Rescheduling debt in default: The Longstaff's proposition revisited


Franck Moraux
Professeur des Universités
franck.moraux@univ-lemans.fr
Université du Maine, GAINS-Argumans and CREM

Patrick Navatte
Professeur des Universités
patrick.navatte@univ-rennes1.fr
Université de Rennes 1, IAE de Rennes and CREM*

Introduction

The maturity extension is a standard way to renegotiate debts and loans in default. This solution is also regularly adopted in many types of financial reorganization such as the troubled debt restructuring. Surprisingly, little attention has been given in the literature to strategies for altering the liability term structure compared to the negotiated debt service reductions and the related strategic behaviours (see, e.g., Anderson-Sundaresan, 1996 and Mella-Barral and Perraudin, 1997).

Maturity extension beyond a first default event is nevertheless conceivable because rescheduling has value for both parties. "*Both the bondholder and the stockholder are better off if the bondholder extends the maturity date optimally*" explains Longstaff (1990). In case of liquidation, indeed, the equity of a defaulting firm is worth nothing[1]. For their part, creditors have many reasons for granting delays. Longstaff (1990) argue that "[because of] *the positive liquidation costs, the bondholders always prefer to extend the maturity of the defaulting bonds rather than instigate bankruptcy proceedings*". Chen, Weston and Altman (1995) recall that lengthening the maturity of all or a portion of the debt enhances the probability of repayment. One can add a couple of arguments. First, when the debtholders are exposed to a tightened network of firms, they may fear an infectious propagation into the network and a contagion of defaults into their debt portfolios. Second, by injecting some new funds, the equityholders may contribute to and incite the rescheduling.

Longstaff (1990) provides insights and formulae for the debtholders gain, the optimal extension period they choose and the equity price. The key assumption is that the extension is decided as soon as the net gain value for debtholders is positive. Simulations show that these latter are ready to grant a very long delay even to safe only a negligible part of their wealth. In addition, his framework assumes that the observed financial distress can be solved just by modifying the liability term structure of the firm. There is no contribution, nor liquidity infusion from equityholders and no further guarantee for debtholders during the granted period.

This paper revisits the Longstaff's analysis and aims at relaxing these assumptions. It argues that rescheduling is especially worthwhile in some specific contexts. Suggested situations are those where the financial distress is not dramatic, those where the stockholders significantly contribute and those where the firm possesses specific assets so that there is no mature second hand market for the firm assets. An immediate liquidation implies indeed a high liquidation cost: the more specific the firm assets are, the lower the instantaneous realization rate is[2]. Assuming that the realization rate is time-varying and increasing as time goes through permits to model the possible improvement of liquidation value if enough time is devoted to assets sale.

Another important innovation of this article rests on the need of a monitoring process of the firm during the extension period. The absence of a monitoring device makes the first default a surprise. After this, the debtholders should monitor the firm more closely and react more promptly to a second default.

The rest of the paper is organized as follows. Section 1 reviews the Longstaff (1990)'s framework, model and results. The following sections present our own contributions. Section 2 depicts a typical scenario at the time of

* CREM is the UMR CNRS 6211. The authors thank two anonymous referees for their suggestions and remarks. An earlier version of this paper has been presented at the 21st AFFI International Conference at Cergy Pontoise and the 2004 EFMA International Conference in Basel. Special thanks are due to participants of these conferences for comments and stimulating discussions. Financial support of the CREM is acknowledged.

default. Section 3 discusses that debtholders may refuse to keep on the business in the most desperate cases. Section 4 investigates situations where the assets are specific and the associated realization rate time-varying. This rate clearly impacts on the rescheduling decision. Section 5 introduces a contribution that debtholders can require from equityholders. Section 6 motivates the need of a monitoring process beyond the first default event i.e. during the extension period. And section 7 offers some concluding remarks.

1. *The Longstaff's model: an overview*

This section groups together the framework, the model and some results of Longstaff (1990). It serves as a benchmark in the rest of the article. Let us consider a perfect and complete financial market in lines with Black, Scholes and Merton. Trading takes place continuously and short positions are possible. There are no tax, nor transaction cost, nor agency cost, nor (for the moment) bankruptcy or liquidation cost. There exists a riskless asset paying a known interest rate denoted *r*. Let it be a risky levered firm with a simple capital structure consisting of equity and a single issue of discount bonds with maturity T_1 and face value F_1. The firm asset's value is denoted *V* and, under the risk neutral measure, its process is correctly described by the dynamics :

$$dV = rVdt + \sigma VdW \qquad (1)$$

where *W* is a Brownian motion and σ denotes the firm volatility.

Notoriously, the payoff function at time F_1 for the equity holders is that of a call option ($max(V_{T_1} - F_1, 0)$). In absence of bankruptcy costs, the one for the bondholder is $min(F_1, V_{T_1})$ where V_{T_1} also stands for the liquidation value of the assets. If there are bankruptcy costs, the liquidation value is then strictly lower than the assets' value of the defaulting firm. And, as claimed by Longstaff (1990), bondholders have an incentive to extend the maturity date of the debt. To understand this, let's denote by β the percentage realization of the firm's assets in case of liquidation ($0 < \beta < 1$). If V_{T_1} is inferior to F_1 at the maturity date T_1 of the bonds, the bondholders take over the firm and receive βV_{T_1} only. If, instead, they choose to extend the maturity of the defaulting debt for an additional period of τ, they swap at T_1 the known payoff βV_{T_1} for a contingent claim that pays $\beta V_{T_{1+\tau}}$ at $T_1 + \tau$ if $V_{T_{1+\tau}}$ is lower than F_1 and F_1 otherwise. Debtholders will rationally extend the maturity of the debt if the value of this latter contingent claim at time T_1 is greater than βV_{T_1}. In addition, they will choose τ the length of the extension period optimally. Longstaff (1990) suggests an optimization procedure to find the optimal length of the extension period.

