Equity to credit pricing

The rapid growth of credit derivatives markets has highlighted the distinction between reduced-form and firm-value models of default. Here, George Pan presents a new, practitioner-tested version of the latter type, providing a closed-form formula linking default probability and equity market data

In recent years, there has been much interest in linking credit risk with equity market data. This idea was originally presented by Merton (1974), and was already suggested in the Black & Scholes (1973) article on option pricing. The simple link between credit spreads and stock prices has been used by many practitioners over the years. But the summer 1998 crisis, which saw a sharp increase in both credit spreads and equity volatility, accentuated the need to treat equity volatility as a crucial component of credit. In light of this, many financial institutions have started to complement the traditional credit analysis and ratings with equity-based models that are now offered by several risk management companies (KMV, Moody's Investors Service, etc). Regulators have also been encouraging the development and use of internal credit models to calculate bank economic capital.

We have also witnessed the exponential growth of credit derivatives in recent years. Initially driven by sovereign markets and balance-sheet management of bank portfolios, they have since evolved in investment-grade and high-yield corporate markets as the instrument needed to bridge the gap among various cash securities (bonds, loans, convertible bonds, etc) as well as between market participants (bank portfolio managers, insurance companies, money managers, collateralised debt obligation managers, hedge funds, etc). This article presents a simple formula that was designed by market practitioners and quantitative researchers at JP Morgan Chase. The use of an approximation for the asset volatility term, as well as for the drift term, leads to a generic closed-form solution that can approximate any sophisticated model relying on similar fundamental assumptions. In addition, we identify the standard deviation of recovery value as a parameter playing an important role in the calculation of the probability of default and its term structure. The closed-form formula presented here uses only observable market data and financial data and parameters supported by many statistical sources and studies (stock price and volatility, debt-per-share, and mean and standard deviation of recovery value of debt).

Model description

The starting point for the model is the approach originally presented by Black & Cox (1976), Leland (1994) and Longstaff & Schwartz (1995). According to this approach (called 'asset model' below), an event of default occurs when the asset value of the firm crosses a predetermined barrier (recovery value of the firm's debt).

Based on the asset model, we derive the closed-form formula for the probability of no default for any given time horizon t. The basic assumptions of our model are shown in figure 1. We introduce a random variable V, which we call asset value although it does not necessarily coincide with the actual asset value of the firm. We then define the default event as the first crossing of the barrier $L \cdot D$ by the variable V, where D is a given debt-per-share. L is the global recovery on the debt as a whole and is itself a random variable.

We consider that the asset V behaves as a lognormal variable with zero drift:

$$\frac{dV_t}{V_t} = \sigma dW_t \tag{1}$$

where W_t is a standard Brownian motion and σ is the asset volatility.

We account for uncertainty of the recovery L by assuming it has a lognormal distribution with mean $\bar{L}$, and a percentage standard deviation λ($\lambda \geq 0$). Specifically:

$$\bar{L} = E[L], \quad \lambda = \text{Stdev}(\ln(L)) \quad \text{and} \quad LD = \bar{L}D\exp\left(\lambda Z - \frac{\lambda^2}{2}\right) \tag{2}$$

where Z is a standard normal random variable that is independent of W_t. The introduction of λ is based on extensive empirical studies of recovery rates upon default (see Portfolio Management Data/Standard & Poor's loss/recovery database). One prevalent finding of these studies is the extreme variance of the distribution of recoveries. In addition to some industrial sector dependence, the recovery rate can be greatly affected by factors such as whether default is triggered by financial or operational difficulties and whether the company will be restructured or liquidated. This finding necessitates the inclusion of λ as an important parameter in pricing credit risk.

For an initial value V_0 of V at time $t = 0$, default is thus defined to occur when:

$$V_0 \exp\left(\sigma W_t - \frac{\sigma^2}{2}t\right) < \bar{L}D\exp\left(\lambda Z - \frac{\lambda^2}{2}\right) \tag{3}$$

Define a process:

$$X_t = \sigma W_t - \lambda Z - \frac{\sigma^2}{2}t - \frac{\lambda^2}{2}$$

so that $X_t \sim N(-1/2\, A_t^2, A_t^2)$ where $A_t^2 = \sigma^2 t + \lambda^2$. Then default may be expressed as:

