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Optimal payoffs under state-dependent preferences

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Most decision theories, including expected utility theory, rank-dependent utility theory and cumulative prospect theory, assume that investors are only interested in the distribution of returns and not in the states of the economy in which income is received. Optimal payoffs have their lowest outcomes when the economy is in a downturn, and this feature is often at odds with the needs of many investors. We introduce a framework for portfolio selection within which state-dependent preferences can be accommodated. Specifically, we assume that investors care about the distribution of final wealth *and* its interaction with some benchmark. In this context, we are able to characterize optimal payoffs in explicit form. Furthermore, we extend the classical expected utility optimization problem of Merton to the state-dependent situation. Some applications in security design are discussed in detail and we also solve some stochastic extensions of the target probability optimization problem.

Keywords: Optimal portfolio selection; State-dependent preferences; Conditional distribution; hedging; State-dependent constraints

1. Introduction

Studies of optimal investment strategies are usually based on the optimization of an expected utility, a target probability or some other (increasing) *law-invariant* measure. Assuming that investors have law-invariant preferences is equivalent to supposing that they care only about the distribution of returns and not about the states of the economy in which the returns are received. This is, for example, the case under expected utility theory, Yaari's dual theory, rank-dependent utility theory, mean-variance optimization and cumulative prospect theory. Clearly, an optimal strategy has some distribution of terminal wealth and must be the cheapest possible strategy that attains this distribution. Otherwise, it is possible to strictly improve the objective and to contradict its optimality. Dybvig (1988) was the first to study strategies that reach a given return distribution at lowest possible cost. Bernard and Boyle (2010) call these strategies cost-efficient and their properties have been examined further in Bernard *et al.* (2014a). In a fairly general market setting, these authors show that the cheapest way to generate a given distribution is obtained by a contract whose payoff is decreasing in the pricing kernel (see also Carlier and Dana 2011). The basic intuition is that investors consume less in states of economic recession because it is more expensive to insure returns under these conditions. This feature is also explicit in a Black–Scholes framework, in which optimal payoffs at time horizon T are shown to be an *increasing* function of the price of the risky asset (as a representation of the economy) at time T. In particular, such payoffs are path-independent.

An important issue with respect to the optimization criteria and the resulting payoffs under most standard frameworks, is that their worst outcomes are obtained when the market declines. Arguably, this property of optimal payoffs does not fit with the aspirations of investors, who may seek protection against declining markets or, more generally, may consider a benchmark when making investment decisions. In other words, two payoffs with the same distribution do not necessarily present the same 'value' for a given investor. Bernard and Vanduffel (2014a) show that insurance contracts can usually be substituted by financial contracts that have the same payoff distribution but are cheaper. The existence of insurance contracts that provide protection against specific events shows that these instruments must present more value for an investor than financial payoffs that lack this feature. This observation supports the general observation that investors are more inclined to receive income in a 'crisis' (for example, when their property burns down or when the economy is in recession) than under 'normal' conditions.

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This paper makes several theoretical contributions to the study of optimal investment strategies and highlights valuable applications of its findings in the areas of portfolio management and security design. First, we clarify the setting under which optimal investment strategies necessarily exhibit path-independence. These findings complement Cox and Leland (1982, 2000) and Dybvig's (1988) seminal results and underscore the important role of path-independence in traditional optimal portfolio selection. Thereafter, as our main contribution, we introduce a framework for portfolio selection that makes it possible to consider the states in which income is received. More precisely, it is assumed that investors target some distribution for their terminal wealth and *additionally* aim for a certain (desired) interaction with a random benchmark.† For example, the investor may want his strategy to be unrelated to the benchmark when it decreases but to follow this benchmark when it performs well. Using our framework, we can characterize optimal payoffs explicitly (theorems 4.2 and 4.4) in this setting. Such explicit characterizations are derived independently in theorems 3.1 and 3.3 of Takahashi and Yamamoto (2013), but proved only for cases in which there is a countable number of states‡. Furthermore, we show that optimal strategies in this setting become *conditionally increasing* functions of the terminal value of the underlying risky asset.

A further main contribution in this part of the paper is the extension of the classical result of portfolio optimization under expected utility (Cox and Huang 1989). Specifically, we determine the optimal payoff for an expected utility maximizer under a dependence constraint, reflecting a desired interaction with the benchmark (theorem 6.2). The proof builds on isotonic approximations and their properties (Barlow *et al.* 1972). We also solve two stochastic generalizations of Browne (1999) and Spivak and Cvitanić's (1999) classical target optimization problem in the given state-dependent context.

Finally, we show how these theoretical results are useful in security design and can help to simplify (and improve) payoffs commonly offered in the financial markets. We show how to substitute highly path-dependent products by payoffs that depend only on two underlying assets, which we refer to as 'twins'. This result is illustrated with an extensive discussion of the optimality of Asian options. We also construct alternative payoffs with appealing properties.

The paper is organized as follows. Section 2 outlines the setting of the investment problem under study. In section 3, we restate basic optimality results for path-independent payoffs for investors with law-invariant preferences. We also discuss in detail the sufficiency of path-independent payoffs when allocating wealth. In section 4, we point out drawbacks of optimal path-independent payoffs and introduce the concept of state-dependence used in the following sections. We show that 'twins', defined as payoffs that depend only on two underlying asset values, are optimal for state-dependent preferences. In section 5, we discuss applications to improve security designs. In particular, we propose several improvements in the design of geometric Asian options. In section 6, we solve the standard Merton problem of maximization of expected utility of final wealth when the investor constrains the interaction of the final wealth with a given benchmark. In this context, we also generalize the results of Browne (1999) and Spivak and Cvitanić (1999) with regard to target probability maximization. Final remarks are presented in section 7. Most of the proofs are provided in the appendix.

2. Framework and notation

Consider investors with a given finite investment horizon T and no intermediate consumption. We model the financial market on a filtered probability space $(\Omega, \mathcal{F}, \mathbb{P})$, in which $\mathbb{P}$ is the real-world probability measure. The market consists of a bank account B paying a constant risk-free rate $r > 0$, so that B_0 invested in a bank account at time 0 yields $B_t = B_0 e^{rt}$ at time t. Furthermore, there is a risky asset (say, an investment in stock) whose price process is denoted by $S = (S_t)_{0 \leq t \leq T}$. We assume that S_t $(0 < t < T)$ has a continuous distribution F_{S_t}. The no-arbitrage price§ at time 0 of a payoff X_T paid at time $T > 0$ is given by

$$c_0(X_T) = \mathbb{E}[\xi_T X_T], \tag{1}$$

where $(\xi_t)_t$ is the state-price density process¶ ensuring that $(\xi_t S_t)_t$ is a martingale. Moreover, based on standard economic theory, we assume throughout this paper that state prices are decreasing with asset prices,‖ i.e.

$$\xi_t = g_t(S_t), \ t \geq 0, \tag{2}$$

where g_t is decreasing (in markets where $\mathbb{E}[S_T] > S_0 e^{rT}$). There is empirical evidence that this relationship may not hold in practice, which is called the pricing kernel puzzle (Brown and Jackwerth 2012, Grith *et al.* 2013). Many explanations have been provided in the literature (Brown and Jackwerth 2012, Hens and Reichlin 2013), including state-dependence of preferences (Chabi-Yo *et al.* 2008). Therefore,

†The paper draws its inspiration from the last section in Bernard *et al.* (2014a), in which a constrained cost-efficiency problem is solved when the joint distribution between the wealth and some benchmark is determined in some specific area (local-dependence constraint).

‡The results of Takahashi and Yamamoto (2013) are stated in a general market, but the proof of their basic theorem 3.1 in appendix A.1 only holds when the number of states is countable. The proof of their main theorem, theorem 3.3 (appendix A.3.), is based on the same idea as in theorem 3.1 (see statement A.3 on page 1571) and is thus also valid in the case of countable states. Their set up also differs from ours in that these authors assume that stock prices follow diffusion processes, and they derive the specific form of the state price density process in this setting (page 1561). In this paper, we do not assume that the underlying stock prices are diffusion processes and hence the state price process does not need to be of a specific form (see also our final remarks).

§The payoffs we consider are all tacitly assumed to be square integrable, to ensure that all expectations mentioned in the paper exist. In particular, $c_0(X_T) < +\infty$ for any payoff X_T considered throughout this paper.

¶The process is commonly so designated. However, strictly speaking, it is not a density that is at issue, but rather the product of a discount factor (generally strictly less than 1) and the Radon–Nikodym derivative between the physical measure and the risk-neutral measure.

‖See e.g. Cox *et al.* (1985) and Bondarenko (2003), who shows that property (2) must hold if the market does not allow for statistical arbitrage opportunities, where a statistical arbitrage opportunity is defined as a zero-cost trading strategy delivering at T, a positive expected payoff unconditionally, and non-negative expected payoffs conditionally on ξ_T.

(2) is not consistent with a market populated by investors with state-dependent preferences. However, we do not tackle the problem of equilibrium and instead study the situation of a small investor whose state-dependent preferences do not influence the pricing kernel that is exogenously given in the market. This is a commonly studied situation since the work of Karatzas *et al.* (1987).

The functional form (2) for $(\xi_t)_t$ allows us to present our results regarding optimal portfolios using $(S_t)_t$ as a reference, which is practical. We will explain in section 7 how the results and characterizations of the optimality of a payoff X_T are tied to its (conditional) anti-monotonicity with ξ_T and do not depend on the functional form (2) per se. Note that assumption (2) is satisfied by many popular pricing models, including the CAPM, the consumption-based models and by exponential Lévy markets in which the market participants use Esscher pricing (Vanduffel *et al.* 2008, von Hammerstein *et al.* 2014). It is also possible to use a market model in which prices are obtained using the Growth Optimal portfolio (GOP) as numéraire (Platen and Heath 2005), as is discussed further in section 7.

The Black–Scholes model can be seen as a special case of this latter setting. Since we will use it to illustrate our theoretical results, we recall here its main properties. In the Black–Scholes market, under the real probability $\mathbb{P}$, the price process $(S_t)_t$ satisfies

$$\frac{dS_t}{S_t} = \mu dt + \sigma dZ_t,$$

with solution $S_t = S_0 \exp\left(\left(\mu - \frac{\sigma^2}{2}\right)t + \sigma Z_t\right)$. Here, $(Z_t)_t$ is a standard Brownian motion, μ ($> r$) the drift and $\sigma > 0$ the volatility. The distribution (cdf) of S_T is given as

$$F_{S_T}(x) = \mathbb{P}(S_T \leq x) = \Phi\left(\frac{\ln\left(\frac{x}{S_0}\right) - \left(\mu - \frac{\sigma^2}{2}\right)T}{\sigma\sqrt{T}}\right), \tag{3}$$

where Φ is the cdf of a standard normal random variable. In the Black–Scholes market, the state-price density process $(\xi_t)_t$ is unique and $\xi_t = e^{-rt}e^{-\theta Z_t - \frac{\theta^2 t}{2}}$ where $\theta = \frac{\mu - r}{\sigma}$. Consequently, ξ_t can also be expressed as a decreasing function of the stock price S_t,

$$\xi_t = \alpha_t \left(\frac{S_t}{S_0}\right)^{-\beta}, \tag{4}$$

where $\alpha_t = \exp\left(\frac{\theta}{\sigma}\left(\mu - \frac{\sigma^2}{2}\right)t - \left(r + \frac{\theta^2}{2}\right)t\right)$, $\beta = \frac{\theta}{\sigma} > 0$ (because we assume that $\mathbb{E}[S_T] = S_0 e^{\mu T} > S_0 e^{rT}$).

3. Law-invariant preferences and optimality of path-independent payoffs

In this section, it is understood that investors have *law-invariant* (state-independent) preferences. This means that they are indifferent between two payoffs having the same payoff distribution (under $\mathbb{P}$). In this case, any random payoff X_T (that possibly depends on the path of the underlying asset price) admits a path-independent alternative with the same price, which is at least as good for (i.e. desirable in the eyes of) these investors. Recall that a payoff is *path-independent* if there exists some function f such that $X_T = f(S_T)$ holds almost surely. Hence, investors with law-invariant preferences only need to consider path-independent payoffs when making investment decisions. Under the additional (typical) assumption that preferences are *increasing*, any path-dependent payoff can be strictly dominated by a path-independent one that is increasing in the risky asset.†

Note that results in this section are related closely to the original work of Cox and Leland (1982, 2000), Dybvig (1988), Bernard *et al.* (2014a) and Carlier and Dana (2011). These overview results are recalled here to facilitate the exposition of the extensions that are developed in the following sections.

3.1. *Sufficiency of path-independent Payoffs*

Proposition 3.1 shows that for any given payoff there exists a path-independent alternative with the same price that is at least as good for investors with law-invariant preferences. Thus, such an investor needs only to consider path-independent payoffs. All other payoffs are indeed redundant in the sense that they are not needed to optimize the investor's objective. The proof of proposition 3.1 provides an explicit construction of an equivalent path-independent payoff.

Proposition 3.1 (Sufficiency of path-independent payoffs) *Let X_T be a payoff with price c and having a cdf F. Then, there exists at least one path-independent payoff $f(S_T)$ with price $c := c_0(f(S_T))$ and cdf F.*

The proof of proposition 3.1 is provided in appendix A.1. □

Proposition 3.1, however, does not conclude that a given path-dependent payoff can be strictly dominated by a path-independent one. The following section shows that the dominance becomes strict as soon as preferences are increasing.

