On cumulative parisian options

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Since the early work of Chesney-Jeanblanc-Yor (1997), there has been a growing interest in occupation time derivatives. In their simplest forms, these claims are path-dependent options whose payout depends on the time spent by the underlying asset above or below a given threshold level (1). Applying the theory of Brownian excursions, Chesney-Jeanblanc-Yor (1997) have derived quasi-closed form solutions for valuing the so-called parisian options. Their formulae involve inverse Laplace transforms. Afterwards, their framework has been extended by Hugonnier (1999) to the pricing and hedging of any kind of occupation time derivatives. Numerical methods have been widely investigated too. One refers (among many others) to Haber-Schönbucher-Wilmott (1999) and Forsyth-Vetzal (1999) for finite difference algorithms, to Avellaneda-Wu (1999) for a lattice method and to Kwok-Lau (2001) for an application of the forward shooting grid methodology (2).

This article focuses on the pricing of cumulative parisian options which are special occupation time derivatives. These claims are designed to knock in/out the contract when the *total* time spent beyond the barrier exceeds a prescribed value (3). These options have already been valued by Hugonnier (1999). It must be stressed, however, that his numerical (and correct) implementation does not correspond to the quasi-analytical pricing formula given in his proposition 14 (p. 166). This article therefore aims at providing the proper closed form solutions needed to perform quasi-analytical pricing of cumulative parisian options. The rest of this article is organized as follows. Section 1 reconsiders the valuation of this kind of options and presents the result. Section 2 gives a short demonstration which follows very closely the framework and arguments of Chesney-Jeanblanc-Yor (1997) and Hugonnier (1999).

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1. PRICING CUMULATIVE PARISIAN OPTIONS

Let's first assume that the standard Black-Scholes economy exists. A financial market with continuous security trading is considered where a risky asset (the stock) and a zero coupon bond are traded. This market is supposed perfect, complete and the riskless interest rate constant. There exists an unique risk neutral probability measure Q such that any discounted price process is a martingale. Under this unique subjective probability, the stock price process is assumed to be correctly described by:

$$dS_t = (r - \delta)\, S_t\, dt + \sigma S_t\, dB_t \tag{1}$$

where r is the instantaneous risk free rate, δ the dividend rate, σ the volatility and B a standard Brownian motion. This stock price process may also be written $S_t = S_0\, e^{\nu\sigma t + \sigma B t}$ where $\nu = \frac{1}{\sigma}\left(r - \delta - \frac{\sigma^2}{2}\right)$. Denoting K the exercise price, T the maturity, $d(t, a, b, c) = \frac{\ln(a\,/\,b)}{\sigma\sqrt{T - t}} + c\sqrt{T - t}$, and Φ the normal probability distribution function, it's well known that the vanilla call option at time 0 is given by

$$C(T) = S_0 e^{-\delta T}\,\Phi(d_1(0, S_0, K)) - Ke^{-rT}\,\Phi(d_2(0, S_0, K))$$

where $d_1(t, a, b) = d(t, a, b, \nu + \sigma)$, $d_2(t, a, b) = d(t, a, b, \nu)$.

A Cumulative Parisian Call Option (CPC hereafter) is *an option whose pay-off is that of a standard call provided that the underlying asset has spent more than a prespecified time* [d] *beyond the barrier level* [L] (Hugonnier, 1999). A superscript ($^+$ or $^-$) precises whether the occupation time is considered above or below the threshold level. Now, let it be an option, maturing at T, that counts the time spent above the threshold L. The price at time $t = 0$ of such a derivative contract can be written:

$$CPC^+(d) = e^{-\left(r + \frac{\nu^2}{2}\right)T}\, C^+_{CPC} \tag{2}$$

where

$$C^+_{CPC} = S_0\,\Psi'^+_{\nu+\sigma}(T, k, l, d) - K\Psi'^+_{\nu}(T, k, l, d) \tag{3}$$

with $l\sigma = \ln(L/S_0)$, $k\sigma = \ln(K/S_0)$ and Ψ' to be precised. This pricing formula is very attractive because not only it resembles to the standard Black-Scholes one but also it allows direct interpretation. For instance, its hedging ratio is simply given by $e^{-\left(r + \frac{\nu^2}{2}\right)T}\,\Psi'^+_{\nu+\sigma}(T, k, l, d)$. Finally, it's important to note that Chesney-Jeanblanc-Yor (1997) have termed $\mathcal{C}_{CPC}$ the (r, ν)-discounted value of the cumulative parisian call option.