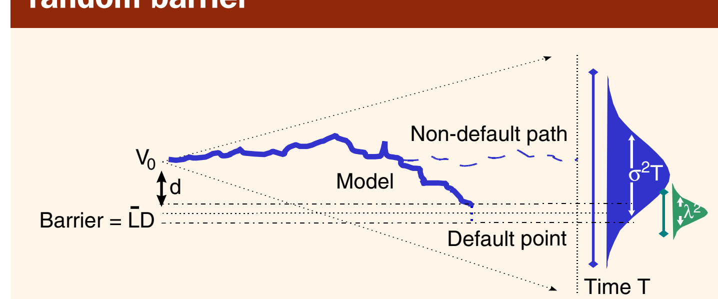
$$X_t < \ln\left(\frac{\bar{L}D}{V_0}\right) - \lambda^2 \tag{4}$$

We approximate the process X_t with the Itô process Y_t where $Y_t \sim N(\mu t, \theta^2 t)$ with variance $\theta^2 t = A_t^2 = \sigma^2 t + \lambda^2$ and drift $\mu t = -1/2\, A_t^2$. We now make use of the distributions for first hitting time of Brownian motion. In particular, we have (Musiela, Marek & Rutkowski, 1998):

$$P(Y_s > y, \forall s < t) = N\left(\frac{\mu t - y}{\theta\sqrt{t}}\right) - \exp\left(\frac{2\mu y}{\theta^2}\right) N\left(\frac{\mu + y}{\theta\sqrt{t}}\right) \tag{5}$$

By substituting θ and μ as above, and setting $y = \ln(\bar{L}D/V_0) - \lambda^2$ we obtain the closed-form formula for the probability of no default $P(t)$ between time 0 and t:

1. Merton framework with a down-and-out random barrier



$$P(t) = N\left[-\frac{A_t}{2} + \frac{\ln(d)}{A_t}\right] - dN\left[-\frac{A_t}{2} - \frac{\ln(d)}{A_t}\right] \tag{6}$$

where $d = V_0 e^{\lambda^2}/\bar{L}D$ and $N[\cdot]$ is the cumulative standard normal distribution.[1] Note that equation (6) implies a non-zero probability of default at $t = 0$ for $\lambda > 0$. Thus the introduction of a time-independent distribution of the barrier results in a non-zero probability of instantaneous default at time 0. This property contrasts with previous asset-based models in that they do not allow for non-zero initial default probabilities and therefore produce short-term spreads that are too low.

It is intuitive that the closer to the barrier (the smaller $V_0/\bar{L}D$) and/or the higher the asset volatility, the higher is the probability of default. In the short end (small t), the main driver of default probability is the uncertainty, represented by λ, surrounding the actual level of the barrier. The higher λ corresponds to a higher probability of default. Moreover, λ becomes an even more important parameter as one moves higher up the credit spectrum.

The par credit spread S_P, which is the fee of a credit default swap whose value is zero, can be found by applying an asset-specific recovery rate R:

$$S_p(t) = (1-R)\frac{-\int_0^t Z_{rf}(\tau)dP(\tau) + 1 - P(0)}{\int_0^t Z_{rf}(\tau)P(\tau)d\tau} \tag{7a}$$

where $Z_{rf}(\tau)$ is the risk-free discount factor.[2] It is important to distinguish the asset-specific recovery R from the global recovery L. While L is the average recovery on the company's overall outstanding debt, which is a parameter in determining the default probability, R is the recovery for a specific debt of the company, which is a parameter in the valuation of the credit spread for this particular bond or loan. The asset-specific recovery R for an unsecured debt is usually lower than $\bar{L}$ as the secured debt will have higher recovery. Define $p(t) = -1/t \ln(P(t))$ as the annualised probability of default. A simple approximate expression for the par credit spread S_P can be obtained from (7a) by assuming a constant $p(t) \cong p$, and $P(0) \cong 1$ such that:

$$S_p(t) \cong (1-R)p = -(1-R)\ln(P(t))/t \tag{7b}$$

We have yet to link the initial asset value V_0 and the asset volatility σ to market observables in order to use equation (6) to calculate the probability of no default. We accomplish this by examining the boundary conditions. In particular, we focus on the long end (large t), since the short end (small t) is mainly driven by λ. Let S be the stock price and σ_S be the stock volatility. For large t, the two regimes that become more and more probable with time, especially for high-yield names, are away from the barrier ($S >> LD$) and near default ($S \to 0$). Consider:

