Stable ground states for the relativistic gravitational Vlasov-Poisson system

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Abstract

We consider the three dimensional gravitational Vlasov-Poisson (GVP) system in both classical and relativistic cases. The classical problem is subcritical in the natural energy space and the stability of a large class of ground states has been derived by various authors. The relativistic problem is critical and displays finite time blow up solutions. Using standard concentration compactness techniques, we however show that the breaking of the scaling symmetry allows the existence of stable relativistic ground states. A new feature in our analysis which applies both to the classical and relativistic problem is that the orbital stability of the ground states does not rely as usual on an argument of uniqueness of suitable minimizers –which is mostly unknown– but on strong rigidity properties of the transport flow, and this extends the class of minimizers for which orbital stability is now proved.

1 Introduction

1.1 Setting of the problem

We consider the three dimensional gravitational Vlasov-Poisson (GVP) system

$$\left\{\begin{array}{l} \partial_t f + \dfrac{v}{\sqrt{1+|v|^2/c^2}}\cdot\nabla_x f - \nabla\phi_f\cdot\nabla_v f = 0, \quad (t,x,v)\in\mathbb{R}_+\times\mathbb{R}^3\times\mathbb{R}^3 \\ f(t=0,x,v) = f_0(x,v)\geq 0, \end{array}\right. \tag{1.1}$$

where

$$\phi_f(x) = -\frac{1}{4\pi|x|}\star\rho_f, \quad \rho_f(x) = \int_{\mathbb{R}^3} f(x,v)\,dv, \tag{1.2}$$

and $c\in]0,+\infty]$ is the dimensionless light speed. The value $c=+\infty$ recovers the classical Vlasov-Poisson system, which is a nonlinear transport equation describing the mechanical state of a stellar system subject to its own gravity (see for instance [4, 10]). In some situations when high velocities occur, relativistic corrections should be introduced, see Van Kampen and Felderhof [35], Glassey and Schaeffer [13] and references therein. A more accurate model is then provided by (1.1) with $0 < c < +\infty$, which is the so called three dimensional relativistic gravitational Vlasov-Poisson system.

These systems are Hamiltonian and all smooth enough solutions to (1.1) satisfy the conservation of the L^q norm and of the total energy (Hamiltonian) on their lifespan:

$$\forall t, \quad \forall q \in [1, +\infty], \qquad |f(t)|_{L^q} = |f_0|_{L^q}, \quad \mathcal{H}_c(f(t)) = \mathcal{H}_c(f(0)), \tag{1.3}$$

with

$$\mathcal{H}_c(f(t)) = \int_{\mathbb{R}^6} \gamma_c(v) f(t,x,v)dxdv - \frac{1}{2}\int_{\mathbb{R}^3} |\nabla \phi_f(t,x)|^2 dx, \tag{1.4}$$

$$\gamma_c(v) = c^2 \left(\sqrt{1 + |v|^2/c^2} - 1\right). \tag{1.5}$$

In particular

$$\mathcal{H}_\infty(f(t)) = \frac{1}{2}\int_{\mathbb{R}^6} |v|^2 f(t,x,v)dxdv - \frac{1}{2}\int_{\mathbb{R}^3} |\nabla \phi_f(t,x)|^2 dx. \tag{1.6}$$

The Cauchy problem for (1.1) in the classical case ($c = +\infty$) is subcritical in the energy space and smooth initial data, say $f_0 \in \mathcal{C}_0^1$ (compactly supported $\mathcal{C}^1$ functions), yield global in time solutions, see Pfaffelmoser [31], Lions, Perthame [28]. The key to global existence is the uniform bound on the kinetic energy which follows from the interpolation estimate:

$$|\nabla \phi_f|_{L^2}^2 \leq K \, \big||v|^2 f\big|_{L^1}^{1/2} |f|_{L^1}^{\frac{7p-9}{6(p-1)}} |f|_{L^p}^{\frac{p}{3(p-1)}}, \tag{1.7}$$

for $9/7 < p < +\infty$. In the relativistic case $0 < c < +\infty$, the Cauchy theory of smooth solutions is so far restricted to smooth radial data

$$f_0 \in \mathcal{C}_{0,rad}^1 = \{f : \mathbb{R}^{2N} \to [0, +\infty) \text{ with radial symmetry, compactly supported and } \mathcal{C}^1\},$$

see Glassey, Schaeffer [13], Kiessling, Tahvildar-Zadeh [21], and then again a uniform bound on the kinetic energy suffices to ensure global existence. However, the relativistic problem is critical according to the interpolation estimate:

$$|\nabla \phi_f|_{L^2}^2 \leq K \, \big||v| f\big|_{L^1} |f|_{L^1}^{\frac{2p-3}{3(p-1)}} |f|_{L^p}^{\frac{p}{3(p-1)}}, \tag{1.8}$$

for $3/2 < p < +\infty$. Glassey and Schaeffer proved in [13] that radially symmetric solutions to (1.1) (for $c < +\infty$) with negative Hamiltonian blow up in finite time. In [25], a *stable* self similar blow up dynamic for the relativistic problem corresponding to a concentration phenomenon is fully described.

We address in this paper the question of the existence and the stability of ground states for the relativistic problem. In the classical case, this question has attracted considerable attention. A large class of stationary solutions has been constructed in [3], among which the functions of the microscopic energy $F(\frac{|v|^2}{2}+\phi)$. The question of stability of such steady states has been addressed in many works in the past and still stimulates a number of research programs. The first work on this subject goes back to Antonov (1960') [1, 2], where a stability criterion for polytropes was established for the *linearized* GVP equation. Then, this linear stability has been improved to a non linear stability by Wolansky [37] for the so-called polytropes and later extended by Guo [14] and, Guo and Rein [16] to more general steady states. These analyses are based on the construction of steady states as minimizers of one-parameter energy-Casimir functionals:

$$\inf_{|f|_{L^1}=M} \mathcal{H}_\infty(f) + \int_{\mathbb{R}^6} j(f)dxdv, \tag{1.9}$$

where j is a suitable strictly convex function on $\mathbb{R}_+$. Extensions of this approach can also be found in [15, 17], completed by Schaeffer [34]. Non variational approaches based on linearization techniques have also been explored in [36] and, more recently in [18].

In [23, 22], we observed that the frame of concentration compactness techniques as introduced by Lions [29], [30] directly allows one to derive the existence of a *two parameters* family of ground states –in accordance with the scaling symmetry of the problem– corresponding to the minimization problem:

$$I(M_1, M_j) = \inf_{|f|_{L^1}=M_1,\ |j(f)|_{L^1}=M_j} \mathcal{H}(f), \quad M_1, M_j > 0 \tag{1.10}$$

for a large class of convex functions j. In the case of polytropes $j(t) = t^p$, this has also been independently observed by Sánchez and Soler [33]. Note also that physical investigations around these minimization problems can be found in [7] and the references therein.

1.2 Statement of the results

Here we propose to extend the variational approach of [23, 22] to the relativistic framework. Very few papers have been devoted to stability issues in this setting. A stability result of some steady states solutions has been obtained recently by Hadžić and Rein [19] for specific perturbations. Our main claim in this paper is that following Lieb, Yau [27], the breaking of the scaling symmetry in the relativistic case allows one to derive a similar variational theory of ground states like in the classical case under an additional subcritical size assumption. This will lead to a stability theory of ground states in the full energy space.