The procedure of Longstaff (1990) exploits a net gain function $H(V_{T_1}, F_1, T_2)$ which is the difference between the value at time T_1 of the contingent claim paying βV_{T_2} at T_2 if $V_{T_2} < F_1$ otherwise F_1 and βV_{T_1}. Consequently, under the risk neutral measure Q:

$$H(V_{T_1}, F_1, T_2) = e^{-r(T_2 - T_1)} E^Q_{T_1}[\beta V_{T_2} 1_{V_{T_2} < F_1} + F_1 1_{V_{T_2} \geq F_1}] - \beta V_{T_1}$$

and, under the Black, Scholes and Merton's setting, we obtain:

$$H(V_{T_1}, F_1, T_2) = -\beta V_{T_1} + \beta V_{T_1} N[-d_{1, V_{T_1}/F_1}(T_2 - T_1)] + F_1 e^{-r(T_2 - T_1)} N[d_{2, V_{T_1}/F_1}(T_2 - T_1)] \qquad (2)$$

where $d_{1,x}(t) = \dfrac{\ln x + (r + \frac{1}{2}\sigma^2)t}{\sigma\sqrt{t}}$

and $d_{2,x}(t) = d_{1,x}(t) - \sigma\sqrt{t}$. Longstaff (1990) then argues that debtholders will extend the maturity of their debt when the net gain function is positive. Among other things, his simulations show that the extension is valuable as the liquidation costs increase. Since the function $H(V_{T_1}, F_1, T_2)$ is a concave function of the maturity T_2, the optimal length of the extension period is obtained at date T_1 by computing

$$\tau(V_{T_1}) = \underset{t \in [0, \infty[}{\arg max} H(V_{T_1}, F_1, t) \qquad (3)$$

Equivalently, a new optimal maturity is obtained by computing:

$T(V_{T_1}) = \underset{t \in [T_1, \infty[}{\arg max} H(V_{T_1}, F_1, t - T_1)$. Of course one has: $T(V_{T_1}) = T_1 + \tau(V_{T_1})$. Both depend on the severity of default $\frac{V_{T_1}}{F_1}$, the realization rate and the firm value. *Table 2* of Longstaff (1990) demonstrates that the optimal length of the extension period is strictly increasing as the firm value at T_1, gets lower. It is therefore an one-to-one function of V_{T_1}.

The above maturity extension has some consequences on the way corporate liabilities are priced. Longstaff (1990) has insisted on the equity price. By deciding to extend the maturity of their claim, debtholders deliver at the time of default a new claim to equityholders. This new claim is a standard call option written on the underlying assets and whose expiration is the optimal extension date $T(V_{T_1})$. Overall, at time T_1, the wealth of equityholders consists of $V_{T_1} - F_1$ if $V_{T_1} > F_1$ and $Eq^{BSM}_{T_1}(V_{T_1}, F_1, T(V_{T_1}) - T_1)$ otherwise. This is the value of this new claim if there is default and debt rescheduling. By discounting the different terms, the equity price at time $t < T_1$ is given by :

$$Eq_t^E = Eq_t^{BSM}(V_t, F_1, T_1 - t) + e^{-r(T_1 - \tau)} \int Eq_{T_1}^{BSM}(V_{T_1}, F_1, T(V_{T_1}) - T_1) I_{V_{T_1} - F_1 < 0} P(V_{T_1}) dV_{T_1} \quad (4)$$

where Eq_t^{BSM} is the price at time t of a standard call option in a Black-Scholes-Merton setting and $P(V_{T_1})$ is the probability density function of V_{T_1}. The price of the "extendible" equity is seen larger than the price of a "standard" equity. It can be denoted $Eq_t^E(V_{T_1}, F_1, T_1, T_2)$.

Longstaff (1990) has posited that debtholders extend the maturity of their debt *as soon as* the net gain function is positive. No matter is the length of the extension period debtholders are ready to face[3]. To a certain extent, the maturity extension of Longstaff (1990) can be viewed as an *automatic* process. This is what we want to relax in this article. Before turning to our own framework, we expose a typical scenario at the time of default. It illustrates a certain combination of the materials considered afterwards.

2. Typical scenarii at the time of default

Lots of different scenarii can occur at time T_1. The following table depicts however a typical one. Under this special scenario, there is no default if the firm value at time T_1 is sufficient to repay the face value. If this is not the case, the debtholders may or may not accept to extend the maturity of their claim.

The extension is motivated either because the financial distress is not dramatic (for $K' \le V_{T_1} < F_1$) or because the stockholders renders this distress not dramatic with some contribution A (for $K \le V_{T_1} < K'$). For instance, an infusion of liquidity is a classical contribution. It can also be motivated by the expected rise of the realization rate (for $V_{T_1} < K$ and small β at time T_1 : $\beta(T_1)$). This is typically the case when the firm assets are specific and need some time to be appropriately sold. If the realization rate $\beta(T_1)$ is large enough, there is no incentive to wait.