$$\eta = \frac{1}{\sigma}\ln\left(\frac{V}{LD}\right) = \frac{V}{\sigma_S S}\frac{\partial S}{\partial V}\ln\left(\frac{V}{LD}\right) \tag{8}$$

which represents the distance (in a lognormal space) between the asset value and the default barrier in terms of standard deviation. Clearly η plays an important role in determining probability of no default (equation (6)). We want to establish the boundary condition for η. Near default ($S \to 0$), we have:

$$V \cong V_{(S=0)} + \frac{\partial V}{\partial S}S + o(S^2) = LD + \frac{1}{\alpha}S + o(S^2) \tag{9}$$

where $\alpha = \partial S/\partial V$ and $V_{(S=0)} = LD$. Substituting V into equation (8), up to the first-order approximation of S, we have:

$$\eta \cong \frac{1}{\sigma_S} \tag{10}$$

for the regime $S \to 0$. Away from the barrier ($S >> LD$), we assume $S/V \to 1$ as a boundary condition for V. Therefore:

$$\eta \cong \frac{1}{\sigma_S}\ln\left(\frac{S}{LD}\right) \tag{11}$$

for the regime $S >> LD$. The simplest expression for η that simultaneously satisfies the boundary conditions (10) and (11) is:

$$\eta = \frac{S+LD}{\sigma_S S}\ln\left(\frac{S+LD}{LD}\right) \tag{12}$$

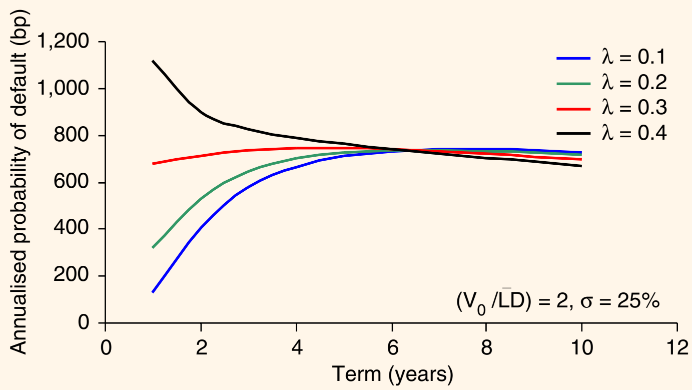
Comparing (12) and (8), we have $V = S + LD$. Thus:

$$V_0 = S_0 + \bar{L}D \tag{13}$$

for the initial value V_0 of V at time $t = 0$ where S_0 is the current stock price, and:

Uncertainty of default barrier

Traditional asset-based models generally produce short-end spreads that are too low, as one cannot reach the barrier by pure diffusion in a short period. Hull & White (2001) proposed to overcome this problem by using a time-dependent default barrier that is calibrated to the market. An alternative approach is to incorporate a jump process. Here, we simply introduce explicitly the uncertainty of the default barrier, as shown by extensive empirical studies of recovery rates in the event of default. The short end of the term structure of default probabilities is primarily driven by λ, as illustrated in the figure. ■



[1] *In deriving equation (6), the Itô process Y_t can be viewed as a process that, for $\lambda \neq 0$, begins in the past at $-\Delta t = -\lambda^2/\sigma^2$ with a constant drift $-\sigma^2/2$ and a constant variance rate σ^2. The probability of no default thus obtained (equation (6)) implicitly includes the possibility of Y_t crossing the barrier in the period $[-\Delta t, 0]$. An alternative to this approximation is to integrate over the barrier distribution. We can then obtain a closed-form solution using the cumulative bivariate normal distribution. The numerical differences between the two approaches are marginal for practical cases. We would like to thank an anonymous referee for pointing out this alternative approach*

