result obtained in [CDG] to non-periodic V's. More importantly, it is worth remarking that the formal structure of the formula which defines the diffusion coefficients a and b coincides with the one of eddy diffusivity in turbulence theory, see [Ta].

Before coming to the formulation of Hypothesis 1, 2, and 3, and to the proof of Theorem 1.1, we wish to make two comments.

Equation (1.1) is the standard setting for the description of an atom in interaction with a light field, say a laser. This is the prototype situation we have in mind: the unperturbed Hamiltonian H_0 is the atomic Hamiltonian, while the perturbation $\varepsilon V(t/\varepsilon^2)$ is the potential energy induced by light in the vicinity of the atom. Equation (1.1) adopts a classical mechanics description of such an interaction.

If a quantum mechanical setting is retained, the kinetic equation (1.1) becomes a quantum Liouville equation, also known as an atomic Bloch equation. It reads in our case

$$i\varepsilon^2\partial_t\rho^\varepsilon(t) = [H_0\,,\,\rho^\varepsilon(t)] + \varepsilon\left[V\left(\frac{t}{\varepsilon^2}\right)\,,\,\rho^\varepsilon(t)\right] + \gamma\mathcal{Q}(\rho^\varepsilon(t)). \tag{1.10}$$

The unknown $\rho^\varepsilon(t)$ now is a time dependent trace class operator, called "density matrix" of the atom, all Poisson brackets in (1.1) have become commutators between operators in (1.10), and $\mathcal{Q}(\rho^\varepsilon)$ is a relaxation operator that plays the same role as $Q(f^\varepsilon)$. The factor $\mathcal{Q}(\rho^\varepsilon)$ describes at a heuristic level the observed trend of various atomic systems to relax towards equilibria of the unperturbed Hamiltonian H_0. The relaxation term $\mathcal{Q}$ is well documented in the physics literature (see e.g. [Lo]), while the operator Q we introduced in [CDG] comes up in mere analogy with $\mathcal{Q}$, as a classical counterpart of the quantum operator $\mathcal{Q}$.

The quantum setting (1.10) has been completely analyzed in [BCD], for potentials that are either (quasi-)periodic in time, or more generally of KBM type (with optimal convergence rates in the first situation, and no better error estimate than $o(1)$ in the second). The main convergence result obtained in [BCD] is similar in spirit to what we prove here and in [CDG]: the atom tends to be well described by a function of the energy only, *i.e.* by a function $N(t, n)$ which describes the occupation probability of the atom's n-th eigenstate at time t. The function $N(t, n)$ plays the role of $F(t, E)$. Yet the main difference with the classical case is that energy levels now are discrete, so that the diffusion process obtained classically for F rather becomes a discrete jump process for N.

Our second comment concerns the role of the relaxation term Q. From the modeling point of view, a mathematically rigorous derivation of this term goes far beyond the scope of this paper. We simply use it as a way to take into account observed relaxation phenomena. On the other hand, the asymptotic analysis of (1.1) or (1.10) is dominated by the resonant interaction between the oscillations of $V(t/\varepsilon^2)$, and those induced by the transport operator $\varepsilon^2\partial_t + \{H_0, .\}$. Technically, the relaxation operators Q or $\mathcal{Q}$ somewhat regularize the situation in this respect: they prevent the possibility of too strong resonances (small denominators), through the introduction of some damping in the model. One may then wonder what happens along the asymptotic process if the relaxation term is set to zero in the original equations (1.1) or (1.10). In the quantum case, and for (quasi-)periodic potentials V, it turns out smaller damping rates of order $\mathcal{O}(\varepsilon^\mu)$ with $\mu < 1/2$ may be considered (see [BCD, BCDG]). The usual (undamped) formulae for the Einstein rate equations [Lo] have been recovered in [BCD, BCDG]. The analysis heavily relies on small denominator estimates, perturbed Diophantine estimates, and other arguments in the same vein, in the spirit of averaging techniques for ODE's. Yet the condition $\mu < 1/2$ still means that damping should not be too small with respect to the other perturbations. In this perspective, a deep gap actually separates the case "with damping" from the case "without damping" (in this article as well as in [CDG] and [BCD, BCDG]). The mathematical and physical situation, as well as the limiting process itself, are completely different when the (possibly small) relaxation term is set to zero from the onset: indeed, the limiting equation (1.9) is time-irreversible, while the associated scaled equation (1.1) only becomes irreversible through the dissipation term. The route we choose here uses a heuristic, and deterministic, relaxation term. The reader may find in [CD, Ca1, Ca2] a similar model "with relaxations" used to rigorously derive the Pauli master equation from the quantum Liouville equation in a deterministic framework. A second, probably more standard approach is the introduction of stochastic averaging in the model, which gives the necessary "loss of memory" in the analysis: to some extent, deterministic relaxation terms play a similar role as stochastic averaging process. The deep role played by stochastic averaging in the derivation of irreversible equations is very well explained in [CIP] (see also [Sp]). More recently, we may mention [EY1], [EY2], [PV], [LV], or also [KPR]. There are actually several other examples of such an alternative: homogenization of convection(-diffusion) equations (see [GP1, GP2] and the references therein), Lorentz gas involving in a billiard (see [BDG] and [BSC]), quantum scattering limit of the Schrödinger equation ([BPR], [EY2], [PR], [PV]...). For the (space-)homogenization of kinetic equations without dissipative term, we refer e.g. to [Al], [FH].

We now state the assumptions we need on the Hamiltonian H_0, and the potential V.

We begin with the assumptions on H_0, which are essentially the same as in [CDG].

Hypothesis 1 (energy levels with finite measure).
We assume that the confining Hamiltonian H_0 satisfies
(i) for almost all $E \in H_0(\mathbb{R}^{2d})$, the set

$$S_E = \{X = (x, v) \in \mathbb{R}^{2d} | H_0(X) = E\},$$

is a smooth orientable $2d - 1$ submanifold of $\mathbb{R}^{2d}$. For any such E, we denote the induced euclidean surface measure by $d\sigma_E(X)$. We also define the (micro-canonical or Liouville) measure $\delta(H_0(X) - E)$ as

$$\delta(H_0(X) - E) = \frac{d\sigma_E(X)}{|\nabla_X H_0(X)|}.$$

(ii) for any E as in (i), the set S_E has finite measure with respect to $\delta(H_0(X) - E)$, namely

$$h_0(E) = \int_{S_E} \delta(H_0(X) - E) < +\infty, \qquad a.e.\ E \in H_0(\mathbb{R}^{2d}).$$

Remark. As noticed in [CDG], the Sard theorem asserts that Hypothesis 1-(i) is generically satisfied (see e.g. [Mi]). Hence the truly important assumption is point (ii). ∎

Hypothesis 2 (stability of the Hamiltonian flow).
Let $\overline{X} : (s, X) \in \mathbb{R} \times \mathbb{R}^{2d} \longmapsto \overline{X}(s, X) \in \mathbb{R}^{2d}$ be the Hamiltonian flow of H_0, namely

$$\frac{d}{ds}\overline{X}(s,X) = \begin{pmatrix} +\nabla_v H_0 \\ -\nabla_x H_0 \end{pmatrix} \Big(\overline{X}(s,X)\Big), \qquad \overline{X}(0,X) = X = (x,v). \tag{1.11}$$

We assume that the linearized flow satisfies, for any $s \geq 0$ and $X \in \mathbb{R}^{2d}$,

$$\left|\frac{D\overline{X}}{DX}(s,X)\right| \leq C(X)\ (1+s)^p, \tag{1.12}$$

$$\left|\frac{D^2\overline{X}}{DX^2}(s,X)\right| \leq C(X)\ (1+s)^p, \tag{1.13}$$

for some $C(X) \geq 0$ which is locally bounded in X, and some exponent $p \geq 0$, independently of s.

Remark. The analysis provided in [CDG] for (quasi-)periodic potentials only requires the first stability estimate (1.12). In the present text, estimate (1.13) is required as well. It may be somewhat relaxed into a locally Lipschitz bound of the form $|D\overline{X}/DX(s, X+Y) - D\overline{X}/DX(s,X)| \leq C(X)\ |Y|\ (1+s)^p$. We do not detail this unnecessary technical point. ∎

Remark. The crucial assumptions on H_0 are Hypothesis 1-(ii) and 2. The former allows to define the relaxation operator Q (see Lemma 1.2 below), while the latter is a strong stability assumption on the Hamiltonian flow of H_0: according to (1.12), any two trajectories starting with nearby initial data should diverge at most polynomially with time. For the standard Hamiltonian $H_0(X) = v^2/2 + V_0(x)$, with V_0 any 'reasonable' potential, we recall that the generic divergence of two nearby trajectories is more likely exponential in time. Naturally, Hypotheses 1 and 2 are fulfilled in the prototype case of the harmonic oscillator $H_{\rm harm}(X) = (x^2 + v^2)/2$.

We mention that the stability Hypothesis 2 may be somewhat relaxed, so as to include the case of exponential divergence, *i.e.* the case when $(1+s)^p$ is replaced by $\exp(C_1\, s)$ for some $C_1 > 0$. However, this can only be done at the (unreasonable) price of considering large enough values of the relaxation parameter γ, namely $\gamma > C_1$. We do not dwell on this aspect of the analysis. ∎

As proved in [CDG], Hypothesis 1 allows to properly define the operators Π and P, as in (1.4) and (1.5). This gives a well-defined relaxation operator $Q = \mathrm{Id} - P$ in (1.1). The basic observation is that the co-area formula reads, with the above notations

$$\forall f \in L^1(\mathbb{R}^{2d}), \qquad \int_{\mathbb{R}^{2d}} f(X)\ dX = \int_{\mathbb{R}} \Pi f(E)\ h_0(E)\ dE. \tag{1.14}$$

Armed with (1.14), one may indeed deduce (see [CDG]) the following

Lemma 1.2 *The operator P defined in (1.4) satisfies the following properties:*
(i) P is a continuous projection operator on L^p spaces:

$$P(Pf) = Pf, \qquad \|Pf\|_{L^p(\mathbb{R}^{2d})} \leq \|f\|_{L^p(\mathbb{R}^{2d})} \quad 1 \leq p \leq \infty.$$

Besides P is conservative: for any integrable function, we have

$$\int_{\mathbb{R}^{2d}} Pf\ dX = \int_{\mathbb{R}^{2d}} f\ dX.$$

(ii) P is self-adjoint with respect to the inner product of $L^2(\mathbb{R}^{2d})$ (denoted $\langle \cdot, \cdot \rangle$ throughout the paper): for any function $f \in L^2(\mathbb{R}^{2d})$ and $\varphi : \mathbb{R} \to \mathbb{R}$ such that $\varphi(H_0(X)) \in L^2(\mathbb{R}^{2d})$, we have

$$\Big\langle \varphi(H_0(X)), (\mathrm{Id} - P)f \Big\rangle = 0.$$

(iii) P is non negative: if $f \geq 0$ a.e. X, then $Pf \geq 0$ a.e. X as well. Moreover, the stronger relation holds:

$$f \geq 0 \text{ a.e. } X, \text{ and } Pf = 0 \text{ a.e. } X \Longrightarrow f = 0 \text{ a.e. } X.$$