To wit, consider a strictly convex and even function $j : \mathbb{R} \to \mathbb{R}^+$ satisfying the following assumptions.
(B1) j is a $\mathcal{C}^1$ strictly convex function with $j(0) = j'(0) = 0$.
(B2) There exists $p > 3/2$ such that:

$$j(t) \geq Ct^p, \quad \forall t \geq 0. \tag{1.11}$$

(B3) There exist $p_1, p_2 > 3/2$, such that

$$p_1 \leq \frac{tj'(t)}{j(t)} \leq p_2 \qquad \forall t > 0. \tag{1.12}$$

Note that this assumption (B3) is equivalent to the usual dichotomy condition

$$b^{p_1} j(t) \leq j(bt) \leq b^{p_2} j(t), \quad \forall\ b \geq 1,\ t \geq 0. \tag{1.13}$$

Indeed, (1.13) is equivalent to

$$\forall b \geq 1, \quad \forall t \geq 0, \qquad (bt)^{-p_1} j(bt) \geq t^{-p_1} j(t), \qquad (bt)^{-p_2} j(bt) \leq t^{-p_2} j(t),$$

which means that $t^{-p_1} j(t)$ is nondecreasing and $t^{-p_2} j(t)$ is nonincreasing on $\mathbb{R}_+$. Taking the derivative yields (1.12).

For a function j satisfying (B1), (B2), (B3), we define the corresponding energy space

$$\mathcal{E}_j = \{f \geq 0 \ \text{ with } \ |f|_{\mathcal{E}_j} = |f|_{L^1} + |j(f)|_{L^1} + |\gamma_c(v) f|_{L^1} < +\infty\}. \tag{1.14}$$

From the interpolation inequality (1.8), one can define a positive constant K_j by

$$K_j = \inf_{f \in \mathcal{E}_j - \{0\}} \frac{||v| f|_{L^1} \, |f|_{L^1}^{\frac{2p-3}{3(p-1)}} \, |j(f)|_{L^1}^{\frac{1}{3(p-1)}}}{|\nabla \phi_f|_{L^2}^2}, \tag{1.15}$$

where $\mathcal{E}_j$ is the relativistic energy space (1.14).

Proposition 1.1 (Existence of relativistic ground states) *Let j be a real function satisfying assumptions (B1), B2), (B3). Let $M_1 > 0, M_j > 0$, and $c \in]0, +\infty]$ be such that*

$$M_1^{\frac{2p-3}{3(p-1)}} \, M_j^{\frac{1}{3(p-1)}} < 2cK_j, \tag{1.16}$$

where K_j is defined by (1.15)*. Let $\mathcal{F}(M_1, M_j) = \{f \in \mathcal{E}_j,\ |f|_{L^1} = M_1,\ \ |j(f)|_{L^1} = M_j\}$. Then every minimizing sequence of the problem:*

$$I_c(M_1, M_j) = \inf_{f \in \mathcal{F}(M_1, M_j)} \mathcal{H}_c(f), \tag{1.17}$$

where $\mathcal{H}_c$ is defined by (1.4)*, is relatively strongly compact in the energy space $\mathcal{E}_j$ up to a translation shift in the space variable x. Moreover, a minimizer Q_j of* (1.17) *is of the form*

$$Q_j(x, v) = (j')^{-1} \left(\frac{\gamma_c(v) + \phi_{Q_j}(x) - \lambda}{\mu} \right)_+, \quad \lambda, \mu < 0, \tag{1.18}$$

where ϕ_{Q_j} is linked to Q_j according to (1.2)*, has radial symmetry up to a translation shift and is a $\mathcal{C}^2$ function on $\mathbb{R}^3$. Eventually, Q_j is a compactly supported stationary solution to (1.1).*

Here we denote $t_+ = \max(t, 0)$ for all $t \in \mathbb{R}$.

Remark 1 *Note that the criterion (1.16) is optimal in the sense that for* $M_1^{\frac{2p-3}{3(p-1)}} M_j^{\frac{1}{3(p-1)}} > 2cK_j$*, one can prove from scaling argument that* $I_c(M_1, M_j) = -\infty$*. Remark also from the definition* (1.15) *of* K_j *and from the conservation laws* (1.3) *that an initial data* $f_0 \in \mathcal{C}^1_{0,rad}$ *satisfying*

$$|f_0|_{L^1}^{\frac{2p-3}{3(p-1)}} |j(f_0)|_{L^1}^{\frac{1}{3(p-1)}} < 2cK_j$$

provides a global in time solution to (1.1)*.*

Following Cazenave, Lions [6], Proposition 1.1 classically implies the orbital stability in the energy space of *the family of ground states* $\{Q_j\}$ only, and not of each Q_j individually. Indeed a standard difficulty occurs here related to the uniqueness of the minimizer to (1.17) –up to the symmetries–. In the classical case and for one-parameter variational problems, this issue was overcome in the analysis by Guo and Rein [17] thanks to a uniqueness result due to Schaeffer [34]. However, uniqueness of the minimizer, even only locally, in the general framework of Proposition 1.1 is a problem of independent importance and up to now mostly open.

We now claim that the orbitally stability of Q_j may be derived even though uniqueness is not known using extra rigidities of the flow provided by the nonlinear transport. In particular, *equimeasurability properties* of the flow will allow us to prove some local isolatedness of the Q_j, and the following theorem follows which completes the analysis of orbital stability of [22] and extends it to the relativistic case:

Theorem 1.2 (Orbital stability of classical and relativistic ground states) *Let j be a real function satisfying assumptions (B1), B2), (B3). Let* $M_1 > 0$, $M_j > 0$ *and* $c \in]0, +\infty]$ *be such that* (1.16) *holds. Then any minimizer* Q_j *of* (1.17) *is orbitally stable under the flow* (1.1) *in the energy space* (1.14)*. More precisely, given* $\varepsilon > 0$*, there exists* $\delta(\varepsilon) > 0$ *such that the following holds true.*

i) Classical case $c = +\infty$*. Let* $f_0 \in \mathcal{C}^1_0$ *with* $|f_0 - Q_j|_{\mathcal{E}_j} \leq \delta(\varepsilon)$*, and let f(t) be the classical solution to (1.1) with initial data* f_0*, then there exists a translation shift* $x(t) \in \mathbb{R}^3$ *such that* $\forall t \in [0, +\infty)$*,*

$$|f(t, x + x(t), v) - Q_j|_{\mathcal{E}_j} < \varepsilon\,.$$

ii) Relativistic case $0 < c < +\infty$*. Let* $f_0 \in \mathcal{C}^1_{0,rad}$ *with* $|f_0 - Q_j|_{\mathcal{E}_j} \leq \delta(\varepsilon)$*, then the classical solution f(t) to (1.1) with initial data* f_0 *is global in time and* $\forall t \in [0, +\infty)$*,*

$$|f(t) - Q_j|_{\mathcal{E}_j} < \varepsilon\,.$$

Remark 2 *In the classical case* $c = +\infty$*, the result of Theorem 1 can also be formulated in the framework of weak (or renormalized) solutions, as the Cauchy theory in this case is completely understood [20, 8, 9].*

Together with the results of [25] on blow up dynamics, Theorem 1.2 shows that the gravitational Vlasov Poisson system displays at least two stable dynamics: a global dynamic near ground state type solutions with a size strictly below the critical size required for blow up, a stable self similar blow up dynamic which may occur just above the critical threshold. This situation seems to be the generic one for critical problems with broken scaling invariance, see for example [32] for similar results in the context of nonlinear Schrödinger equations. Let us also mention also the works [11], [12] on the pseudo-relativistic Boson star equation which are somehow connected both on the physical and mathematical side and where stable subcritical ground states are exhibited while a critical type finite time blow up regime is expected for larger masses.

