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A closed form solution for pricing defaultable bonds

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Abstract

Cathcart and El-Jahel [Journal of Fixed Income 8 (1998)] have formalized the "signaling approach" for modeling the default risk of some risky bonds. Their pricing formula requires a numerical method to invert the Laplace transform of the default probability. This letter rather provides a closed form formula based on standard results of the theory of exotic barrier options. One verifies that the original numerical method implemented by Cathcart and El-Jahel [Journal of Fixed Income 8 (1998)] does not lead to significant computational errors.

Keywords: Bond pricing; Default risk; Signaling approach

Introduction

The signaling approach, formulated by Cathcart and El-Jahel (1998), assumes that, for any issuer (corporate or country), there exists a signaling variable which captures the factors that affect its default probability. Default occurs when an hypothetical process hits a lower constant default barrier. This setting generalizes the approach of Longstaff and Schwartz (1995) whose signaling variable is the value of the firm's assets.

The signaling approach is especially relevant for sovereign issuers and also municipalities. Indeed, in these cases, no underlying asset is clearly identifiable. In fact, a similar framework had implicitly been considered by Claessens and Pennacchi (1996) to price Mexican Brady bonds. The signaling variable had been identified there as an hypothetical "measure of repayment capacity" and the associated default threshold as zero. Hui and Lo

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(2002) also develop a signaling approach by suggesting that an exchange rate could be useful to price Korean and Brazilian Government bonds.

Cathcart and El-Jahel (1998) manage to compute a risk neutral default probability based on a signaling variable. From a technical viewpoint, they first demonstrate that the Laplace transform of this default probability is the solution of a particular partial differential equation. Then, they solve it analytically. Finally, they compute numerically the default probability by inverting the Bromwich integral with the Gaussian quadrature method of Piessens (1969).[1]

The present letter demonstrates that the problem admits a closed-form solution. It is shown that the (default) probability searched by Cathcart and El-Jahel (1998) is similar to some well-known in the theory of barrier options pioneered by Merton (1973).[2] With an analytical formula, one avoids the coding of a numerical procedure. One can also evaluate the Piessens' numerical method used by the authors. Numerical experiments reveal that this Gaussian quadrature approach is especially relevant for the considered Laplace transform but also that it is not necessary the case.

The rest of this paper is organized as follows. Section 2 shortly presents Cathcart and El-Jahel (1998) model. Section 3 provides its analytical solution and offers some arguments. Section 3 evaluates the Piessens (1969) Gaussian quadrature technique.

1. The original model

For reader's convenience, one shortly presents the framework of Cathcart and El-Jahel (1998) for pricing defaultable bonds. Cathcart and El-Jahel (1998) present a credit risk model where there are two underlying state variables: the instantaneous interest rate and the signaling variable. Following Cox et al. (1985), Cathcart and El-Jahel (1998) assume that (1) the instantaneous interest rate governs the term structure behavior and that (2) it follows a square-root process. The dynamics of the instantaneous interest rate r under the risk neutral probability is thus described by

$$\mathrm{d}r_t = \kappa_r(\theta_r - r_t)\,\mathrm{d}t + \sigma_r\sqrt{r_t}\,\mathrm{d}W_t^r, \tag{1}$$

where $W^r = (W_t^r)_t$ is a Brownian motion. Hence, the interest rate level impacts on its volatility, it also converges instantaneously to its long-run mean θ_r with a coefficient speed κ_r. The associated condition $2\kappa_r\theta_r \geqslant \sigma_r^2$ ensures that zero is not attainable. Any riskless bond promising an amount of B at maturity T is then valued at time t by

$$p_0(r_t, t, T) = B \cdot \mathrm{e}^{C(T-t)-D(T-t)r_t},$$

[1] A natural starting point for the numerical inversion of a Laplace transform is indeed the so-called Bromwich integral. See the text below for some details and Doetsch (1974, Theorem 24.4, p. 257). A recent and valuable reference is Abate et al. (1999).