(iv) The operators $f \longmapsto Pf$ and $f \longmapsto \big\{H_0, f\big\}$ are orthogonal, i.e. *the relation*

$$P\big\{H_0, f\big\} = 0,$$

holds for any $f \in L^2(\mathbb{R}^{2d})$ such that $\{H_0, f\} \in L^2(\mathbb{R}^{2d})$. Consequently, for any $f, g \in L^2(\mathbb{R}^{2d})$ such that $\{H_0, f\}$ and $\{H_0, g\}$ in $L^2(\mathbb{R}^{2d})$, we have

$$P(\big\{H_0, f\big\}g) = -P(f\big\{H_0, g\big\}).$$

(v) The operator $Q = \mathrm{Id} - P$ is a bounded operator on $L^2(\mathbb{R}^{2d})$ and the relation

$$-\int_{\mathbb{R}^{2d}} Q(f)f \; dX = \int_{\mathbb{R}^{2d}} \big|Pf - f\big|^2 \; dX \geq 0$$

holds for any $f \in L^2(\mathbb{R}^{2d})$.
(vi) Let $\mathcal{V} : \mathbb{R} \times \mathbb{R}^{2d} \to \mathbb{R}$ be a C^1 function. Let $\varphi \in C_c^\infty(\mathbb{R})$. We have

$$P\big\{\mathcal{V}(t), \varphi(H_0)\big\} = 0.$$

Remark. Point (v) asserts that Q relaxes towards solutions to $Pf = f$, and the rate of convergence is unity. The proof of point (vi) relies on the equality

$$P\big\{\mathcal{V}, \varphi(H_0)\big\}(X) = P\Big(\big\{\mathcal{V}, H_0\big\}\Big)(X) \; \partial_E \varphi(H_0(X)) = 0.$$

∎

There remains to specify the behavior of the perturbing potential $V(s, x)$ in the fast time variable s. This is the main new point in the present paper.

To motivate our approach, let us describe the situation in the quantum case, *i.e.* when analyzing (1.10) instead of (1.1). The key point in [BCD, BCDG] is the following: one may transform the original PDE (1.10) into an infinite dimensional ODE, which in turn very much behaves like a finite dimensional ODE with oscillatory coefficients (up to small remainder terms), of the form

$$\frac{d}{dt} y_\varepsilon(t) = \psi\left(\frac{t}{\varepsilon^2}, y_\varepsilon(t)\right). \tag{1.15}$$

As a consequence, the quantum problem (1.10) reduces, in essence, to performing an averaging procedure on the ODE (1.15). Now, the basic averaging theorem for ODE's (see e.g. [SV]) asserts that the solution $y_\varepsilon(t)$ to any oscillatory ODE of the form (1.15), converges in some topology towards the solution of the averaged ODE

$$\frac{d}{dt} y(t) = \langle\psi\rangle\big(y(t)\big) \qquad \left(= \lim_{T\to\infty} \frac{1}{T} \int_0^T \psi\big(s, y(t)\big)\, ds\right), \tag{1.16}$$

provided the function $\psi(s, z)$ has KBM dependence in the time variable s. We recall that a function $\psi(s, z)$ is called a KBM function whenever the limit

$$\langle\psi\rangle(z) = \lim_{T\to\infty} \frac{1}{T} \int_0^T \psi(s, z)\, ds \tag{1.17}$$

exists, for every z. We refer to e.g. [SV] for the precise statements concerning averaging of ODE's. Periodic, quasi-periodic, or even almost-periodic functions of time, are obvious examples of KBM functions.

In the present paper, the analysis shows that an argument similar to the one used in the quantum context [BCD, BCDG] allows to transform the original PDE (1.1) into a PDE with oscillatory coefficients, of the form (roughly)

$$\frac{\partial}{\partial t} y_\varepsilon(t,E) = \alpha\left(\frac{t}{\varepsilon^2}, E\right) \partial_E y_\varepsilon(t,E) + \beta\left(\frac{t}{\varepsilon^2}, E\right) \partial^2_{E,E} y_\varepsilon(t,E), \tag{1.18}$$

for some coefficients α and β. This result is very much in the spirit of observation (1.15) above. The quantitative statement is Proposition 2.5 below, where the relevant coefficients are the two functions $A(t/\varepsilon^2, X)$ and $B(t/\varepsilon^2, X)$ entering Hypothesis 3. Keeping in mind the paradigm (1.15)-(1.16), the natural assumption is that the time dependent coefficients $\alpha(s,E)$, $\beta(s,E)$ (or $A(s,E)$, $B(s,E)$) should have well defined long time averages in s. When V is periodic or quasi-periodic in time, this requirement is easily met, since the relevant coefficients then are (quasi-)periodic in time as well [CDG]. The natural extension, which is the purpose of this paper, is the case when the coefficients α, β (or A, B) are merely KBM.

All these considerations motivate our main assumption on V:

Hypothesis 3 (existence of long-time averages of V).
(i) We assume that $V \in C^3_b(\mathbb{R}^+ \times \mathbb{R}^d)$ is bounded with bounded derivatives up to third order.

(ii) Introduce the family of transport operators associated to the Hamiltonian flow of H_0,

$$\mathcal{S}_u(\varphi)(X) := \varphi\big(\overline{X}(u,X)\big).$$

For any $t \geq 0$, define the functions

$$A(t,X) := \int_0^t e^{-\gamma u} \left\{ V(t)\,,\, \mathcal{S}_u \left\{ V(t+u)\,,\, H_0 \right\} \right\} du,$$

$$B(t,X) := \int_0^t e^{-\gamma u} \left\{ V(t), H_0 \right\} \times \mathcal{S}_u \left\{ V(t+u)\,,\, H_0 \right\} du.$$

We assume that A and B admit long-time averages, in that the following limits exist in $L^\infty_{\mathrm{loc}}(\mathbb{R}^{2d})$-weak-$\star$,

$$\langle A \rangle(X) := \lim_{T\to+\infty} \frac{1}{T} \int_0^T A(s,X)\,ds, \quad \langle B \rangle(X) := \lim_{T\to+\infty} \frac{1}{T} \int_0^T B(s,X)\,ds. \tag{1.19}$$

(iii) Furthermore, we assume that the following limits hold true

$$A\Big(\frac{t}{\varepsilon^2}, X\Big) \rightharpoonup \langle A \rangle(X), \qquad B\Big(\frac{t}{\varepsilon^2}, X\Big) \rightharpoonup \langle B \rangle(X), \qquad \text{in } L^\infty_{\mathrm{loc}}(\mathbb{R}^+ \times \mathbb{R}^{2d}) - \text{weak} - \star. \tag{1.20}$$

Remark. Assumption (i) may be somewhat relaxed in that V may be assumed C^2_b, with a Lipschitz second derivative, globally in time, locally in X. We do not detail this technical aspect. ∎

The above assumption requires some comments.

An important hypothesis is point (i). Indeed, we show below (Proposition 2.2) that the assumed regularity on V and H_0 implies the coefficients A and B are bounded, globally in time $t \geq 0$, and locally in X. Hence the sequences $1/T \int_0^T \cdots$ are bounded in $L^\infty_{\mathrm{loc}}(\mathbb{R}^{2d})$, and they possess (up to subsequences) weak-$\star$ limits as $T \to \infty$, limits that are automatically independent of time. This ensures the validity of statement (ii), of KBM type. The same argument shows the sequences $A(t/\varepsilon^2, X)$ and $B(t/\varepsilon^2, X)$ also are bounded in $L^\infty_{\mathrm{loc}}(\mathbb{R}^+ \times \mathbb{R}^{2d})$, hence possess weak-$\star$ limits as $\varepsilon \to 0$. The point is, assertion (iii) requires these limits are independent of t. This in turn implies that $A(t/\varepsilon^2, X)$ automatically goes to $\langle A \rangle(X)$, and similarly for B. In summary, assumption (iii) is considerably stronger than (ii). It should also be noted that, in the context of ODE's, the averaging of (1.15) only requires the long time averages $1/T \int_0^T \psi(s,z)ds$ converge, while in the present PDE context, we do need the reinforced assumption (iii).

The remainder part of this paper is devoted to the proof of Theorem 1.1.

2 Proof of Theorem 1.1

2.1 Two uniform bounds

Our analysis starts with the

Proposition 2.1 *Suppose the initial data f_0^ε is bounded in $L^2(\mathbb{R}^{2d})$. Then,*

(i) The family $\left(f^\varepsilon\right)_{\varepsilon>0}$ is bounded in $L^\infty(\mathbb{R}^+; L^2(\mathbb{R}^{2d}))$,

(ii) The family $\left(g^\varepsilon\right)_{\varepsilon>0} := \left(\dfrac{f^\varepsilon - Pf^\varepsilon}{\varepsilon}\right)_{\varepsilon>0}$ is bounded in $L^2(\mathbb{R}^+ \times \mathbb{R}^{2d})$.

Proof of Proposition 2.1
We readily observe

$$\frac{1}{2}\,\frac{d}{dt}\int_{\mathbb{R}^{2d}} |f^\varepsilon|^2\,dX = \gamma\int_{\mathbb{R}^{2d}} Q(f^\varepsilon)f^\varepsilon\,dX = -\frac{\gamma}{\varepsilon^2}\int_{\mathbb{R}^{2d}} |Pf^\varepsilon - f^\varepsilon|^2\,dX \leq 0,$$

where Lemma 1.2-(vi) has been used. ∎

For later convenience, we also state and prove the following easy, yet crucial, uniform bound on the coefficients $A(t/\varepsilon^2, X)$ and $B(t/\varepsilon^2, X)$ entering Hypothesis 3.

Proposition 2.2 *Suppose Hypotheses 2 and 3 are fulfilled. Let $0 < R < \infty$ and set*

$$\mathcal{E}(R) := \{X \in \mathbb{R}^{2d},\, |H_0(X)| \leq R\}, \qquad B(R) := \{X \in \mathbb{R}^{2d},\, |X| \leq R\}.$$

(i) There exists $0 < \rho(R) < \infty$ such that any $X \in \mathcal{E}(R)$ belongs to $B(\rho(R))$ as well. Moreover, for any $X \in \mathcal{E}(R)$ and $u \in \mathbb{R}$, we have $\overline{X}(u, X) \in B(0, \rho(R))$ too.

(ii) There exists a function $C(R)$, bounded for bounded values of $R > 0$, such that

$$\sup_{\varepsilon>0}\ \sup_{t\geq 0}\ \sup_{X\in\mathcal{E}(R)} \left|\left\{V(t/\varepsilon^2 - u)\ ,\ H_0\right\}\,\mathcal{S}_u\,\left\{V(t/\varepsilon^2)\ ,\ H_0\right\}\right| \leq C(R),$$

$$\sup_{\varepsilon>0}\ \sup_{t\geq 0}\ \sup_{X\in\mathcal{E}(R)} \left|\left\{V(t/\varepsilon^2 - u)\ ,\ \mathcal{S}_u\,\left\{V(t/\varepsilon^2)\ ,\ H_0\right\}\right\}\right| \leq C(R)\,(1+u)^p,$$

whenever $u \geq 0$. The exponent p is as in Hypothesis 3.