[2] *In the case of a constant risk-free rate, we can obtain a closed-form formula for the par spread:*

$$S_P(t) = r(1-R)\frac{1 - P(0) + e^{r\xi}\left(G(t+\xi) - G(\xi)\right)}{P(0) - P(t)e^{-rt} - e^{r\xi}\left(G(t+\xi) - G(\xi)\right)}$$

where r is the continuous compounded risk-free rate, $\xi = \lambda^2/\sigma^2$, and the function $G(u)$ is given by Rubinstein & Reiner (1991):

$$G(u) = d^{\frac{1}{2}+z}N\left(-\frac{\ln(d)}{\sigma\sqrt{u}} - z\sigma\sqrt{u}\right) + d^{\frac{1}{2}-z}N\left(-\frac{\ln(d)}{\sigma\sqrt{u}} + z\sigma\sqrt{u}\right)$$

where:

$$z = \sqrt{\frac{1}{4} + \frac{2r}{\sigma^2}}$$

$$\sigma = \sigma_S^* \frac{S^*}{S^* + \bar{L}D} \tag{14}$$

for the asset volatility using some given share price S^* and implied share price volatility σ_S^*. Here, we limit ourselves to an assessment of the share price volatility σ_S^*, obtained from either historical or implied data, which is representative for a given stock price S^*. With (13) and (14), the closed-form formula involves only market observable parameters.[3]

Another assumption that has been made is that the asset has zero drift ($\mu_V = 0$), which again can be validated at the boundary where $S \to 0$, and $S >> LD$. Keep in mind that for pricing credit, it is not the asset drift itself, but rather the drift of the asset relative to the default boundary that is relevant. Near the default barrier, we have:

$$\mu_V dt = \frac{dV}{V} = \frac{S}{\alpha V}\frac{dS}{S} = (r - p)\frac{S}{\alpha V}dt$$

where r is the stock financing rate and p is the dividend yield, respectively. Thus $\mu_V \cong 0$ for $S \to 0$. On the other hand, $\mu_D = 0$ since the debt notional should be steady when the company is near default. Away from the barrier, it is reasonable to assume that the company would issue more debt as the equity value grows, or pay a dividend to keep the leverage level, D/V, steady. Thus, $dV/V = dD/D$, ie, $\mu_V = \mu_D$. Therefore, the drift of the assets relative to the default barrier would in fact become zero.

The debt-per-share D is determined based on financial data from consolidated statements as follows. One first calculates all the liabilities that participate in the financial leverage of the firm. These include the principal value of all financial debts, short-term and long-term borrowings, convertible bonds, as well as quasi-financial debts such as capital leases, or underfunded pension liabilities or preferred shares depending on their financial burden to the firm. Non-financial liabilities such as account payables, deferred taxes, reserves, etc, are not included. Debt-per-share is then calculated by dividing the value of the liabilities by the equivalent number of shares. This includes the number of common shares outstanding as well as any equivalent number of shares necessary to account for other classes of shares or other instruments of the firm's equity capital. The financial data used in the debt-per-share calculation must be adjusted for recent past events or future predictable events that are already priced in by the market.

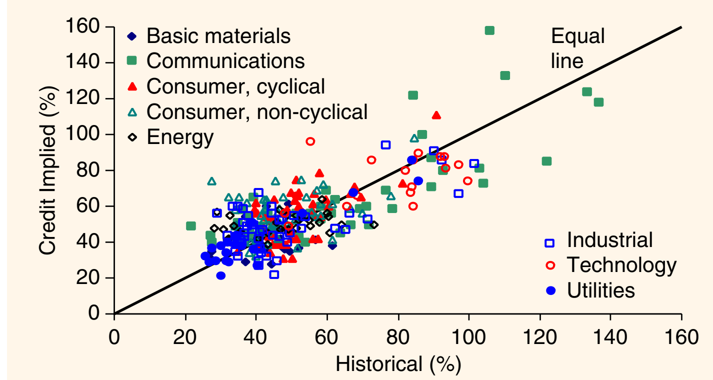
Empirical data

The mean ($\bar{L}$) and the percentage standard deviation (λ) of the global recovery L have been estimated using the Portfolio Management Data/Standard & Poor's database. The database contains actual recovery data for about 300 non-financial US companies that defaulted from 1987 to 1997. Defaulted instruments include bonds and bank loans. Based on the study of this historical data, $\bar{L}$ and λ are estimated to be 0.5 and 0.3, respectively. A lower λ is expected for the financial sector due to the sector-specific government regulations. In the empirical studies below, we assume $\bar{L} = 0.5$ and $\lambda = 0.3$, and we exclude banks, brokerages and other financial firms.