No default	if	$F_1 \le V_{T_1}$
Automatic monitored extension	if	$K' \le V_{T_1} < F_1$
Monitored Extension conditional on contribution	if	$K \le V_{T_1} < K'$ and A
Technical extension	if	$V_{T_1} < K$ and small $\beta(T_1)$
Default	if	$V_{T_1} < K$ and large $\beta(T_1)$

The above typical scenario exposes some reasons motivating debtholders to extend the maturity of their debt. Note that the debtholders can rationally refuse to extend the maturity because the situation is too desperate. The following sections discuss each elementary condition. If these requirements jointly form the above scenario, they can also yield to many other ones.

3. A continuation threshold

In desperate situations, the continuation success may appear quite improbable and, rationally, debtholders will refuse the continuation. In the present setting, the severe economic distress means that the (financial) value of the firm's assets at time T_1 is very low. In absence of any other incentive, assets should be liquidated as soon as possible. As a result, the value of the firm cannot be less than a given threshold at the default time T_1. One therefore assumes that *there exists a level K for the firm asset value at time T_1 beyond which the bankruptcy is required by debtholders.*

The threshold K can be chosen arbitrarily and equal, say, to a given proportion of the face value of the debt. It can also be computed endogenously[4]… For instance, if debtholders refuse to wait for too long beyond the default date, they admit implicitly a maximum delay of τ^*. Recalling that the optimal extension period is a one-to-one function of the first argument of the net gain function H, debtholders can deduce an endogenous continuation level of K^* such that:

$$\tau^* = \underset{t \in [0, \infty[}{\arg max} H(K^*, F_1, t) \quad (5)$$

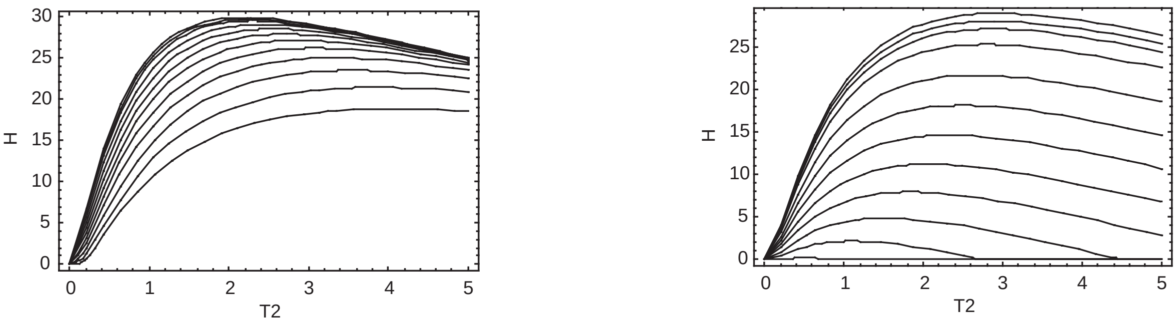
With no more assumption than the existence of the threshold K, the net gain function of Longstaff (1990) remains unchanged except that it is now defined for $K \le V_{T_1} \le F_1$ only. The optimal extended maturity is then obtained for this range of values along the lines previously described. The interval $K \le V_{T_1} \le F_1$ may be interpreted as an effective area for negotiation.

This first condition must however be further refined. First, the rescheduling decision may be influenced by many factors such as the liquidity infusion from equityholders and some specific guarantee beyond default. And, even below the level K, extending the maturity of a debt in default may be worthwhile to increase the value of the liquidated asset. This is the case when the realization rate is very low at the default date and when one expects it to increase sharply. These are the cases that are exposed in the following sections.

4. A time-dependent realization rate and the pure "technical" extension

A key variable governing the decision to liquidate is formed by the level of specialization of the firm assets and the knowledge of the associated second-hand market. It is well known that the liquidation costs are far from being negligible (Franks and Torous, 1989 ; Weiss, 1990). Granting a delay can mean increasing the realization rate of the liquidated assets. This will be of practical matter if the firm assets are particularly specialized and not very liquid. Even in cases of a second or subsequent definitive failure, the recovery rate could be higher. During the extended period, indeed, the firm

Figure 1. **The net gain function and the realization rate parameters:** $H(\beta(T_1), a)$



On the left graph, $\beta(T_1) = 5\%$, a ranges from 10% to 100% by 10% with in addition 1% and 95% and the lowest line stands for $H(5\%,1\%)$. On the right graph: $a = 50\%$, $\beta(T_1)$ ranges from 0% to 90% by 10% with in addition 2.5% and 5% and The lowest line stands for $H(90\%,50\%)$. Other 6Fparameters are $V_{T_1} = 40$, $\sigma = 20\%$, $F_1 = 50$, $r = 10\%$ and $\beta = 90\%$. For ease of representation, useless negative values have been omitted.

may look for the better way to sell its assets so as to get the lowest realization costs possible. In some cases, the whole firm may also be sold in one part. Ideally, as time gets to infinity, the realization rate increases to a maximum (β^*) or, equivalently, the replacement costs decrease to a minimum ($1 - \beta^*$). This implies that the overall liquidation costs decrease as well. For short, one assumes that *the realization rate* $\beta(t)$ *is time-dependent. It is an increasing function and converges to* β^**, as time goes through infinity.*