(iii) There exists a function $C(R)$, bounded for bounded values of $R > 0$, such that[1]

$$\sup_{\varepsilon>0}\ \sup_{t\geq 0}\ \sup_{X\in\mathcal{E}(R)} \left|A\left(\frac{t}{\varepsilon^2}, X\right)\right| \leq \frac{C(R)}{\gamma^{p+1}}, \qquad \sup_{\varepsilon>0}\ \sup_{t\geq 0}\ \sup_{X\in\mathcal{E}(R)} \left|B\left(\frac{t}{\varepsilon^2}, X\right)\right| \leq \frac{C(R)}{\gamma}.$$

Here, functions A and B are as in Hypothesis 3.

Proof of Proposition 2.2.
The fact that H_0 is confining (1.3) readily implies point (i).

Next, point (ii) comes as an immediate consequence of the regularity assumptions we have made on H_0 and V. Indeed, we may write, whenever $\varepsilon > 0$, $t \geq 0$, and $X \in \mathcal{E}(R)$,

$$\left|\left\{V(t/\varepsilon^2 - u)\ ,\ H_0\right\}\,\mathcal{S}_u\,\left\{V(t/\varepsilon^2)\ ,\ H_0\right\}\right| \leq \|\nabla_x V\|^2_{L^\infty(\mathbb{R}\times B(\rho(R)))}\ \|\nabla_v H_0\|^2_{L^\infty(B(\rho(R)))} \leq C(R),$$

[1] Note that we implicitly assume here $\gamma < 1$ – this is no loss of generality

where $C(R)$ is a locally bounded function of R. Similarly, we have

$$\begin{aligned}
&\Big|\Big\{V(t/\varepsilon^2 - u)\ ,\ \mathcal{S}_u\ \big\{V(t/\varepsilon^2)\ ,\ H_0\big\}\Big\}\Big| \\
&\leq \|\nabla_x V\|_{L^\infty(\mathbb{R}\times B(\rho(R)))} \times \Big\|\frac{D\overline{X}(u,X)}{DX}\Big\|_{L^\infty(B(\rho(R)))} \\
&\quad \times \Big(\|D_x^2 V\|_{L^\infty(\mathbb{R}\times B(\rho(R)))}\ \|\nabla_v H_0\|_{L^\infty(B(\rho(R)))} + \|\nabla_x V\|_{L^\infty(\mathbb{R}\times B(\rho(R)))}\|\ D^2_{x,v} H_0\|_{L^\infty(B(\rho(R)))}\Big) \\
&\quad\ \leq C(R) \times (1+u)^p.
\end{aligned}$$

The last line uses Hypothesis 3.

Point (iii) is easily deduced, since, for any $\varepsilon > 0$, $t \geq 0$, and $X \in \mathcal{E}(R)$, we have the two upper bounds

$$\begin{aligned}
|B(t/\varepsilon^2, X)| &\leq C(R)\ \int_0^{+\infty} e^{-\gamma u}\ du \leq C(R)\ \gamma^{-1}, \\
|A(t/\varepsilon^2, X)| &\leq C(R)\ \int_0^{+\infty} e^{-\gamma u}\ (1+u)^p\ du \leq C(R)\ \gamma^{-p-1}.
\end{aligned}$$

∎

2.2 Obtaining an equation for Pf^ε

The following result completes the previous Proposition 2.1, according to which $f^\varepsilon = Pf^\varepsilon + O(\varepsilon)$: we derive here an equation for Pf^ε. We mention that the splitting $f^\varepsilon = Pf^\varepsilon + \varepsilon\ g^\varepsilon$ into a (dominant) function of the energy, and a (smaller) fluctuation term, is partly motivated by a similar decomposition occurring in the quantum context, see [BCD, BCDG]. The crucial point is that the dominant function of the energy turns out to satisfy a closed equation (up to small remainder terms). Note that [CDG] follows a completely different approach, based on a double-scale convergence analysis for f^ε itself.

Proposition 2.3 *The function $Pf^\varepsilon = Pf^\varepsilon(t, X)$ satisfies the equation*

$$\partial_t\left(Pf^\varepsilon\right)(t,X) = L^\varepsilon\left(Pf^\varepsilon\right)(t,X) + \varepsilon\ \big[R^\varepsilon(t,X) + I^\varepsilon(t,X)\big]. \tag{2.1}$$

The Leading term $L^\varepsilon(Pf^\varepsilon)$ is a second order, linear operator, with memory term and oscillatory coefficients. It has the value

$$L^\varepsilon\left(Pf^\varepsilon\right)(t,X) = \int_0^{t/\varepsilon^2} e^{-\gamma u}\ P\ \Big\{V(t/\varepsilon^2)\ ,\ \mathcal{S}_{-u}\ \big\{V(t/\varepsilon^2 - u)\ ,\ Pf^\varepsilon(t - \varepsilon^2 u)\big\}\Big\}\, du.$$

The Remainder and Initial terms R^ε and I^ε are

$$\begin{aligned}
R^\varepsilon(t,X) &= \int_0^{t/\varepsilon^2} e^{-\gamma u}\ P\ \Big\{V(t/\varepsilon^2)\ ,\ \mathcal{S}_{-u}\big\{V(t/\varepsilon^2 - u)\ ,\ g^\varepsilon(t-\varepsilon^2 u)\big\}\Big\}\, du, \\
I^\varepsilon(t,X) &= -\varepsilon^{-1}\ e^{-\gamma t/\varepsilon^2}\ P\ \big\{V(t/\varepsilon^2)\ ,\ \mathcal{S}_{-t/\varepsilon^2}\ g_0^\varepsilon\big\}.
\end{aligned}$$

The picture given in Proposition 2.3 is made complete through the next technical statement, according to which operators L^ε, R^ε, and I^ε are bounded in some weak topology.

Proposition 2.4 *Let $\varphi : \mathbb{R} \to \mathbb{R}$ be a C^∞ compactly supported function. Then, there exists a constant $C > 0$, depending on γ, $\mathrm{supp}(\varphi)$, H_0 and V, such the following estimates hold true*

$$\sup_{t\geq 0}\ \Big|\int_{\mathbb{R}^{2d}} L^\varepsilon\left(P\ f^\varepsilon\right)(t,X)\ \varphi(H_0(X))\, dX\Big| \leq C\ \|\varphi\|_{W^{2,\infty}(\mathbb{R})}, \tag{2.2}$$

$$\int_0^\infty \Big|\int_{\mathbb{R}^{2d}} R^\varepsilon(t,X)\ \varphi(H_0(X))\, dX\Big|^2 dt \leq C\ \|\varphi\|^2_{W^{2,\infty}(\mathbb{R})}, \tag{2.3}$$

$$\int_0^\infty \Big|\int_{\mathbb{R}^{2d}} I^\varepsilon(t,X)\ \varphi(H_0(X))\, dX\Big|^2 dt \leq C\ \|\varphi\|^2_{W^{1,\infty}(\mathbb{R})}. \tag{2.4}$$

Proof of Proposition 2.3.
Proposition 2.1 naturally leads to splitting the original kinetic equation (1.1) on f^ε into an equation for Pf^ε, and an equation on $g^\varepsilon = (f^\varepsilon - Pf^\varepsilon)/\varepsilon$. We have on the one hand

$$\partial_t Pf^\varepsilon = -\frac{1}{\varepsilon}\; P\big\{V(t/\varepsilon^2), f^\varepsilon\big\}, = -\frac{1}{\varepsilon} P\big\{V(t/\varepsilon^2), Pf^\varepsilon\big\} - \big\{V(t/\varepsilon^2), g^\varepsilon\big\} = -P\big\{V(t/\varepsilon^2), g^\varepsilon\big\},$$

(Lemma 1.2 has been used, which allows to cancel the $O(1/\varepsilon)$ contribution), while, on the other hand

$$\varepsilon^2 \partial_t g^\varepsilon + \big\{H_0, g^\varepsilon\big\} + \gamma g^\varepsilon = -\big\{V(t/\varepsilon^2), Pf^\varepsilon\big\} - \varepsilon\; (\mathrm{Id} - P)\; \big\{V(t/\varepsilon^2), g^\varepsilon\big\}.$$

Hence we arrive at a system with a particular triangular structure, namely,

$$\partial_t Pf^\varepsilon = -P\big\{V(t/\varepsilon^2), g^\varepsilon\big\}, \tag{2.5}$$
$$\varepsilon^2 \partial_t g^\varepsilon + \big\{H_0, g^\varepsilon\big\} + \gamma g^\varepsilon = -\big\{V(t/\varepsilon^2), Pf^\varepsilon\big\} - \varepsilon\; (\mathrm{Id} - P)\; \big\{V(t/\varepsilon^2), g^\varepsilon\big\}. \tag{2.6}$$

The similar (and crucial) structure is involved in the quantum case (see [BCD, BCDG]).

Exploiting this observation, one may first express g^ε as a function of Pf^ε: the method of characteristics readily gives

$$\begin{aligned} g^\varepsilon(t, X) &= e^{-\gamma t/\varepsilon^2} g_0^\varepsilon\big(\overline{X}(-t/\varepsilon^2, X)\big) - \int_0^{t/\varepsilon^2} e^{-\gamma u}\; \big\{V(\cdot/\varepsilon^2)\;,\; Pf^\varepsilon\big\}\big(t - \varepsilon^2 u, \overline{X}(-u, X)\big)\, du \\ &\qquad - \varepsilon \int_0^{t/\varepsilon^2} e^{-\gamma u}\; (\mathrm{Id} - P)\; \big\{V(\cdot/\varepsilon^2)\;,\; g^\varepsilon\big\}\big(t - \varepsilon^2 u, \overline{X}(-u, X)\big)\, du. \end{aligned} \tag{2.7}$$

Now, inserting (2.7) into (2.5) roughly gives a closed equation on Pf^ε, namely

$$\partial_t\, (Pf^\varepsilon) = L^\varepsilon\, (Pf^\varepsilon) + \varepsilon\; \widetilde{R^\varepsilon} + \varepsilon\; I^\varepsilon,$$
$$\text{with}\;\; \widetilde{R^\varepsilon} = \int_0^{t/\varepsilon^2} e^{-\gamma u}\; P\; \Big\{V(t/\varepsilon^2)\;,\; (\mathrm{Id} - P)\; \mathcal{S}_{-u}\; \big\{V(t/\varepsilon^2 - u)\;,\; g^\varepsilon(t - \varepsilon^2 u)\big\}\Big\}\, du.$$

There remains to observe that, for any u, we have

$$P\; \Big\{V(t/\varepsilon^2)\;,\; P\; \mathcal{S}_{-u}\; \big\{V(t/\varepsilon^2 - u)\;,\; g^\varepsilon(t - \varepsilon^2 u)\big\}\Big\} = 0,$$

by virtue of Lemma 1.2. Hence $\widetilde{R^\varepsilon} = R^\varepsilon$, and the Proposition is proved. ∎

Proof of Proposition 2.4.
Let us pick some $R > 0$ such that supp $\varphi \subset (-R, +R)$. Proposition 2.2-(i) implies $\varphi(H_0(X)) = 0$ whenever $|X| \geq \rho(R)$. Actually, using the energy conservation, the function $\mathcal{S}_u \varphi(H_0) \equiv \varphi(H_0)$ is supported in $B(\rho(R))$ for any $u \geq 0$. This observation is used repeatedly below.