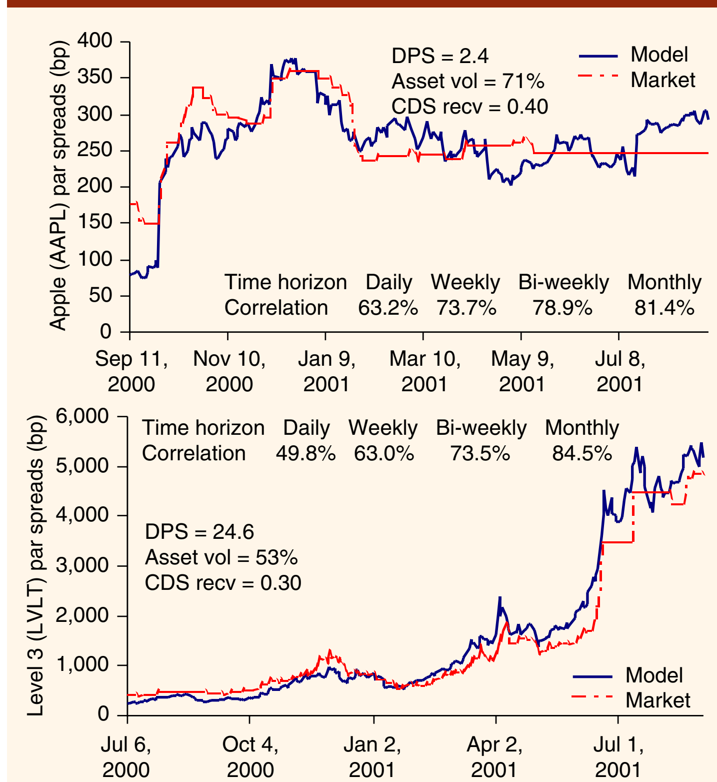
We now examine statistically how well the model works for a portfolio of names. We took a snapshot of the JP Morgan Chase closing par spreads of three-year (the shortest tenor that is liquid) credit default swaps (CDS) as of July 2, 2001 for 298 names, which represents all the JP Morgan Chase North America actively traded credit derivatives, of both investment grade and high yield, excluding banks, brokerages and other financial companies as well as companies without public equity. We then calculated the CDS implied stock volatilities and plotted those against the 360-day (the longest horizon available from Bloomberg) historical stock volatilities (see figure 2). The CDS implied stock volatility is found by solving equation (7a) to match the CDS market spread for a given stock price and debt-per-share of the company. Note that the data points are scattered fairly evenly across the 45° line. This holds even within the same industrial sector and across different credit ratings.

We have compared time series of model spreads with those of market spreads. Figure 3 shows two examples that compare the five-year CDS market par spreads (JP Morgan Chase closing CDS spreads) against the model par spreads, as calculated using equation (7a). We calculate time correlation between variation of the model spreads and that of the market spreads over different time intervals, namely, $\rho(u_i, v_i)$, where $u_i = \ln(C^{mod}_{i+\Delta i}/C^{mod}_i)$, $v_i = \ln(C^{mkt}_{i+\Delta i}/C^{mkt}_i)$ and C^{mod}_i and C^{mkt}_i are the model par spread and the market par spread, respectively, with Δi = one day, one week, two weeks and one month. The figure clearly shows that, even though the variations may be poorly correlated on a daily basis, they tend to catch up with each other over a period of two weeks to a month. Note that the

2. Historical v. credit implied stock volatility for 298 investment-grade and high-yield corporates

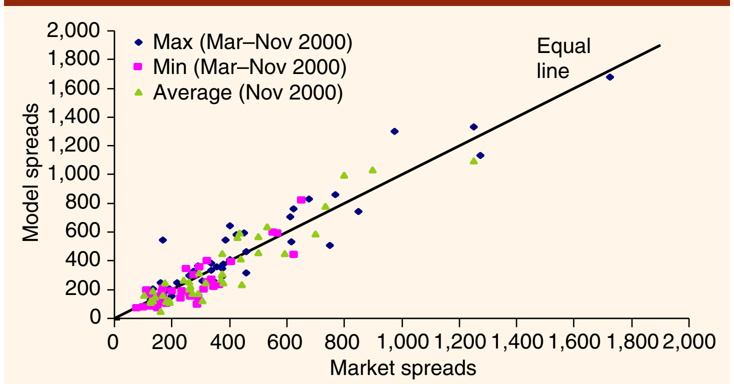


3. Comparison of the five-year CDS market par spreads with the model par spreads



[3] *Note V_0 does not necessarily correspond to the real initial asset value, nor does σ necessarily correspond to the real asset volatility of the firm. Nevertheless, these simple expressions are able to approximate the distance to default η in equation (12)*