To fix ideas, β is supposed to be well described by:

$$\begin{cases} \beta(T_1) = \beta_1 \\ d\beta(t) = a(\beta^* - \beta(t))dt \end{cases}$$

whose solution[5] is :

$$\beta(t) = \beta^*(1 - e^{-a(t-T_1)}) + \beta_1 e^{-a(t-T_1)},\ t > T_1 .$$

The larger a is, the faster the realization rate grows to β^*. The parameter a is therefore quite intuitive as it can model the knowledge of the second-hand market. If $\beta(T_1)$ is rather small, the increasing property of β may incite debtholders to let the firm survive in order to appreciate the liquidated value. Hence, a *technical* extension is artificially created.

Under the risk neutral measure Q, the associated net gain function is given by :

$$H(V_{T_1}, F_1, T_2; \beta(t))$$
$$= e^{-r(T_2 - T_1)} E^Q_{T_1}[\beta(T_2) V_{T_2} 1_{V_{T_2} < F_1} + F_1 1_{V_{T_2} \geq F_1}] - \beta(T_1) V_{T_1}$$

for some extension date T_2. So we obtain under the Black, Scholes and Merton setting :

$$\begin{aligned} H(V_{T_1}, F_1, T_2; \beta(t)) &= \\ -\beta(T_1)V_{T_1} &+ \beta(T_2)V_{T_1}N[-d_{1, V_{T_1}/F_1}(T_2 - T_1)] \\ &+ F_1 e^{-r(T_2 - T_1)} N[d_{2, V_{T_1}/F_1}(T_2 - T_1)] \end{aligned} \quad (6)$$

To find this net gain function, the former constant realization rate β has just been replaced by its time-varying but deterministic values: $\beta(T_1)$ and $\beta(T_2)$.

Figure 1 shows that the increasing property of the realization rate has a major impact on the wealth expected by debtholders. One plots $H \equiv H(\beta(T_1), a)$ as a function of T_2 for different value of $\beta(T_1)$ and a. The left graph keeps $\beta(T_1)$ equal to 5% and varies a from 10% to 100% by 10% (with 1% and 95%). The lowest line stands for $H(5\%, 1\%)$. The right graph keeps the a equal to 50% and ranges $\beta(T_1)$ from 0% to 90% by 10% (with 2.5% and 5%). The lowest line stands for $H(90\%,50\%)$. Other parameters are identical for the two graphs, these are: $V = 40$, $\sigma = 20\%$, $F_1 = 50$, $r = 10\%$ and $\beta^* = 90\%$.

Figure 1 indicates that the net gain function significantly increases, as a grows. It almost doubles for the chosen range of a. This increasing feature is confirmed by the positivity of the first derivative of H with respect to a. The second derivative of H with respect to a is negative meaning that it is vain to develop a closed to perfect knowledge of the second-hand market. The left graph also shows that, as a grows, the optimal extension date is earlier. The right graph sheds lights on the specific impacts of $\beta(T_1)$. The highest is $\beta(T_1)$, the smallest is the net gain function. Other way writing, the lowest is the liquidation cost, the fewest is the incentive to extend. In case of very specific assets, debtholders are rather motivated to extend the maturity of their debt.

This section claims that debtholders are ready to extend the maturity of their debt, at least, to increase the value of the liquidated assets. It is clear that other incentives exist as a direct contribution from equityholders.

5. *A contribution from equityholders*

A contribution from equityholders may significantly influence the debtholders decision to extend their maturity. And there can be many kinds of subjective motivations for the firm owners to avoid the bankruptcy at any price[6].

Figure 2. **The effect of the equityholders contribution on the debtholders net gain function**

The left graph plots H for a contribution A invested in the firm. The right graph plots H for contribution A devoted to a partial reimbursement of the debt. The value of the contribution A ranges from 1 to 10 with in addition 9.5 from the bottom to the top. Other parameters are $V_{T_1} = 40$, $\sigma = 20\%$, $F_1 = 50$, $r = 10\%$.

However we follow in this section a pure quantitative analysis.

A direct consequence of the maturity extension is that debtholders offer a positive wealth to equityholders (see equation 4). As a result, debtholders can require equityholders to concede some substantial contribution to restructure either the financing structure or the business portfolio. Rationally, shareholders cannot refuse this condition but only to a certain extent. So one assumes here that *there exists a intermediate level* K' *within the negotiation interval* ($K' \in [K, F_1]$) *under which the shareholders* (*willing to continue to run the firm*) *are constrained to inject an amount of money* A *at the default time for continuation. More practically, they are forced either to invest the money in the firm or to redeem part of the debt.*

Re-investing in the firm or paying back to debtholders for partially reimbursement has several consequences. First, either the firm asset value at T_1 is increased or the debt is lowered. In the former case, the firm asset value grows to be $\tilde{V}_{T_1} = V_{T_1} + A$ and this variable can be written $\tilde{V}_t = \frac{V_{T_1} + A}{V_{T_1}} V_t$ for any future date $t > T_1$ [7]. In the latter case, the remaining due face value becomes : $F_1 - A \equiv F_2$. In both cases, the leverage ratio beyond the default time is lowered and the probabilities of recovery and complete repayment of the debt in the future is increased. Second, the contribution lowers the wealth of equityholders. Third, it has a direct impact on the net gain function of debtholders and consequently on the optimal extension maturity.