3.2. *Optimality of path-independent payoffs*

Let F be a payoff distribution with (left-continuous) inverse defined as

$$F^{-1}(p) = \inf\{x \mid F(x) \geq p\}. \tag{5}$$

The basic result provided here was originally derived by Dybvig (1988) and was presented more generally in Bernard *et al.* (2014a). It shows how to construct a payoff that generates the distribution F at minimal price. Such payoff is referred to as *cost-efficient* by Bernard and Boyle (2010).

Theorem 3.2 (Cost optimality of path-independent payoffs) *Let F be a cdf. The optimization problem*

$$\min_{X_T \sim F} c_0(X_T) \tag{6}$$

has an almost surely unique solution X_T^* that is path-independent, almost surely increasing in S_T and given by

$$X_T^* = F^{-1}(F_{S_T}(S_T)) \tag{7}$$

†This dominance can easily be implemented in practice, as all path-independent payoffs can be replicated statistically with European call and put options as shown e.g. by Carr and Chou (1997) and by Breeden and Litzenberger (1978).

This theorem can be seen as an application of the Hoeffding–Fréchet bounds recalled in lemma A.1, which is presented in the appendix. This result implies that investors with *increasing* law-invariant preferences may restrict their optimization *strictly* to the set of path-independent payoffs when making investment decisions.† The payoff (7) is obviously *increasing* in S_T. In fact, this property characterizes cost-efficiency because of the a.s. uniqueness of the cost-efficient payoff established in theorem 3.2. Consequently, this implies the following corollary.

COROLLARY 3.3 (Cost-efficient payoffs) *A payoff is cost-efficient if and only if it is almost surely increasing in S_T.*

Theorem 3.2 also implies that investors with increasing law-invariant preferences only invest in path-independent payoffs that are increasing in S_T. This is consistent with the literature on optimal investment problems in which optimal payoffs derived using various techniques always turn out to exhibit this property.

COROLLARY 3.4 (Optimal payoffs for increasing law-invariant preferences) *For any payoff Y_T at price c that is not almost surely increasing in S_T there exists a path-independent payoff Y_T^* at price c that is a strict improvement for any investor with increasing and law-invariant preferences.*

A possible choice for Y_T^* is given by $Y_T^* := F^{-1}(F_{S_T}(S_T)) + (c - c_0^*)e^{rT}$, in which c_0^* denotes the price of 7. Note that the payoff Y_T^* has price c and is almost surely increasing in S_T. It consists in investing an amount $c_0^* < c$ in the cost-efficient payoff (also distributed with F) and leaving the remaining funds $c - c_0^* > 0$ in the bank account, so that it is a strict improvement of the payoff Y_T.

4. Optimal payoffs under state-dependent preferences

Many of the contracts chosen by law-invariant investors do not offer protection in times of economic hardship. In fact, due to the observed monotonicity property with S_T, the lowest outcomes for an optimal (thus, cost-efficient) payoff occur when the stock price S_T reaches its lowest levels. More specifically, denote by $f(S_T)$ a cost-efficient payoff (with an increasing function f) and by X_T another payoff such that both are distributed with F at maturity. Then, $f(S_T)$ delivers low outcomes when S_T is low and it holds‡ for all $a \geq 0$ that

$$\mathbb{E}[f(S_T)|S_T < a] \leq \mathbb{E}[X_T|S_T < a]. \tag{8}$$

Let F be the distribution of a put option with payoff $X_T := (K - S_T)^+ = \max(K - S_T, 0)$. Bernard *et al.* (2014a) show that the payoff of the cheapest strategy with cdf F can be computed as in (7). It is given by $X_T^* = (K - a\, S_T^{-1})^+$ with $a := S_0^2 \exp(2\left(\mu - \sigma^2/2\right) T)$ and is a power put option (with power -1). X_T^* is the cheapest way to achieve the distribution F, whereas the first 'ordinary' put strategy (with payoff X_T) is actually the most expensive way to do so. These payoffs interact with S_T in fundamentally different ways, as one payoff is increasing in S_T while the other is decreasing in it. A put option protects the investor against a declining market, in which consumption is more expensive than is otherwise typical, whereas the cost-efficient counterpart X_T^* provides no protection but rather emphasizes the effect of a market deterioration on the wealth received.

As mentioned in the introduction, the use of put options and the demand for insurance (Bernard and Vanduffel 2014a) are signals that many investors care about states of the economy in which income derived from investment strategies is received. In particular, they may seek strategies that provide protection against declining markets or, more generally, that exhibit a desired dependence with some benchmark.

Hence, in the remainder of this paper, we consider investors who exhibit state-dependent preferences in the sense that they seek a payoff X_T with a desired distribution *and* a desired dependence with a benchmark asset A_T. In other words, they fix the joint distribution G of the random couple (X_T, A_T). The optimal state-dependent strategy is the one that solves for

$$\min_{(X_T, A_T) \sim G} c_0(X_T). \tag{9}$$

Note that the setting also includes law-invariant preferences as a special (limiting) case when A_T is deterministic. In this case, we effectively revert to the framework of state-independent preferences that we discussed in the previous section. In what follows, we consider as benchmark the underlying risky asset or any other asset in the market, considered at final or intermediate time(s). Moreover, to ensure that the impact of state-dependent preferences on the structure of optimal payoffs is clear, we have organized the rest of the present section along similar lines to those of section 3.

Remark 4.1 One can use a copula as a device to model the interaction between payoffs and benchmarks. The joint distribution G of the couple (X_T, A_T) can be written using a copula C. From Sklar's theorem, $G(x, a) = C(F_{X_T}(x), F_{A_T}(a))$, where C is a copula (this representation is unique for continuously distributed random variables). It is then clear that the determination of optimal strategies in (9) can also be formulated as

$$\min_{\substack{X_T \sim F, \\ \mathcal{C}_{(X_T, A_T)} = C}} c_0(X_T), \tag{10}$$

where '$\mathcal{C}_{(X_T, A_T)} = C$' means that the copula between the payoff X_T and the benchmark A_T is C. In particular, (10) shows that knowledge of the distribution of A_T is not necessary in order to determine optimal state-dependent strategies.

4.1. *Sufficiency of twins*

In this paper, any payoff that writes as $f(S_T, A_T)$ or $f(S_T, S_t)$ is called a *twin*. We show first that, in our state-dependent setting, for any payoff there exists a twin that is at least as good. When also assuming that preferences are increasing, we find that optimal payoffs write as twins, and we are able to characterize them explicitly. Conditionally on A_T, optimal twins are increasing in the terminal value of the risky asset S_T.

†Similar optimality results to those in theorem 3.2 have been given in the class of admissible claims X_T that are smaller than F in convex order in Dana and Jeanblanc (2005) and in Burgert (2006).

‡We provide here a short proof of (8). It is clear that the couple $(f(S_T), \mathbb{1}_{S_T<a})$ has the same marginal distributions as $(X_T, \mathbb{1}_{S_T<a})$, but $\mathbb{E}[f(S_T)\mathbb{1}_{S_T<a}] \leq \mathbb{E}[X_T \mathbb{1}_{S_T<a}]$ because $f(S_T)$ and $\mathbb{1}_{S_T<a}$ are anti-monotonic (from lemma A.1).

The following theorems show that for any given payoff there is a twin that is at least as good for investors with state-dependent preferences.

THEOREM 4.2 (Twins as payoffs with a given joint distribution with a benchmark A_T and price c). *Let X_T be a payoff with price c having joint distribution G with some benchmark A_T, where (S_T, A_T) is assumed to have a joint density with respect to the Lebesgue measure. Then, there exists at least one twin $f(S_T, A_T)$ with price $c = c_0(f(S_T, A_T))$ having the same joint distribution G with A_T.*

Theorem 4.2 does not cover the case in which S_T plays the role of the benchmark (because (S_T, S_T) has no density). This interesting case is considered in the following theorem (theorem 4.3).

THEOREM 4.3 (Twins as payoffs with a given joint distribution with S_T and price c) *Let X_T be a payoff with price c having joint distribution G with the benchmark S_T. Assume that (S_T, S_t) for some $0 < t < T$ has a joint density with respect to the Lebesgue measure. Then, there exists at least one twin $f(S_t, S_T)$ with price $c = c_0(f(S_t, S_T))$ having a joint distribution G with S_T. An example is given by*

$$f(S_t, S_T) := F^{-1}_{X_T|S_T}(F_{S_t|S_T}(S_t)). \tag{11}$$

The proofs for theorems 4.2 and 4.3 are in appendix A.3 and A.4. □

Theorems 4.2 and 4.3 imply that investors who care about the *joint distribution of terminal wealth with some benchmark A_T* need only consider the twins in both cases, i.e. when (A_T, S_T) is continuously distributed, as in theorem 4.2, or when A_T is equal to S_T, as in theorem 4.3. These results extend proposition 3.1 to the presence of a benchmark and state-dependent preferences. All other payoffs are useless in the sense that they are not needed for these investors per se.†

Note that in theorem 4.3, t can be chosen freely in $(0, T)$ and the dependence with respect to S_t is not fixed. So, for instance, replacing $F_{S_t}(S_t)$ with $1 - F_{S_t}(S_t)$ in (11) would also lead to the appropriate properties. Hence, there is an infinite number of twins $f(S_t, S_T)$ having the joint distribution G with S_T. All of them have the same price.‡ The question then arises: how does one select one among them. A natural possibility is to determine the optimal twin $X_T = f(S_t, S_T)$ by imposing an additional criterion. For example, one could define the best twin X_T as the one that minimizes

$$\mathbb{E}\left[(X_T - H_T)^2\right], \tag{12}$$

where H_T is another payoff that is not a function of S_T. This approach appears natural in the context of simplifying the design of contracts. For instance, start with a geometric Asian option and compute its joint distribution G with S_T. Then, all twins as in (11) have the same price but one of them may be closer to the original Asian derivative (in the sense of minimizing the distance, as in (12)). Note that since all marginal distributions are fixed, the criterion (12) is equivalent to maximizing the correlation between X_T and H_T. We use this criterion in one of our applications (see section 5.1).

†This finding is consistent with the result obtained by Takahashi and Yamamoto (2013), who apply it to replicate a joint distribution in the hedge fund industry.

‡To see this, recall that the joint distribution between the twin $f(S_t, S_T)$ and S_T is fixed, and thus also the joint distribution between the twin and ξ_T (as ξ_T is a decreasing function of S_T due to (2)). All twins $f(S_t, S_T)$ with such a property have the same price $\mathbb{E}[\xi_T f(S_t, S_T)]$.

4.2. Optimality of twins

Next, we investigate the cost optimality of twins. As discussed above, if the benchmark A_T coincides with S_T, then all twins that satisfy $(X_T, A_T) \sim G$ have the same cost and the problem of searching for the cheapest one is not meaningful. However, this observation is no longer true when the benchmark A_T has a density with S_T. In this case, the cheapest twin is determined by theorem 4.4 that extends theorem 3.2 to the state-dependent case. Theorem 3.2 finds that among the infinite number of payoffs with a given distribution F, the cheapest one is increasing in S_T. In the state-dependent setting, one has that optimal payoffs are increasing in S_T, conditionally on A_T.

THEOREM 4.4 (Cost optimality of twins) *Assume that (S_T, A_T) has joint density with respect to the Lebesgue measure. Let G be a bivariate cumulative distribution function. The optimal state-dependent strategy determined by*

$$\min_{(X_T, A_T)\sim G} c_0(X_T) \tag{13}$$

has an almost surely unique solution X_T^* which is a twin of the form $f(S_T, A_T)$. X_T^* is almost surely increasing in S_T, conditionally on A_T, and given by

$$X_T^* := F^{-1}_{X_T|A_T}(F_{S_T|A_T}(S_T)). \tag{14}$$

The proof of theorem 4.4 is provided in appendix A.5. □

Recall from section 3 that when preferences are law-invariant, optimal payoffs are path-independent and increasing in S_T. When preferences are state-dependent, we observe from expression (14) that optimal state-dependent payoffs may become path-dependent, and are increasing in S_T, *conditionally* on A_T. We end this section with a corollary derived from theorem 4.4. The result echoes the one established for investors with law-invariant preferences in the previous section (corollary 3.4)

COROLLARY 4.5 (Cheapest twin) *Assume that (S_T, A_T) has joint density with respect to the Lebesgue measure. Let G be a bivariate cumulative distribution function. Let X_T be a payoff such that $(X_T, A_T) \sim G$. Then, X_T is the cheapest payoff if and only if, conditionally on A_T, X_T is (almost surely) increasing in S_T.*

The proof of corollary 4.5 is provided in appendix A.6. □

5. Improving security design

In this section, we show that the results above are useful in designing balanced and transparent investment policies for retail investors as well as financial institutions:

(1) If the investor who buys the financial contract has law-invariant preferences and if the contract is not increasing in S_T, then there exists a strictly cheaper derivative

(cost-efficient contract) that is strictly better for this investor. We find its design by applying theorem 3.2.

(2) If the investor buys the contract because of the interaction with the market asset S_T, and the contract depends on another asset, then we can apply theorem 4.3 to simplify its design while keeping it 'at least as good.' The contract then depends, for example, on S_T and S_t for some $t \in (0, T)$.

(3) If the investor buys the contract because he likes the dependence with a benchmark A_T, which is not S_T, and if the contract does not only depend on A_T and S_T, then we use theorem 4.2 to construct a simpler one that is 'at least as good' and that writes as a function of S_T and A_T. Finally, if the obtained contract is not increasing in S_T conditionally on A_T, then it is also possible to construct a strictly cheaper alternative using theorem 4.4 and corollary 4.5.

We now use the Black–Scholes market to illustrate these three situations. We begin with the example of an Asian option with fixed strike, followed by the example of one with floating strike.