[2] One refers to Reiner and Rubinstein (1991a, b) for a simple exposition of barrier options. Ingersoll (2000) highlights the key role of digital options for pricing complex financial contracts.

where

$$D(\tau) = D(\zeta,\tau)W^{-1}(\zeta,\kappa_r,\tau), \qquad C(\tau) = -\frac{\kappa_r\theta_r}{\sigma_r^2}\big[(\zeta-\kappa_r)\tau + 2\ln W(\zeta,\kappa_r,\tau)\big]$$

with

$$D(a,\tau) = \frac{1-\mathrm{e}^{-a\tau}}{a}, \qquad W(\zeta,\kappa_r,\tau) = 1 - \frac{1}{2}(\zeta-\kappa_r)D(\zeta,\tau), \quad \text{and}$$
$$\zeta^2 = \kappa_r^2 + 2\sigma_r^2.$$

$\tau = T - t$ is the time to maturity.

Cathcart and El-Jahel (1998) then introduce a signaling variable X which captures factors affecting the probability of default. It could be some macro-economic variables such as the GDP growth rate, the long-term interest rate, or a specific foreign exchange rate. Cathcart and El-Jahel (1998) assume that the risk neutral behavior of X is well-described by

$$\mathrm{d}X_t = mX_t\,\mathrm{d}t + \sigma_X X_t\,\mathrm{d}W_t^X, \tag{2}$$

where m is a constant drift and σ_X a constant volatility. $W^X = (W_t^X)_t$ is a Brownian motion whose increments $((\mathrm{d}W_t^X)_t)$ are supposed uncorrelated with those of the instantaneous interest rate process. Cathcart and El-Jahel (1998) argue that the constant risk neutral drift is justified since X is not a value process. They also claim that the no-correlation assumption is in line with many reduced-form models. As soon as the signaling variable reaches a lower threshold value X_1, default of the underlying issuer is declared on all its obligations. Cathcart and El-Jahel (1998) assume that "*bondholders receive* $1-\delta$ *default-free discount bonds*." In this recovery scheme, the write down δ is supposed constant and exogenously available.

Denoting $p(r_t, X_t, t, T)$ the defaultable bond, Cathcart and El-Jahel (1998) have shown that p must satisfy the fundamental partial differential equation:

$$rp + \frac{\partial p}{\partial\tau} = \frac{1}{2}\sigma_r^2 r\frac{\partial^2 p}{\partial r^2} + \frac{1}{2}\sigma_X^2 X\frac{\partial^2 p}{\partial X^2} + \kappa_r(\theta_r - r)\frac{\partial p}{\partial r} + mX\frac{\partial p}{\partial X} \tag{3}$$

subject to some appropriate boundary conditions. Guessing a solution of the form

$$p(r_t, X_t, t, T) = p_0(r_t, t, T) - \delta p_0(r_t, t, T)f(X_t, \tau), \tag{4}$$

where $p_0(r_t, t, T)$ is a default-free zero-coupon bonds and $\tau = T - t$, they demonstrate that the default probability f must solve

$$\frac{1}{2}\sigma_X^2 X\frac{\partial f}{\partial X^2} + mX\frac{\partial f}{\partial X} = \frac{\partial f}{\partial\tau} \tag{5}$$

with $f(X,0) = 0$, $f(X_1,\tau) = 1$, $f(\infty,\tau) = 0$. The Laplace transform associated to f $(F(q) = \int_0^\infty \mathrm{e}^{-q\tau} f(X,\tau)\,\mathrm{d}\tau)$ then verifies a partial differential equation which can be solved analytically. The authors find that

$$F(q) = \frac{1}{q}\frac{X}{X_1}^{\alpha_1+\sqrt{\alpha_2 q+\alpha_3}}$$

for some $\alpha_1, \alpha_2, \alpha_3$. In order to invert this Laplace transform (i.e., compute $\int_0^\infty e^{q\tau} F(q)\,dq \equiv f(X,\tau)$), they suggest to compute the so-called Bromwich integral given by