4. Maximum, minimum and average model spreads v. those of market spreads for 39 high-yield names



Probability of no default

The probability of no default formula (Lardy, Finkelstein, Khuong-Huu & Yang, 2000):

$$P(t) = N\left[-\frac{A_t}{2} + \frac{\ln(d)}{A_t}\right] - dN\left[-\frac{A_t}{2} - \frac{\ln(d)}{A_t}\right]$$

is expressed as a function of market observable parameters:

$$A_t^2 = \left(\sigma_S^* \frac{S^*}{S^* + \bar{L}D}\right)^2 t + \lambda^2 \qquad d = \frac{S_0 + \bar{L}D}{\bar{L}D} e^{\lambda^2}$$

where S_0 is the initial stock price; S^* is the reference stock price; σ_S^* is the reference stock volatility; D is the debt per share; $\bar{L}$ is the global debt recovery; λ is the percentage standard deviation of the default barrier; and N[·] is the cumulative standard normal distribution function. ■

model par spreads in figure 3 are calculated using daily closing stock prices. However, a fixed asset volatility for the whole period is used because of the lack of historical implied volatility. Adjusting the asset volatility along the path according to equation (14), as one would do in practice as the market moves, should result in higher time correlation.

Figure 4 compares, for 39 high-yield names from various industry sectors, the maximum and minimum model spreads against those of the CDS market (JP Morgan Chase closing CDS spreads) from March to November 2000. The figure also shows the comparison of the average spreads for November. The model spreads are calculated using a fixed volatility for the whole period. Note again that the data points are located evenly across the 45° line, indicating that the difference between the model spreads and the market spreads goes in both directions for a diversified portfolio of names. The model demonstrates robust capabilities in difficult market conditions and over time.

Define $\varepsilon^i = (C^{mod} - C^{mkt})/C^{mkt}$ as the basis for reference name i. The results of figures 3 and 4 suggest that for a diversified portfolio of names, ε is fairly evenly distributed around zero. We have examined the pair-wise correlation of the variation of basis, ie, $\rho(\Delta\varepsilon^i, \Delta\varepsilon^j)$, where $\Delta\varepsilon^i_k = \varepsilon^i_{k+1} - \varepsilon^i_k$ from July 6, 2000 to February 8, 2001 between any two names in a diversified portfolio of 15 companies from different industry sectors, including technology, energy, cable, airline, healthcare, chemical, auto parts and retail. The basis varies partly because the credit market does not always react in the same way and to the same degree as the equity market would react to any given piece of information. Not surprisingly, companies from the same or similar industry sector tend to have higher correlation for basis variations due to the fact that, from time to time, the securities of these companies are driven by sector-specific news. Nevertheless, the average correlation for the diversified portfolio is reasonably low, at about 5%, with a standard deviation of 10%. Thus, the basis risk is diversifiable.

Conclusion

We have presented a simple closed-form formula that provides a robust relationship between default probability, equity price and volatility. The formula involves only market observable parameters and is capable of producing various shapes for the term structure of default probability. Back testing using historical market data suggests that the model is not biased, that it tracks the trends of the spread movements in a robust manner and that the basis risk is diversifiable.

This approach allows the use of equity derivatives to hedge credit risks on a name-by-name basis in a diversified portfolio of exposures. It can be applied as a price discovery tool to determine a fair market price of the credit risk for illiquid credits that have public equity, or as an arbitrage tool to build a proprietary portfolio of relative value of debt, equities and equity derivatives. The closed-form formula of default probability can be used as a credit rating tool that feeds off of observable parameters, and more frequently updated market data such as stock prices. In addition, the simplicity and transparency of the closed-form solution makes it an ideal candidate to perform risk management and analysis of large portfolios of credit exposures. The computational efficiency of such a closed-form formula would allow a large portfolio to be recalculated daily, or even more frequently during the day on a real-time basis, and allow the computation of economic capital, value-at-risk and stress tests. ■

George Pan is a vice-president in credit derivatives research at JP Morgan Chase in New York. **He would like to thank Jean-Pierre Lardy, Lionnel Pradier and Vladimir Finkelstein for their contribution to this work and for helpful discussions in writing this article**

Comments on this article may be posted on the technical discussion forum on the Risk website at http://www.risk.net

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