5.1. The geometric Asian twin with fixed strike

Consider a fixed strike (continuously monitored) geometric Asian call with payoff given by

$$Y_T := (G_T - K)^+ . \tag{15}$$

Here, K denotes the fixed strike and G_T is the geometric average of stock prices from 0 to T, defined as

$$\ln(G_T) := \frac{1}{T}\int_0^T \ln(S_s)\, ds. \tag{16}$$

We can now apply the results derived above to design products that improve upon Y_T.

5.1.1. Use of cost-efficiency payoff for investors with increasing law-invariant preferences. By applying theorem 3.2 to the payoff Y_T (15), one finds that the cost-efficient payoff associated with a fixed strike (continuously monitored) geometric Asian call is

$$Y_T^* = d\left(S_T^{1/\sqrt{3}} - \frac{K}{d}\right)^+, \tag{17}$$

where $d = S_0^{1-\frac{1}{\sqrt{3}}} e^{\left(\frac{1}{2}-\sqrt{\frac{1}{3}}\right)\left(\mu-\frac{\sigma^2}{2}\right)T}$. This is also the payoff of a power call option, with well-known price

$$c_0\left(Y_T^*\right) = S_0 e^{(\frac{1}{\sqrt{3}}-1)rT+(\frac{1}{2}-\frac{1}{\sqrt{3}})\mu T-\frac{\sigma^2 T}{12}}\Phi(h_1) - Ke^{-rT}\Phi(h_2) \tag{18}$$

where

$$h_1 = \frac{\ln\left(\frac{S_0}{K}\right) + (\frac{1}{2}-\frac{1}{\sqrt{3}})\mu T + \frac{r}{\sqrt{3}}T + \frac{1}{12}\sigma^2 T}{\sigma\sqrt{\frac{T}{3}}},$$

$$h_2 = h_1 - \sigma\sqrt{\frac{T}{3}}.$$

While the above results can also be found in Bernard *et al.* (2014a), they are worth considering here for the purpose of comparison with what follows. Note that letting K go to zero provides a cost-efficient payoff that is equivalent to the geometric average G_T.

5.1.2. A twin that is useful for investors who care about the dependence with S_T. By applying theorem 4.3 to the payoff G_T, we can find a twin payoff $R_T(t) = f(S_t, S_T)$ such that

$$(S_T, R_T(t)) \sim (S_T, G_T) . \tag{19}$$

By definition, this twin preserves existing dependence between G_T and S_T. However, compared to the original contract it is simpler and 'less' path-dependent, as it depends only on two values of the path of the stock price. Interestingly, the call option written on $R_T(t)$ and the call option written on G_T have the same joint distribution with S_T. Consequently,

$$\left(S_T, (R_T(t) - K)^+\right) \sim \left(S_T, (G_T - K)^+\right). \tag{20}$$

$(R_T(t) - K)^+$ is therefore a twin equivalent to the fixed strike geometric Asian call (as in theorem 4.3). We can compute $R_T(t)$ by applying theorem 4.3, and we find that

$$R_T(t) = S_0^{\frac{1}{2}-\frac{1}{2\sqrt{3}}\sqrt{\frac{T-t}{t}}} S_t^{\frac{T}{t}\frac{1}{2\sqrt{3}}\sqrt{\frac{t}{T-t}}} S_T^{\frac{1}{2}-\frac{1}{2\sqrt{3}}\sqrt{\frac{t}{T-t}}}, \tag{21}$$

where t is freely chosen in $(0, T)$. Details on how (11) becomes (21) are provided in appendix B.1.† The equality of joint distributions exposed in (20) implies that the call option written on $R_T(t)$ has the same price as the original fixed strike (continuously monitored) geometric Asian call (15). The time−0 price of both contracts is therefore

$$c_0((R_T(t) - K)^+) = S_0 e^{-\frac{rT}{2}-\frac{\sigma^2 T}{12}}\Phi(\tilde{d}_1) - Ke^{-rT}\Phi(\tilde{d}_2), \tag{22}$$

where $\tilde{d}_1 = \frac{\ln(S_0/K)+rT/2+\sigma^2 T/12}{\sigma\sqrt{T/3}}$ and $\tilde{d}_2 = \tilde{d}_1 - \sigma\sqrt{T/3}$ (see Kemna and Vorst 1990).

5.1.3. Choosing among twins. The construction in theorem 4.3 depends on t. Maximizing the correlation between $\ln(R_T(t))$ and $\ln(G_T)$ is nevertheless a possible way to select a specific t. The covariance between $\ln(R_T(t))$ and $\ln(G_T)$ is provided by

$$\operatorname{cov}(\ln(R_T(t)), \ln(G_T)) = \frac{\sigma^2}{2}\left(\frac{T}{2} + \frac{\sqrt{t}\sqrt{T-t}}{2\sqrt{3}}\right)$$

and, by construction of $R_T(t)$, the standard deviations of $\ln(R_T(t))$ and $\ln(G_T)$ are both equal to $\sigma\sqrt{\frac{T}{3}}$. Maximizing the correlation coefficient is therefore equivalent to maximizing the covariance, and thus of $f(t) = (T - t)t$. This maximum is obtained for $t^* = \frac{T}{2}$, and the maximal correlation ρ_{max} between $\ln(R_T(t))$ and $\ln(G_T)$ is

†Formula (21) is based on the expression (11) for a twin dependent on S_t and S_T. Note that there is no uniqueness. For example, $1 - F_{S_t|S_T}(S_t)$ is also independent of S_T, and we can thus also consider $H_T(t) := F^{-1}_{X_T|S_T}(1 - F_{S_t|S_T}(S_t))$ as a suitable twin ($0 < t < T$) satisfying the joint distribution, as in (19). In this case, one obtains $H_T(t) = S_0^{\frac{1}{2}+\frac{1}{2\sqrt{3}}\sqrt{\frac{T-t}{t}}} S_t^{-\frac{T}{t}\frac{1}{2\sqrt{3}}\sqrt{\frac{t}{T-t}}} S_T^{\frac{1}{2}+\frac{1}{2\sqrt{3}}\sqrt{\frac{t}{T-t}}}$.

$$\rho_{\max} = \frac{3}{4} + \frac{\sqrt{3}\sqrt{(T-t^*)\,t^*}}{4T} = \frac{3}{4} + \frac{\sqrt{3}}{8} \approx 0.9665,$$

which shows that the optimal twin is highly correlated to the initial Asian, while being considerably simpler. Note that both the maximum correlation and the optimum $R_T\left(\frac{T}{2}\right)$ are robust to changes in market parameters.

5.2. *The geometric Asian twin with floating strike*

Consider now a floating strike (continuously monitored) Asian put option defined by

$$Y_T = (G_T - S_T)^+ . \tag{23}$$

For increasing law-invariant preferences, corollary 3.4 may be used to find a cheaper contract that depends on S_T only. The cheapest contract with cdf F_{Y_T} is known to be $F_{Y_T}^{-1}\left(\Phi\left(\frac{\ln\left(\frac{S_T}{S_0}\right)-(\mu-\frac{\sigma^2}{2})T}{\sigma\sqrt{T}}\right)\right)$. Notice that $F_{Y_T}^{-1}$ can only be numerically approximated because the distribution of the difference between two lognormal distributions is unknown.

If investors care about the dependence with S_T, by applying theorem 4.3, one can find twins $F_{Y_T|S_T}^{-1}(F_{S_t|S_T}(S_t))$ as functions of S_t and S_T, which are explicitly given as

$$\left(S_0^{\frac{1}{2}-\frac{1}{2\sqrt{3}}\sqrt{\frac{T-t}{t}}}\, S_t^{\frac{T}{t}\frac{1}{2\sqrt{3}}\sqrt{\frac{t}{T-t}}}\, S_T^{\frac{1}{2}-\frac{1}{2\sqrt{3}}\sqrt{\frac{t}{T-t}}} - S_T\right)^+ . \tag{24}$$

Details can be found in appendix B.2.

Finally, if investors care about the dependence with G_T, then it is possible to construct a cheaper twin because the payoff (23) is not conditionally increasing in S_T. Therefore, it can be strictly improved using theorem 4.4. The reason is that we can improve the payoff (23) by making it cheaper, while maintaining dependence with G_T. Hence, we invoke theorem 4.4 (expression 14) to exhibit another payoff $X_T = F_{Y_T|G_T}^{-1}\left(F_{S_T|G_T}(S_T)\right)$ such that

$$(Y_T, G_T) \sim (X_T, G_T),$$

but so that X_T is strictly cheaper. After some calculations, we find that X_T writes as

$$X_T = \left(G_T - a\frac{G_T^3}{S_T}\right)^+ , \tag{25}$$

where $a = \frac{e^{\left(\mu-\frac{\sigma^2}{2}\right)\frac{T}{2}}}{S_0}$. Details can be found in appendix B.3.

Finally, one can easily assess the extent to which the twin (25) is cheaper than the initial payoff Y_T. To do so, we recall the price of a geometric Asian option with floating strike (the no-arbitrage price of Y_T):

$$\begin{aligned} c_0(Y_T) &= e^{-rT}\mathbb{E}_{\mathbb{Q}}(G_T - S_T)^+ \\ &= S_0 e^{-\frac{rT}{2}}\left(\Phi(f)\, e^{-\frac{\sigma^2 T}{12}} - e^{\frac{rT}{2}}\Phi\left(f - \sigma\sqrt{\frac{T}{3}}\right)\right), \end{aligned} \tag{26}$$

where $f = \frac{\frac{\sigma^2}{12}T - r\frac{T}{2}}{\sigma\sqrt{\frac{T}{3}}}$. Similarly, one finds that

$$\begin{aligned} c_0(X_T) &= e^{-rT}\mathbb{E}_{\mathbb{Q}}\left(G_T - a\frac{G_T^3}{S_T}\right)^+ \\ &= S_0 e^{-\frac{rT}{2}}\left(\Phi(d)\, e^{-\frac{\sigma^2 T}{12}} - e^{\frac{\mu T}{2}}\Phi\left(d - \sigma\sqrt{\frac{T}{3}}\right)\right) \end{aligned} \tag{27}$$

where $d = \frac{\frac{\sigma^2 T}{12} - \mu\frac{T}{2}}{\sigma\sqrt{\frac{T}{3}}}$, which we need to compare numerically to (26). For example, when $\mu = 0.06$, $r = 0.02$, $\sigma = 0.3$ and $T = 1$, one has $c_0(Y_T) = 6.74$ and $c_0(X_T) = 5.86$, indicating that cost savings can be substantial. Also note the close correspondence between formulas (26) and (27). The proofs for these formulas are provided in appendix B.4.

6. Portfolio management

This section provides several contributions to the field of portfolio management. We first derive the optimal investment for an expected utility maximizer who has a constraint on the dependence with a given benchmark. Next, we revisit optimal strategies for target probability maximizers (see Browne 1999 and Spivak and Cvitanić 1999), and we extend this problem in two directions by adding dependence constraints and by considering a random target. In both cases, we derive analytical solutions that are given by twins. From now on, we denote by W_0 the initial wealth.

6.1. *Expected utility maximization with dependence constraints*

The most prominent decision theory used in various fields of economics is the expected utility theory (EUT) of Von Neumann and Morgenstern (1947). In the expected utility, framework investors assign a utility $u(x)$ to each possible level of wealth x. Increasing preferences are equivalent to an increasing utility function $u(\cdot)$. Assuming that $u(\cdot)$ is concave is equivalent to assuming that investors are risk averse in the sense that for a given budget they prefer a sure income over a random one with the same mean. In their seminal paper on optimal portfolio selection, Cox and Huang (1989) showed how to obtain the optimal strategy for a risk averse expected utility maximizer; see also Merton (1971) and He and Pearson (1991a,b). We recall this classical result in the following theorem.

THEOREM 6.1 (Optimal payoff in EUT) *Consider a utility function $u(\cdot)$ defined on (a, b) such that $u(\cdot)$ is continuously differentiable and strictly increasing, $u'(\cdot)$ is strictly decreasing, $\lim_{x\searrow a} u'(x) = +\infty$ and $\lim_{x\nearrow b} u'(x) = 0$. Consider the following portfolio optimization problem:*

$$\max_{\mathbb{E}[\xi_T X_T]=W_0} \mathbb{E}[u(X_T)]. \tag{28}$$

The optimal solution to this problem is given by

$$X_T^* = (u')^{-1}(\lambda\xi_T), \tag{29}$$

where λ is such that $\mathbb{E}\left[\xi_T (u')^{-1}(\lambda\xi_T)\right] = W_0$.

Note that the optimal EUT payoff X_T^* is decreasing in ξ_T and thus increasing in S_T (illustration of the results derived in section 3), which highlights the lack of protection of optimal portfolios when markets decline. To account for this, we give the investor the opportunity to maintain a desired dependence with a benchmark portfolio (e.g. representing the financial market). This extends earlier results on expected utility maximization with constraints, such as those of Brennan and Solanki (1981), Brennan and Schwartz (1989), He and Pearson (1991a,b), Basak (1995), Grossman and Zhou (1996), Sorensen (1999) and Jensen and Sorensen (2001). These studies were for the most part concerned with the expected utility maximization problem when investors want a lower bound on their optimal wealth either at maturity or throughout some time interval. When this bound is deterministic, this corresponds to classical portfolio insurance. Boyle and Tian (2007) extend and unify the various results by allowing the benchmark to be beaten with some confidence. They consider the following maximization problem over all payoffs X_T:

$$\max_{\substack{\mathbb{P}(X_T \geq A_T)\geq\alpha,\\ c_0(X_T)=W_0}} \mathbb{E}[u(X_T)], \tag{30}$$

where A_T is some benchmark (e.g. the portfolio of another manager in the market). In theorem 2.1 (page 327) of Boyle and Tian (2007), the optimal contract X_T^* is derived explicitly (under some regularity conditions ensuring feasibility of the stated problem), and it is an optimal twin.†

This also follows from our results. Assume that the solution to (30) exists, and denote it by X_T^*. Then let G be the bivariate cdf of (X_T^*, A_T). The cheapest way to preserve this joint bivariate cdf is obtained by a twin $f(A_T, S_T)$, which is increasing in S_T conditionally on A_T (see corollary 4.5). Hence, X_T^* must also be of this form, otherwise one can easily contradict the optimality of X_T^* to the problem. Thus, the solution to optimal expected utility maximization with the additional probability constraint (when it exists) is an optimal twin. By similar reasoning, this result also holds when there are several probability constraints involving the joint distribution of terminal portfolio value X_T and benchmark A_T.