All other call options are then given by:

$$CPC^{\mp}(d) + CPC^{\pm}(T-d) = C(T).$$

Roughly speaking, if the occupation time under a barrier level is limited to d (until maturity T), the stock price is equivalentely expected to spend more than $T-d$ time above it. A similar relation holds for put options:

$$CPP^{\mp}(d) + CPP^{\pm}(T - d) = P(T).$$

In addition, some parity relations between cumulative calls and puts exist (Chesney-Jeanblanc-Yor, 1997 ; Hugonnier, 1999). Detailing parameters, one has:

$$CPP^{\mp}(T, S_0, K, L; r, \delta) = S_0 K CPC^{\pm}\left(T, \frac{1}{S_0}, \frac{1}{K}, \frac{1}{L}; \delta, r\right).$$

The derivation of Ψ^{+} appears thus critical for pricing cumulative parisian options. Its expression is as follows.

Proposition 1. *If* $S_0 < L$,

$$\Psi_{\mu}^{+}(t, k, l, d) =$$

$$\int_d^T ds\left[\int_{k \wedge l}^{l} e^{\mu x}\Upsilon(2l - x, 0, s, t - s)\, dx + \int_{k \vee l}^{\infty} e^{\mu x}\Upsilon(l, x - l, s, t - s)\, dx\right].$$

If $S_0 > L$,

$$\Psi_{\mu}^{+}(t, k, l, d) = e^{\mu^2 t/2}\left(\Phi(d_{\Xi(\mu)}(t, S_0, L \vee K)) - \left(\frac{L}{S_0}\right)^{2\mu/\sigma}\Phi\left(d_{\Xi(\mu)}\left(t, \frac{L^2}{S_0}, L \vee K\right)\right)\right)$$

$$+ \int_d^T ds\left[\int_{k \wedge l}^{l} e^{\mu x}\Upsilon(l - x, -l, s, t - s)\, dx + \int_{k \vee l}^{\infty} e^{\mu x}\Upsilon(0, x - 2l, s, t - s)\, dx\right]$$

where

$$\Upsilon(a, b, u, v) = \int_0^{\infty} \frac{(z + a)(z + b)}{\pi(uv)^{1,5}} \exp\left(-\frac{(z + a)^2}{2v}\right)\exp\left(-\frac{(z + b)^2}{2u}\right) dz$$

and

$$\Xi(\mu) = \begin{cases} 1 & \text{if } \mu = \nu + \sigma, \\ 2 & \text{if } \mu = \nu \end{cases}. \tag{4}$$

This proposition is partly similar to Hugonnier's result (proposition 14, p. 106). It differs however for $S_0 > L$. Here, Ψ_{μ}^{+} is shown to be a (much more) complex fonction of μ. First, $\Xi(\mu)$ must be introduced. Second, the power of $\frac{L}{S_0}$ depends on it.

2. A SHORT DEMONSTRATION

In order to demonstrate this proposition, more materials are needed. First, let's consider:

$$A_t^{L,\mp} = \int_0^t 1_{\{S_v - L \in \mathbb{R}^{\mp}\}} \, dv$$

the occupation time below or above the level L and let's denote $Z_t = \nu t + B_t$ and T_l the first time Z hits l. The risky asset price may now be written as $S_t = S_0 \, e^{\sigma Z_t}$ and the density Λ defined by $\Lambda_l(T, x)\, dx = \text{prob}(Z_t \in dx, T_l > T)$ verifies by symmetric arguments or direct formulation $\Lambda_{-l}(t, -x) = \Lambda_l(t, x)$. In the chosen framework, the price of a Cumulative Parisian Call Option is given by:

$$CPC(d) = e^{-rT} E^{Q}\left[(S_T - K) 1_{\{S_T - K \geq 0\}} 1_{\{A_T^{L,\mp} \geq d\}} \right]$$

Let's now introduce, following Chesney-Jeanblanc-Yor (1997, p. 170), a new probability $\tilde{Q}$ (equivalent to Q) that makes $Z = (Z_t)_t$ a standard $\tilde{Q}$-Brownian motion.