If the amount A is injected in the firm, the net gain function is defined by the equation :

$$H(V_{T_1}, F_1, T_2; A) =$$
$$= e^{-r(T_2 - T_1)} E^Q_{T_1}\left[\beta \tilde{V}_{T_2} 1_{\tilde{V}_{T_2} < F_1} + F_1 1_{\tilde{V}_{T_2} \ge F_1}\right] - \beta V_{T_1}$$

where $V_{T_1} \in [K, K']$. If instead the contribution A serves to reimburse the debtholders partially, the net gain function is described by :

$$H(V_{T_1}, F_1, T_2; A)$$
$$= e^{-r(T_2 - T_1)} E^Q_{T_1}[\beta V_{T_2} 1_{V_{T_2} < F_1 - A} + (F_1 - A) 1_{V_{T_2} \ge F_1 - A}] - \beta V_{T_1} + A$$

for $V_{T_1} \in [K, K']$. Under the Black, Scholes and Merton setting, we then obtain respectively:

$$H(V_{T_1}, F_1, T_2; A)$$
$$= -\beta V_{T_1} + \beta V_{T_1} N[-d_{1 (V_{T_1} + A)/F_1}(T_2 - T_1)] \tag{7}$$
$$+ F_1 e^{-r(T_2 - T_1)} N[d_{2, (V_{T1} + A)/F_1}(T_2 - T_1)]$$

and

$$H(V_{T_1}, F_1, T_2; A) =$$
$$A - \beta V_{T_1} + \beta V_{T_1} N[-d_{1, V_{T1}/(F_1 - A)}(T_{2} - _{T_1})] \tag{8}$$
$$+ (F_1 - A) e^{-r(T_2 - T_1)} N[d_{2, V_{T1}/(F_1 - A)}(T_2 - T_1)]$$

In both cases, the optimal extension maturity is computed by maximizing the net gain function with respect to T_2.

Graphs in *Figure 2* show the effects of the equityholders contribution on the net gain function. The left graph plots H for a contribution A invested in the firm whereas the right one considers that A is devoted to a partial reimbursement of the debt. A is supposed constant and ranges from 1 to 10 with a supplement figure of 9.5. In both graphs, the lowest line is associated with the lowest contribution. Other parameters value are $V_{T_1} = 40$, $\sigma = 20\%$, $F_1 = 50$, $r = 10\%$.

Both graphs show that the contribution increases the net gain function and that it shortens the optimal extension period. A contribution dedicated to the partial reimbursement of the face value slightly increases the total wealth of the bondholders (compared to the right situation). This alternative does not however change dramatically the shape of the function H and the optimal delay. Overall, one concludes that the participation of stockholders is an important factor

Figure 3. **The largest contributions possible.**

The continuous line stands for $\tilde{A}$, the other one for $\tilde{\tilde{A}}$. On the left graph, $V_{T_1} = 25$. On the left graph $V_{T_1} = 26$. Other parameters are $\sigma = 20\%$, $F_1 = 50$, $r = 10\%$.

for the extension of the firm and that debtholders may prefer a partial reimbursement to liquidity infusion.

In view of this, the debtholders could want to maximize the amount A and receive it as a partial reimbursement. But this is not so simple. The amount A is indeed not totally exogenous since stockholders can always refuse to contribute. From their viewpoint, they will refuse with certainty to give more than the value of the claim they implicitly receive.

Let's denote by C the value of the claim stockholders receive in default[8]. As explained above, stockholders won't accept to contribute more than the value they get. This amount depends on the way stockholders will contribute too. If the amount is injected in the firm for a given delay $T(V_{T_1})$, the maximum value denoted A verifies :

$$C(V_{T_1} + \tilde{A}, F_1, T(V_{T_1}) - T_1) - \tilde{A} = 0 \qquad (9)$$

If the contribution serves to reimburse partially the debtholders, the maximum value denoted $\tilde{\tilde{A}}$ is :

$$C(V_{T_1}, F_1 - \tilde{\tilde{A}}, T(V_{T_1}) - T_1) - \tilde{\tilde{A}} = 0 \qquad (10)$$

Figure 3 plots the two maximum contributions $\tilde{A}$ and $\tilde{\tilde{A}}$ attainable for debtholders as function of the extension period and for two different firm values at default. On the left graph $V_{T_1} = 25$ whereas $V_{T_1} = 26$ on the right one. The contribution from stockholders is required because the firm value at the default point is half the due face value.

Some remarkable features appear on these graphs. First, in both cases, $\tilde{A} > \tilde{\tilde{A}}$. In other words, debtholders may force stockholders to contribute at a higher level if these latter are allowed to invest in the firm. Second, compared to $\tilde{\tilde{A}}$, $\tilde{A}$ appears quite sensible to the level of the financial distress. A rise of V_{T_1} increases $\tilde{A}$ at least by a factor 1.25 (up to 2). It could be shown that, for more desperate situations (say, for $V_{T_1} = 20$), $\tilde{A}$ and $\tilde{\tilde{A}}$ are very low figures and the capture of wealth is rather limited (less than 2). Considering these graphs alone incites debtholders to let the stockholders invest in the firm. This conclusion contrasts with the one of the previous graph. Overall, the arbitrage for debtholders is not so straightforward.