The following theorem extends theorem 6.1 and the referenced literature above by considering an expected utility maximization problem in which the investor fixes the dependence with a benchmark. Doing so amounts to specifying up front the joint copula of (X_T, A_T). Hence, let us assume that the copula between X_T and A_T is specified to be C, i.e. $\mathcal{C}_{(X_T,A_T)} = C$. We formulate the following portfolio optimization problem

$$\max_{\substack{c_0(X_T)=W_0\\ \mathcal{C}_{(X_T,A_T)}=C}} \mathbb{E}\left(u(X_T)\right). \tag{31}$$

In order to solve the expected utility optimization problem with dependence constraints (31), we denote by $C_{1|A_T}$ the conditional distribution of the first component, given A_T (or equivalently given $F_{A_T}(A_T)$) and define

$$U_T = F_{S_T|A_T}(S_T) \text{ and } Z_T = C^{-1}_{1|A_T}(U_T). \tag{32}$$

†The observation that in the given context optimal payoffs write as twins is also consistent with the solutions of the constrained portfolio optimization problems considered in Bernard *et al.* (2014b) and Bernard and Vanduffel (2014b).

Note that when (A_T, S_T) has a joint density, then U_T and Z_T are uniformly distributed on (0, 1) and (Z_T, A_T) has copula C (see also lemma A.2). Theorem 6.2 makes also use of the projection on the convex cone

$$M_\downarrow := \{f \in L^2[0,1];\ f \text{ decreasing}\}, \tag{33}$$

which is a subset of $L^2[0, 1]$ equipped with the Lebesgue measure and the standard $||\cdot||_2$ norm. For an element $\varphi \in L^2[0, 1]$, we denote by $\hat{\varphi} = \pi_{M_\downarrow}(\varphi)$ the projection of φ on $M_\downarrow$. $\hat{\varphi}$ can be interpreted as the best approximation of φ by a decreasing function for the $||\cdot||_2$ norm.

Theorem 6.2 (Optimal payoff in EUT with dependence constraint) *Consider a utility function $u(\cdot)$ as in theorem 6.1 and assume that (A_T, S_T) has a joint density. Let $H_T = \mathbb{E}(\xi_T|Z_T) = \varphi(Z_T)$ and $\hat{H}_T = \hat{\varphi}(Z_T)$ in which Z_T is defined as in (32). Then, the solution to the optimization problem (31) is given by*

$$\hat{X}_T = \left(u'\right)^{-1}\left(\lambda \hat{H}_T\right), \tag{34}$$

where λ is such that $\mathbb{E}\left[\xi_T \left(u'\right)^{-1}\left(\lambda \hat{H}_T\right)\right] = W_0$.

The proof of theorem 6.2 is provided in appendix C.1. □

Remark 6.3 In the case that $H_T = \mathbb{E}(\xi_T|Z_T)$ is decreasing in Z_T, we obtain, as solution to (31),

$$\hat{X}_T = \left(u'\right)^{-1}(\lambda H_T). \tag{35}$$

In this case, the proof of theorem 6.2 can be simplified and reduced to the classical optimization result in theorem 6.1 since by theorem 4.4 an optimal solution X_T is unique and satisfies

$$X_T = F^{-1}_{X_T|A_T}(F_{S_T|A_T}(S_T)).$$

By lemma A.2 one can conclude that $X_T = F^{-1}_{X_T}(Z_T)$, i.e. X_T is an increasing function of Z_T. Theorem 6.1 then allows one to find the optimal element in this class.

Remark 6.4 The determination of the isotonic approximation $\hat{\varphi}$ of φ is a well-studied problem (see theorem 1.1 in Barlow *et al.* 1972). $\hat{\varphi}$ is the slope of the smallest concave majorant $SCM(\varphi)$ of φ, i.e. $\hat{\varphi} = (SCM(\varphi))'$. In Barlow *et al.* (1972), the projection on $M_\uparrow$ is given as the slope of the greatest convex minorant $GCM(\varphi)$ of φ. Fast algorithms are known to determine $\hat{\varphi}$.

Remark 6.5 Some special cases of interest concern the study of the optimum when the copula constraint is the lower or upper Fréchet bound. If in theorem 6.2 the copula C is the upper Fréchet bound, then $Z_T = F_{A_T}(A_T)$. When $A_T = S_T$, then $H_T = E[\xi_T|A_T] = \xi_T$ and we find that $\hat{X}_T$ is equal to the optimal portfolio when there is no dependence constraint (theorem 6.1). This result is intuitive because the dependence constraint that we impose implies that that the optimum is increasing in S_T, which is a feature that arises naturally in the unconstrained problem. If $A_T = S_t$, then $H_T = E[\xi_T|S_t]$ is decreasing in S_t. Thus, $\hat{H}_T = H_T$ and the optimum can be explicitly calculated (see also the example below). Finally, if in theorem 6.2, the copula C is the lower Fréchet bound, then $Z_T = 1 - F_{A_T}(A_T)$. Assume that $A_T = S_T$, then $H_T = E[\xi_T|Z_T] = \xi_T$, which is increasing in S_T and therefore decreasing in Z_T. The isotonic approximation is the constant.

Hence, the optimal portfolio is also a constant, i.e. the budget is entirely invested in the risk-free asset.

6.1.1. Example (CRRA investor). Next, we illustrate theorem 6.2 by a comparison of the optimal wealth $\hat{X}_T$ derived under a dependence constraint (theorem 6.2) with the optimal wealth $X_T^\star$ derived with no constraints on dependence (theorem 6.1). W_0 stands for the initial wealth and we set the benchmark A_T equal to S_t for some $0 < t < T$. We assume also that the dependence between S_t and the final wealth is described by a Gaussian copula C with correlation coefficient $\rho \in \left[-\sqrt{1-\frac{t}{T}}, 1\right)$. Consider a CRRA utility function with risk aversion $\eta > 0$:

$$u(x) := \begin{cases} \frac{x^{1-\eta}}{1-\eta} & \text{when } \eta \neq 1 \\ \ln(x) & \text{when } \eta = 1 \end{cases}.$$

The standard Merton problem (28) exposed in theorem 6.1 involves no dependence constraint on the final wealth. The solution is $X_T^\star = \left(u'\right)^{-1}(\lambda \xi_T)$ where λ is found to meet the initial wealth constraint ($\mathbb{E}\left[\xi_T X_T^\star\right] = W_0$). It is straightforward to verify that for all $\eta > 0$ the optimal wealth is given by

$$\begin{aligned} X_T^\star(\eta) &= \lambda^{-\frac{1}{\eta}} \xi_T^{-\frac{1}{\eta}} \\ &= W_0 e^{rT} e^{-\frac{1}{\eta}\frac{\theta}{\sigma}\left(\mu - \frac{\sigma^2}{2}\right)T + \left(\frac{1}{\eta} - \frac{1}{2\eta^2}\right)\theta^2 T} \left(\frac{S_T}{S_0}\right)^{\frac{1}{\eta}\frac{\theta}{\sigma}} \end{aligned} \tag{36}$$

Observe that the dependence between $X_T^*(\eta)$ and S_t is characterized by the Gaussian copula with correlation parameter

$$\operatorname{corr}\left(\ln(X_T^*(\eta)), \ln(S_t)\right) = \sqrt{\frac{t}{T}}. \tag{37}$$

When there is a constraint on the dependence, we show in appendix C.2 that the solution to the optimization problem (31) (that is the optimal wealth satisfying the initial budget and the dependence constraint) is given as

$$\begin{aligned} \hat{X}_T(\eta) = {} & W_0 e^{rT} e^{-\frac{1}{\eta}\frac{\theta}{\sigma}\left(\mu - \frac{\sigma^2}{2}\right)A^2 + \left(\frac{1}{\eta} - \frac{1}{2\eta^2}\right)\theta^2 A^2} \\ & \times \left(\frac{S_T}{S_0}\right)^{\frac{\theta}{\eta\sigma} A \frac{\sqrt{1-\rho^2}}{\sqrt{T-t}}} \left(\frac{S_t}{S_0}\right)^{\frac{\theta}{\eta\sigma} A \left[\frac{\rho}{\sqrt{t}} - \frac{\sqrt{1-\rho^2}}{\sqrt{T-t}}\right]}. \end{aligned} \tag{38}$$

where

$$A = \rho\sqrt{t} + \sqrt{(1-\rho^2)(T-t)}$$

Note that the expressions (36) and (38) coincide when $\rho = \sqrt{\frac{t}{T}}$. The basis reason for this feature is that the unconstrained optimum has correlation $\sqrt{\frac{t}{T}}$ with S_t. When $\eta \neq 1$, the expected utilities of $X_T^\star(\eta)$ and $\hat{X}_T(\eta)$ are given by

$$\mathbb{E}\left[u\left(X_T^\star(\eta)\right)\right] = \frac{1}{1-\eta} W_0^{1-\eta} e^{(1-\eta)rT + \frac{1}{2}\frac{1-\eta}{\eta}\theta^2 T}$$

and

$$\begin{aligned} & \mathbb{E}\left[u\left(\hat{X}_T(\eta)\right)\right] \\ & = \frac{1}{1-\eta} W_0^{1-\eta} e^{(1-\eta)rT + \frac{1}{2}\frac{1-\eta}{\eta}\theta^2\left(\rho\sqrt{t} + \sqrt{(1-\rho^2)(T-t)}\right)^2}, \end{aligned}$$

respectively. In the case that $\eta = 1$, i.e. the log-utility case $u(x) := \ln(x)$, we find that

$$\mathbb{E}\left[u\left(\hat{X}_T\right)\right] = \ln(W_0) + rT + \frac{1}{2}\theta^2\left(\rho\sqrt{t} + \sqrt{1-\rho^2}\sqrt{T-t}\right)^2$$

and

$$\mathbb{E}\left[u\left(X_T^\star\right)\right] = \ln(W_0) + rT + \frac{1}{2}\theta^2 T,$$

respectively.

Assume that $t = T/2$ for the numerical application so that $S_{T/2}$ is the benchmark. Using an initial wealth $W_0 = 100$ and the same set of parameters as in the previous section, $\mu = 0.06$, $r = 0.02$, $\sigma = 0.3$ and $T = 1$. Figure 1 plots the expected utility as a function of ρ for the constrained payoff ($\hat{X}_T$) and we have an horizontal line corresponding to the expected utility of X_T^*. Note that they share exactly one common point corresponding to the level of correlation found in (37).

6.2. Target probability maximization

Target probability maximizers are investors who, for a given budget (initial wealth) and a given time frame, want to maximize the probability that the final wealth achieves some fixed target b. In a Black–Scholes financial market model, Browne (1999) and Spivak and Cvitanić (1999) derive the optimal investment strategy for these investors using stochastic control theory and show that it is optimal to purchase a digital option written on the risky asset. We show that their results follow from theorem 3.2 in a more straightforward way.

PROPOSITION 6.6 (Browne's original problem) *Let W_0 be the initial wealth and let $b > W_0 e^{rT}$ be the desired target.*† *The solution to the following target probability maximization problem,*

$$\max_{X_T \geq 0,\ c_0(X_T) = W_0} \mathbb{P}[X_T \geq b], \tag{39}$$

is given by the payoff

$$X_T^* = b\,\mathbb{1}_{\{S_T > \lambda\}}, \tag{40}$$

in which λ is given by $b\mathbb{E}\left(\xi_T \mathbb{1}_{\{S_T > \lambda\}}\right) = W_0$.

The proof of this proposition is provided in appendix C.3. In a Black–Scholes market one easily verifies that $\lambda = S_0 \exp\left((r - \frac{\sigma^2}{2})T - \sigma\sqrt{T}\Phi^{-1}\left(\frac{W_0 e^{rT}}{b}\right)\right)$. □

A target probability maximizing strategy is essentially an all-or-nothing strategy. Intuitively, investors might not be attracted by the design of the optimal payoff, which maximizes the probability beating a fixed target. The obtained wealth depends solely on the ultimate value of the underlying risky asset, which makes it highly dependent on final market behaviour and thus prone to unexpected and brutal changes. Our first extension concerns the case of a stochastic target, so that preferences become state-dependent.

THEOREM 6.7 (Target probability maximization with a random target) *Let W_0 be the initial wealth and let B be the*

†If $b \leq W_0 e^{rT}$, then the problem is not interesting since an investment in the risk-free asset allows the investor to reach a 100% probability of beating the target b.