$$f(X,\tau) = \frac{1}{2i\pi} \int_{c-i\infty}^{c+i\infty} e^{q\tau} q^{-1} e^{-(\mu+\sqrt{\mu^2+2\sigma_X^2 q})/\sigma_X^2 \ln(X/X_1)}\,dq, \tag{6}$$

where $\mu = m - \frac{1}{2}\sigma_X^2$ and c is chosen arbitrarily but larger than any real part of the complex singularities of F. They note however that Eq. (6) requires some numerical computation.

The following section proposes a closed-form formula that involves no more than the standard Gaussian cumulative density function. The numerical computation of the above equation may thus be avoided.

2. The analytical pricing formula

This section demonstrates that the risk neutral default probability is given by

$$f(X,\tau) = N\left[-\frac{\ln(X/X_1)+\mu\tau}{\sigma_X\sqrt{\tau}}\right] + \left(\frac{X}{X_1}\right)^{-(2\mu)/\sigma_X^2} N\left[-\frac{\ln(X/X_1)-\mu\tau}{\sigma_X\sqrt{\tau}}\right], \tag{7}$$

where $\mu = m - \frac{1}{2}\sigma_X^2$. Equivalent to Eq. (6), this analytical expression of the default probability nests components that are simple to compute. These are key blocks of the theory of option pricing. Some arguments are considered now.

A closer look at the underlying assumptions provides the first way to prove Eq. (7). In presence of stochastic interest rates, the risky bond is indeed valued by

$$p(r_t, X_t, t, T) = p_0(r_t, t, T) - \delta p_0(r_t, t, T) Q^T[\tau < T], \tag{8}$$

where the default probability $Q^T(\tau < T)$ is the probability that X reaches a threshold (X_1) before maturity under the forward neutral measure. Under the specific assumptions of Cathcart and El-Jahel (1998), this latter probability may be further simplified for a couple of reasons. On one hand, the risk neutral drift of the signaling process is constant and does not depend on the interest rate behavior. On the other hand, the two Brownian motions are assumed uncorrelated. As a result, the signaling variable (i.e., the default event process) and the interest rates are independent. One therefore has

$$Q^T[\tau < T] = Q[\tau < T] \tag{9}$$

with Q the risk neutral probability.[3] The dynamics of X under Q is known and described in Eq. (2). The probability that X reaches a threshold (X_1) before maturity may then be found by standard results of stochastic calculus and the Eq. (7) above is obtained.

[3] On the one hand, by using the concept of forward neutral probability, one finds

$$E^Q\left[e^{-\int_0^T r_s\,ds} 1_{\{\exists t \leqslant T:\, X_t < X_1\}}\right] = p_0(0,T) E^{Q^T}\left[1_{\{\exists t \leqslant T:\, X_t \leqslant X_1\}}\right] = p_0(0,T) Q^T[\exists t \leqslant T:\ X_t \leqslant X_1].$$

A financial interpretation of the risk neutral default probability or equivalently of the above partial differential equation suggests a second way to prove Eq. (7). Both describe a financial contract that pays one unit of cash at T if the underlying variable X reaches the barrier X_1 before T. This is exactly what the European Down-and-In Cash (at expiry)-or-Nothing options promise, if the amount of cash at expiry is unity.[4] Readers familiar with barrier options may also recognize one minus the (unit) rebate of a standard Down-and-In barrier option.[5] The theory of barrier options is therefore relevant. It provides the analytical Eq. (7). As a by-product of this derivation, one can evaluate any numerical method used to compute Eq. (6).