The (r, ν)-discounted price of the cumulative parisian call option is then:

$$C_{CPC} = E^{\tilde{Q}}\left[e^{\nu Z_T} (S_0 e^{\sigma Z_T} - K) \, 1_{\{S_0 e^{\sigma Z_T} - K \geq 0\}} 1_{\{A_T^{L,\mp} \geq d\}} \right] \tag{5}$$

$$= S_0 E^{\tilde{Q}}\left[e^{(\nu + \sigma) Z_T} 1_{\{Z_T \geq k\}} 1_{\{A_T^{l,\mp} \geq d\}} \right] \tag{6}$$

$$- K \, E^{\tilde{Q}}\left[e^{\nu Z_T} 1_{\{Z_T \geq k\}} 1_{\{A_T^{l,\mp} \geq d\}} \right]. \tag{7}$$

The following result allows one to compute many occupation time derivatives among which the ones considered here.

Lemma 2 (Proposition 9, Hugonnier, 1999). *Let f be the pay-off function of an occupation time derivative. Let $\hat{f}$: $\mathbb{R}^+ \times [0, T] \to \mathbb{R}^+$ be the function defined by $\hat{f}(z, s) = e^{\nu z} f(S_0 \, e^{\sigma z}, s)$. The (r, ν)-discounted price at time $t = 0$ of the underlying contract is given by:*

— *for* $l > 0$: $\displaystyle\int_{-\infty}^{l} \hat{f}(z, 0) \, \Lambda_l(T, z) \, dz \, +$

$$\int_0^T ds \left[\int_{-\infty}^{l} \hat{f}(z, s) \Upsilon(2l - z, 0, s, T - s) \, dz + \int_l^{\infty} \hat{f}(z, s) \Upsilon(l, z - l, s, T - s) \, dz \right]$$

— *for* $l < 0$: $\int_l^{\infty} \hat{f}(z, T)\Lambda_{-l}(T, -z)\, dz +$

$$\int_0^T ds \left[\int_{-\infty}^{l} \hat{f}(z, s)\Upsilon(l - z, -l, s, T - s)\, dz + \int_l^{\infty} \hat{f}(z, s)\Upsilon(0, z - 2l, s, T - s)\, dz\right].$$

To succeed in computing terms (6) and (7) with lemma 2 ([4]), two hypothetical pay-off functions (f) must be identified. To the expectation in (6), that provides $\Psi_{\nu+\sigma}$ in the pricing formula (3), one associates the pay-off $f(u, v) = 1_{u \geq K} 1_{v \geq d} \frac{u}{S_0}$. To the expectation in (7), providing Ψ_ν in (3), $f(u, v) = 1_{u \geq K} 1_{v \geq d}$ is chosen. Let's denote respectively $\hat{f}_{\nu+\sigma}$, $\hat{f}_\nu$, the associated $\hat{f}$ functions. Lemma 2 then tells us that must be considered:

$$\int_{-\infty}^{l} \hat{f}_\mu(z, 0)\Lambda_l(T, z)\, dz, \quad l < 0 \tag{8}$$

$$\int_l^{\infty} \hat{f}_\mu(z, T)\Lambda_{-l}(T, -z)\, dz, \quad l > 0 \tag{9}$$

where either $\mu = \nu + \sigma$ or $\mu = \nu$. Recalling that by definition d is strictly positive and that, whatever μ is, $\hat{f}_\mu(., t)$ involves $1_{t \geq d}$, one concludes that $\hat{f}_\mu(z, 0) = 0$ in Equation (8). As a result, when $S_0 < L$, the only term in Ψ_μ is the double integral. This has already been stated by Hugonnier (1999).