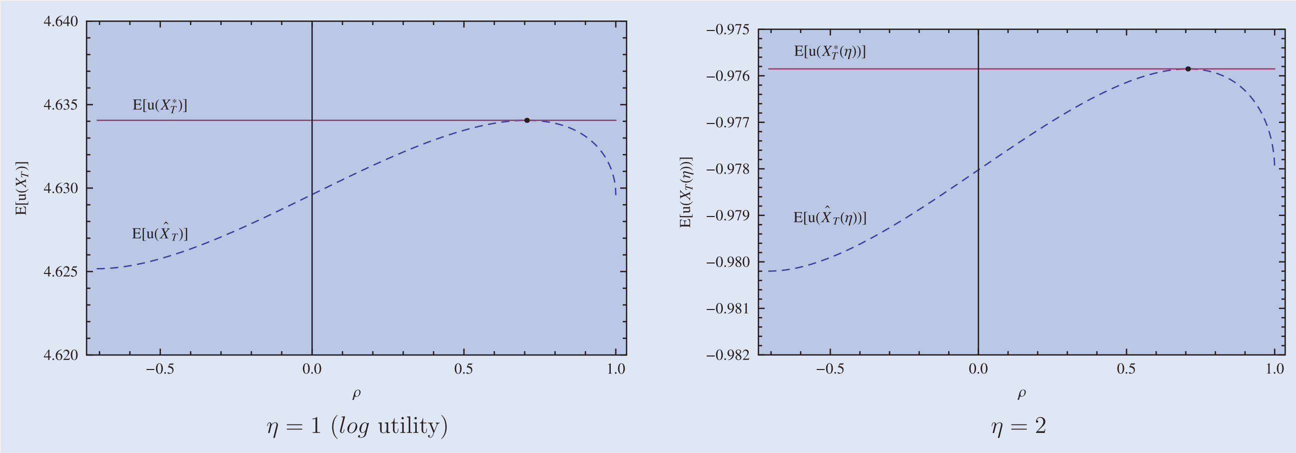

Figure 1. Expected utility as a function of ρ for a CRRA investor, with and without dependence constraint.

random target such that (B, S_T) has a density. The solution to the random target probability maximization problem,

$$\max_{X_T \geq 0,\ c_0(X_T)=W_0} \mathbb{P}[X_T \geq B], \tag{41}$$

is given by the payoff

$$X_T^* = B\mathbb{1}_{\{B\xi_T<\lambda\}}, \tag{42}$$

in which λ is implicitly given by $\mathbb{E}\left[B\xi_T \mathbb{1}_{\{B\xi_T<\lambda\}}\right] = W_0$.

The proof of this proposition is provided in appendix C.4. □

Our second extension assumes a fixed dependence with a benchmark in the financial market. We now consider the problem of an investor who, for a given budget, aims to maximize the probability that the final wealth will achieve some fixed target while preserving a certain dependence with a benchmark.

Theorem 6.8 (Target probability maximization with a random benchmark) *Let W_0 be the initial wealth and let $b > W_0e^{rT}$ the desired target for final wealth. Assume that the pair (A_T, S_T) has a density. Then the solution to the target probability optimization problem with random benchmark A_T,*

$$\max_{\substack{X_T \geq 0, c_0(X_T)=W_0,\\ \mathcal{C}_{(X_T,A_T)}=C}} \mathbb{P}[X_T \geq b], \tag{43}$$

is given by

$$X_T^* = b\mathbb{1}_{\{Z_T>\lambda\}}, \tag{44}$$

in which λ is determined by $b\mathbb{E}\left[\xi_T \mathbb{1}_{\{Z_T>\lambda\}}\right] = W_0$ *and Z_T is defined as in (32).*

The proof of this result is provided in appendix C.5. □

The result derived in theorem 6.8 holds in particular when $A_T = S_t$ $(0 < t < T)$ and when C is a Gaussian copula with correlation coefficient ρ. Then, the optimal solution is explicit and equal to

$$X_T^* = b\mathbb{1}_{\{S_t^\alpha S_T>\lambda\}}, \tag{45}$$

with $\alpha = \sqrt{\frac{T-t}{t(1-\rho^2)}}\rho - 1$, and $\lambda = S_0^{\alpha+1}\exp\left((r - \frac{\sigma^2}{2})(\alpha t+T) - \sigma\sqrt{k}\Phi^{-1}\left(\frac{W_0e^{rT}}{b}\right)\right)$ with $k = (\alpha+1)^2 t + (T-t) = \frac{T-t}{1-\rho^2}$. The proof of (45) is provided in appendix C.6.

6.2.1. Illustration of target probability maximization. Let us compare the payoffs that arise from the unconstrained target probability maximization problem in theorem 6.6 and the constrained maximization problem in theorem 6.8. We use the same set of parameters as in section 6.1, i.e. $\mu = 0.06$, $r = 0.02$, $\sigma = 0.3$ and $T = 1$. We also take $S_0 = 100$ and $b = 106$. In figure 2, we plot for both payoffs their expected value as a function of ρ. The optimum for the unconstrained target optimization problem in theorem 6.6 is given by $b\mathbb{1}_{\{S_T>\lambda_1\}}$ in which λ_1 is such that the budget constraint is satisfied. Its expected value is given as

$$\mathbb{E}\left[b\mathbb{1}_{\{S_T>\lambda_1\}}\right] = b\Phi\left[\theta\sqrt{T} + \Phi^{-1}\left[\frac{W_0e^{rT}}{b}\right]\right].$$

By similar reasoning, we find for the expected value of the optimum of theorem 6.8,

$$\mathbb{E}\left[b\mathbb{1}_{\{S_t^\alpha S_T>\lambda_2\}}\right] = b\Phi\left[\theta\frac{\alpha t+T}{\sqrt{k}} + \Phi^{-1}\left[\frac{W_0e^{rT}}{b}\right]\right],$$

in which $\alpha = \sqrt{\frac{T-t}{t(1-\rho^2)}}\rho - 1$, $k = \frac{T-t}{1-\rho^2}$ and λ_2 is such that the budget constraint is satisfied. Note that the expected values are proportional to the probabilities to beat the target value b. We observe that in the constrained target probability maximization

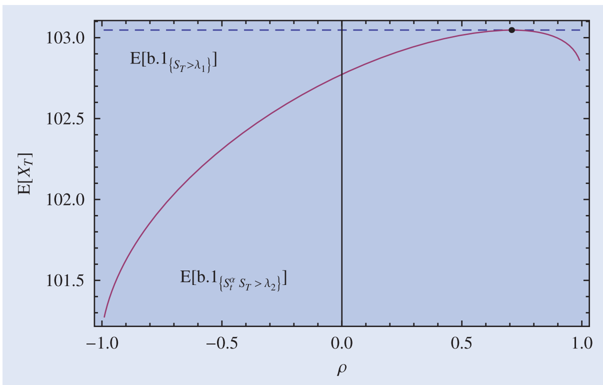

Figure 2. Expected payoff as a function of ρ for the different target probability maximization strategies considered in theorem 6.6 and theorem 6.8.

problem the expected value (and the corresponding success probability) is smaller than in the unconstrained problem.

7. Final remarks

In this paper, we introduce a state-dependent version of the optimal investment problem. We deal with investors who target a known wealth distribution at maturity (as in the traditional setting) and *additionally* desire a particular interaction with a random benchmark. We show that optimal contracts depend at most on two underlying assets, or on one asset evaluated at two different dates, and we are able to characterize and determine them explicitly. Our characterization of optimal strategies allows us to extend the classical expected utility optimization problem of Merton to the state-dependent situation. Throughout the paper, we have assumed that the state-price density process ξ_T is a decreasing functional of the risky asset price S_T and that there is a single risky asset. It is possible to relax these assumptions and yet still to provide explicit representations of optimal payoffs. However, the optimality is then no longer related to path-independence properties.

Throughout the paper, we assumed that ξ_T is decreasing in S_T (in (2)). Moreover, we use the one-dimensional Black-Scholes model to illustrate our findings. However, the case of multidimensional markets described by a price process $\left(S_t^{(1)}, \ldots, S_t^{(d)}\right)_t$ is essentially included in the results presented in this paper, assuming that the state-price density process $(\xi_t)_t$ of the risk-neutral measure chosen for pricing is of the form $\xi_t = g_t\left(h_t\left(S_t^{(1)}, \ldots, S_t^{(d)}\right)\right)$ with some real functions g_t, h_t (as in Bernard *et al.* 2011 who considered the state-independent case). All results in the paper apply by replacing the one-dimensional stock price process S_t by the one-dimensional process $h_t\left(S_t^{(1)}, \ldots, S_t^{(d)}\right)$. In addition, we have assumed that asset prices are continuously distributed, which amounts essentially to assuming that the state-price density process ξ_t is continuously distributed at any time. An extension to the case in which ξ_t may have atoms is possible but not in the scope of the present paper.

A straightforward extension of the results presented in this paper is to consider the market model of Platen and Heath (2005) using the growth optimal portfolio (GOP). Its origins can be traced back to Kelly (1956). It consists of replacing the state-price density process ξ_t by $1/S_t^*$, where S_t^* denotes the value of the GOP at time t. In the Black–Scholes setting, S_t^* is simply the value of one unit investment in a constant-mix strategy, where a fraction $\frac{\theta}{\sigma}$ is invested in the risky asset and the remaining fraction $1 - \frac{\theta}{\sigma}$ in the bank account. It is easy to prove that this strategy is optimal for an expected log-utility maximizer. Using a milder notion of arbitrage, Platen and Heath (2005) argue that, in general, the price of (non-negative) payoffs could be achieved using the pricing rule (1) where the role of ξ_T is now played by the inverse of the GOP. Hence, our results are also valid in their setting, where the GOP is taken as the reference (see Bernard *et al.* 2014b for an example). Other dependence constraints can be considered, e.g. a constraint on the correlation between the terminal wealth and a benchmark (developed in the context of mean-variance optimization by Bernard and Vanduffel (2014b)).

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Appendix 1. Proofs

Throughout the paper and the different proofs, we make repeatedly use of the following lemmas. The first lemma gives a restatement of the classical Hoeffding–Fréchet bounds going back to the early work of Hoeffding (1940) and Fréchet (1940).

Lemma A.1 (Hoeffding–Fréchet bounds) *Let (X, Y) be a random pair and U uniformly distributed on $(0, 1)$. Then*

$$\mathbb{E}\left[F_X^{-1}(U)F_Y^{-1}(1-U)\right] \le \mathbb{E}[XY] \le \mathbb{E}\left[F_X^{-1}(U)F_Y^{-1}(U)\right]. \tag{A1}$$

The upper bound for $\mathbb{E}[XY]$ is attained if and only if (X, Y) is comonotonic, i.e. $(X, Y) \sim (F_X^{-1}(U), F_Y^{-1}(U))$. Similarly, the lower bound for $\mathbb{E}[XY]$ is attained if and only if (X, Y) is anti-monotonic, i.e. $(X, Y) \sim (F_X^{-1}(U), F_Y^{-1}(1-U))$.

The following lemma combines special cases of two classical construction results. The Rosenblatt transformation describes a transform of a random vector to iid uniformly distributed random variables (see Rosenblatt 1952). The second result is a special form of the standard recursive construction method for a random vector with given distribution out of iid uniform random variables due to O'Brien (1975), Arjas and Lehtonen (1978) and Rüschendorf (1981).

Lemma A.2 (Construction method) *Let (X, Y) be a random pair and assume that $F_{Y|X=x}(\cdot)$ is continuous $\forall x$. Denote $V = F_{Y|X}(Y)$. Then V is uniformly distributed on $(0, 1)$ and independent of X. It is also increasing in Y conditionally on X. Furthermore, for every variable Z, $(X, F_{Z|X}^{-1}(V)) \sim (X, Z)$.*

For the proof of the first part note that by the continuity assumption on $F_{Y|X=x}$ we get from the standard transformation

$$(V \mid X = x) \sim \left(F_{Y|X=x}(Y) \mid X = x\right) \sim U(0, 1), \quad \forall x.$$

Clearly $V \sim U(0, 1)$. Furthermore, the conditional distribution $F_{V|X=x}$ does not depend on x and thus V and X are independent. For the second part one gets by the usual quantile construction that $F_{Z|X=x}^{-1}(V)$ has distribution function $F_{Z|X=x}$. This implies that $(X, F_{Z|X}^{-1}(V)) \sim (X, Z)$ since both sides have the same first marginal distribution and the same conditional distribution.