First step: proof of estimate (2.2) on L^ε

Upon using the definition of L^ε, performing the obvious integration by parts, and systematically exploiting the identity $\{V\;,\; \varphi(H_0)\} = \{V\;,\; H_0\}\; \partial_E \varphi(H_0)$ and so on, we obtain,

$$\begin{aligned} &\int_{\mathbb{R}^{2d}} L^\varepsilon(Pf^\varepsilon)(t, X)\; \varphi(H_0(X))\, dX \\ &= \int_{\mathbb{R}^{2d}} \int_0^{t/\varepsilon^2} e^{-\gamma u} P\Big\{V(t/\varepsilon^2), \mathcal{S}_{-u}\big\{V(t/\varepsilon^2 - u), Pf^\varepsilon(t - \varepsilon^2 u)\big\}\Big\}\; \varphi(H_0)\, du\; dX \\ &= \int_{\mathbb{R}^{2d}} \int_0^{t/\varepsilon^2} e^{-\gamma u}\; Pf^\varepsilon(t - \varepsilon^2 u)\; \Big\{V(t/\varepsilon^2 - u)\;,\; \mathcal{S}_u\; \big\{V(t/\varepsilon^2)\;,\; \varphi(H_0)\big\}\Big\}\, du\; dX \end{aligned}$$

$$
\begin{aligned}
&= \int_{\mathbb{R}^{2d}} \int_0^{t/\varepsilon^2} e^{-\gamma u}\ Pf^\varepsilon(t-\varepsilon^2 u) \Big\{ V(t/\varepsilon^2 - u)\ ,\ \mathcal{S}_u \left\{ V(t/\varepsilon^2)\ ,\ H_0 \right\} \partial_E \varphi(H_0) \Big\}\, du\ dX \\
&= \int_{\mathbb{R}^{2d}} \int_0^{t/\varepsilon^2} e^{-\gamma u}\ Pf^\varepsilon(t-\varepsilon^2 u) \Bigg[\left\{ V(t/\varepsilon^2 - u)\ ,\ H_0 \right\} \mathcal{S}_u \left\{ V(t/\varepsilon^2)\ ,\ H_0 \right\} \partial^2_{EE} \varphi(H_0) \\
&\qquad\qquad + \Big\{ V(t/\varepsilon^2 - u)\ ,\ \mathcal{S}_u \left\{ V(t/\varepsilon^2)\ ,\ H_0 \right\} \Big\} \partial_E \varphi(H_0) \Bigg]\, du\ dX.
\end{aligned}
$$

This together with Proposition 2.2-(ii) allows to estimate

$$
\begin{aligned}
&\left| \int_{\mathbb{R}^{2d}} L^\varepsilon(Pf^\varepsilon)(t,X)\ \varphi(H_0)\, dX \right| \\
&\quad \le C(R)\ \|\varphi\|_{W^{2,\infty}(\mathbb{R})} \int_{\mathbb{R}^{2d}} \int_0^{t/\varepsilon^2} e^{-\gamma u}\ Pf^\varepsilon(t-\varepsilon^2 u)\ \mathbf{1}_{B(\rho(R))}\ (1+u^p)\, du\, dX \\
&\quad \le C(R)\ \|\varphi\|_{W^{2,\infty}(\mathbb{R})}\ \|f^\varepsilon\|_{L^\infty(\mathbb{R}^+;L^2(\mathbb{R}^{2d}))} |B(\rho(R))|^{1/2} \int_0^\infty e^{-\gamma u}(1+u^p)\, du \\
&\quad \le C(R)\ \gamma^{-p}\ \|\varphi\|_{W^{2,\infty}(\mathbb{R})},
\end{aligned}
$$

where the indicator function $\mathbf{1}_{B(\rho(R))}$ is used to keep track of the compact support of φ, while the last estimate uses Proposition 2.1.

Second step: proof of estimate (2.3) on R^ε

The proof follows the same lines as before. Multiplying R^ε by the trial function $\varphi(H_0)$ and performing the obvious integration by parts, we obtain

$$
\begin{aligned}
&\int_{\mathbb{R}^{2d}} R^\varepsilon(t,X)\ \varphi(H_0(X))\, dX \\
&\quad = \int_{\mathbb{R}^{2d}} \int_0^{t/\varepsilon^2} g^\varepsilon(t-\varepsilon^2 u)\ e^{-\gamma u} \Big\{ V(t/\varepsilon^2 - u)\ ,\ \mathcal{S}_u \big\{ V(t/\varepsilon^2)\ ,\ \varphi(H_0) \big\} \Big\}\, du\, dX. \\
&\quad = \int_{\mathbb{R}^{2d}} \int_0^{t/\varepsilon^2} g^\varepsilon(t-\varepsilon^2 u)\ e^{-\gamma u} \Bigg[\Big\{ V(t/\varepsilon^2 - u)\ ,\ H_0 \big\} \mathcal{S}_u \left\{ V(t/\varepsilon^2)\ ,\ H_0 \right\} \Big\} \partial^2_{EE} \varphi(H_0) \\
&\qquad\qquad + \Big\{ V(t/\varepsilon^2 - u)\ ,\ \mathcal{S}_u \left\{ V(t/\varepsilon^2)\ ,\ H_0 \right\} \Big\} \partial_E \varphi(H_0) \Bigg]\, du\, dX.
\end{aligned}
$$

As a consequence, using Proposition 2.2-(ii), we recover

$$
\begin{aligned}
&\int_0^\infty \Big| \int_{\mathbb{R}^{2d}} R^\varepsilon(t,X)\ \varphi(H_0)\, dX \Big|^2 dt \\
&\qquad \le C(R)\ \|\varphi\|^2_{W^{2,\infty}(\mathbb{R})} \int_0^\infty \Big| \int_{\mathbb{R}^{2d}} \int_0^{t/\varepsilon^2} |g^\varepsilon(t-\varepsilon^2 u)|\ e^{-\gamma u}\ (1+u^p)\ \mathbf{1}_{B(\rho(R))}\, du\, dX \Big|^2 dt \\
&\le C(R)\ \|\varphi\|^2_{W^{2,\infty}(\mathbb{R})}\ \|g^\varepsilon\|^2_{L^2(\mathbb{R}\times\mathbb{R}^{2d})}\ \gamma^{-2p} \le C(R)\ \gamma^{-2p}\ \|\varphi\|^2_{W^{2,\infty}(\mathbb{R})}.
\end{aligned}
$$

The last estimate uses Proposition 2.1.

Third step: proof of estimate (2.4) on I^ε

We write

$$
\begin{aligned}
&\left| \int_{\mathbb{R}^{2d}} I^\varepsilon(t,X)\ \varphi(H_0)\, dX \right| = \left| \int_{\mathbb{R}^{2d}} \varepsilon^{-1}\ e^{-\gamma t/\varepsilon^2}\ g_0^\varepsilon(X)\ \mathcal{S}_{t/\varepsilon^2} \big\{ V(t/\varepsilon^2), H_0 \big\}\ \partial_E \varphi(H_0(X))\, dX \right| \\
&\quad \le C(R)\ \|\varphi\|_{W^{1,\infty}(\mathbb{R})} \int_{\mathbb{R}^{2d}} \varepsilon^{-1}\ e^{-\gamma t/\varepsilon^2}\ |g_0^\varepsilon(X)| \mathbf{1}_{B(\rho(R))}\, dX \\
&\quad \le C(R)\ \|\varphi\|_{W^{1,\infty}(\mathbb{R})}\ \varepsilon^{-1}\ e^{-\gamma t/\varepsilon^2}\ \|f_0^\varepsilon\|_{L^2(\mathbb{R}^{2d})}.
\end{aligned}
$$

Hence

$$\int_0^{+\infty} \left| \int_{\mathbb{R}^{2d}} I^\varepsilon(t,X)\ \varphi(H_0)\, dX \right|^2 dt \le C(R)\ \|\varphi\|_{W^{1,\infty}(\mathbb{R})} \int_0^{+\infty} \varepsilon^{-2}\ e^{-2\gamma t/\varepsilon^2}\ dt$$
$$\le C(R)\ \gamma^{-1}\ \|\varphi\|^2_{W^{1,\infty}(\mathbb{R})}.$$

This ends the proof of Proposition 2.4. ∎

2.3 Analysis of the memory effect

Proposition 2.3 establishes the equation $\partial_t(Pf^\varepsilon) = L^\varepsilon(Pf^\varepsilon) + O(\varepsilon)$, where the operator L^ε is a second order differential operator involving both a memory effect (this is the term $Pf^\varepsilon(t-\varepsilon^2 u)$) and coefficients that oscillate in time (these are the terms $V(t/\varepsilon^2)$ etc.). The next Proposition allows to get rid of the memory effect, and to somewhat put in evidence the true difficulty of the present analysis, namely the presence of time oscillatory terms.

Proposition 2.5 *For any smooth test functions $\zeta(t) \in C_c^\infty(\mathbb{R}^+)$ and $\varphi(E) \in C_c^\infty(\mathbb{R})$, we have*

$$\int_{\mathbb{R}^{2d}} Pf_0^\varepsilon(X)\ \varphi(H_0(X))\ \zeta(0)\ dX + \int_0^\infty\!\!\int_{\mathbb{R}^{2d}} Pf^\varepsilon(t,X)\ \varphi(H_0(X))\ \zeta'(t)\ dt\, dX$$
$$+ \int_0^\infty\!\!\int_{\mathbb{R}^{2d}} Pf^\varepsilon(t,X)\ \left[A\Big(\frac{t}{\varepsilon^2},X\Big)\ \partial_E\varphi(H_0) + B\Big(\frac{t}{\varepsilon^2},X\Big)\ \partial^2_{EE}\varphi(H_0)\right]\ \zeta(t)\ dX\ dt = O(\varepsilon), \tag{2.8}$$

where $O(\varepsilon)$ is estimated by $C\varepsilon$ for some C, independent of ε, which depends on the test functions ζ and φ. Here, the coefficients $A(t/\varepsilon^2, X)$ and $B(t/\varepsilon^2, X)$ are defined in Hypothesis 3.