2 Existence of relativistic ground states

This section is devoted to the proof of Proposition 1.1 which is a consequence of the standard concentration compactness techniques, [29]. We shall adapt the proof of [22].

2.1 Properties of the infimum

Let us start by summarizing some monotonicity properties of the infimum (1.17).

Proposition 2.1 (Monotonicity properties of the infimum $I_c(M_1, M_j)$) *Let j be a real function satisfying assumptions (B1)-(B3) and let $M_1 > 0, M_j > 0$ and $c \in]0, +\infty]$ be such that* (1.16) *holds. Let $I_c(M_1, M_j)$ be the infimum defined by* (1.17)*, then we have:*

$$-\infty < I_c(M_1, M_j) < 0 \tag{2.19}$$

and there holds the nondichotomy condition: for all $0 < \alpha < 1$ and $0 \leq \beta \leq 1$,

$$I_c(\alpha M_1, \beta M_j) + I_c((1-\alpha)M_1, (1-\beta)M_j) > I_c(M_1, M_j). \tag{2.20}$$

Proof. The case $c = +\infty$ is treated in [22, 23] and we therefore assume $0 < c < +\infty$.

Step 1. The infimum is negative.

We first prove (2.19). Let $\mathcal{H}_c$ given by (1.4) and $f \in \mathcal{E}_j$ with $|f|_{L^1} = M_1$ and $|j(f)|_{L^1} = M_j$, then from the definition (1.15) of K_j, we have:

$$\begin{aligned}
\mathcal{H}_c(f) &\geq \left(\int_{\mathbb{R}^6} \gamma_c(v) f\right) - \frac{1}{2K_j}\left(\int |v| f\right) M_1^{\frac{2p-3}{3(p-1)}} M_j^{\frac{1}{3(p-1)}} \\
&\geq \left(\int_{\mathbb{R}^6} \gamma_c(v) f\right) - \frac{1}{2cK_j}\left(\int (\gamma_c(v) + c^2) f\right) M_1^{\frac{2p-3}{3(p-1)}} M_j^{\frac{1}{3(p-1)}} \\
&\geq \left(\int_{\mathbb{R}^6} \gamma_c(v) f\right)\left(1 - \frac{1}{2cK_j} M_1^{\frac{2p-3}{3(p-1)}} M_j^{\frac{1}{3(p-1)}}\right) - \frac{c}{2K_j} M_1^{\frac{5p-6}{3(p-1)}} M_j^{\frac{1}{3(p-1)}} \\
&\geq -\frac{c}{2K_j} M_1^{\frac{5p-6}{3(p-1)}} M_j^{\frac{1}{3(p-1)}},
\end{aligned} \tag{2.21}$$

where we have used condition (1.16) and the fact that $\sqrt{1+|v|^2/c^2} \geq |v|/c$. This proves that $I_c(M_1, kM_j)$ is bounded from below. To prove that it is negative, we use a rescaling argument. Assume that f is moreover compactly supported and let $\tilde{f}(x,v) = f(\frac{x}{\lambda}, \lambda v)$, $\lambda > 0$, then:

$$\begin{aligned}
\mathcal{H}_c(\tilde{f}) &= \frac{1}{\lambda}\left(\int_{\mathbb{R}^6} c^2\left(\sqrt{\lambda^2+|v|^2/c^2}-\lambda\right)f - \frac{1}{2}|\nabla\phi_f|^2_{L^2}\right) \\
&= \frac{1}{\lambda}\int_{\mathbb{R}^6} \frac{|v|^2 f}{\sqrt{\lambda^2+\frac{|v|^2}{c^2}}+\lambda} - \frac{1}{2\lambda}|\nabla\phi_f|^2_{L^2} \\
&\sim -\frac{1}{2\lambda}|\nabla\phi_f|^2_{L^2} \quad \text{as} \quad \lambda \to +\infty
\end{aligned}$$

and (2.19) follows.

Step 2. Monotonicity properties of the infimum.

We now claim the following monotonicity properties: for all $0 < k \leq 1$,

$$I_c(M_1, kM_j) \geq k^{\frac{1}{3(p_2-1)}} I_c(M_1, M_j), \tag{2.22}$$

$$I_c(kM_1, M_j) \geq k^{\frac{5p_1-6}{3(p_1-1)}} I_c(M_1, M_j). \tag{2.23}$$

Proof of (2.22). We fix a real number $0 < k \leq 1$ and consider $f \in \mathcal{E}_j$ such that $|f|_{L^1} = M_1$ and $|j(f)|_{L^1} = kM_j$. We introduce the rescaled function

$$\tilde{f}(x,v) = \alpha f(\alpha^{1/3}x, v),$$

then $|\tilde{f}|_{L^1} = M_1$ and $|j(\tilde{f})|_{L^1} = h(\alpha, f)kM_j$ where:

$$h(\alpha, f) = \frac{|j(\alpha f)|_{L^1}}{\alpha |j(f)|_{L^1}}. \tag{2.24}$$

Observe that $h(1,f) = 1$ and $h(\alpha, f) \to +\infty$ as $\alpha \to +\infty$ from (1.13), hence from $k \leq 1$, we can find $\alpha \geq 1$ such that

$$h(\alpha, f) = \frac{1}{k} \quad \text{and thus} \quad |j(\tilde{f})|_{L^1} = M_j.$$

Moreover, from $\alpha \geq 1$ and (1.13):

$$\frac{1}{k} \leq \alpha^{p_2-1}. \tag{2.25}$$

We now compute

$$\mathcal{H}_c(\tilde{f}) = \int_{\mathbb{R}^6} \gamma_c(v) f - \frac{1}{2}\alpha^{1/3}|\nabla\phi_f|^2_{L^2},$$

and thus from (2.25):

$$I_c(M_1, M_j) \leq \mathcal{H}_c(\tilde{f}) \leq \int_{\mathbb{R}^6} \gamma_c(v) f - \frac{1}{2}\left(\frac{1}{k}\right)^{\frac{1}{3(p_2-1)}} |\nabla \phi_f|^2_{L^2} \leq \left(\frac{1}{k}\right)^{\frac{1}{3(p_2-1)}} \mathcal{H}_c(f),$$

where we have used that $k \leq 1$. This concludes the proof of (2.22).
Proof of (2.23). Similarly, we take f such that $|f|_{L^1} = kM_1$ and $|j(f)|_{L^1} = M_j$ and set

$$\tilde{f}(x, v) = \alpha f(\alpha^{1/3} k^{1/3} x, v).$$

We have $|\tilde{f}|_{L^1} = M_1$ and $|j(\tilde{f})|_{L^1} = h(\alpha, f) k^{-1} M_j$. Now from (B1), $h(0, f) = 0$ and we may find $\alpha \leq 1$ such that

$$h(\alpha, f) = k \ \text{ and thus } \ |j(\tilde{f})|_{L^1} = M_j.$$

Moreover, from $\alpha \leq 1$ and (1.13):

$$k \leq \alpha^{p_1 - 1}$$

from which:

$$\begin{aligned} I_c(M_1, M_j) \leq \mathcal{H}_c(\tilde{f}) &= k^{-1} \int_{\mathbb{R}^6} \gamma_c(v) f - \frac{1}{2} k^{-5/3} \alpha^{1/3} |\nabla \phi_f|^2_{L^2} \\ &\leq k^{-1} \int_{\mathbb{R}^6} \gamma_c(v) f - \frac{1}{2} k^{-5/3} k^{\frac{1}{3(p_1-1)}} |\nabla \phi_f|^2_{L^2} \\ &\leq k^{-\frac{5p_1-6}{3(p_1-1)}} \mathcal{H}_c(f), \end{aligned}$$

where we have used the fact that $\frac{5p_1-6}{3(p_1-1)} > 1$ from $p_1 > 3/2$. Hence (2.23) follows.