Lemma A.3 *Let (X, Y) be jointly normally distributed. Then, conditionally on Y, X is normally distributed and,*

$$\mathbb{E}(X|Y) = \mathbb{E}(X) + \frac{\text{cov}(X, Y)}{\text{var}(Y)}(Y - \mathbb{E}(Y)$$
$$var(X|Y) = (1-\rho^2)\,\text{var}(X).$$

Denote the density of Y by $f_Y(y)$. One has,

$$\int_{-\infty}^{c} e^{a+by} f_Y(y)dy = e^{a+b\mathbb{E}(Y)+\frac{b^2}{2}\text{var}(Y)} \frac{1}{\sqrt{2\pi\,\text{var}(Y)}} \int_{-\infty}^{c} \times e^{-\frac{1}{2}\left(\frac{y-(\mathbb{E}(Y)+b\,\text{var}(Y))}{\sqrt{\text{var}(Y)}}\right)^2} dy.$$

The results in this lemma are well known and we omit its proof. □

A.1. *Proof of proposition 3.1*

Let $U = F_{S_T}(S_T)$ a uniformly distributed variable on $(0, 1)$. Consider a payoff X_T. One has,

$$c_0(X_T) = \mathbb{E}[X_T\xi_T] \ge \mathbb{E}\left[F_{X_T}^{-1}(U)\xi_T\right] = c_0(X_T^*),$$

where the inequality follows from the fact that $F_{X_T}^{-1}(U)$ and ξ_T are anti-monotonic and using the Hoeffding–Fréchet bounds in lemma A.1. Hence, $X_T^* = F^{-1}(F_S(S_T))$ is the cheapest payoff with cdf F. Similarly, the most expensive payoff with cdf F writes as $Z_T^* = F^{-1}(1 - F_S(S_T))$. Since c is the price of a payoff X_T with cdf F, one has

$$c \in [c_0(X_T^*), c_0(Z_T^*)].$$

If $c = c_0(X_T^*)$ then X_T^* is a solution. Similarly, if $c = c_0(Z_T^*)$ then Z_T^* is a solution. Next, let $c \in (c_0(X_T^*), c_0(Z_T^*))$ and define the payoff $f_a(S_T)$ with $a \in \mathbb{R}$,

$$f_a(S_T) = F^{-1}\left[(1 - F_{S_T}(S_T))\mathbb{1}_{S_T \le a} + (F_{S_T}(S_T) - F_{S_T}(a))\mathbb{1}_{S_T > a}\right].$$

Then $f_a(S_T)$ is distributed with cdf F. The price $c_0(f_a(S_T))$ of this payoff is a continuous function of the parameter a. Since $\lim_{a\to 0^+} c_0(f_a(S_T)) = c_0(X_T^*)$ and $\lim_{a\to+\infty} c_0(f_a(S_T)) = c_0(Z_T^*)$, using the theorem of intermediary values for continuous functions, there exists a^* such that $c_0(f_{a^*}(S_T)) = c$. This ends the proof. □

A.2. *Proof of corollary 3.3*

Let $X_T \sim F$ be cost-efficient. Then X_T solves (6) and theorem 3.2 implies that $X_T = F^{-1}(F_{S_T}(S_T))$ almost surely. Reciprocally, let $X_T \sim F$ be increasing in S_T. Then, by our continuity assumption, $X_T = F^{-1}(F_{S_T}(S_T))$ almost surely and thus, X_T is cost-efficient. □

A.3. *Proof of theorem 4.2*

The idea of the proof is very similar to the proof of proposition 3.1. Let U be given by $U = F_{S_T|A_T}(S_T)$. It is uniformly distributed over (0, 1) and independent of A_T (see lemma A.2). Furthermore, conditionally on A_T, U is increasing in S_T. Consider next a payoff X_T and note that $F^{-1}_{X_T|A_T}(U) \sim X_T$. We find that

$$\begin{aligned} c_0(X_T) &= \mathbb{E}[X_T\xi_T] = \mathbb{E}[\mathbb{E}[X_T\xi_T \mid A_T]] \\ &\ge \mathbb{E}\left[\mathbb{E}\left[F^{-1}_{X_T|A_T}(U)\xi_T \,\middle|\, A_T\right]\right] = \mathbb{E}\left[F^{-1}_{X_T|A_T}(U)\xi_T\right], \end{aligned} \tag{A2}$$

where the inequality follows from the fact that $F^{-1}_{X_T|A_T}(U)$ and ξ_T are conditionally (on A_T) anti-monotonic and using (A1) in lemma A.1 for the conditional expectation (conditionally on A_T). Similarly, one finds that

$$c_0(X_T) \le \mathbb{E}\left[F^{-1}_{X_T|A_T}(1-U)\,\xi_T\right].$$

Next we define the uniform (0, 1) distributed variable,

$$g_a(S_T) = (1 - F_{S_T}(S_T))\mathbb{1}_{S_T \le a} + (F_{S_T}(S_T) - F_{S_T}(a))\mathbb{1}_{S_T > a}.$$

We observe that thanks to lemma A.2, $F_{g_a(S_T)|A_T}(g_a(S_T))$ is independent of A_T and also that $f_a(S_T, A_T)$ given as

$$f_a(S_T, A_T) = F^{-1}_{X_T|A_T}(F_{g_a(S_T)|A_T}(g_a(S_T)))$$

is a twin with the desired joint distribution G with A_T. Denote by $X_T^* = F^{-1}_{X_T|A_T}(U)$ and by $Z_T^* = F^{-1}_{X_T|A_T}(1-U)$. Note that $X_T^* = f_0(S_T, A_T)$ and $Z_T^* = f_1(S_T, A_T)$ almost surely. The same discussion as in the proof of proposition 3.1 applies here. When $c = c_0(X_T^*)$ then X_T^* is a twin with the desired properties. Similarly, when $c = c_0(Z_T^*)$ then Z_T^* is a twin with the desired properties. Otherwise, when $c \in (c_0(X_T^*), c_0(Z_T^*))$ then the continuity of $c_0(f_a(S_T, A_T))$ with respect to a ensures that there exists a^* such that $c := c_0(f_{a^*}(S_T, A_T))$. Thus, $f_{a^*}(S_T, A_T)$ is a twin with the desired joint distribution G with A_T and with cost c. This ends the proof. □

A.4. *Proof of theorem 4.3*

Let $0 < t < T$. It follows from lemma A.2 that $F_{S_t|S_T}(S_t)$ is uniformly distributed on (0, 1) and independent of S_T. Let the twin $f(S_t, S_T)$ be given as

$$f(S_t, S_T) := F^{-1}_{X_T|S_T}(F_{S_t|S_T}(S_t)).$$

Using lemma A.2 again, one finds that $(f(S_t, S_T), S_T) \sim (X_T, S_T) \sim G$. This also implies,

$$c_0(f(S_t, S_T)) = \mathbb{E}[f(S_t, S_T)\xi_T] = \mathbb{E}[X_T\xi_T] = c_0(X_T),$$

and this ends the proof. □

A.5. *Proof of theorem 4.4*

It follows from lemma A.2 that $U = F_{S_T|A_T}(S_T)$ is uniformly distributed on (0, 1), stochastically independent of A_T and increasing in S_T conditionally on A_T. Let the twin X_T^* be given as

$$X_T^* = F^{-1}_{X_T|A_T}(U).$$

Invoking lemma A.2 again, $(X_T^*, A_T) \sim (X_T, A_T) \sim G$. Moreover,

$$\begin{aligned} c_0(X_T) = \mathbb{E}[X_T\xi_T] &= \mathbb{E}[\mathbb{E}[X_T\xi_T \mid A_T]] \\ &\ge \mathbb{E}\left[\mathbb{E}\left[F^{-1}_{X_T|A_T}(U)\xi_T \,\middle|\, A_T\right]\right] \\ &= \mathbb{E}\left[F^{-1}_{X_T|A_T}(U)\xi_T\right] = c_0(X_T^*) \end{aligned}$$

where the inequality follows from the fact that $F^{-1}_{X_T|A_T}(U)$ and S_T are conditionally (on A_T) comonotonic and using (A1) in lemma A.1 for the conditional expectation (conditionally on A_T). □

A.6. *Proof of corollary 4.5*

Let us first assume that X_T is a cheapest twin. By theorem 4.4, X_T is (almost surely) equal to X_T^* as defined by (14) which is, conditionally on A_T, increasing in S_T. Reciprocally, we now assume that $X_T = f(S_T, A_T)$ is conditionally on A_T increasing in S_T. Hence, $X_T = F^{-1}_{X_T|A_T}\left(F_{S_T|A_T}(S_T)\right)$ almost surely, which means it is a solution to (13) and thus a cheapest twin. □

Appendix 2. Security design

B.1. *Twin of the fixed strike (continuously monitored) geometric Asian call option*

Expression (11) allows us to find twins satisfying the constraint (19) on the dependence with the benchmark S_T. Using lemma A.3, we find that

$$\ln(S_t/S_0)|\ln(S_T/S_0) \sim \mathcal{N}\left(\frac{t}{T}\ln\left(\frac{S_T}{S_0}\right), \sigma^2 t\left(1 - \frac{t}{T}\right)\right),$$

and thus

$$F_{S_t|S_T}(S_t) = \Phi\left(\frac{\ln\left(\frac{S_t S_0^{\frac{t}{T}-1}}{S_T^{\frac{t}{T}}}\right)}{\sigma\sqrt{\frac{tT-t^2}{T}}}\right).$$

Furthermore, the couple $(\ln(G_T), \ln(S_T))$ is bivariate normally distributed with mean and variance for the marginal distributions that are given as $\mathbb{E}[\ln(G_T)] = \ln S_0 + \left(\mu - \frac{1}{2}\sigma^2\right)\frac{T}{2}$, $\mathrm{var}[\ln(G_T)] = \frac{\sigma^2 T}{3}$ and $\mathbb{E}[\ln(S_T)] = \ln S_0 + \left(\mu - \frac{1}{2}\sigma^2\right)T$, $\mathrm{var}[\ln(S_T)] = \sigma^2 T$. For the correlation coefficient one has $\rho(\ln(S_T), \ln(G_T)) = \frac{\sqrt{3}}{2}$. Applying lemma A.3 Again, one finds that,

$$\ln(G_T)|\ln(S_T) \sim \mathcal{N}\left(\ln\left(S_0^{1/2}S_T^{1/2}\right), \frac{\sigma^2 T}{12}\right), \tag{B1}$$

and thus,

$$F_{G_T|S_T}(x) = \Phi\left(\frac{\ln(x) - \ln\left(S_0^{1/2}S_T^{1/2}\right)}{\frac{\sigma\sqrt{T}}{2\sqrt{3}}}\right).$$

Therefore,

$$F^{-1}_{G_T|S_T}(y) = \exp\left(\ln\left(S_0^{1/2}S_T^{1/2}\right) + \frac{\sigma\sqrt{T}}{2\sqrt{3}}\Phi^{-1}(y)\right).$$

The expression of $R_T(t)$ given in (21) is then straightforward to derive.

For choosing a specific twin among others, we suggest to maximize $\rho(\ln R_T(t), \ln G_T)$. First, we calculate,

$$\operatorname{cov}\left(\ln S_T, \frac{1}{T}\int_0^T \ln(S_s)\,ds\right) = \frac{1}{T}\int_0^T \operatorname{cov}(\ln S_T, \ln(S_s))\,ds = \frac{\sigma^2}{T}\int_0^T (s\wedge T)\,ds = \frac{\sigma^2 T}{2}.$$

Furthermore, by denoting $a = \frac{1}{2} - \frac{1}{2\sqrt{3}}\sqrt{\frac{T-t}{t}}$, $b = \frac{T}{t}\frac{1}{2\sqrt{3}}\sqrt{\frac{t}{T-t}}$ and $c = \frac{1}{2} - \frac{1}{2\sqrt{3}}\sqrt{\frac{t}{T-t}}$, equation (21) may be rewritten as $\ln R_T(t) = a\ln S_0 + b\ln S_t + c\ln S_T$. The covariance being bilinear, one then has,

$$\begin{aligned}\operatorname{cov}(\ln R_T(t), \ln G_T) &= b\operatorname{cov}\left(\ln S_t, \frac{1}{T}\int_0^T \ln(S_s)\,ds\right) \\ &\quad + c\operatorname{cov}\left(\ln S_T, \frac{1}{T}\int_0^T \ln(S_s)\,ds\right) \\ &= \frac{\sigma^2}{2}\left(\frac{T}{2} + \frac{\sqrt{t}\sqrt{T-t}}{2\sqrt{3}}\right).\end{aligned}$$

Denote by $\sigma_{\ln R_T(t)}$ and by $\sigma_{\ln G_T}$ the respective standard deviations. For the correlation, we find that

$$\rho(\ln R_T(t), \ln G_T) = \frac{\operatorname{cov}(\ln R_T(t), \ln G_T)}{\sigma_{\ln R_T(t)}\sigma_{\ln G_T}} = \frac{3}{4} + \frac{\sqrt{3}\sqrt{(T-t)\,t}}{4T}.$$

Hence $\rho(\ln R_T(t), \ln G_T)$ is maximized for $t = \frac{T}{2}$. □

B.2. Twin of the floating strike (continuously monitored) geometric Asian put option

We first recall from equation (B1) that,

$$\ln(G_T)|\ln(S_T) \sim \mathcal{N}\left(\ln\left(S_0^{\frac{1}{2}}S_T^{\frac{1}{2}}\right), \frac{\sigma^2 T}{12}\right).$$

Therefore, $Y_T = (G_T - S_T)^+$ has the following conditional cdf

$$\mathbb{P}(Y_T \le y|S_T = s) = \Phi\left(\frac{\ln(s+y) - \ln\left(S_0^{1/2}s^{1/2}\right)}{\frac{\sigma\sqrt{T}}{2\sqrt{3}}}\right)\mathbb{1}_{y\ge 0}$$

Then

$$F^{-1}_{Y_T|S_T}(z) = \left(S_0^{\frac{1}{2}}S_T^{\frac{1}{2}}e^{\frac{\sigma}{2}\sqrt{\frac{T}{3}}\Phi^{-1}(z)} - S_T\right)^+.$$

Therefore, $F^{-1}_{Y_T|S_T}\left(F_{S_t|S_T}(S_t)\right)$ can then easily be computed and after some calculations it simplifies to (24). □

B.3. Cheapest twin of the floating strike (continuously monitored) geometric Asian put option

Applying lemma A.3, we find

$$\ln(S_T)|\ln(G_T) \sim \mathcal{N}\left(\ln\left(\frac{G_T^{3/2}}{S_0^{\frac{1}{2}}}\right) + \frac{1}{4}\left(\mu - \frac{\sigma^2}{2}\right)T, \frac{\sigma^2 T}{4}\right).$$

Hence,

$$F_{S_T|G_T}(S_T)) = \Phi\left(\frac{\ln\left(\frac{S_T S_0^{\frac{1}{2}}}{G_T^{\frac{3}{2}}}\right) - \left(\mu - \frac{\sigma^2}{2}\right)\frac{T}{4}}{\frac{\sigma\sqrt{T}}{2}}\right). \tag{B2}$$