Let's now turn to Equation (9) i.e. when $S_0 > L$. Since Λ_l verifies $\sqrt{2\pi t}\Lambda_l(t, x) = e^{-\frac{x^2}{2t}} - e^{-\frac{(2l - x)^2}{2t}}$, the equation may also be written:

$$\int_{k \vee l}^{\infty} e^{\mu z} e^{-\frac{z^2}{2t}} \frac{dz}{\sqrt{2\pi t}} - \int_{k \vee l}^{\infty} e^{\mu z} e^{-\frac{(2l - z)^2}{2t}} \frac{dz}{\sqrt{2\pi t}} = \Gamma_1(\mu) - \Gamma_2(\mu) = \Gamma_\mu .$$

Each term can be integrated thanks to well known quadrature relations. Hence, denoting by n and Φ respectively the gaussian probability and gaussian cumulative density functions, one obtains:

$$\Gamma_1(\mu) = e^{\mu^2 t/2} \int_{k \vee l}^{\infty} e^{-\frac{(z - \mu t)^2}{2t}}\, d\left(\frac{z}{\sqrt{2\pi t}}\right) = e^{\mu^2 t/2} \int_{\frac{k \vee l - \mu t}{\sqrt{t}}}^{\infty} n(z)\, dz = e^{\mu^2 t/2}\Phi\left[-\frac{k \vee l - \mu t}{\sqrt{t}}\right]$$

$$= e^{\mu^2 t/2}\Phi\left[\frac{\ln(S/(K \vee L)) + \mu\sigma t}{\sigma\sqrt{t}}\right],$$

$$\Gamma_2(\mu) = e^{\mu^2 t/2} e^{2l\mu} \int_{k \vee l}^{\infty} e^{-\frac{(z - 2l - \mu t)^2}{2t}}\, d\left(\frac{z}{\sqrt{2\pi t}}\right) = e^{\mu^2 t/2}\left(\frac{L}{S}\right)^{2\mu/\sigma} \int_{\frac{k \vee l - 2l - \mu t}{\sqrt{t}}}^{\infty} n(z)\, dz$$

$$= e^{\mu^2 t/2}\left(\frac{L}{S}\right)^{2\mu/\sigma} \Phi\left[-\frac{k \vee l - 2l - \mu t}{\sqrt{t}}\right]$$

$$= e^{\mu^2 t/2}\left(\frac{L}{S}\right)^{2\mu/\sigma} \Phi\left[\frac{\ln(L^2/S(K \vee L)) + \mu\sigma t}{\sigma\sqrt{t}}\right].$$

As a result, one has:

$$\Gamma_{\nu+\sigma} = e^{(\nu+\sigma)^2 t/2}\left(\Phi[d_1(0, S, K \vee L)] - \left(\frac{L}{S}\right)^{2(\nu+\sigma)/\sigma} \Phi[d_1(0, L^2/S, K \vee L)]\right)$$

$$\Gamma_{\nu} = e^{\nu^2 t/2}\left(\Phi[d_2(0, S, K \vee L)] - \left(\frac{L}{S}\right)^{2\nu/\sigma} \Phi[d_2(0, L^2/S, K \vee L)]\right).$$

Or, for short, $\Gamma_{\mu} = e^{\mu^2 t/2}\left(\Phi[d_{\Xi(\mu)}(.)] - \left(\frac{L}{S}\right)^{2\mu/\sigma} \Phi[d_{\Xi(\mu)}(.)]\right)$ with

$$\Xi(\mu) = \begin{cases} 1 & \text{if } \mu = \nu + \sigma, \\ 2 & \text{if } \mu = \nu \end{cases}$$

This ends the proof.

NOTES

(1) Hence, both the switch options discussed in Pechtl (1995) and the step options described in Linetsky (1999) are occupation time derivatives.

(2) As shown by Fusai-Tagliani (2001), the underlying monitoring frequency may be of critical importance when choosing a pricing methodology.

(3) These options are referred to as "cumulative barrier options" in Hugonnier (1999), "delayed barrier options" in Linetsky (1999) and "parasian options" in Haber-Schönbucher-Wilmott (1999, p. 73). One follows here the terminology of Chesney-Jeanblanc-Yor (1997) who have first considered them.

(4) Another way could be to directly apply lemma 2 to Equation (5).

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