Step 3. The nondichotomy property.

Let $0 < \alpha < 1$, $0 \leq \beta \leq 1$, then (2.22) and (2.23) imply:

$$I_c(\alpha M_1, \beta M_j) \geq \alpha^{\frac{5p_1-6}{3(p_1-1)}} \beta^{\frac{1}{3(p_2-1)}} I_c(M_1, M_j),$$

and a similar inequality by exchanging α and β with $1-\alpha$ and $1-\beta$ respectively. As $I_c(M_1, M_j) < 0$, (2.20) is equivalent to the following

$$\alpha^{\frac{5p_1-6}{3(p_1-1)}} \beta^{\frac{1}{3(p_2-1)}} + (1-\alpha)^{\frac{5p_1-6}{3(p_1-1)}} (1-\beta)^{\frac{1}{3(p_2-1)}} < 1$$

which holds true since $p_2 > 1$ and $\frac{5p_1-6}{3(p_1-1)} > 1$ (since $p_1 > 3/2$). This concludes the proof of Proposition 2.1. $\square$

2.2 The concentration compactness argument

We now prove Proposition 1.1. We adapt the argument from [22, 23] to which we refer for more details.

Proof of Proposition 1.1

Step 1. Compactness of the minimizing sequences.

Let $M_1, M_j > 0$, then from Lemma 2.1 we know that $I_c(M_1, M_j)$ is finite and negative. Take then a minimizing sequence f_n of (1.17):

$$|f_n|_{L^1} = M_1, \quad |j(f_n)|_{L^1} = M_j \quad \text{and} \quad \lim_{n\to+\infty} \mathcal{H}(f_n) = I_c(M_1, M_j). \tag{2.26}$$

Let

$$\rho_n(x) = \int_{\mathbb{R}^3} f(x,v)dv.$$

We know from the concentration compactness principle developed in [29, 30] that there exists a subsequence ρ_{n_k} for which one of the three possibilities occurs (B_R being the ball of radius R centered at the origin in $\mathbb{R}^3$):

- *Compactness*: there exists $y_k \in \mathbb{R}^3$ such that

$$\forall \varepsilon > 0, \quad \exists R < +\infty \quad \text{such that for all} \quad k \geq 1 \quad \int_{y_k+B_R} \rho_{n_k}(x)dx \geq M_1 - \varepsilon\,; \tag{2.27}$$

- *Vanishing*:

$$\forall R < +\infty, \quad \lim_{k\to+\infty} \sup_{y\in\mathbb{R}^3} \int_{y+B_R} \rho_{n_k}(x)dx = 0\,;$$

- *Dichotomy:* there exists $m \in (0, M_1)$ such that for all $\varepsilon > 0$, there exist subsequences $(\rho_k^1)_{k\geq1},\ (\rho_k^2)_{k\geq1} \in L^1_+(\mathbb{R}^3)$ and $k_0 \geq 1$ such that for all $k \geq k_0$,

$$\begin{cases} \rho_{n_k} = \rho_k^1 + \rho_k^2 + w_k \quad \text{with} \quad 0 \leq \rho_k^1, \rho_k^2, w_k \leq \rho_{n_k}, \quad \rho_k^1\rho_k^2 = \rho_k^1 w_k = \rho_k^2 w_k = 0 \quad \text{a.e.,} \\ dist(Supp(\rho_k^1), Supp(\rho_k^2)) \to +\infty \quad \text{as} \quad k \to +\infty, \\ \left|\rho_{n_k} - \rho_k^1 - \rho_k^2\right|_{L^1} \leq \varepsilon, \quad \left|\int_{\mathbb{R}^3} \rho_k^1(x)dx - m\right| + \left|\int_{\mathbb{R}^3} \rho_k^2(x)dx - (M_1 - m)\right| < \varepsilon. \end{cases}$$

We claim that only compactness can occur. Indeed, if vanishing occurs, then from Lemma 3.1 in [22],

$$|\nabla\phi_{f_{n_k}}|_{L^2} \to 0, \quad as \quad k \to 0.$$

Passing to the limit into

$$\mathcal{H}_c(f_{n_k}) = \int_{\mathbb{R}^6} \gamma_c(v) f_{n_k} - \frac{1}{2}|\nabla\phi_{f_{n_k}}|^2_{L^2} \geq -|\nabla\phi_{f_{n_k}}|^2_{L^2},$$

leads to $I_c(M_1, M_j) \geq 0$ which contradicts property (2.19). Dichotomy cannot occur as it would violate the nondichotomy property (2.20), see again [22] for further details.

We conclude that the compactness occurs on a subsequence. Now observe from (2.26), (B2) that f_n is a bounded sequence in $L^p \cap L^1$. Moreover, the kinetic energy $\int_{\mathbb{R}^6} \gamma_c(v) f_n(x,v)dxdv$ is also bounded thanks to the lower bound (2.21). This implies the compactness in the velocity variable v. We deduce that we have L^1 compactness in x up to a translation shift and that no concentration can occur in x, v due to the L^p boundedness. Thus the Dunford-Pettis criterion ensures:

$$f_{n_k}(. + y_k) \rightharpoonup f \quad \text{in} \quad L^1, L^p$$

and from (2.27):

$$\int_{\mathbb{R}^6} f(x,v)dxdv = M_1.$$

This implies from a standard compactness argument –see [22]–:

$$|\nabla \phi_{f_{n_k}}|_{L^2}^2 \to |\nabla \phi_f|_{L^2}^2,$$

and thus by lower semi-continuity of the L^p norms:

$$\mathcal{H}_c(f) \leq I_c(M_1, M_j), \qquad |j(f)|_{L^1} \leq M_j. \tag{2.28}$$

Then from (2.22),

$$I_c(M_1, M_j) \geq \mathcal{H}_c(f) \geq I_c(M_1, |j(f)|_{L^1}) \geq \left(\frac{|j(f)|_{L^1}}{M_j}\right)^{\frac{1}{3(p_2-1)}} I_c(M_1, M_j).$$

This implies from (2.19) that $|j(f)|_{L^1} \geq M_j$ and thus, from (2.28), we obtain

$$|f|_{L^1} = M_1 \quad \text{and} \quad |j(f)|_{L^1} = M_j$$

which together with (2.28) implies that f is a minimizer of (1.17). Moreover, we get:

$$|f_{n_k}|_{L^1} \to |f|_{L^1}, \quad |\gamma_c(v)(f_{n_k})|_{L^1} \to |\gamma_c(v) f|_{L^1}, \quad |j(f_{n_k})|_{L^1} \to |j(f)|_{L^1}.$$

We now conclude from standard convexity arguments, see [22], [5], that $f_{n_k}(. + y_k) \to f$ in L^1, $|v|^2 f_{n_k}(. + y_k) \to |v|^2 f$ in L^1 and $j(f_{n_k}(. + y_k) - f)) \to 0$ in L^1, hence the strong convergence in the energy space $\mathcal{E}_j$.

Step 2. Euler-Lagrange equation for the minimizer.