As final words, one can investigate the effect of the contribution on the price at time 0 of the equity. The different pay-offs for the equityholders at time T_1 are now:

$V_{T_1} - F_1$	if	$V_{T_1} \geq F_1$
$C(V_{T_1}, F_1, \tau(V_{T_1}))$	if	$F_1 > V_{T_1} \geq K'$
$C(X, F, \tau'(V_{T_1})) - A$	if	$K' > V_{T_1} \geq K$ (11)

where ($X = V_{T_1} + A$ and $F = F_1$) or ($X = V_{T_1}$ and $F = F_2$) depending on the way they contribute. The equity price at time 0 is obtained by discounting the different terms in the Black-Scholes-Merton setting. One obtains respectively:

$$\begin{aligned} Eq(A) = {} & Eq^{BSM} + e^{-rT_1}\int_{K'}^{F_1} C\left(V_{T_1}, F_1, T\left(V_{T_1}\right) - T_1\right)P\left(V_{T_1}\right)dV_{T_1} \\ & + e^{-rT1}\int_{K}^{K'} C\left(V_{T_1} + A, F_1, T\left(V_{T_1}\right) - T_1\right)P\left(V_{T_1}\right)dV_{T_1} \\ & - Ae^{-rT_1}\left(N\left(d_{2, V_{T_1}/K(T_1)}\right)\right) - N\left(d_{2, V_{T_1}/K'(T_1)}\right) \end{aligned} \qquad (12)$$

and

$$\begin{aligned} Eq(A) = {} & Eq^{BSM} + e^{-rT_1}\int_{K'}^{F_1} C\left(V_{T_1}, F_1, T\left(V_{T_1}\right) - T_1\right)P\left(V_{T_1}\right)dV_{T_1} \\ & + e^{-rT_1}\int_{K}^{K'} C\left(V_{T_1}, F_1 - A, T''\left(V_{T_1}\right) - T_1\right)P\left(V_{T_1}\right)dV_{T_1} \\ & - Ae^{-rT_1}\left(N\left(d_{2, V_{T_1}/K(T_1)}\right)\right) - N\left(d_{2, V_{T_1}/K'(T_1)}\right) \end{aligned} \qquad (12')$$

where T, T' and T'' stand for the optimal extension dates computed with the appropriate net gain functions. If the equityholders can be forced to pay a maximum contribution, their current wealth becomes:

$$Eq(\bar{A}) = Eq^{BSM} + e^{-rT_1}\int_{K'}^{F_1} C\left(V_{T_1}, F_1, T\left(V_{T_1}\right) - T_1\right)P\left(V_{T_1}\right)dV_{T_1}$$

with $\bar{A} = \tilde{A}$ or $\bar{A} = \tilde{\tilde{A}}$. The third and fourth terms of equations (12) and (12') collapse because the third payoff of the system of equation (11) drops to zero (due to the equations 9 and 10). Interestingly, this wealth is equivalent to the one held by equityholders if the extension was automatic and limited to $F_1 > V_{T_1} \geq K'$. In fact, the additional value has been captured by the debtholders. If, in addition, no automatic extension has been considered (i.e. $K' = F_1$), the equation becomes: $Eq(\bar{A}) = Eq^{BSM}$.

And the equityholders are not better off by the extension of the debt.

To conclude this section, we note that there exists (at least) a scenario under which the decision to extend the maturity has no consequence on the current wealth of equityholders. Under some scenario, the wealth generated by the extension of the debt maturity is entirely captured by debtholders. The following section introduces another condition for extension, as last innovation of the paper.

6. *After the default: a monitoring device*

Once there is a first default event, it is more than probable that debtholders want to monitor the firm more closely with the objectives to secure their capital, due at the extended maturity, and, if necessary, to react promptly to a second default. This can be considered within our setting by introducing a monitoring threshold V_B and by assuming that, as the firm asset value reaches this barrier, debtholders may force stockholders to bankrupt the firm. So, once declared, the second default leads to an immediate liquidation of the firm assets[9].

To sum up: one assumes that, *after accepting to extend the maturity of their debt beyond the first default event, debtholders will supervise the firm with the help of a monitoring barrier denoted* V_B. *Debtholders aim at being fully repaid at the extended maturity of their claim but if the firm value reaches the barrier* V_B, *this causes an "early" bankruptcy and the immediate liquidation of the firm assets. The assets are then sold and the debtholders are repaid with, eventually, some specific realization costs with rate* $(1 - \beta_{V_B})$[10]

The bargaining power of the bondholders is assumed strong enough to impose the shareholders a monitoring device. The threshold V_B may be imposed (exogenously) by the debtholders. Or it may result from a bargain between shareholders and creditors. Whatever it is, because of its monitoring meaning, the barrier is assumed deterministic. For instance, its level is a constant or a function of the renegotiated face value of the debt and of the time to the extended maturity.