Proof of Proposition 2.5
Propositions 2.3 and 2.4 readily give, upon testing equation (2.1) against $\zeta(t)\,\varphi(H_0(X))$,

$$\int_{\mathbb{R}^{2d}} (Pf^\varepsilon)(0,X)\ \varphi(H_0(X))\ \zeta(0) + \int_0^\infty\!\!\int_{\mathbb{R}^{2d}} (Pf^\varepsilon)(t,X)\ \varphi(H_0(X))\ \zeta'(t)$$
$$+ \int_0^\infty\!\!\int_{\mathbb{R}^{2d}} L^\varepsilon(Pf^\varepsilon)(t,X)\ \varphi(H_0(X))\ \zeta(t)\ dt\ dX = O(\varepsilon).$$

Now, the proof of Proposition 2.4 (estimate (2.2)) provides the weak form of the operator L^ε, namely

$$\int_0^\infty \int_{\mathbb{R}^{2d}} L^\varepsilon(Pf^\varepsilon)(t,X)\ \varphi(H_0(X))\ \zeta(t)\ dt\ dX$$
$$= \int_0^\infty \int_{\mathbb{R}^{2d}} \int_0^{t/\varepsilon^2} e^{-\gamma u}\ Pf^\varepsilon(t,X)\ \Bigg[\Big\{V(t/\varepsilon^2)\ ,\ H_0\Big\}\ \mathcal{S}_u\ \big\{V(t/\varepsilon^2+u)\ ,\ H_0\big\}\ \partial^2_{EE}\varphi(H_0)$$
$$+ \Big\{V(t/\varepsilon^2)\ ,\ \mathcal{S}_u\ \big\{V(t/\varepsilon^2+u)\ ,\ H_0\big\}\Big\}\ \partial_E\varphi(H_0)\Bigg]\ \zeta(t+\varepsilon^2 u)\, du\ dX\ dt. \tag{2.9}$$

In order to replace $\zeta(t+\varepsilon^2 u)$ by $\zeta(t)$ in (2.9), we evaluate the difference (here, we do not rewrite the exact value of the term between brackets),

$$\left| \int_0^\infty \int_{\mathbb{R}^{2d}} \int_0^{t/\varepsilon^2} e^{-\gamma u}\ Pf^\varepsilon(t,X)\ \Big[\cdots\Big]\ \Big[\zeta(t+\varepsilon^2 u) - \zeta(t)\Big]\, du\ dX\ dt \right|$$
$$\le \varepsilon^2 \times T\ \|\zeta'\|_{L^\infty} \times \int_0^{t/\varepsilon^2} u\ e^{-\gamma u}\ \Big\|Pf^\varepsilon\Big\|_{L^\infty(\mathbb{R}^+;L^2(\mathbb{R}^{2d}))}\ \Big\|\Big[\cdots\Big]\Big\|_{L^2(\mathbb{R}^{2d})}\, du$$
$$\le \varepsilon^2 \times T\ \|\zeta'\|_{L^\infty} \times C(R)\ \|\varphi\|_{W^{2,\infty}} \times \int_0^{t/\varepsilon^2} u\ (1+u)^p\ e^{-\gamma u}\, du$$
$$\le C\ \varepsilon^2,$$

where we used Propositions 2.1 and 2.2, and the last constant C depends on T, R, γ, φ and ζ. The Proposition is proved. ∎

2.4 Compactness Properties

According to Proposition 2.5, the limiting dynamics of Pf^ε may be obtained upon averaging out the coefficients $A(t/\varepsilon^2, X)$ and $B(t/\varepsilon^2, X)$ in equation (2.8). The averaging procedure naturally needs bounds together with compactness properties. These elements are essentially gathered in Proposition 2.6. Yet the compactness at hand is not enough to conclude at once, since equation (2.8) involves products of weakly convergent sequences $Pf^\varepsilon(t,X)\times A(t/\varepsilon^2, X)$ and $Pf^\varepsilon(t,X)\times B(t/\varepsilon^2, X)$. The crucial point, which allows to deal with such products, is that the function $Pf^\varepsilon(t,X)$ possesses some compactness in the time variable t, while $A(t/\varepsilon^2, X)$ and $B(t/\varepsilon^2, X)$ have some compactness in the phase-space variable X. Based on this observation Proposition 2.7 shows one can pass to the weak limit in the above products. The idea of exploiting this particular structure in products of weakly convergent sequences is borrowed from a similar observation made in [Li] in a different context. It may also be seen as a version of the compensated-compactness principle.

Let us come to the details. We first state the

Proposition 2.6 *Take any time $T > 0$.*

(i) The sequence $\Big(\partial_t\big[h_0(E)\ (\Pi f^\varepsilon)(t,E)\big]\Big)_{\varepsilon>0}$ *is bounded in* $L^2(0,T; W^{-2,1}_{\rm loc}(\mathbb{R}))$.
As a corollary, the sequence $\Big(h_0(E)\ (\Pi f^\varepsilon)(t,E)\Big)_{\varepsilon>0}$ *is relatively compact in* $C^0([0,T]; L^2_{\rm loc}(\mathbb{R}) - \text{weak})$, *and the sequence* $\Big((Pf^\varepsilon)(t,X)\Big)_{\varepsilon>0}$ *is relatively compact in* $C^0([0,T]; L^2_{\rm loc}(\mathbb{R}^{2d}) - \text{weak})$.

(ii) There exists a function $F(t,E) \in C^0(\mathbb{R}^+; L^2(\mathbb{R}))$ such that, up to extracting subsequences, the family $h_0(E)\ (\Pi f^\varepsilon)(t,E)$ converges to $h_0(E)\ F(t,E)$ in $C^0([0,T]; L^2_{\rm loc}(\mathbb{R}) - \text{weak})$, while the sequence $(Pf^\varepsilon)(t,X)$ converges to $F(t, H_0(X))$ in $C^0([0,T]; L^2_{\rm loc}(\mathbb{R}^{2d}) - \text{weak})$.

Remark. We draw the reader's attention to the following important difficulty. Proposition 2.6 asserts the sequence $\Big((Pf^\varepsilon)(t,X)\Big)_{\varepsilon>0}$ has some compactness in time. This is due to the fact that, when going to the energy variable E, the associated sequence $\Big(h_0(E)\ (\Pi f^\varepsilon)(t,E)\Big)_{\varepsilon>0}$ is once differentiable in time, with values lying in the negative Sobolev space $W^{-2,1}(\mathbb{R})$, uniformly in t and ε. Note however that, though both sequences $\Big((Pf^\varepsilon)(t,X)\Big)_{\varepsilon>0}$ and $\Big(h_0(E)\ (\Pi f^\varepsilon)(t,E)\Big)_{\varepsilon>0}$ are roughly the same object, yet the time derivative of the sequence $\Big((Pf^\varepsilon)(t,X)\Big)_{\varepsilon>0}$ does not belong to a negative Sobolev space in any obvious way. ∎

From Proposition 2.6, we are able to deduce the

Proposition 2.7 *Take two arbitrary test functions $\zeta(t) \in C^\infty_c(\mathbb{R}^+)$ and $\varphi(E) \in C^\infty_c(\mathbb{R})$.*
The following convergence result holds

$$\int_0^\infty\!\!\int_{\mathbb{R}^{2d}} (Pf^\varepsilon)(t,X) A\left(\frac{t}{\varepsilon^2}, X\right) \zeta(t)\varphi(H_0(X))\, dtdX \underset{\varepsilon\to 0}{\to} \int_0^\infty\!\!\int_{\mathbb{R}^{2d}} F(t,H_0(X))\langle A\rangle(X)\zeta(t)\varphi(H_0(X))\, dtdX.$$

Similarly, we have,

$$\int_0^\infty\!\!\int_{\mathbb{R}^{2d}} (Pf^\varepsilon)(t,X) B\left(\frac{t}{\varepsilon^2}, X\right) \zeta(t)\varphi(H_0(X))\, dtdX$$
$$\underset{\varepsilon\to 0}{\longrightarrow} \int_0^\infty\!\!\int_{\mathbb{R}^{2d}} F(t,H_0(X))\langle B\rangle(X)\zeta(t)\varphi(H_0(X))\, dtdX.$$

Proof of Proposition 2.6

The whole Proposition is essentially a consequence of Proposition 2.3 combined with the co-area formula (1.14). We write, taking a compactly supported test function $\varphi \in W^{2,\infty}(\mathbb{R})$,

$$\int_{\mathbb{R}} \partial_t\Big(h_0(E)\ (\Pi f^\varepsilon)(t,E)\Big)\ \varphi(E)\ dE$$
$$= \int_{\mathbb{R}^{2d}} \partial_t\big(P\ f^\varepsilon\big)(t,X)\ \varphi(H_0(X))\ dX \qquad \text{(using the co-area formula)}$$
$$= \int_{\mathbb{R}^{2d}} \left[L^\varepsilon\left(P\ f^\varepsilon\right) + \varepsilon\ R^\varepsilon + \varepsilon I^\varepsilon\right](t,X)\ \varphi(H_0(X))\ dX \qquad \text{(using Proposition 2.3).}$$

Hence, using Proposition 2.4, we may upper-bound,

$$\int_0^T \Big[\int_{\mathbb{R}} \partial_t\Big(h_0(E)\ (\Pi f^\varepsilon)(t,E)\Big)\ \varphi(E)\ dE\Big]^2\ dt \leq C\ \|\varphi\|^2_{W^{2,\infty}(\mathbb{R})},$$

and the sequence $\Big(\partial_t\big[h_0(E)\ (\Pi f^\varepsilon)(t,E)\big]\Big)_{\varepsilon>0}$ is bounded in $L^2(0,T;W^{-2,1}_{\rm loc}(\mathbb{R}))$.

On the other hand, Proposition 2.1 and the co-area formula allow to similarly establish that the sequence $\big(h_0(E)\ (\Pi f^\varepsilon)(t,E)\big)_{\varepsilon>0}$ is bounded in $L^\infty(\mathbb{R}^+;L^2_{\rm loc}(\mathbb{R}))$. Indeed, taking a compactly supported test function $\varphi \in L^2(\mathbb{R})$, we may write

$$\Big|\int_{\mathbb{R}} h_0(E)\ (\Pi f^\varepsilon)(t,E)\ \varphi(E)\ dE\Big| = \Big|\int_{\mathbb{R}^{2d}} \big(P\ f^\varepsilon\big)(t,X)\ \varphi(H_0(X))\ dX\Big|$$
$$\leq C\ \|\varphi(H_0(X))\|_{L^2(\mathbb{R}^{2d})} \leq C\ \|\varphi\|_{L^2(\mathbb{R})},$$

where the first inequality uses Proposition 2.1 and the second estimate uses once again that H_0 is confining.

Therefore, standard compactness results give that the sequence $\big(h_0(E)\ (\Pi f^\varepsilon)(t,E)\big)_{\varepsilon>0}$ is relatively compact in $C^0([0,T];L^2_{\rm loc}(\mathbb{R}) - \text{weak})$. Hence the existence of $F(t,E) \in C^0([0,T];L^2_{\rm loc}(\mathbb{R}))$ such that $h_0(E)\ (\Pi f^\varepsilon)(t,E)$ goes to $h_0(E)\ F(t,E)$ in $C^0([0,T];L^2_{\rm loc}(\mathbb{R}) - \text{weak})$, as $\varepsilon \to 0$.

Besides, using the co-area formula again gives the similar compactness for the sequence $\Big((Pf^\varepsilon)(t,X)\Big)_{\varepsilon>0}$. Indeed, taking a compactly supported test function $\psi(X) \in L^2(\mathbb{R}^{2d})$, we may write

$$\int_{\mathbb{R}^{2d}} (Pf^\varepsilon)(t,X)\ \psi(X)\ dX = \int_{\mathbb{R}} h_0(E)\ (\Pi f^\varepsilon)(t,E)\ (\Pi\psi)(E)\ dE$$
$$\longrightarrow \int_{\mathbb{R}} h_0(E)\ F(t,E)\ (\Pi\psi)(E)\ dE = \int_{\mathbb{R}^{2d}} F(t,H_0(X))\ \psi(X)\ dX,$$

where the convergence is uniform as t varies in compact subsets of $\mathbb{R}^+$ (thanks to the previous result). The Proposition is now proved. ∎

Proof of Proposition 2.7

We only prove the first convergence result, the other one being completely similar.

Our proof closely follows that of a similar statement, given in [Li] (Lemma 5.1 page 12).