Let Q_j be a minimizer of (1.17). Let Q_j^* be the nondecreasing symmetric rearrangement of Q_j in x –see [26] for instance–, then $|Q_j^*|_{L^1} = M_1$, $|j(Q_j^*)|_{L^1} = M_j$ and $\mathcal{H}_c(Q_j^*) \leq \mathcal{H}_c(Q_j)$, this inequality being strict unless ϕ_{Q_j} is radial in x up to a space translation shift. From now on, without loss of generality, we assume that ϕ_{Q_j} is radial around 0.

From standard Euler-Lagrange theory –see [22] for further details–, there exist constants λ and μ such that:

$$\gamma_c(v) + \phi_{Q_j}(x) = \lambda + \mu j'(Q_j), \quad \text{on the support of } Q_j\,. \tag{2.29}$$

We now claim that $\lambda, \mu < 0$. We first multiply (2.29) by Q_j and integrate over $(x, v) \in \mathbb{R}^6$:

$$\int_{\mathbb{R}^6} \gamma_c(v) Q_j + \int_{\mathbb{R}^3} \phi_{Q_j} Q_j = \lambda M_1 + \mu \int_{\mathbb{R}^6} j'(Q_j) Q_j. \tag{2.30}$$

Next, we multiply (2.29) by $v \cdot \nabla_v Q_j$ and integrate by parts to get:

$$\int_{\mathbb{R}^6} \gamma_c(v) Q_j + \int_{\mathbb{R}^3} \phi_{Q_j} Q_j + \frac{1}{3} \int_{\mathbb{R}^6} \frac{|v|^2}{\sqrt{1 + |v|^2/c^2}} Q_j = \lambda M_1 + \mu M_j. \tag{2.31}$$

Similarly, we multiply (2.29) by $x \cdot \nabla_x Q_j$ and get:

$$\int_{\mathbb{R}^6} \gamma_c(v) Q_j + \frac{5}{6} \int_{\mathbb{R}^3} \phi_{Q_j} Q_j = \lambda M_1 + \mu M_j. \tag{2.32}$$

Combining (2.31) with (2.32), we get the relativistic Viriel identity:

$$\int_{\mathbb{R}^6} \frac{|v|^2}{\sqrt{1 + |v|^2/c^2}} Q_j = -\frac{1}{2} \int_{\mathbb{R}^3} \phi_{Q_j} Q_j. \tag{2.33}$$

Therefore, we have

$$I_c(M_1, M_j) = \mathcal{H}_c(Q_j) = -c^2 \int_{\mathbb{R}^6} \left(1 - \frac{1}{\sqrt{1 + |v|^2/c^2}} \right) Q_j. \tag{2.34}$$

Let

$$E_{kin}(Q_j) = \int_{\mathbb{R}^6} \gamma_c(v) Q_j, \tag{2.35}$$

then, we have from (2.33)

$$\int_{\mathbb{R}^6} \frac{|v|^2}{\sqrt{1 + |v|^2/c^2}} Q_j = E_{kin}(Q_j) - I_c(M_1, M_j). \tag{2.36}$$

Now, reporting (2.36) and (2.33) in relations (2.30) and (2.32), we get

$$-E_{kin}(Q_j) + 2 I_c(M_1, M_j) = \lambda M_1 + \mu \int_{\mathbb{R}^6} j'(Q_j) Q_j \tag{2.37}$$

and

$$-\frac{2}{3} E_{kin}(Q_j) + \frac{5}{3} I_c(M_1, M_j) = \lambda M_1 + \mu M_j \tag{2.38}$$

Subtracting these two last relations gives:

$$3\mu \int_{\mathbb{R}^6} \big(j'(Q_j)Q_j - j(Q_j)\big) = I_c(M_1, M_j) - E_{kin}(Q_j) < 0 \tag{2.39}$$

and thus the strict convexity of j implies $\mu < 0$. Now reporting the expression of μ in (2.38) leads to

$$\begin{aligned} &3\lambda M_1 \int_{\mathbb{R}^6} \big(j'(Q_j)Q_j - j(Q_j)\big) = \\ &\quad -E_{kin}(Q_j) \int_{\mathbb{R}^6} \big(2j'(Q_j)Q_j - 3j(Q_j)\big) + I_c(M_1, M_j) \int_{\mathbb{R}^6} \big(5j'(Q_j)Q_j - 6j(Q_j)\big), \end{aligned} \tag{2.40}$$

thus $\lambda < 0$ from (B3).

Step 3. Regularity of the potential and compact support of Q_j.

Let us now prove the $\mathcal{C}^2$ regularity of ϕ_{Q_j} on $[0, +\infty)$. Using the Euler-Lagrange equation (2.29), we have

$$\rho_{Q_j}(x) = \int_{\mathbb{R}^3} (j')^{-1} \left(\frac{\gamma_c(v) + \phi_{Q_j}(x) - \lambda}{\mu} \right)_+ dv.$$

We then pass to the spherical velocity coordinate $u = |v|$ and perform the change of variable $q = \gamma_c(v)/|\mu|$, to get

$$\rho_{Q_j}(x) = 4\pi \int_0^{+\infty} (j')^{-1} \left(\frac{\phi_{Q_j}(r) - \lambda}{\mu} - q \right)_+ |\mu| c \left(1 + \frac{|\mu| q}{c^2}\right) \left[\left(1 + \frac{|\mu| q}{c^2}\right)^2 - 1 \right]^{1/2} dq. \tag{2.41}$$

Using this expression, we shall now control ρ_{Q_j} by a power of the potential ϕ_{Q_j}. We first recall that the support of Q_j is contained in the set of (x, v) such that

$$\gamma_c(v) + \phi_{Q_j}(r) - \lambda \leq 0, \quad \text{with} \quad r = |x|. \tag{2.42}$$

This in particular implies that $0 \leq \lambda - \phi_{Q_j}(r) \leq |\phi_{Q_j}(r)|$ and $|\phi_{Q_j}| \geq |\lambda| > 0$, on the support of ρ_{Q_j}. From (1.12) we also have $(j')^{-1}(s) \leq C s^{1/(p-1)}$, hence we straightforwardly get from (2.41)

$$\begin{aligned} \rho_{Q_j}(x) &\leq K \int_0^{+\infty} \left(\frac{\phi_{Q_j}(r) - \lambda}{\mu} - q \right)_+^{\frac{1}{p-1}} (1 + q^2) dq. \\ &\leq K \left(\frac{\phi_{Q_j}(r) - \lambda}{\mu} \right)_+^{1 + \frac{1}{p-1}} + K \left(\frac{\phi_{Q_j}(r) - \lambda}{\mu} \right)_+^{3 + \frac{1}{p-1}} \leq K |\phi_{Q_j}(r)|^{3 + \frac{1}{p-1}} \end{aligned}$$

on the support of ρ_{Q_j} (recall indeed that $|\phi_{Q_j}| \geq |\lambda| > 0$ on this support). Now we follow a bootstrap argument as in [25] (proof of Proposition 2): if $\rho_{Q_j} \in L^{q_k}$, then $\phi_{Q_j} = \frac{1}{4\pi|x|} \star \rho_{Q_j} \in L^{r_k}$ with $r_k = \frac{3q_k}{3-2q_k}$ from Hardy, Littlewood, Sobolev so that $\rho_{Q_j} \in L^{q_{k+1}}$ with

$$q_{k+1}\left(3 + \frac{1}{p-1}\right) = r_k \quad \text{ie} \quad q_{k+1} = \frac{3(p-1)q_k}{(3p-2)(3-2q_k)}.$$

A simple analysis of the sequence q_k with $q_0 = \frac{6}{5}$ shows that $p > \frac{3}{2}$ implies that there exists $k_0 = k_0(p)$ such that $q_{k_0} > \frac{3}{2}$ and thus $\rho_{Q_j} \in L^{q_{k_0}}$. From Sobolev embeddings, this implies that $\phi_{Q_j} \in \mathcal{C}^{0,\alpha}$ for some $0 < \alpha < 1$. Finally, by (2.41), ρ_{Q_j} is continuous on $[0, +\infty)$, which is enough to deduce from the Laplace equation

$$(r^2\phi'_{Q_j}(r))' = r^2\rho_{Q_j}(r), \tag{2.43}$$

that ϕ_{Q_j} is a strictly increasing $\mathcal{C}^2$ function on $[0, +\infty)$. Furthermore, as $\rho_{Q_j} \in L^1 \cap L^{q_{k_0}}$ with $q_{k_0} > \frac{3}{2}$, the potential $\phi_{Q_j} = \frac{1}{4\pi|x|} \star \rho_{Q_j}$ goes to 0 at infinity.