3. Evaluating the Piessens Gaussian quadrature method

Although several methods are available for finding the inverse Laplace transform of F ($\int_0^\infty e^{qt} F(X,q)\,dq = f(X,t)$), Cathcart and El-Jahel (1998) recall that the best known approach for deriving values of $f(X,t)$ from values of $F(q)$ is the numerical evaluation of the Bromwich integral. This integral is given by

$$f(X,\tau) = \frac{1}{2i\pi} \int_{c-i\infty}^{c+i\infty} e^{q\tau} F(q)\,dq,$$

where c is chosen so that the line $Re(p) = c$ lies to the right of all singularities of $F(q)$, but is otherwise arbitrary. Trying different techniques, they finally adopt a Gaussian quadrature method of Piessens (1969) because "*it gives good results and accuracy*." Gaussian quadrature methods approximate the integral (or a modification of it) by a finite summation of the form

$$f(X,\tau) \approx \sum_{k=1}^{N} \omega_k H(\alpha_k), \tag{10}$$

where some optimal weights ω_k, abscissas α_k, N, and the associated function H must be found. The Piessens' method is just one method to compute these elements. The optimal

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See Geman et al. (1995) for more details on the changes of probability measure. On the other hand, the independence leads to

$$E^Q\big[e^{-\int_0^T r_s\,ds} 1_{\{\exists t \leqslant T:\, X_t < X_1\}}\big] \overset{\text{ind.}}{=} E^Q\big[e^{-\int_0^T r_s\,ds}\big] E^Q[1_{\{\exists t \leqslant T:\, X_t \leqslant X_1\}}]$$
$$= p_0(0,T)\,Q[\exists t \leqslant T:\, X_t \leqslant X_1].$$

[4] Down-and-In Cash (at expiry)-or-Nothing options are exotic barrier contingent claims (expiring at T) that pay K unit(s) of cash at T if a lower threshold is touched by the underlying asset before T (and nothing otherwise). At T, its pay-off is also K minus the pay-off of a Down-and-Out Cash-or-Nothing option. See Reiner and Rubinstein (1991b) for further discussion.

[5] Equivalently, this is the (unit) rebate of a *special* Down-and-Out barrier option that pays one unit of cash at T if the lower threshold is reached by the underlying asset before. Recall that standard rebates for Down-and-Out options are used to be paid as the barrier is knocked. See Reiner and Rubinstein (1991a).

weights, abscissas, and number of summation may be found in Piessens (1971). Piessens (1971) informs that this approximate formula (for the inversion of the Laplace transform F) is exact whenever $F(q)$ is a linear combination of q^{k-s}, $k = 0, 1, \ldots, 2N - 1$, with s an arbitrary positive real number. This is however not the case in our context.

Numerical experiments are then run. All of them conclude that Piessens' method implemented by Cathcart and El-Jahel (1998) does not lead to significant computational