The monitoring barrier has different consequences on the net gain function, denoted for short $H(V_B)$, and the debtholders' behaviour. To exhibit them, one will focus on an automatic but monitored extension. The net gain function is once again the discounting value of the different payoffs received by the debtholders minus the value given up at time T_1. If the monitoring barrier is not reached during the granted period (if $\forall\tau \in [T_1, T_2]$, $V_\tau > V_B$), the debtholders will, at T_2, be fully repaid (if $V_{T_2} > F_1$) or they will receive the firm assets (if $V_{T_2} > F_1$). If the barrier is attained during the period ($\exists\tau \in [T_1, T_2], V_\tau \leq V_B$), then there is a second default and a precipitated bankruptcy. The assets are immediately liquidated at the amount $\beta_{V_B} V_B$. This amount can be received either immediately or latter (and we assume that this future date is the extended maturity T_2) causing an alternative. More formally, one has:

$$\begin{aligned}H = {}& e^{-r(T_2 - T_1)}E_{T_1}^{Q} \\ & [\beta(T_2)V_{T_2}1_{\{V_B < V_{T_2} < F_1;\forall\tau \in [T_1, T_2], V_\tau > V_B\}}] \\ & + F_1 1_{\{F_1 < V_{T_2};\forall\tau \in [T_1, T_2], V_\tau > V_B\}} \\ & + \beta_{V_B} V_B R - \beta(T_1)V_{T_1}\end{aligned}$$

where $\beta_{V_B} V_B R$ stands for the value received in case of a second default. Denoting Q the risk neutral probability and $\tilde{Q}$ the equivalent probability which used V as numéraire, one can write:

$$\begin{aligned}H = {}& -\beta(T_1)V_{T_1} \\ & + \beta(T_2)V_{T_1}\tilde{Q}\left[V_B < V_{T_2} < F_1;\forall\tau \in [T_1, T_2], V_\tau > V_B\right] \\ & + F_1 e^{-r(T_2 - T_1)}Q\left[F_1 < V_{T_2};\forall\tau \in [T_1, T_2], V_\tau > V_B\right] \\ & + \beta_{V_B} V_B R\end{aligned}$$

If the amount $\beta_{V_B} V_B$ is paid at T_2, then $R = e^{-r(T_2 - T_1)}Q[\exists\tau \in [T_1, T_2] V\tau \leq V_B]$. If it is paid immediately, then $R = \int_{T_1}^{T_2} e^{-rt}h(t)dt$, where h is the standard first hitting time density[11]. Within a Black-Scholes-Merton, this expression can be computed analytically. One finds:

$$\begin{aligned}H = {}& -\beta(T_1)V_{T_1} + \beta_{V_B} V_B R \\ & + \beta(T_2)V_{T_1}\left\{\left(N\left[d_{1, V_{T_1}/F_2}\right] + \left(\frac{V_B}{V_{T_1}}\right)^{2\lambda} N\left[d_{1, V_B^2/(F_2 V_{T1})}\right]\right)\right. \\ & \left. -\left(N\left[-d_{1, V_{T_1}/V_B}\right] + \left(\frac{V_B}{V_{T_1}}\right)^{2\lambda} N[d_{1, V_B/V_{T1}}]\right)\right\} \\ & + F_1 e^{-r(T_2 - T_1)}\left(N\left[d_{2, V_{T_1}/F_2}\right] - \left(\frac{V_B}{V_{T_1}}\right)^{2\lambda - 2} N\left[d_{2, V_B^2/(F_2 V_{T_1})}\right]\right)\end{aligned} \tag{13}$$

Figure 4. **The net gain function and the monitoring barrier**

The graphs plot the function $H(T_2)$ for different value of T_2. The monitoring barrier V_B ranges from 50% of the firm assets value $V(T_1)$ (the continuous line) to 90% ()10% by 10%) with in addition 95% and 99% (the dotted one). One has $\beta(T_1) = \beta(T_2) = \beta_{V_B} = 50\%$ in the left graph. In the right graph, the realization is time-varying, one has $\beta_{V_B} = \beta(T_2)$, $\beta = 90\%$ and $a = 0, 75$. Other parameters are $V_{T_1} = 40$, $\sigma = 20\%$, $F_1 = 50$ and $r = 10\%$.

Figure 5. **The optimal extension period and the monitoring barrier**

These graphs plot optimal extension periods as function of the monitoring barrier. They explore different values of $\beta(T_1)$ ranging from 50% (the continuous line) to 90% (the dotted one) 10% by 10%. The left graph assumes that β remains constant through time. In the right graph, it is increasing to $\beta^* = 90\%$ with speed $a = 0, 75$. Other parameters are $V_{T_1} = 40$, $\sigma = 20\%$, $F_1 = 50$ and $r = 10\%$.

where, depending on the way the debtholders recover the value of the liquidated assets either at maturity or straight, one has respectively:

$$R = \begin{cases} N[-d_{2,\, V_{T_1}/V_B}] + \left(\dfrac{V_B}{V_{T_1}}\right)^{2\lambda - 2} N[d_{2,\, V_B/V_{T_1}}] \\ \left(\dfrac{V_B}{V_{T_1}}\right)^{2\lambda - 1} N[d_{1,\, V_B/V_{T_1}}] + \dfrac{V_{T_1}}{V_B} N[-d_{1,\, V_{T_1}/V_B}] \end{cases} \qquad (13')$$

Here, $\lambda\sigma^2 = r + \frac{1}{2}\sigma^2$, $d_{1,x} = \dfrac{\ln x + \lambda\sigma^2(T_2 - T_1)}{\sigma\sqrt{T_2 - T_1}}$,

and $d_{2,x} = d_{1,x} - \sigma\sqrt{T_2 - T_1}$

The following figures illustrate the effect of the monitoring barrier on the debtholders wealth and behaviour. Some graphs will show that the above revisited net gain function is concave. As a result, the optimization procedure of Longstaff (1990) to choose the optimal maturity extension of the debt in default may be adapted to our setting. It will be computed for plotting some optimal extension period.