We choose two cutoff functions in time, resp. in space, $\Phi(X) \in C_c^\infty(\mathbb{R}^{2d})$ and $\chi(t) \in C_c^\infty(\mathbb{R}^+)$, such that $\Phi \geq 0$, $\chi \geq 0$, $\int \Phi = 1$, $\int \chi = 1$. Associated with these regularizing function, we take a small parameter $\delta > 0$, and set

$$\chi_\delta(t) := \frac{1}{\delta}\ \chi\left(\frac{t}{\delta}\right), \quad \Phi_\delta(X) := \frac{1}{\delta^{2d}}\ \Phi\left(\frac{X}{\delta}\right),$$

together with

$$(Pf^\varepsilon)_\delta(t,X) := \int_{\mathbb{R}^+} (Pf^\varepsilon)(t+s,X)\ \chi_\delta(s)\ ds, \quad F_\delta(t,X) := \int_{\mathbb{R}^+} F(t+s,X)\ \chi_\delta(s)\ ds,$$
$$A_\delta\left(\frac{t}{\varepsilon^2},X\right) := \int_{\mathbb{R}^{2d}} A\left(\frac{t}{\varepsilon^2},X-Y\right)\ \Phi_\delta(Y)\ dY, \quad \langle A\rangle_\delta(X) := \int_{\mathbb{R}^{2d}} \langle A\rangle(X-Y)\ \Phi_\delta(Y)\ dY.$$

With these notations, the argument, borrowed from [Li] and developed in the subsequent steps below, is the following. First, standard functional analysis shows that Proposition 2.7 is true when (Pf^ε), $A(t/\varepsilon^2, X)$, F, $\langle A\rangle$ are replaced by $(Pf^\varepsilon)_\delta$, $A_\delta(t/\varepsilon^2, X)$, F_δ and $\langle A\rangle_\delta$, respectively. Second, the compactness of Πf^ε in time (Proposition 2.6) turns out to imply some compactness of Pf^ε in time as well. Similarly, the coefficients $A(t/\varepsilon^2, X)$, $B(t/\varepsilon^2, X)$ are smooth in X, uniformly in t/ε^2, and for this reason they possess some compactness in X. From this it follows that the regularized function $(Pf^\varepsilon)_\delta$, $A_\delta(t/\varepsilon^2, X)$, $B_\delta(t/\varepsilon^2, X)$ go to Pf^ε, $A(t/\varepsilon^2, X)$, $B(t/\varepsilon^2, X)$ in some strong topology as δ goes to zero. The conclusion follows.

First step: Proof of a regularized version of the Proposition

We write, for any fixed value of $\delta > 0$,

$$\int_0^\infty \int_{\mathbb{R}^{2d}} (Pf^\varepsilon)_\delta(t,X)\ A_\delta\left(\frac{t}{\varepsilon^2}, X\right)\ \zeta(t)\ \varphi(H_0(X))\ dt\ dX =$$

$$\int_0^\infty\!\!\int_{\mathbb{R}^{2d}} \underbrace{\left(\int_{\mathbb{R}^{2d}} (Pf^\varepsilon)(s,X)\ \Phi_\delta(X-Y)\ \varphi(H_0(X))\ dX\right)}_{=:p_{\delta,\varepsilon}(s,Y)} \underbrace{\left(\int_0^\infty A\left(\frac{t}{\varepsilon^2}, Y\right)\ \chi_\delta(s-t)\ \zeta(t)\ dt\right)}_{=:a_{\delta,\varepsilon}(s,Y)} ds\ dY,$$

and all integrals actually carry over compact compact sets, say $t \in [0, T_0]$, $s \in [0, T_0]$, $|X| \le R_0$, $|Y| \le R_0$. Now, given $\delta > 0$, and given arbitrary values of the variables s, Y, we have, using the known weak convergence of Pf^ε (Proposition 2.6),

$$p_{\delta,\varepsilon}(s,Y) \underset{\varepsilon\to 0}{\longrightarrow} p_\delta(s,Y) := \int_{\mathbb{R}^{2d}} F(s, H_0(X))\ \Phi_\delta(X-Y)\ \varphi(H_0(X))\ dX.$$

(We used here that the convergence of Pf^ε is weak in space but pointwise in time). Besides, we have the uniform bound

$$\left|p_{\delta,\varepsilon}(s,Y)\right| \le C\ \|\Phi_\delta(X-Y)\|_{L^2(\mathbb{R}^{2d}_Y)} \le C\ \delta^{-d},$$

where C is independent of ε and δ, but it does depend on φ. As a consequence, the dominated convergence Theorem gives

$$\forall \delta > 0,\ p_{\delta,\varepsilon}(s,Y) \underset{\varepsilon\to 0}{\longrightarrow} \int_{\mathbb{R}^{2d}} F(s, H_0(X))\ \Phi_\delta(X-Y)\ \varphi(H_0(X))\ dX \ \text{ strongly in } L^2_{\rm loc}(\mathbb{R}^+ \times \mathbb{R}^{2d}).$$

On the other hand, the function $a_{\delta,\varepsilon}(s,Y)$ satisfies the uniform bound

$$\left|a_{\delta,\varepsilon}(s,Y)\right| \le C(R)\ \gamma^{-p-1}\ \ \|\chi_\delta(s-t)\|_{L^1(\mathbb{R}^{2d}_t)} \le C,$$

where C is independent of ε and δ (we used Proposition 2.2). All these informations allow to deduce

$$\left|\int_0^{T_0}\int_{B(R_0)} \left[p_{\delta,\varepsilon}(s,Y) - p_\delta(s,Y)\right]\ a_{\delta,\varepsilon}(s,Y)\ ds\ dY\right| \le C\ \|p_{\delta,\varepsilon} - p_\delta\|_{L^2([0,T_0]\times B(R_0))} = o_\delta(1),$$

where C is independent of ε and δ, and $o_\delta(1)$ denotes a term which goes to zero with ε, for any fixed $\delta > 0$.

Using these notations, we arrive at

$$\begin{aligned}
&\int_0^\infty\!\!\int_{\mathbb{R}^{2d}} (Pf^\varepsilon)_\delta(t,X)\, A_\delta\left(\frac{t}{\varepsilon^2}, X\right) \zeta(t)\, \varphi(H_0(X))\, dt\, dX \\
&\quad = \int_0^{T_0}\!\!\int_{B(R_0)} p_\delta(s,Y)\, a_{\delta,\varepsilon}(s,Y)\, ds\, dY + o_\delta(1) \\
&\quad = \int_0^{T_0}\!\!\int_{B(R_0)} A\left(\frac{t}{\varepsilon^2}, Y\right) \left(\int_0^{T_0}\!\!\int_{B(R_0)} F(s, H_0(X))\, \Phi_\delta(X-Y)\, \varphi(H_0(X))\, \chi_\delta(s-t)\, \zeta(t)\, ds\, dX\right) dt\, dY \\
&\qquad\quad + o_\delta(1) \\
&\quad = \int_0^{T_0}\!\!\int_{B(R_0)} \langle A\rangle(Y) \left(\int_0^{T_0}\!\!\int_{B(R_0)} F(s, H_0(X))\, \Phi_\delta(X-Y)\, \varphi(H_0(X))\, \chi_\delta(s-t)\, \zeta(t)\, ds\, dX\right) dt\, dY \\
&\qquad\quad + o_\delta(1)
\end{aligned}$$

where the last equality uses the assumed weak convergence of $A(t/\varepsilon^2, Y)$ in $L^\infty_{\rm loc}(\mathbb{R}^+ \times \mathbb{R}^{2d})$-weak-$\star$ (Hypothesis 3). Hence, we may conclude

$$\int_0^\infty \int_{\mathbb{R}^{2d}} (Pf^\varepsilon)_\delta(t,X)\, A_\delta\left(\frac{t}{\varepsilon^2}, X\right)\, \zeta(t)\, \varphi(H_0(X))\, dt\, dX$$
$$= \int_0^\infty \int_{\mathbb{R}^{2d}} F_\delta(t,X)\, \langle A\rangle_\delta\,(X)\, \zeta(t)\, \varphi(H_0(X))\, dt\, dX + o_\delta(1).$$

<u>Second step: estimating the effect of the regularization</u>

The previous step allows to write

$$\begin{aligned}
&\int_0^\infty\!\!\int_{\mathbb{R}^{2d}} \Big[(Pf^\varepsilon)(t,X)\, A\left(\frac{t}{\varepsilon^2}, X\right) \,-\, F(t, H_0(X))\, \langle A\rangle(X) \Big]\, \zeta(t)\, \varphi(H_0(X))\, dt\, dX \\
&= o_\delta(1) + \int_0^\infty\!\!\int_{\mathbb{R}^{2d}} \Big[(Pf^\varepsilon)(t,X)\, A\left(\frac{t}{\varepsilon^2}, X\right) \,-\, (Pf^\varepsilon)_\delta(t,X)\, A_\delta\left(\frac{t}{\varepsilon^2}, X\right)\, \Big]\, \zeta(t)\, \varphi(H_0(X))\, dt\, dX \\
&\quad + \int_0^\infty\!\!\int_{\mathbb{R}^{2d}} \Big[F_\delta(t, H_0(X))\, \langle A\rangle_\delta(X) \,-\, F(t, H_0(X))\, \langle A\rangle(X) \Big]\, \zeta(t)\, \varphi(H_0(X))\, dt\, dX \\
&=: o_\delta(1) + \mathrm{I}_{\delta,\varepsilon} + \mathrm{II}_\delta.
\end{aligned} \tag{2.10}$$

This serves as a definition of the two terms $\mathrm{I}_{\delta,\varepsilon}$ and II_δ. We now prove that $\mathrm{I}_{\delta,\varepsilon}$ and II_δ go to zero with δ, uniformly in ε.