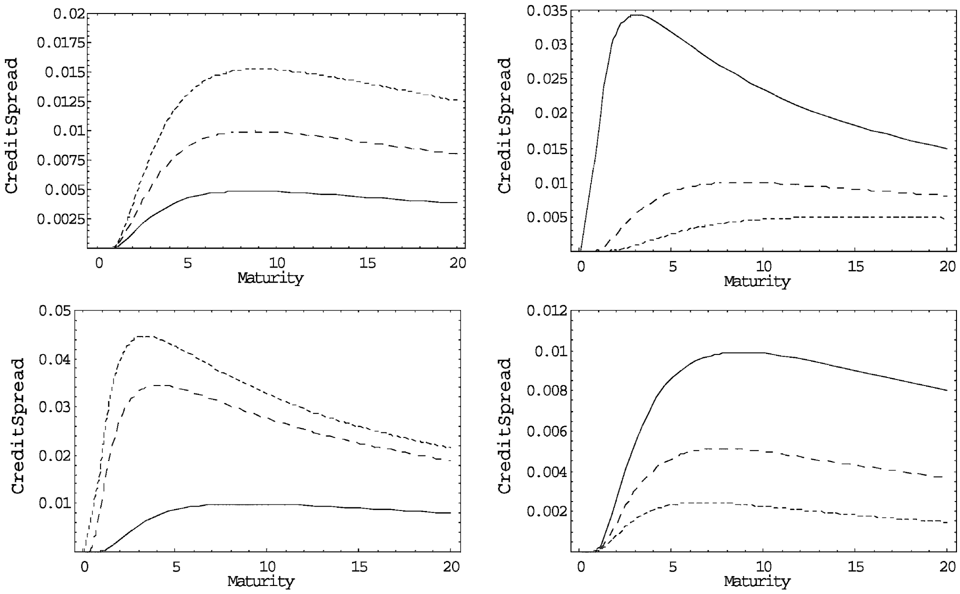

Fig. 1. Term structures of credit spreads. Base case parameters are $\alpha = 4\%$, $\sigma_X = 20\%$, $X/X_1 = 200\%$. Credit spreads are plotted for different levels of loss given default ($\delta = 25$, 50, 75%) in the upper left graph, for different creditworthiness ratios ($X/X_1 = 150$, 200, 250%) in the upper right graph, for different variances ($\sigma_X^2 = 4$, 9, 11%) in the lower left graph, and for different risk neutral drifts ($m = 4$, 7, 10%) in the lower right graph.

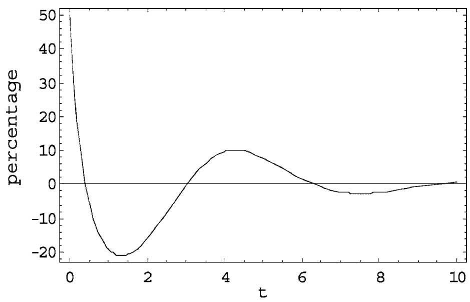

Fig. 2. Percentage of computational errors for a classical test function. This figure plots the computational errors obtained by using the Gaussian quadrature method of Piessens to invert $F(q) = 1/(q + 0.5) + 1/q^2 + 1/(1 + (q + 0.2)^2)$. The function F is known to be the Laplace transform of $f(t) = \mathrm{e}^{-0.5t} + t + \mathrm{e}^{-0.2t} \sin t$. Computational errors are defined as $100((\tilde{f}(t) - f(t))/f(t))$ where $\tilde{f}$ is the numerical approximate of $\int_0^\infty \mathrm{e}^{qt} F(X, q)\,\mathrm{d}q$, i.e., the Piessens estimator of f.

errors. To illustrate this, Fig. 1 plots term structures of credit spreads ($-\frac{1}{\tau}\ln[1 - \delta.f(X,\tau)]$) obtained by using our formula and the value of parameters considered by Cathcart and El-Jahel (1998). These graphs are undoubtedly similar to that of the above authors (in their Figs. 4–7). Computation errors appear to be far less than 1%! It is important to claim that this could have not been the case. Indeed, as suggested above, there exist some Laplace transforms whose inverse is not accurately computed by the considered numerical method. To illustrate this, Fig. 2 graphs the percentage of computational errors of a classical test function ($F(q) = 1/(q+0.5) + 1/q^2 + 1/(1+(q+0.2)^2)$) whose analytical inverse is known to be $f(t) = \mathrm{e}^{-0.5t} + t + \mathrm{e}^{-0.2t}\sin t$. One can see that the errors can be as large as 50%.

4. Conclusion

The signaling approach, formulated by Cathcart and El-Jahel (1998) is especially relevant for pricing risky sovereign bonds and other munies. This letter demonstrates that there exists a closed form solution for the bond pricing formula derived by Cathcart and El-Jahel (1998). This formula involves no more than the Gaussian cumulative density function. Complex numerical procedures for inverting Laplace transforms can therefore be avoided. One also verifies that the original numerical method implemented by Cathcart and El-Jahel (1998) does not lead to significant computational errors.

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