The graphs of *figure 4* plot the net gain function $H(T_2)$. The realization rate remains constant through time on the left graph, it is time-varying on the right one. The monitoring barrier V_B ranges from 50% of the firm assets value $V(T_1)$ (the continuous line) to 90% (10% by 10%), one also adds 95% and 99 % (the dotted line). Recall that a higher monitoring barrier implies a higher probability of a second default.

The graphs show that the maximum net profit is higher on the right i.e. when a rise in the realization rate is expected. The left graph indicates that the (maximum) net gain strictly decreases as the monitoring barrier grows for a constant realization rate. This is not the same for a time-varying β, on the right graph. When the monitoring barrier is high, the probability of a precipitated liquidation in the near future is high. The liquidation is just postponed. When in addition the realization rate remains constant, the obtained liquidated value is not much different. When the monitoring barrier is low, the extension may "save" the firm and the debtholders may be fully repaid. When the realization rate is time varying, a second and precipitated default caused by the monitoring barrier may be worth for the debtholders.

The *figure 5* plots some optimal extension periods τ in presence of a monitoring device. The monitoring barrier is expressed with respect to the firm's assets value at date T_1. The realization rate $\beta(T_1)$ ranges from 50% (the continuous line) to 90% (the dotted one), 10% by 10%. The left graph explores the cases where β remains constant through time. The right one assumes that $\beta(t)$ is increasing to $\beta = 90\%$ at a speed $a = 0,75$.

Both graphs agree that the optimal extension period diminishes as the realization rate gets higher for a given monitoring barrier. But the way debtholders fix the monitoring barrier greatly affects the rescheduling parameter. The left graph is a direct extension of the Longstaff (1990) setting. Globally, the presence of a monitoring barrier decreases the optimal extension period. For very low monitoring barrier, one finds the Longstaff's optimal value. When the β is not constant, this is not the case. A higher monitoring barrier induces a longer extension period. The way debtholders anticipate the realization rate and the way they choose the monitoring barrier are thus intimately related. This suggests that the monitoring device cannot be considered independently from the expected realization rate.

The monitoring device impacts on the debtholders net gain function and the decision to extend. It has also some consequences on the way the liabilities are priced. E.g. the function C used in the equations (12) and (12') for the equity price needs to be reconsidered. In presence of a monitoring device, this is nothing else than the price of a down-and-out call option. So the current price of the equity has much in common with an option written on a knock out barrier option. The global effects will nevertheless depend on the definitive scenario retained at T_1 and we don't pursue this analysis further.

7. Concluding remarks

This paper has revisited the analysis of Longstaff (1990) by considering a continuation threshold, a technical postponed liquidation, a contribution from equityholders and a monitoring device beyond default. These elements describe a number of favourable cases in which debtholders might choose to reschedule defaulting debt rather than foreclosing. Numerical simulations have shown how these conditions modify the wealth and the behaviour of the debtholders. We found a scenario under which the decision to extend the maturity has no consequence on the current wealth of equityholders. Under this scenario, the wealth generated by the extension of the debt maturity is entirely captured by debtholders.

Obviously, the present study calls itself some further extensions. Clearly the stochastic behaviour of the interest rates may be considered in this context of corporate debt analysis. It can also be argued that the business risk should be restructured beyond the first default time. Nothing prevents our framework to account for a new asset volatility for the extension period. But the relative differentiation with the pre-default period cannot be highlighted nor captured by the present research… This is left to further investigation. ■

1. One neglects here potential deviation from the absolute priority rule.
2. In what follows, the realization rate is one minus the liquidation cost.
3. Simulations of Longstaff (1990) confirm that debtholders must sometimes grant a very long delay to safe only a negligible amount. In some dramatic situations (e.g. when the firm value at date T_1 is worth less than 50% of the promised face value), they must wait more than 5 years to safe only 1.5 % of their face value.
4. Empirical research could be informative too.
5. The above specification is equivalent to assume that the costs function upon realization $\gamma(t) = 1 - \beta(t)$ is the solution of $d\gamma(t) = -a(\gamma^* - \gamma(t))dt$.
In many cases, generalization to a stochastic recovery is a straightforward but useless exercise.
6. E.g., some of them are perhaps the original entrepreneurs…
7. By doing so, one assumes no change in the drift nor in the business risk.
8. In the present context, the new claim is a standard call option so $C = Eq^{BSM}$.
9. This barrier may take the form of a covenant on the future cash-flows of the firm in lines of Anderson and Sundaresan (1996). It serves the role of an "early default" threshold in the spirit of the one considered in Black-Cox (1976) and Longstaff and Schwartz (1995). An important difference however is that it is used beyond a first default event.
10. β_{V_B} is a constant realization rate in case of subsequent default. Nothing prevents one to assume a time-varying realization rate and eventually a further technical extension.
11. One may recognize here for the first expression one minus the rebate of down-and-in barrier options and for the second one the rebate of down and out barrier options.

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