To complete the proof, we now show that Q_j is compactly supported. Observe from (2.42) and the monotonicity of ϕ_{Q_j} on $[0, +\infty[$ that $\gamma_c(v) \leq \lambda - \phi_{Q_j}(0)$ and $\phi_{Q_j}(r) \leq \lambda < 0$ on the support of Q_j. These two last inequalities, together with the fact that $\phi_{Q_j}(r)$ goes to 0 as $r \to +\infty$, imply that the support of Q_j is compact in (x, v). □

3 Orbital stability of the ground states

We now prove the orbital stability of the ground states Q_j without the knowledge of the uniqueness of the minimizer for the problem (1.17). The key is a uniqueness statement of the minimizer under a condition of equimeasurability which is inherited from the transport evolution, see Proposition 3.1.

3.1 Proof of theorem 1.2

We treat the case $c < \infty$ which forces us to restrict to compactly supported smooth radial solutions due to the Cauchy theory. The case $c = +\infty$ would be treated similarly without the radial assumption restriction.

Proof of Theorem 1.2

Let Q_0 be a radial minimizer of (1.17) and assume that the result of Theorem 1.2 is false. Then there exist $\varepsilon > 0$ and sequences $f_0^n \in \mathcal{C}^1_{0,rad}$, $t_n > 0$, such that

$$\lim_{n \to +\infty} |f_0^n - Q_0|_{\mathcal{E}_j} = 0, \tag{3.44}$$

and

$$\forall n \geq 0, \quad |f^n(t_n, x, v) - Q_0|_{\mathcal{E}_j} \geq \varepsilon, \tag{3.45}$$

where $f^n(t,x,v)$ is a solution to (1.1) with initial data f_0^n. This means in particular that

$$\lim \mathcal{H}_c(f_0^n) = I_c(M_1, M_j), \qquad \lim |f_0^n|_{L^1} = M_1, \qquad \lim |j(f_0^n - Q_0)|_{L^1} = 0. \tag{3.46}$$

In particular, f_0^n converges to Q_0 in the strong L^p topology and hence almost everywhere, up to a subsequence. From assumptions (B1)–(B3) and the convexity of j, this implies form classical argument –see Theorem 2 in Brézis and Lieb [5]–:

$$|j(f_0^n)|_{L^1} - |j(Q_0)|_{L^1} \to 0 \ \text{ as } \ n \to +\infty. \tag{3.47}$$

Let now $g_n(x,v) = f^n(t_n,x,v)$. Then from the conservation properties of the Vlasov-Poisson flow, there holds

$$\lim_{n\to+\infty} \mathcal{H}_c(g_n) = I_c(M_1, M_j), \quad |g_n|_{L^1} = M_1, \qquad |j(g_n)|_{L^1} = M_j\,,$$

which means that g_n is a minimizing sequence of (1.17). From Proposition 1.1, g_n is relatively strongly compact in $\mathcal{E}_j$:

$$g_n \to Q_1 \ \text{ in } \ \mathcal{E}_j \tag{3.48}$$

for some minimizer Q_1 –without any translation due to the radial assumption–. Moreover, for all smooth compactly supported θ, there holds the conservation law:

$$\int_{\mathbb{R}^6} \theta(g_n) = \int_{\mathbb{R}^6} \theta(f_0^n), \tag{3.49}$$

and hence passing to the limit $n \to +\infty$ from (3.44) and (3.48), we get:

$$\int_{\mathbb{R}^6} \theta(Q_1) = \int_{\mathbb{R}^6} \theta(Q_0). \tag{3.50}$$

Now, from standard arguments, (3.50) implies the equimeasurability of Q_1 and Q_0:

$$\forall t > 0, \quad meas\{Q_0(x,v) > t\} = meas\{Q_1(x,v) > t\}.$$

Let now $h_n = f^n(t'_n, x, v)$ where t'_n is the time such that

$$\left|f^n(t'_n,x,v) - Q_0\right|_{\mathcal{E}_j} = \frac{\varepsilon}{2}, \tag{3.51}$$

which is well defined by the continuity of the flow. Then arguing as above for Q_1, $h_n \to Q_2$ in $\mathcal{E}_j$ where Q_2 is a minimizer of (1.17) which satisfies the equimeasurability property:

$$\forall t > 0, \quad meas\{Q_0(x,v) > t\} = meas\{Q_1(x,v) > t\} = meas\{Q_2(x,v) > t\}. \tag{3.52}$$

In particular, from (3.45) and (3.51), $\{Q_0, Q_1, Q_2\}$ are three distinct solutions to the Euler-Lagrange equation (1.18) for some a priori distinct Euler-Lagrange multipliers $(\lambda_j, \mu_j)_{0\le j\le 2}$, and satisfying (3.52). This now contradicts the following uniqueness statement of equimeasurable steady solutions to (1.1) which is the core of our argument.

Proposition 3.1 (Isolatedness of equimeasurable steady states) *Let $c \in]0,+\infty]$ and G be a given continuous and strictly increasing function on $\mathbb{R}_+$ such that $G(0) = 0$. Let $\mathcal{A}$ be the set of all functions of the form:*

$$Q(x,v) = G\left(\frac{\gamma_c(v) + \phi(x) - \lambda}{\mu}\right)_+, \quad \forall (x,v) \in \mathbb{R}^6, \tag{3.53}$$

where $\lambda < 0$, $\mu < 0$ are arbitrary constants and where ϕ is a $\mathcal{C}^2$ radial solution to the elliptic equation:

$$\Delta\phi = \int_{\mathbb{R}^3} G\left(\frac{\gamma_c(v) + \phi(x) - \lambda}{\mu}\right)_+ dv = \rho_Q(x), \quad \phi(r) \to 0 \quad as \quad r \to +\infty. \tag{3.54}$$

Let $Q_0 \in \mathcal{A}$, then the following set

$$\mathcal{S}(Q_0) = \{Q \in \mathcal{A}, \ s.t. \ \ meas\,\{Q(x,v) > t\} = meas\{Q_0(x,v) > t\}\,, \ \ \forall t > 0\}\,, \tag{3.55}$$

cannot contain more than two elements.