We begin with the most difficult term $\mathrm{I}_{\delta,\varepsilon}$. We first split in the obvious way

$$\begin{aligned}
\mathrm{I}_{\delta,\varepsilon} = &\int_0^\infty\!\!\int_{\mathbb{R}^{2d}} (Pf^\varepsilon)_\delta(t,X) \Big[A\left(\frac{t}{\varepsilon^2}, X\right) - A_\delta\left(\frac{t}{\varepsilon^2}, X\right) \Big]\, \zeta(t)\, \varphi(H_0(X))\, dt\, dX \\
&+ \int_0^\infty\!\!\int_{\mathbb{R}^{2d}} \Big[(Pf^\varepsilon)(t,X) - (Pf^\varepsilon)_\delta(t,X) \Big]\, A\left(\frac{t}{\varepsilon^2}, X\right)\, \zeta(t)\, \varphi(H_0(X))\, dt\, dX.
\end{aligned}$$

Next, going to the energy variable in order to treat the second term later, we split further

$$\begin{aligned}
\mathrm{I}_{\delta,\varepsilon} = &\int_0^\infty\!\!\int_{\mathbb{R}^{2d}} (Pf^\varepsilon)_\delta(t,X) \Big[A\left(\frac{t}{\varepsilon^2}, X\right) - A_\delta\left(\frac{t}{\varepsilon^2}, X\right) \Big]\, \zeta(t)\, \varphi(H_0(X))\, dt\, dX \\
&+ \int_0^\infty\!\!\int_{\mathbb{R}} h_0(E) \Big[(\Pi f^\varepsilon)(t,E) - (\Pi f^\varepsilon)_\delta(t,E) \Big]\, (\Pi A)\left(\frac{t}{\varepsilon^2}, E\right)\, \zeta(t)\, \varphi(E)\, dt\, dE \\
= &\int_0^\infty\!\!\int_{\mathbb{R}^{2d}} (Pf^\varepsilon)_\delta(t,X) \Big[A\left(\frac{t}{\varepsilon^2}, X\right) - A_\delta\left(\frac{t}{\varepsilon^2}, X\right) \Big]\, \zeta(t)\, \varphi(H_0(X))\, dt\, dX \\
+ &\int_0^\infty\!\!\int_{\mathbb{R}} h_0(E) \Big[(\Pi f^\varepsilon)(t,E) - (\Pi f^\varepsilon)_\delta(t,E) \Big]\, (\Pi A)_\delta\left(\frac{t}{\varepsilon^2}, E\right)\, \zeta(t)\, \varphi(E)\, dt\, dE \\
+ &\int_0^\infty\!\!\int_{\mathbb{R}} h_0(E) \Big[(\Pi f^\varepsilon)(t,E) - (\Pi f^\varepsilon)_\delta(t,E) \Big] \Big[(\Pi A)\left(\frac{t}{\varepsilon^2}, E\right) - (\Pi A)_\delta\left(\frac{t}{\varepsilon^2}, E\right) \Big]\, \zeta(t)\, \varphi(E)\, dt\, dE.
\end{aligned}$$

Here, we have used a regularization of $(\Pi A)(t,E)$ in the energy variable E, namely

$$(\Pi A)_\delta\left(\frac{t}{\varepsilon^2}, E\right) := \int_{\mathbb{R}} (\Pi A)\left(\frac{t}{\varepsilon^2}, E - E'\right)\, \chi_{\delta^s}(E')\, dE' = \int_{\mathbb{R}} (\Pi A)\left(\frac{t}{\varepsilon^2}, E - \delta^s\, E'\right)\, \chi(E')\, dE', \tag{2.11}$$

and $s \in]0, 1/2[$ is a parameter than can be chosen arbitrarily (the need for $s \in]0, 1/2[$ becomes clear later). At this point, we have split $\mathrm{I}_{\delta,\varepsilon}$ into

$$\mathrm{I}_{\delta,\varepsilon} = \mathrm{I}^{(1)}_{\delta,\varepsilon} + \mathrm{I}^{(2)}_{\delta,\varepsilon} + \mathrm{I}^{(3)}_{\delta,\varepsilon},$$

with the obvious notations. We prove that each term $\mathrm{I}^{(i)}_{\delta,\varepsilon}$, $i = 1, 2, 3$, goes to zero with δ, independently of ε.

The first term $\mathrm{I}^{(1)}_{\delta,\varepsilon}$ is easily bounded by, say,

$$\left|\mathrm{I}^{(1)}_{\delta,\varepsilon}\right| \leq C \sup_{|Y|\leq\delta} \sup_{\varepsilon>0} \sup_{0\leq t\leq T_0} \left\| A\left(\frac{t}{\varepsilon^2}, X+Y\right) - A\left(\frac{t}{\varepsilon^2}, X\right) \right\|_{L^\infty(B(R_0))},$$

where C is independent of ε and δ. Here, we implicitly assumed that supp $\Phi \subset B(1)$. We also used the uniform boundedness of Pf^ε in L^2_{loc} and the fact that ζ and φ have compact supports. Now, our assumptions on the potential V, the Hamiltonian H_0, and the flow $\overline{X}$, readily imply the following uniform Lipschitz bounds, valid for any compact set $K \subset \mathbb{R}^+ \times \mathbb{R}^{2d}$,

$$\sup_{|Y|\leq\delta} \sup_{\varepsilon>0} \sup_{(t,X)\in K} \left| \frac{D^2V}{Dx^2}\left(\frac{t}{\varepsilon^2}, X+Y\right) - \frac{D^2V}{Dx^2}\left(\frac{t}{\varepsilon^2}, X\right) \right| \leq C\ \delta,$$

$$\sup_{|Y|\leq\delta} \sup_{\varepsilon>0} \sup_{(t,X)\in K} \left| \frac{DV}{Dx}\left(\frac{t}{\varepsilon^2}, X+Y\right) - \frac{DV}{Dx}\left(\frac{t}{\varepsilon^2}, X\right) \right| \leq C\ \delta,$$

$$\sup_{|Y|\leq\delta} \sup_{(t,X)\in K} \left| \frac{D^2H_0}{DX^2}(X+Y) - \frac{D^2H_0}{DX^2}(X) \right| \leq C\ \delta,$$

$$\sup_{|Y|\leq\delta} \sup_{\varepsilon>0} \sup_{(t,X)\in K} \left| \frac{D\overline{X}}{DX}(u, X+Y) - \frac{D\overline{X}}{DX}(u, X) \right| \leq C\ \delta\ (1+u)^p,$$

where C is independent of ε, δ, and $u \geq 0$, but it does depend on the compact set K. An easy adaptation of the proof of Proposition 2.2 then allows to deduce from these bounds that the function $A(t/\varepsilon^2, X)$ is Lipschitz, locally in X but uniformly in the first variable, *i.e.*

$$\sup_{|Y|\leq\delta} \sup_{\varepsilon>0} \sup_{0\leq t\leq T_0} \left\| A\left(\frac{t}{\varepsilon^2}, X+Y\right) - A\left(\frac{t}{\varepsilon^2}, X\right) \right\|_{L^\infty(B(R_0))} \leq C\ \delta, \tag{2.12}$$

where C is independent of ε and δ (but it does depend on R_0, T_0). The conclusion is

$$\left|\mathrm{I}^{(1)}_{\delta,\varepsilon}\right| \leq C\ \delta,$$

where C is independent of ε and δ.

The analysis of $\mathrm{I}^{(2)}_{\delta,\varepsilon}$ follows roughly the same idea: we write

$$\mathrm{I}^{(2)}_{\delta,\varepsilon} = \int_0^\infty \zeta(t) \left\langle h_0(E)\,(\Pi f^\varepsilon)(t,E) - h_0(E)\,(\Pi f^\varepsilon)_\delta(t,E)\,,\, (\Pi A)_\delta\left(\frac{t}{\varepsilon^2}, E\right)\ \varphi(E) \right\rangle dt,$$

where $\langle .\,,.\rangle$ denotes the duality bracket in the energy variable E. Hence we may safely estimate

$$\left|\mathrm{I}^{(2)}_{\delta,\varepsilon}\right| \leq C\,\delta \sup_{\varepsilon>0} \sup_{0\leq\theta\leq1} \left\| \left\langle \partial_t\big[h_0(E)\,(\Pi f^\varepsilon)(t+\theta\delta, E)\big]\,,\, (\Pi A)_\delta\left(\frac{t}{\varepsilon^2}, E\right)\ \varphi(E) \right\rangle \right\|_{L^2([0,T_0])},$$

where C is independent of ε and δ. Using now the uniform boundedness of $\partial_t\big[h_0(E)\,(\Pi f^\varepsilon)\big]$ in $L^2_{\mathrm{loc}}(W^{-2,1}_{\mathrm{loc}})$, we recover, for some $E_0 > 0$ that depends on the support of φ,

$$\begin{aligned}\left|\mathrm{I}^{(2)}_{\delta,\varepsilon}\right| &\leq C\ \delta \sup_{\substack{\varepsilon>0\\0\leq t\leq T_0}} \left\| (\Pi A)_\delta\left(\frac{t}{\varepsilon^2}, E\right)\ \varphi(E) \right\|_{W^{2,\infty}(\mathbb{R})} \leq C\ \delta \sup_{\substack{\varepsilon>0\\0\leq t\leq T_0}} \left\| (\Pi A)_\delta\left(\frac{t}{\varepsilon^2}, E\right) \right\|_{W^{2,\infty}(]0,E_0[)} \\ &\leq C\ \delta^{1-2s} \sup_{\substack{\varepsilon>0\\0\leq t\leq T_0}} \left\| (\Pi A)\left(\frac{t}{\varepsilon^2}, E\right) \right\|_{L^\infty(]0,E_0+1[)} \leq C\ \delta^{1-2s} \sup_{\substack{\varepsilon>0\\0\leq t\leq T_0}} \left\| A\left(\frac{t}{\varepsilon^2}, X\right) \right\|_{L^\infty(B(R_0))}.\end{aligned}$$

Note that the need for using a regularized version of $(\Pi A)(t,E)$, namely $(\Pi A)_\delta)(t,E)$, in the splitting $\mathrm{I}_{\delta,\varepsilon} = \mathrm{I}^{(1)}_{\delta,\varepsilon} + \mathrm{I}^{(2)}_{\delta,\varepsilon} + \mathrm{I}^{(3)}_{\delta,\varepsilon}$ is clear in this estimate: it is enforced by the low regularity of $\partial_t\big[h_0(E)\,(\Pi f^\varepsilon)\big](t,E)$ in the E variable. The choice $0<s<1/2$ also appears clearly now. The conclusion is, using Proposition 2.2-(iii)

$$\big|\mathrm{I}^{(2)}_{\delta,\varepsilon}\big| \le C\ \delta^{1-2s},$$

with C independent of δ and ε.

Last the term $\mathrm{I}^{(3)}_{\delta,\varepsilon}$ is obviously bounded by, say,

$$\begin{aligned}\big|\mathrm{I}^{(3)}_{\delta,\varepsilon}\big| &\le C \sup_{|E'|\le\delta^s}\ \sup_{\varepsilon>0}\ \sup_{0\le t\le T_0} \left\|(\Pi A)\left(\frac{t}{\varepsilon^2},E+E'\right)-(\Pi A)\left(\frac{t}{\varepsilon^2},E\right)\right\|_{L^2([0,E_0],h_0(E)\,dE)}\\ &\le C \sup_{|E'|\le\delta^s}\ \sup_{\varepsilon>0}\ \sup_{0\le t\le T_0} \left\|(PA)\left(\frac{t}{\varepsilon^2},E+E'\right)-(PA)\left(\frac{t}{\varepsilon^2},E\right)\right\|_{L^2(B(R_0)},\end{aligned}$$

for some given $E_0>0$ (that depends on R_0), and C is independent of ε and δ. Here we used the co-area formula. Now, estimate (2.12) establishes that the functions $A(t/\varepsilon^2,X)\in L^2(B(R_0))$, parametrized by $t\in[0,T_0]$ and $\varepsilon>0$, satisfy a uniform equi-integrability criterion, hence belong to a relatively compact set of $L^2(B(R_0))$. The continuity of the projection operator P readily implies that the functions $(PA)(t/\varepsilon^2,X)\in L^2(B(R_0))$ also belong to a relatively compact set of $L^2(B(R_0))$. This in turn implies that the functions $(PA)(t/\varepsilon^2,X)$, parametrized by t and ε, satisfy a uniform equi-integrability criterion, namely

$$\sup_{|E'|\le\delta^s}\ \sup_{\varepsilon>0}\ \sup_{0\le t\le T_0} \left\|(PA)\left(\frac{t}{\varepsilon^2},E+E'\right)-(PA)\left(\frac{t}{\varepsilon^2},E\right)\right\|_{L^2(B(R_0)} \underset{\delta\to 0}{\longrightarrow} 0. \tag{2.13}$$