Furthermore, $Y_T = (G_T - S_T)^+$ has the following conditional cdf,

$$P(Y_T \le y|G_T = g) = \begin{cases} 1 & \text{if } y \ge g, \\ \Phi\left(\frac{\ln\left(\frac{g^{3/2}}{S_0^{\frac{1}{2}}}\right) + \frac{1}{4}\left(\mu - \frac{\sigma^2}{2}\right)T - \ln(g-y)}{\frac{\sigma\sqrt{T}}{2}}\right) & \text{if } 0 \le y \le g, \\ 0 & \text{if } y < 0. \end{cases}$$

Then

$$F^{-1}_{Y_T|G_T}(z) = \left(G_T - \frac{G_T^{\frac{3}{2}}}{S_0^{\frac{1}{2}}}e^{\frac{1}{4}\left(\mu - \frac{\sigma^2}{2}\right)T - \frac{\sigma}{2}\sqrt{T}\Phi^{-1}(z)}\right)^+.$$

Replacing z by the expression (B2) for $F_{S_T|G_T}(S_T))$ derived above, then gives rise to expression (25). □

B.4. Derivation of prices (26) and (27)

B.4.1. Price (26). Let us observe that,

$$(G_T - S_T)^+ = G_T\left(1 - \frac{S_T}{G_T}\right)^+ = S_0 e^Y\left(1 - e^Z\right)^+,$$

where $Z = X - Y$, $Y = \ln\left(\frac{G_T}{S_0}\right)$, $X = \ln\left(\frac{S_T}{S_0}\right)$. We find, with respect to the *risk neutral measure* $\mathbb{Q}$,

$$\begin{aligned}\mathbb{E}_{\mathbb{Q}}\left[(G_T - S_T)^+\right] &= S_0\mathbb{E}_{\mathbb{Q}}\left(\mathbb{E}_{\mathbb{Q}}\left[e^Y|Z\right]\left(1 - e^Z\right)^+\right) \\ &= S_0\mathbb{E}_{\mathbb{Q}}\Big[\Big(e^{\mathbb{E}_{\mathbb{Q}}(Y|Z) + \frac{1}{2}\operatorname{var}_{\mathbb{Q}}(Y|Z)} \\ &\quad - e^{\mathbb{E}_{\mathbb{Q}}(Y|Z) + \frac{1}{2}\operatorname{var}_{\mathbb{Q}}(Y|Z) + Z}\Big)^+\Big].\end{aligned}$$

We now compute (still with respect to $\mathbb{Q}$),

$$\begin{aligned}\mathbb{E}_{\mathbb{Q}}(Y|Z) &= \mathbb{E}_{\mathbb{Q}}(Y) + \frac{\operatorname{cov}_{\mathbb{Q}}(Y,Z)}{\operatorname{var}_{\mathbb{Q}}(Z)}(Z - \mathbb{E}_{\mathbb{Q}}(Z)) \\ &= \left(r - \frac{\sigma^2}{2}\right)\frac{T}{4} + \frac{1}{2}Z\end{aligned}$$

$$\operatorname{var}_{\mathbb{Q}}(Y|Z) = (1-\rho^2)\operatorname{var}_{\mathbb{Q}}(Y) = \frac{3}{4}\frac{\sigma^2 T}{3} = \frac{\sigma^2 T}{4}.$$

Hence,

$$\begin{aligned}\mathbb{E}_{\mathbb{Q}}(G_T - S_T)^+ &= S_0\mathbb{E}_{\mathbb{Q}}\left(e^{r\frac{T}{4} + \frac{1}{2}Z} - e^{r\frac{T}{4} + \frac{3}{2}Z}\right)^+ \\ &= S_0\int_{-\infty}^0 e^{r\frac{T}{4} + \frac{1}{2}Z} f_Z(z)dz \\ &\quad - S_0\int_{-\infty}^0 e^{r\frac{T}{4} + \frac{3}{2}Z} f_Z(z)dz,\end{aligned}$$

where $f_Z(z)$ is now denoting the density of Z under $\mathbb{Q}$. Here Z is normally distributed with parameters $\left(r - \frac{\sigma^2}{2}\right)\frac{T}{2}$ and variance $\frac{\sigma^2 T}{3}$. Hence, taking into account lemma A.3,

$$\begin{aligned}\mathbb{E}_{\mathbb{Q}}(G_T - S_T)^+ &= S_0 e^{r\frac{T}{2} - \sigma^2\frac{T}{12}}\Phi\left(\frac{-\left(r - \frac{\sigma^2}{2}\right)\frac{T}{2} - \frac{\sigma^2 T}{6}}{\sqrt{\frac{\sigma^2 T}{3}}}\right) \\ &\quad - S_0 e^{rT}\Phi\left(\frac{-\left(r - \frac{\sigma^2}{2}\right)\frac{T}{2} - \frac{\sigma^2 T}{2}}{\sqrt{\frac{\sigma^2 T}{3}}}\right)\end{aligned}$$

Choose $f = \frac{-r\frac{T}{2} + \frac{\sigma^2 T}{12}}{\sigma\sqrt{\frac{T}{3}}}$ to obtain (26).

B.4.2. Price (27). One has,

$$\left(G_T - a\frac{G_T^3}{S_T}\right)^+ = G_T\left(1 - a\frac{G_T^2}{S_T}\right)^+ = S_0 e^Y\left(1 - ce^Z\right)^+$$

where $Z = 2Y - X$, $Y = \ln\left(\frac{G_T}{S_0}\right)$, $X = \ln\left(\frac{S_T}{S_0}\right)$, $c = e^{\left(\mu - \frac{\sigma^2}{2}\right)\frac{T}{2}}$. Hence, with respect to the *risk neutral measure* $\mathbb{Q}$,

$$\begin{aligned}\mathbb{E}_{\mathbb{Q}}\left(G_T - a\frac{G_T^3}{S_T}\right)^+ &= S_0\mathbb{E}_{\mathbb{Q}}\left(\mathbb{E}_{\mathbb{Q}}\left(e^Y|Z\right)\left(1 - ce^Z\right)^+\right)\\ &= S_0\mathbb{E}_{\mathbb{Q}}\Big(e^{\mathbb{E}_{\mathbb{Q}}(Y|Z) + \frac{1}{2}\operatorname{var}_{\mathbb{Q}}(Y|Z)}\\ &\quad - ce^{\mathbb{E}_{\mathbb{Q}}(Y|Z) + \frac{1}{2}\operatorname{var}_{\mathbb{Q}}(Y|Z) + Z}\Big)^+.\end{aligned}$$

We now compute,

$$\mathbb{E}_{\mathbb{Q}}(Y|Z) = \left(r - \frac{\sigma^2}{2}\right)\frac{T}{2} + \frac{1}{2}Z \quad \text{and} \quad \operatorname{var}_{\mathbb{Q}}(Y|Z) = \frac{\sigma^2 T}{4}.$$

Hence,

$$\begin{aligned}\mathbb{E}_{\mathbb{Q}}\left(G_T - a\frac{G_T^3}{S_T}\right)^+ &= S_0\mathbb{E}_{\mathbb{Q}}\left(e^{r\frac{T}{2} - \frac{\sigma^2 T}{8} + \frac{1}{2}Z} - ce^{r\frac{T}{2} - \frac{\sigma^2 T}{8} + \frac{3}{2}Z}\right)^+\\ &= S_0\int_{-\infty}^{\ln(c)} e^{r\frac{T}{2} - \frac{\sigma^2 T}{8} + \frac{1}{2}Z} f_Z(z)dz\\ &\quad - S_0 c\int_{-\infty}^{\ln(c)} e^{r\frac{T}{2} - \frac{\sigma^2 T}{8} + \frac{3}{2}Z} f_Z(z)dz,\end{aligned}$$

where $f_Z(z)$ is the density of Z, under $\mathbb{Q}$. Note that Z is normally distributed with parameters 0 and variance $\frac{\sigma^2 T}{3}$. Taking into account lemma A.3,

$$\begin{aligned}&\mathbb{E}_{\mathbb{Q}}\left(G_T - a\frac{G_T^3}{S_T}\right)^+\\ &= S_0 e^{\frac{rT}{2}}\left(\Phi(d)e^{-\frac{\sigma^2 T}{12}} - e^{\frac{\mu T}{2}}\Phi\left(d - \frac{\sigma\sqrt{T}}{\sqrt{3}}\right)\right)\end{aligned}$$

where $d = \frac{-\ln(c) - \frac{\sigma^2 T}{6}}{\sigma\sqrt{\frac{T}{3}}} = \frac{\frac{\sigma^2 T}{12} - \mu\frac{T}{2}}{\sigma\sqrt{\frac{T}{3}}}$. □

Appendix 3. Portfolio management

C.1. Proof of theorem 6.2

Let $H_T = \mathbb{E}(\xi_T|Z_T) = \varphi(Z_T)$ and let $\hat{\varphi}$ denote the projection of φ on the cone $M_\downarrow$ defined as in (33) with respect to $L^2(\lambda_{[0,1]})$. Then we define $\hat{X}_T$ and $k(\cdot)$ by

$$u'(\hat{X}_T) := \lambda\hat{\varphi}(Z_T),$$

i.e. $\hat{X}_T = (u')^{-1}(\lambda\hat{\varphi}(Z_T)) =: k(Z_T)$ with λ such that $\mathbb{E}[\xi_T\hat{X}_T] = \mathbb{E}[\varphi(Z_T)k(Z_T)] = \int_0^1 \varphi(t)k(t)dt = \varphi\cdot k = W_0$. By definition, $\hat{X}_T$ is increasing in Z_T since $(u')^{-1}$ is decreasing and $\hat{\varphi}$ is decreasing (it belongs to $M_\downarrow$). As a consequence $\hat{X}_T$ is increasing in S_T, conditionally on A_T. For any $Y_T = h(Z_T)$ with a increasing function h, we have by concavity of u

$$u(Y_T) - u(\hat{X}_t) \le u'(\hat{X}_T)(Y_T - \hat{X}_T) = \lambda\hat{\varphi}(Z_T)(h(Z_T) - k(Z_T)).$$

Thus, we obtain

$$\mathbb{E}[u(Y_T)] - \mathbb{E}[u(\hat{X}_T)] \le \lambda\int_0^1 \hat{\varphi}(t)(h(t) - k(t))dt = \lambda\hat{\varphi}\cdot(h - \Psi(\hat{\varphi})), \tag{C1}$$

where $\Psi(\hat{\varphi}) = (u')^{-1}(\lambda\hat{\varphi}) = k$ is increasing and $\Psi(t) = (u')^{-1}(\lambda t)$ is decreasing.

Now we use some properties of isotonic approximations (see Barlow *et al.* 1972) and obtain

$$\begin{aligned}&\hat{\varphi}\cdot(h - \Psi(\hat{\varphi}))\\ &= \hat{\varphi}\cdot((-\Psi)(\hat{\varphi}) - (-h))\\ &= \varphi\cdot(-\Psi)(\hat{\varphi}) - \hat{\varphi}\cdot(-h)\\ &\quad \text{(see theorem 1.7 in Barlow et al. (1972))}\\ &= \varphi\cdot(-h) - \hat{\varphi}\cdot(-h) \ \text{ both claims have price } W_0\\ &= (\varphi - \hat{\varphi})\cdot(-h) \le 0\end{aligned}$$

by the projection equation (see theorem 7.8 in Barlow *et al.* (1972)) using that $-h \in M_\downarrow$. As a result we obtain from (C1) that

$$\mathbb{E}[u(Y_T)] \le \mathbb{E}[u(\hat{X}_T)],$$

i.e. $\hat{X}_T$ is an optimal claim. □

C.2. Proofs of equations (36) and (38) in the example of section 6.1

We apply theorem 6.1 to an investor with a power-utility. Then,

$$X_T^\star(\eta) = (u')^{-1}(\lambda\xi_T) = (\lambda\xi_T)^{-\frac{1}{\eta}} \tag{C2}$$

where λ is chosen to meet the budget constraint, i.e.

$$\mathbb{E}[\xi_T(u')^{-1}(\lambda\xi_T)] = \mathbb{E}\left[\xi_T(\lambda\xi_T)^{-\frac{1}{\eta}}\right] = \lambda^{-\frac{1}{\eta}}\mathbb{E}\left[\xi_T^{1-\frac{1}{\eta}}\right] = W_0 \tag{C3}$$

Since $\xi_T = \exp\left\{-rT - \frac{1}{2}\theta^2 T - \theta Z_T\right\}$, we find that $\lambda^{-\frac{1}{\eta}} = W_0\exp\left\{-r\left(\frac{1-\eta}{\eta}\right)T - \frac{1}{2}\theta^2 T\left(\frac{1-\eta}{\eta}\right)\frac{1}{\eta}\right\}$ and

$$\begin{aligned}X_T^\star(\eta) &= (\lambda\xi_T)^{-\frac{1}{\eta}}\\ &= W_0 e^{-r\left(\frac{1-\eta}{\eta}\right)T - \frac{1}{2}\theta^2 T\left(\frac{1-\eta}{\eta}\right)\frac{1}{\eta} - \frac{1}{\eta}\left[\frac{\theta}{\sigma}\left(\mu - \frac{\sigma^2}{2}\right)T - \left(r + \frac{\theta^2}{2}\right)T\right]}\\ &\quad \times\left(\frac{S_T}{S_0}\right)^{\frac{\theta}{\sigma\eta}},\end{aligned}$$

which can be simplified to find (36).