Applying Proposition 3.1 with $G = (j')^{-1}$ yields a contradiction and concludes the proof of Theorem 1.2. $\square$

3.2 Isolatedness of equimeasurable steady state solutions

We now turn to the proof of Proposition 3.1. Note that the difficulty is that the Lagrange multiplier (λ,μ) in (3.53) are a priori assumed to be different for the solutions we consider. If they were the same, uniqueness in the ODE sense for (3.53) would immediately conclude the proof (see below). Even in the more restricted setting of solutions being minimizers of (1.17), the Lagrange multipliers cannot be simply connected to quantities conserved by the flow due to the breaking of the scaling symmetry –this would be the case for $c = +\infty$ where the proof can therefore be simplified–.

Proof of Proposition 3.1

Let Q_0, Q be two radially symmetric solutions of (3.53) for some parameters $(\lambda_0,\mu_0) < 0$, $(\lambda,\mu) < 0$.

Step 1. Comparison of the L^∞ norms.

Let ϕ_0, ϕ be the Poisson potentials associated respectively to Q_0, Q. We claim the first relation:

$$\left(\frac{\phi(0) - \lambda}{\mu}\right)_+ = \left(\frac{\phi_0(0) - \lambda_0}{\mu_0}\right)_+. \tag{3.56}$$

Proof of (3.56): As ϕ_0 is radial, we shall use the abuse of notation $\phi_0(x) = \phi_0(r)$ with $r = |x|$, and write the Laplace equation $\Delta\phi_0 = \rho_0(x) = \int_{\mathbb{R}^3} Q_0(x, v)dv$, as

$$r^2\phi_0'(r) = \int_0^r s^2\rho_0(s)ds.$$

In particular this implies that ϕ_0 is a nondecreasing function in r and we have: $\phi_0(r) \geq \phi_0(0)$. Now, since G is nondecreasing and since $\mu < 0$, we have from (3.53):

$$\forall(x, v) \in \mathbb{R}^6, \quad Q_0(x, v) \leq G\left(\frac{\phi_0(0) - \lambda_0}{\mu_0}\right) = |Q_0|_{L^\infty}. \tag{3.57}$$

We now observe that

$$|Q|_{L^\infty} = |Q_0|_{L^\infty}. \tag{3.58}$$

Indeed, if $|Q_0|_{L^\infty} < |Q|_{L^\infty}$, we may choose t such that $|Q_0|_{L^\infty} < t < |Q|_{L^\infty}$ to get

$$meas\{Q_0(x, v) > t\} = 0 \ \text{ and } \ meas\{Q(x, v) > t\} > 0$$

contradicting (3.55). (3.57)) and (3.58) now imply (3.56).

Step 2. Two possible values for μ

From (3.56) and (3.57),

$$a = \frac{\phi(0) - \lambda}{\mu} = \frac{\phi_0(0) - \lambda_0}{\mu_0} > 0 \tag{3.59}$$

or otherwise $Q = Q_0 = 0$ and the proof is over. From (3.55),

$$meas\{Q_0(x, v) > t\} = meas\{Q(x, v) > t\} > 0, \quad \forall t \in [0, G(a)],$$

which we rewrite equivalently:

$$meas\left\{\gamma_c(v) + \phi_0(r) - \phi_0(0) < |\mu_0|(a - \tau)\right\} = meas\left\{\gamma_c(v) + \phi(r) - \phi(0) < |\mu|(a - \tau)\right\},$$

for all $\tau \in [0, a]$. We now identify the leading terms in this equality when $\tau \to a$. First remark that the set $\{\gamma_c(v) + \phi_0(r) - \phi_0(0) < |\mu_0|(a - \tau)\}$ goes to $\{0\}$ when $\tau \to a$. Therefore, by expanding $\gamma_c(v)$ near $v = 0$ and $\phi_0(r)$ near $r = 0$ and taking the leading terms, we get

$$meas\left\{\gamma_c(v) + \phi_0(r) - \phi_0(0) < |\mu_0|(a - \tau)\right\} \sim meas\left\{\frac{|v|^2}{2} + \phi_0''(0)\frac{|x|^2}{2} < |\mu_0|(a - \tau)\right\},$$

when $t \to a$. The measure (in $\mathbb{R}^6$) of the above rhs can be computed explicitly leading to

$$meas\left\{\gamma_c(v) + \phi_0(r) - \phi_0(0) < |\mu_0|(a - \tau)\right\} \sim K\left(\frac{-\mu_0}{\sqrt{\phi_0''(0)}}\right)^3 (a - \tau)^3, \tag{3.60}$$

for some universal constant $K > 0$, and thus:

$$\frac{\phi''(0)}{\mu^2} = \frac{\phi_0''(0)}{\mu_0^2}. \tag{3.61}$$

Now, we shall use the elliptic equations satisfied by ϕ_0 and ϕ. The Laplace equation (3.54) for ϕ can be written is spherical coordinates as

$$\phi''(r) + 2\frac{\phi'(r)}{r} = \int_{\mathbb{R}^3} G\left(\frac{\gamma_c(v) + \phi(r) - \lambda}{\mu}\right)_+ dv.$$

Taking $r = 0$ in this equation and using (3.59), we get:

$$3\phi''(0) = \int_{\mathbb{R}^3} G\left(\frac{\gamma_c(v)}{\mu} + a\right)_+ dv.$$

Let us now focus onto the case $c < +\infty$. Passing to the spherical velocity coordinate $u = |v|$ and performing the change of variable

$$q = \frac{\gamma_c(u)}{|\mu|},$$

we obtain:

$$3\phi''(0) = 4\pi \int_0^{+\infty} G\left((a-q)_+\right)|\mu| c \left(1 + \frac{|\mu| q}{c^2}\right)\left[\left(1 + \frac{|\mu| q}{c^2}\right)^2 - 1\right]^{1/2} dq.$$

A similar identity holds for ϕ_0 and μ_0. Combining this with (3.61), we get that μ must be a solution to

$$F(|\mu|) = F(|\mu_0|), \tag{3.62}$$

where F is defined on $]0, +\infty[$ by

$$F(s) = c\int_0^a \frac{1}{s}\left(1 + \frac{sq}{c^2}\right)\left[\left(1 + \frac{sq}{c^2}\right)^2 - 1\right]^{1/2} G(a-q)dq. \tag{3.63}$$

We compute the derivatives of $F(s)$ and find

$$F''(s) = \int_0^a \frac{q^2}{c^3}\left(\frac{q}{c^2} + \frac{3}{s}\right)\left[\left(1 + \frac{sq}{c^2}\right)^2 - 1\right]^{-3/2} G(a-q)dq > 0, \tag{3.64}$$

and thus F is strictly convex on $\mathbb{R}_+$. Therefore, for given μ_0, equation (3.62) in $|\mu|$ has at most two solutions and thus $\mu_0, \mu < 0$ ensures that μ can take at most two values.

Remark 3 *If $c = +\infty$, then F given by* (3.63) *simplifies into $F(s) = Ks^{-1/2}$ for some universal constant $K > 0$. In particular, F is a nonincreasing function in this case and thus* (3.62) *implies $\mu = \mu_0$. In this case, the set $\mathcal{S}(Q_0)$ defined by* (3.55) *is reduced to $\{Q_0\}$.*

Step 3. Conclusion

We now claim from an ODE type argument that $\mu_0 = \mu$ implies $Q_0 = Q$. Indeed, let then

$$\psi_0(r) = \phi_0(r) - \lambda_0, \quad \text{and} \quad \psi(r) = \phi(r) - \lambda,$$

then from (3.59) and the radial symmetry, we have

$$\psi(0) = \psi_0(0), \quad \text{and} \quad \psi'(0) = \psi_0'(0) = 0. \tag{3.65}$$

Moreover, ψ and ψ_0 solve the same radial Laplace equation (since $\mu = \mu_0$):

$$\Delta\psi = \int_{\mathbb{R}^3} G\left(\frac{\gamma_c(v) + \psi(r)}{\mu}\right)_+ dv = h(\psi(r)), \tag{3.66}$$

with

$$h(u) = 4\pi \int_0^u \left(1 + \frac{u-q}{c^2}\right)\left[\left(1 + \frac{u-q}{c^2}\right)^2 - 1\right]^{1/2} G\left(\frac{q}{\mu}\right) dq.$$

It is clear that h is a $\mathcal{C}^1$ function of u thus, together with (3.65), uniqueness for the ODE (3.66) implies that $\psi = \psi_0$. Hence $Q = Q_0$ from (3.53). This concludes the proof of Proposition 3.1. □

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References

[1] Antonov, A. V., Remarks on the problem of stability in stellar dynamics. *Soviet Astr., AJ.,* **4**, 859-867 (1961).