The conclusion is

$$\mathrm{I}^{(3)}_{\delta,\varepsilon} \underset{\delta\to0}{\longrightarrow} 0, \quad \text{uniformly with respect to } \varepsilon.$$

Summarizing, the above estimates on $\mathrm{I}^{(i)}_{\delta,\varepsilon}$ $(i=1,2,3)$ give

$$\mathrm{I}_{\delta,\varepsilon} \underset{\delta\to0}{\longrightarrow} 0, \quad \text{uniformly with respect to } \varepsilon.$$

Let us now come to the easier term II_δ. We write as before

$$\begin{aligned}\mathrm{II}_\delta &= \int_0^\infty\!\!\int_{\mathbb{R}^{2d}} F_\delta(t,H_0(X))\,\Big[\langle A\rangle_\delta(X)-\langle A\rangle(X)\Big]\,\zeta(t)\,\varphi(H_0(X))\,dt\,dX\\ &\quad+\int_0^\infty\!\!\int_{\mathbb{R}} h_0(E)\,\Big[F(t,E)-F_\delta(t,E)\Big]\,(\Pi\langle A\rangle)_\delta(E)\,\zeta(t)\,\varphi(H_0(X))\,dt\,dE\\ &\quad+\int_0^\infty\!\!\int_{\mathbb{R}} h_0(E)\,\Big[F(t,E)-F_\delta(t,E)\Big]\,\Big[(\Pi\langle A\rangle)(E)-(\Pi\langle A\rangle)_\delta(E)\Big]\,\zeta(t)\,\varphi(E)\,dt\,dE,\end{aligned}$$

where we defined the regularization of $(\Pi\langle A\rangle)(E)$ in the energy variable E,

$$(\Pi\langle A\rangle)_\delta(E) := \int_{\mathbb{R}} (\Pi\langle A\rangle)(E-E')\,\chi_{\delta^s}(E')\,dE' = \int_{\mathbb{R}} (\Pi\langle A\rangle)(E-\delta^s\,E')\,\chi(E')\,dE', \tag{2.14}$$

and $s\in]0,1/2[$ is a parameter that can be chosen arbitrarily. At this point, we have split II_δ into

$$\mathrm{II}_\delta = \mathrm{II}^{(1)}_\delta + \mathrm{II}^{(2)}_\delta + \mathrm{II}^{(3)}_\delta,$$

with the obvious notations. A fairly easy adaptation of the estimates produced while analyzing the term $\mathrm{I}_{\delta,\varepsilon}$ gives successively

$$\big|\mathrm{II}^{(1)}_\delta\big| \le C\,\delta, \quad \big|\mathrm{II}^{(2)}_\delta\big| \le C\,\delta,$$

with C independent of δ, and

$$\mathrm{II}_\delta^{(3)} \underset{\delta\to 0}{\longrightarrow} 0.$$

The adaptation is obtained upon observing that the various functions $A(t/\varepsilon^2, X)$ etc. entering the analysis of $\mathrm{I}_{\delta,\varepsilon}$, and their various weak limits (in the relevant spaces) $\langle A\rangle(X)$ etc. that enter the analysis of II_δ, admit the same boundedness properties (typically, if $u_n \rightharpoonup u$ in L^2, then $\|u\|_{L^2} \leq \inf_n \|u_n\|_{L^2}$). This is enough to reproduce the arguments given for $\mathrm{I}_{\delta,\varepsilon}$.

Third step: conclusion

Summarizing, we have proved up to now that

$$\int_0^\infty\int_{\mathbb{R}^{2d}} \Big[(Pf^\varepsilon)(t,X)\ A\left(\frac{t}{\varepsilon^2}, X\right)\ - F(t, H_0(X))\ \langle A\rangle(X)\ \Big]\ \zeta(t)\ \varphi(H_0(X))\ dt\ dX = o_\delta(1) + O(\delta),$$

where $o_\delta(1)$ goes to zero as $\varepsilon \to 0$ for any fixed value of $\delta > 0$, and $O(\delta)$ is estimated by $C\ \delta$ with C independent of ε and δ. This proves the Proposition. ∎

2.5 The homogenization procedure: proof of the Theorem

We are now in position to end the proof of the main Theorem.

First step: homogenization procedure

Take two smooth test functions $\zeta(t) \in C_c^\infty(\mathbb{R}^+)$ and $\varphi(E) \in C_c^\infty(\mathbb{R})$. Proposition 2.5 asserts

$$\int_{\mathbb{R}^{2d}} Pf_0^\varepsilon(t,X)\ \varphi(H_0(X))\ \zeta(0)\ dX + \int_0^\infty Pf^\varepsilon(t,X)\ \varphi(H_0(X))\ \zeta'(t)\ dt\, dX$$
$$+ \int_0^\infty\int_{\mathbb{R}^{2d}} Pf^\varepsilon(t,X)\ \Big[A\Big(\frac{t}{\varepsilon^2}, X\Big)\ \partial_E\varphi(H_0) + B\Big(\frac{t}{\varepsilon^2}, X\Big)\ \partial^2_{EE}\varphi(H_0)\Big]\ \zeta(t)\ dX\ dt = O(\varepsilon).$$

Hence, Propositions 2.6 and 2.7 allow to pass to the limit and eventually obtain

$$\int_{\mathbb{R}^{2d}} F(0, H_0(X))\ \varphi(H_0(X))\ \zeta(0)\ dX + \int_0^\infty\int_{\mathbb{R}^{2d}} F(t, H_0(X))\ \varphi(H_0(X))\ \zeta'(t)\ dt\, dX$$
$$+ \int_0^\infty\int_{\mathbb{R}^{2d}} F(t, H_0(X))\ \Big[\langle A\rangle(X)\ \partial_E\varphi(H_0) + \langle B\rangle(X)\ \partial^2_{EE}\varphi(H_0)\Big]\ \zeta(t)\ dX\ dt = 0.$$

The co-area formula next provides

$$\int_{\mathbb{R}} h_0(E)\, F(0,E)\, \varphi(E)\, \zeta(0)\, dE + \int_0^\infty\int_{\mathbb{R}} h_0(E)\, F(t,E)\, \varphi(E)\, \zeta'(t)\, dt\, dE$$
$$+ \int_0^\infty\int_{\mathbb{R}} h_0(E)\, F(t,E)\, \Big[a(E)\partial_E\varphi(E) + b(E)\partial^2_{EE}\varphi(E)\Big]\, \zeta(t)\, dE\, dt = 0, \tag{2.15}$$

where

$$a(E) := \Pi\,\langle A\rangle\,(E), \quad b(E) := \Pi\,\langle B\rangle\,(E).$$

Note that both coefficients a and b are well-defined and belong to $L^\infty(\mathbb{R})$ since the two functions $\langle A\rangle$ and $\langle B\rangle$ belong to $L^\infty(\mathbb{R}^{2d})$. Note also that equation (2.15) is the weak formulation of the second order equation

$$\partial_t\big[h_0(E)\, F(t,E)\big] + \partial_E\big[a(E)\, h_0(E)\, F(t,E)\big] - \partial^2_{E,E}\big[b(E)\, h_0(E)\, F(t,E)\big] = 0. \tag{2.16}$$

There remains to establish the properties of the coefficients $a(E)$ and $b(E)$. This is done in the next step.

Second step: properties of the effective coefficients

The proof we give now establishes the claimed formula $h_0 a = \partial_E(h_0 b)$. It also sheds some light on the connection between the present setting and the case when the potential V is assumed (quasi-)periodic in time.

As a preliminary, let $h \in L^\infty_{\rm loc}(\mathbb{R}\times\mathbb{R}^{2d})$. We associate to any such function the quantity

$$\chi_h(t,X) := \int_0^\infty h\big(t+s, \overline{X}(s,X)\big)\; e^{-\gamma s}\, ds,$$

which obviously also belongs to $L^\infty_{\rm loc}(\mathbb{R}\times\mathbb{R}^{2d})$. An immediate computation shows χ_h satisfies the backward transport equation (or: adjoint equation) $-\partial_t \chi_h - \big\{H_0, \chi_h\big\} + \gamma\chi_h = h$, with vanishing initial datum. With this notation, we may introduce the profile $\chi = \chi(t,X)$ given by

$$\chi := \chi_{\{V\,,\,H_0\}}.$$

The profile χ allows to express the coefficients a and b in the following way

$$a(E) = \Pi\left[⨍_0^\infty \big\{V(\tau)\,,\,\chi(\tau)\big\}\, d\tau\right](E), \qquad b(E) = \Pi\left[⨍_0^\infty \big\{V(\tau)\,,\,H_0\big\}\;\chi(\tau)\, d\tau\right](E),$$

where $⨍_0^\infty \cdots$ stands for the weak limit in $L^\infty_{\rm loc} - \star$ as $T\to\infty$ of $\dfrac{1}{T}\displaystyle\int_0^T \cdots$ This observation allows to both prove the non-negativity of b and the relation $h_0 a = \partial_E(h_0 b)$. Indeed, we may write, on the one hand

$$\begin{aligned}
⨍_0^\infty \big\{V(\tau)\,,\,H_0\big\}\;\chi(\tau)\,d\tau &= \lim_{T\to\infty} \frac{1}{T}\int_0^T -\Big(\partial_\tau\chi(\tau) + \big\{H_0\,,\,\chi(\tau)\big\} - \gamma\chi(\tau)\Big)\,\chi(\tau)\,d\tau \\
&= \lim_{T\to\infty}\left(\frac{\chi^2(0)-\chi^2(T)}{2T} - \frac{1}{T}\int_0^T \big\{H_0\,,\,\chi^2/2\big\}\,d\tau + \frac{\gamma}{T}\int_0^T \chi^2(\tau)\,d\tau\right) \\
&= \lim_{T\to\infty}\left(-\frac{1}{T}\int_0^T \big\{H_0\,,\,\chi^2/2\big\}\,d\tau + \frac{\gamma}{T}\int_0^T \chi^2(\tau)\,d\tau\right),
\end{aligned}$$

from which it follows, using Lemma 1.2, that

$$P ⨍_0^\infty \big\{V(\tau)\,,\,H_0\big\}\;\chi(\tau)\,d\tau = P\left(\lim_{T\to\infty}\frac{\gamma}{T}\int_0^T \chi^2(\tau)\,d\tau\right) \geq 0.$$

Hence $b(E)\geq 0$. On the other hand, using the co-area formula and integration by parts, we have, for any trial function $\psi\in C^\infty_c(\mathbb{R})$,

$$\begin{aligned}
\int_{\mathbb{R}} h_0 a\;\psi\,dE &= \int_{\mathbb{R}^{2d}} ⨍_0^\infty \big\{V,\chi\big\}\;\psi(H_0(X))\,dX\,d\tau = -\int_{\mathbb{R}^{2d}} ⨍_0^\infty \chi\;\big\{V, \psi(H_0(X))\big\}\,dX\,d\tau \\
&= -\int_{\mathbb{R}^{2d}} ⨍_0^\infty \chi\;\big\{V, H_0(X)\big\}\;(\partial_E\psi)(H_0(X))\,dX\,d\tau = -\int_{\mathbb{R}} h_0 b\;\partial_E\psi\,dE,
\end{aligned}$$

which proves $h_0 a = \partial_E(h_0 b)$.

This ends the proof of our main Theorem. ∎

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