Next, we apply theorem 6.2 with $A_T = S_t$, for some t such that $t < T$. From lemma A.3, we know

$$\ln(S_T)|\ln(S_t) \sim \mathcal{N}\left(\ln(St) + \left(\mu - \frac{\sigma^2}{2}\right)(T - t), \sigma^2(T - t)\right)$$

so that

$$F_{S_T|S_t}(S_T) = \Phi\left(\frac{\ln\left(\frac{S_T}{S_t}\right) - \left(\mu - \frac{\sigma^2}{2}\right)(T - t)}{\sigma\sqrt{T - t}}\right).$$

Because C is a Gaussian copula, one has

$$C_{1|S_t}(x) = \Phi\left[\frac{\Phi^{-1}[x] - \rho\left(\frac{\ln\left(\frac{S_t}{S_0}\right) - \left(\mu - \frac{\sigma^2}{2}\right)t}{\sigma\sqrt{t}}\right)}{\sqrt{1 - \rho^2}}\right]$$

and

$$C_{1|S_t}^{-1}(y) = \Phi\left[\sqrt{1 - \rho^2}\Phi^{-1}[y] + \rho\left(\frac{\ln\left(\frac{S_t}{S_0}\right) - \left(\mu - \frac{\sigma^2}{2}\right)t}{\sigma\sqrt{t}}\right)\right].$$

This implies

$$\zeta_T = C_{1|S_t}^{-1}(F_{S_T|S_t}(S_T)) = \Phi[\varpi_T],$$

where ϖ_T is a function of S_T and S_t given by

$$\varpi_T = \sqrt{1-\rho^2}\left(\frac{\ln\left(\frac{S_T}{S_t}\right) - \left(\mu - \frac{\sigma^2}{2}\right)(T-t)}{\sigma\sqrt{T-t}}\right) + \rho\left(\frac{\ln\left(\frac{S_t}{S_0}\right) - \left(\mu - \frac{\sigma^2}{2}\right)t}{\sigma\sqrt{t}}\right). \quad \text{(C4)}$$

Since $\xi_T = \alpha_T\left(\frac{S_T}{S_0}\right)^{-\beta}$ where $\alpha_T = \exp\left(\frac{\theta}{\sigma}\left(\mu - \frac{\sigma^2}{2}\right)T - \left(r + \frac{\theta^2}{2}\right)T\right)$, $\beta = \frac{\theta}{\sigma}$ and $\theta = \frac{\mu - r}{\sigma}$ (from (4)), one has

$$H_T = \mathbb{E}(\xi_T|\zeta_T) = \mathbb{E}(\xi_T|\varpi_T) = \delta e^{-\beta\,\mathrm{cov}(\ln(S_T),\varpi_T)\varpi_T},$$

for some $\delta > 0$ and we find

$$H_T = \delta e^{-\theta\left(\rho\sqrt{t}+\sqrt{(1-\rho^2)(T-t)}\right)\varpi_T}.$$

Note that conditions on the correlation coefficient imply that H_T is decreasing in ϖ_T and thus H_T is decreasing in Z_T. The optimal contract thus writes as

$$\hat{X}_T := (u')^{-1}\left(\lambda e^{-\theta\left(\rho\sqrt{t}+\sqrt{(1-\rho^2)(T-t)}\right)\varpi_T}\right), \quad \text{(C5)}$$

where λ is chosen to meet the budget constraint.

When the investor has a power-utility, i.e. $u(x) = \frac{x^{1-\eta}}{1-\eta}$ so that $(u')^{-1}(x) = x^{-\frac{1}{\eta}}$ we find that equation (C5) reads as

$$\hat{X}_T(\eta) := \lambda^{-\frac{1}{\eta}} e^{\frac{1}{\eta}\theta\left(\rho\sqrt{t}+\sqrt{(1-\rho^2)(T-t)}\right)\varpi_T} \quad \text{(C6)}$$

and the budget constraint $\left(\text{i.e. } \mathbb{E}\left[\xi_T \hat{X}_T(\eta)\right] = W_0\right)$ requires that

$$\mathbb{E}\left[e^{-rT}e^{-\frac{\theta^2}{2}T-\theta Z_T}\lambda^{-\frac{1}{\eta}} \exp\left[\frac{1}{\eta}\theta\left(\rho\sqrt{t}+\sqrt{(1-\rho^2)(T-t)}\right)\varpi_T\right]\right] = W_0,$$

where we have used the expressions for ξ_T and $\hat{X}_T(\eta)$. We find that

$$\lambda^{-\frac{1}{\eta}} = W_0 e^{rT} e^{\theta^2\left(\rho\sqrt{t}+\sqrt{(1-\rho^2)(T-t)}\right)^2\left(\frac{1}{\eta}-\frac{1}{2\eta^2}\right)}.$$

The optimal solution is then derived by using this expression into (C6). □

C.3. Proof of proposition 6.6

Assume that there exists an optimal solution to the target probability maximization problem. It is a maximization of a law-invariant objective and therefore, it is path-independent. Denote it by $X_T^* := f^*(S_T)$. Define $A_0 = \{x \mid f^*(x) = 0\}$, $A_1 = \{x \mid f^*(x) = b\}$, $A_2 = \{x \mid f^*(x) \in]0, b[\}$ and $A_3 = \{x \mid f^*(x) > b\}$. We show that $\mathbb{P}(S_T \in A_0 \cup A_1) = 1$ must hold. Assume $\mathbb{P}(S_T \in A_0 \cup A_1) < 1$ so that $\mathbb{P}(S_T \in A_2 \cup A_3) > 0$. Define

$$Y = \begin{cases} f^*(S_T) & \text{for } S_T \in A_0 \cup A_1, \\ 0 & \text{for } S_T \in A_2, \\ b & \text{for } S_T \in A_3. \end{cases}$$

Then we observe that $Y = f^*(S_T)$ on $A_0 \cup A_1$ and $Y < f^*(S_T)$ on $A_2 \cup A_3$. Since $\mathbb{P}(S_T \in A_2 \cup A_3) > 0$ also $\mathbb{Q}(S_T \in A_2 \cup A_3) > 0$ because $\mathbb{P}$ and the risk neutral probability $\mathbb{Q}$ are equivalent. Hence $c_0(Y) < W_0$. Next we define $Z = b\mathbb{1}_{S_T \in C} + Y$ where we have chosen $C \subseteq A_2 \cup A_0$ such that $c_0(b\mathbb{1}_{S_T\in C}) = W_0 - c_0(Y)$. Since $\mathbb{P}(S_T \in C) > 0$ one has that $\mathbb{P}(Z \geq b) > \mathbb{P}(Y \geq b) = \mathbb{P}(f^*(S_T) \geq b)$. Hence Z contradicts the optimality of $f^*(S_T)$. Therefore, $\mathbb{P}(S_T \in A_0 \cup A_1) = 1$. Hence $f^*(S_T)$ can take only the values 0 or b. Since it is increasing in S_T almost surely (by cost-efficiency) it must write as

$$f^*(S_T) = b\mathbb{1}_{S_T > a},$$

where a is chosen such that the budget constraint is satisfied. □

C.4. Proof of theorem 6.7

The (random) target probability maximization problem is given as

$$\max_{X_T \geq 0, c_0(X_T) = W_0} \mathbb{P}[X_T \geq B].$$

Assume that there exists an optimal solution X_T^* to this optimization problem. There are three steps in the proof.

(1) The optimal payoff is of the form $f(S_T, B)$.
(2) The optimal payoff is of the form $B\mathbb{1}_{h(S_T,B)\in A}$.
(3) The optimal payoff is of the form $B\mathbb{1}_{B\xi_T<\lambda^*}$ for $\lambda^* > 0$.

Step 1 We observe that X_T^* has some joint distribution G with B. Theorem 4.2 implies there exists a twin $f(B, S_T)$ such that $(f(B, S_T), B_T) \sim (X_T^*, B) \sim G$ and $c_0(f(B, S_T)) = c_0(X_T^*) = W_0$. Therefore, $\mathbb{P}(f(B, S_T) \geq B) = \mathbb{P}(X_T^* \geq B)$ and $\mathbb{P}(f(B, S_T) \geq 0) = \mathbb{P}(X_T^* \geq 0) = 1$. Thus, $f(B, S_T)$ is also an optimal solution.
Step 2 This is similar to the proof of proposition 6.6, applied conditionally on B. Define the sets $A_0 = \{s, f(B, s) = 0\}$, $A_1 = \{s, f(B, s) = B\}$, then $\mathbb{P}(S_T \in A_0 \cup A_1|B) = 1$ and Therefore, $\mathbb{P}(S_T \in A_0 \cup A_1) = 1$. Thus, there exists a set A and a function h such that

$$f(B, S_T) = B\mathbb{1}_{h(S_T,B)\in A}.$$

Step 3 Define $\lambda > 0$ such that

$$\mathbb{P}(h(S_T, B) \in A) = \mathbb{P}(B\xi_T < \lambda).$$

Observe that $\mathbb{1}_{h(S_T,B)\in A}$ and $\mathbb{1}_{B\xi_T<\lambda}$ have the same distribution and that in addition, $B\xi_T$ is anti-monotonic with $\mathbb{1}_{B\xi_T<\lambda}$. Therefore by applying lemma A.1, one has that

$$c_0(B\mathbb{1}_{B\xi_T<\lambda}) = \mathbb{E}[B\xi_T\mathbb{1}_{B\xi_T<\lambda}] \leq \mathbb{E}[B\xi_T\mathbb{1}_{h(S_T,B)\in A}]$$

and therefore the optimum must be of the form $B\mathbb{1}_{B\xi_T<\lambda^*}$ where $\lambda^* > \lambda$ is determined such that $c_0(B\mathbb{1}_{B\xi_T<\lambda^*}) = W_0$. □

C.5. Proof of theorem 6.8

The target probability maximization problem is given by

$$\max_{\substack{X_T \geq 0,\ c_0(X_T)=W_0, \\ \mathcal{C}_{(X_T,A_T)}=C}} \mathbb{P}[X_T \geq b]$$

Assume that there exists an optimal solution X_T^* to this optimization problem. There are three steps in the proof.

(1) The optimal payoff is of the form $f(S_T, A_T)$.
(2) The optimal payoff is of the form $b\mathbb{1}_{h(S_T,A_T)\in A}$.
(3) The optimal payoff is of the form $A_T\mathbb{1}_{Z_T>\lambda^*}$ for $\lambda^* > 0$.

Step 1 We observe that X_T^* has some joint distribution G with A_T. Theorem 4.2 implies there exists a twin $f(S_T, A_T)$ such that $(f(S_T, A_T), A_T) \sim (X_T^*, A_T) \sim G$ and $c_0(f(S_T, A_T)) = c_0(X_T^*) = W_0$. Therefore, $\mathbb{P}(f(S_T, A_T) \geq b) = \mathbb{P}(X_T^* \geq b)$ and $P(f(S_T, A_T) \geq 0) = \mathbb{P}(X_T^* \geq 0) = 1$. Thus, $f(S_T, A_T)$ is also an optimal solution.
Step 2 This is similar to the proof of proposition 6.6. Define the sets $A_0 = \{(s, t), f(s, t) = 0\}$, $A_1 = \{(s, t), f(s, t) = b\}$, then $\mathbb{P}(S_T \in A_0 \cup A_1) = 1$. Thus, there exists a set A and a function h such that

$$f(S_T, A_T) = b\mathbb{1}_{h(S_T,A_T)\in A}.$$

Step 3 Define $\lambda > 0$ such that

$$\mathbb{P}(h(S_T, A_T) \in A) = \mathbb{P}(Z_T > \lambda).$$

Observe that $b\mathbb{1}_{h(S_T,A_T)\in A}$ and $b\mathbb{1}_{Z_T>\lambda}$ have the same joint distribution G with distribution A_T. Therefore, theorem 4.4 shows that,

$$c_0(b\mathbb{1}_{Z_T>\lambda}) \leq c_0(b\mathbb{1}_{h(S_T,A_T)\in A}).$$

Hence, $b\mathbb{1}_{Z_T>\lambda^*}$ where λ^* such that $c_0(b\mathbb{1}_{Z_T>\lambda^*}) = W_0$ is the optimum. □

C.6. *Proof of formula (45)*

We know that $b\mathbb{1}_{Z_T>\lambda^*}$ where λ^* is such that $c_0(b\mathbb{1}_{Z_T>\lambda^*}) = W_0$ is the optimal solution. We find that

$$\begin{aligned} Z_T &= C^{-1}_{1|S_t}(F_{S_T|S_t}(S_T)) \\ &= \Phi\left[\sqrt{1-\rho^2}\left(\frac{\ln\left(\frac{S_T}{St}\right) - \left(\mu - \frac{\sigma^2}{2}\right)(T-t)}{\sigma\sqrt{T-t}}\right)\right. \\ &\quad \left. +\rho\left(\frac{\ln\left(\frac{S_t}{S_0}\right) - \left(\mu - \frac{\sigma^2}{2}\right)t}{\sigma\sqrt{t}}\right)\right]. \end{aligned}$$

It is then straightforward that $X_T^* = b\mathbb{1}_{\{S_t^\alpha S_T>\lambda^*\}}$ is the optimal solution, with α and λ given by

$$\begin{aligned} \alpha &= \sqrt{\frac{T-t}{t(1-\rho^2)}}\rho - 1 \\ \lambda &= S_0^{1+\alpha}\exp\left(\left(r - \frac{\sigma^2}{2}\right)(\alpha t + T)\right. \\ &\quad \left. -\sigma\sqrt{(\alpha+1)^2 t + (T-t)}\,\Phi^{-1}\left(\frac{W_0 e^{rT}}{b}\right)\right). \end{aligned}$$

□