[2] Antonov, A. V., Solution of the problem of stability of a stellar system with the Emden density law and spherical velocity distribution. *J. Leningrad Univ.Se. Mekh. Astro.* **7**, 135-146 (1962).

[3] Batt, J.; Faltenbacher, W.; Horst, E., Stationary spherically symmetric models in stellar dynamics, Arch. Rat. Mech. Anal. 93, 159-183 (1986).

[4] Binney, J.; Tremaine, S., Galactic Dynamics, Princeton University Press, 1987.

[5] Brézis, H.; Lieb, E.: A relation between pointwise convergence of functions and convergence of functionals. Proceedings of the American Mathematical Society. Volume 88, no 3, July 1983.

[6] Cazenave, T.; Lions, P.-L. Orbital stability of standing waves for some nonlinear Schrödinger equations, Comm. Math. Phys. 85 (1982), no. 4, 549–561.

[7] Chavanis P. H. Dynamical stability of collisionless stellar systems and barotropic stars: the nonlinear Antonov first law. A&A, 451, 109-123, 2006.

[8] DiPerna, R. J.; Lions, P.-L., Solution globale de type Vlasov-Poisson. C. R. Acad. Sci. Paris Sér I math. 307 (1988), no 12, 655-658.

[9] DiPerna, R. J.; Lions, P.-L., Global weak solutions of kinetic equations. Rend. Sem. Mat. Univ. Politec. Torino 46 (1988), no. 3, 259–288 (1990).

[10] Fridmann, A. M.; Polyachenko, V. L., Physics of gravitating systems, Springer-Verlag

[11] Fröhlich, J.; Jonsson, B.; Lars G.; Lenzmann, E., Boson stars as solitary waves, Comm. Math. Phys. 274 (2007), no. 1, 1–30.

[12] Fröhlich, J.; Lenzmann, E., Blowup for nonlinear wave equations describing boson stars, Comm. Pure Appl. Math. 60 (2007), no. 11, 1691–1705

[13] Glassey, R.T.; Schaeffer, J., On symmetric solutions of the relativistic Vlasov-Poisson system, Comm. Math. Phys. 101 (1985), no. 4, 459–473.

[14] Guo, Y., Variational method for stable polytropic galaxies, Arch. Rat. Mech. Anal. 130 (1995), 163-182.

[15] Guo, Y., On the generalized Antonov's stability criterion. *Contem. Math.* **263**, 85-107 (2000)

[16] Guo, Y.; Rein, G., Stable steady states in stellar dynamcics, Arch. Rat. Mech. Anal. 147 (1999), 225–243.

[17] Guo, Y.; Rein, G., Isotropic steady states in galactic dynamics, Comm. Math. Phys. 219 (2001), 607–629.

[18] Guo, Y.; Rein, G., A non-variational approach to nonlinear stability in stellar dynamics applied to the King model. Comm. Math. Phys. 271 (2007), no. 2, 489–509.

[19] Hadžić, M.; Rein, G., Global existence and nonlinear stability for the relativistic Vlasov-Poisson system in the gravitational case , Indiana Univ. Math. J. 56, 2453-2488 (2007).

[20] Horst, E.; Hunze, R., Weak solutions of the initial value problem for the unmodified nonlinear Vlasov equation. Math. Methods Appl. Sci. 6. (1984), no. 2, 262-279.

[21] Kiessling M. K. H, Tahvildar-Zadeh A. S, On the relativistic Vlasov-Poisson system, arXiv preprint 0708.1760, to appear in Indiana Univ. Math. J.

[22] Lemou, M.; Méhats, F.; Raphaël, P., On the orbital stability of the ground states and the singularity formation for the gravitational Vlasov-Poisson system, Arch. Ration. Mech. Anal. **189** (2008), no. 3, 425–468.

[23] Lemou, M.; Méhats, F.; Raphaël, P., Orbital stability and singularity formation for Vlasov-Poisson systems. C. R. Math. Acad. Sci. Paris 341 (2005), no. 4, 269–274.

[24] Lemou, M.; Méhats, F.; Raphaël, P., Structure of the linearized gravitational Vlasov-Poisson system close to a polytropic ground state, SIAM J. Math. Anal., **39** (2008), no. 6, 1711–1739.

[25] Lemou, M.; Méhats, F.; Raphaël, P., Stable self-similar blow-up dynamics for the three dimensional gravitational Vlasov-Poisson system, J. Amer. Math. Soc. **21** (2008), no. 4, 1019-1063.

[26] Lieb, E. H.; Loss, M., Analysis. Second edition. Graduate Studies in Mathematics, 14. American Mathematical Society, Providence, RI, 2001.

[27] Lieb, E.H.; Yau, H.T., The Chandrasekhar theory of stellar collapse as the limit of quantum mechanics, Comm. Math. Phys. 112 (1987), no. 1, 147–174.

[28] Lions, P.-L.; Perthame, B., Propagation of moments and regularity for the 3-dimensional Vlasov-Poisson system. Invent. Math. 105 (1991), no. 2.

[29] Lions, P.-L., The concentration-compactness principle in the calculus of variations. The locally compact case. I. Ann. Inst. H. Poincaré Anal. Non Linéaire 1 (1984), no. 2, 109–145.

[30] Lions, P.-L., The concentration-compactness principle in the calculus of variations. The locally compact case. II. Ann. Inst. H. Poincaré Anal. Non Linéaire 1 (1984), no. 4, 223–283.

[31] Pfaffelmoser, K.; Global classical solutions of the Vlasov-Poisson system in three dimensions for general initial data, J. Diff. Eq. **95** (1992), 281-303.

[32] Planchon, F.; Raphaël, P., Existence and stability of the log-log blow-up dynamics for the L^2-critical nonlinear Schrödinger equation in a domain, Ann. Henri Poincaré 8 (2007), no. 6, 1177–1219

[33] Sánchez, Ó.; Soler, J., Orbital stability for polytropic galaxies. Ann. Inst. H. Poincaré Anal. Non Linéaire 23 (2006), no. 6, 781–802.

[34] Schaeffer, J., Steady States in Galactic Dynamics, Arch. Rational, Mech. Anal. **172** (2004), 1–19.

[35] Van Kampen, N.G.; Felderhof, B.V., Theoretical methods in plasma physics, Amsterdam, North Holland 1967.

[36] Wan, Y-H, On nonlinear stability of isotropic models in stellar dynamics, Arch. Ration. Mech. Anal 147 (1999), no.3, 245-268.

[37] Wolansky, G., On nonlinear stability of polytropic galaxies. *Ann. Inst. Henri Poincaré,* **16**, 15-48 (1999).