Uniqueness of the critical mass blow up solution for the four dimensional gravitational Vlasov-Poisson system

Mohammed Lemou[(1),(4)], Florian Méhats[(2)] and Pierre Raphaël[(3),(4)]

[(1)] *Université Paul Sabatier, MIP, Toulouse, France*
[(2)] *IRMAR, Université Rennes 1, France*
[(3)] *Université Paris-Sud, Orsay, France*
[(4)] *CNRS*

Abstract

We study the gravitational Vlasov-Poisson system $\partial_t f + v \cdot \nabla_x f - E \cdot \nabla_v f = 0$, $E(x) = \nabla\phi(x)$, $\Delta\phi(x) = \int f(x,v)dv$, in dimension $N = 4$ where the problem is L^1 critical. We proved in [16] a sharp criterion for the global existence of weak solutions based on the variational characterization of the polytropic steady states solutions. From the existence of a pseudo-conformal symmetry, this criterion is sharp and there exist critical mass blow up solutions. We prove in this paper the uniqueness of the critical mass blow up solution. This gives in particular a first dynamical classification of the polytropic stationary solutions. The proof is an adaptation of a similar result by Frank Merle [21] for the L^2 critical nonlinear Schrödinger equation.

Unicité de la solution explosive de masse critique pour le système de Vlasov-Poisson gravitationnel en dimension 4

Resumé

Nous considérons le système de Vlasov-Poisson $\partial_t f + v \cdot \nabla_x f - E \cdot \nabla_v f = 0$, $E(x) = \nabla\phi(x)$, $\Delta\phi(x) = \int f(x,v)dv$, en dimension $N = 4$ où le problème est L^1 critique. En se basant sur une caractérisation variationnelle des solutions stationnairs polytropiques, nous avons établi dans [16] un critère optimal garantissant l'existence globale de solutions faibles pour ce système. L'optimalité de ce critère est une conséquence directe de l'existence d'une symétrie pseudo-conforme qui permet d'exhiber des solutions explosives de masse critique. Nous démontrons ici l'unicité de la solution explosive de masse critique. La preuve est une adaptation d'un résultat similaire de Frank Merle [21] pour l'équation de Schrödinger non linéaire L^2 critique.

1 Introduction

1.1 Setting of the problem

We consider the gravitational Vlasov-Poisson system in dimension 4:

$$(VP) \quad \begin{cases} \partial_t f + v \cdot \nabla_x f - E_f \cdot \nabla_v f = 0, \quad (t,x,v) \in \mathbb{R}_+ \times \mathbb{R}^4 \times \mathbb{R}^4 \\ f(t=0,x,v) = f_0(x,v) \geq 0, \\ E_f(x) = \nabla_x \phi_f \,, \qquad \phi_f(x) = -\frac{1}{4\pi^2} \int_{\mathbb{R}^4} \frac{1}{|x-y|^2} \rho_f(y)\, dy, \\ \rho_f(t,x) = \int_{\mathbb{R}^4} f(t,x,v)\, dv. \end{cases} \tag{1.1}$$

It is a generalization of the three dimensional Vlasov-Poisson system which describes the mechanical state of a stellar system subject to its own gravity, see for instance [5, 10] and [7]. From the mathematical point of view, the four dimensional case is a model problem to study the singularity formation.

From Horst, Hunze [13] and Diperna, Lions [8, 9], this system is locally well-posed in the energy space

$$\mathcal{E}_p = \{f(x,v) \geq 0 \ \text{ with } \ |f|_{\mathcal{E}_p} = |f|_{L^1} + |f|_{L^p} + ||v|^2 f|_{L^1} < +\infty\} \qquad \text{for } p \in (2, +\infty]$$

in the sense that given an initial data $f_0 \in \mathcal{E}_p$, there exists a maximal time $T > 0$ and a weak solution $f \in L^\infty_{loc}([0,T), \mathcal{E}_p)$, and either $T = +\infty$, we say the solution is global, or $T < +\infty$ and then $\lim_{t \to T} \left||v|^2 f(t)\right|_{L^1} = +\infty$, we say the solution blows up in finite time. Moreover, this solution satisfies the conservation of the L^p norms:

$$\forall t \in [0,T), \quad \forall q \in [1,p], \qquad |f(t)|_{L^q} = |f_0|_{L^q}, \tag{1.2}$$

the conservation of the total momentum:

$$\forall t \in [0,T), \quad \iint_{\mathbb{R}^4 \times \mathbb{R}^4} v f(t,x,v) dx dv = \iint_{\mathbb{R}^4 \times \mathbb{R}^4} v f(0,x,v) dx dv \tag{1.3}$$

and a uniform bound on the Hamiltonian:

$$\forall t \in [0,T), \quad \mathcal{H}(f(t)) = \iint_{\mathbb{R}^4 \times \mathbb{R}^4} |v|^2 f(t,x,v) dx dv - \int_{\mathbb{R}^4} |E_f(t,x)|^2 \, dx \leq \mathcal{H}(f_0). \tag{1.4}$$

In this text, by a weak solution we will always mean one constructed in $\mathcal{E}_p$ from the standard regularization procedure and satisfying (1.2), (1.3) and (1.4).

Let us recall that in the framework of weak solutions, the exact conservation of The Hamiltonian is related to the uniqueness of the solution in the energy space which is a major open problem and is known only for more regular initial data, see in particular Lions, Perthame [18].

A large group of symmetries in the energy space $\mathcal{E}_p$ leaves (1.1) invariant: if $f(t,x,v)$ solves (1.1), then $\forall \mu_0, \lambda_0, x_0, v_0 \in \mathbb{R}^*_+ \times \mathbb{R}^*_+ \times \mathbb{R}^4 \times \mathbb{R}^4$, so does

$$\left(\frac{\mu_0}{\lambda_0}\right)^2 f\left(\frac{t}{\lambda_0 \mu_0}, \frac{x + x_0 + v_0 t}{\lambda_0}, \mu_0(v + v_0)\right).$$

Note in particular that the two parameter scaling symmetry leaves invariant the quantity

$$|f|_{L^1}|f|_{L^p}^{\frac{p}{p-2}}.$$

Moreover, (1.1) admits another symmetry which is *not* in the energy space $\mathcal{E}_p$ but in the virial space

$$\Sigma = \mathcal{E}_p \cap \{|x|^2 f \in L^1\}, \tag{1.5}$$

the socalled pseudo-conformal symmetry: if $f(t,x)$ solves (1.1), then $\forall T \in \mathbb{R}^*$, so does:

$$f_T(\tau, x, v) = f\left(\frac{T\tau}{T+\tau}, \frac{Tx}{T+\tau}, \frac{(T+\tau)v - x}{T}\right). \tag{1.6}$$

The pseudo-conformal symmetry (1.6) formally induces the conservation of the conformal Hamiltonian:

$$\begin{aligned} \mathcal{H}\left(f_T\left(\frac{Tt}{T-t}\right)\right) &= \left(\frac{T-t}{T}\right)^2 \mathcal{H}(f(t)) + \frac{2(T-t)}{T^2}\iint_{\mathbb{R}^4\times\mathbb{R}^4} v\cdot x f(t,x,v)dxdv \\ &+ \frac{1}{T^2}\iint_{\mathbb{R}^4\times\mathbb{R}^4} |x|^2 f(t,x,v)dxdv = \mathcal{H}(f(0)). \end{aligned} \tag{1.7}$$

This property is more often expressed as the virial law for regular solutions [11]:

$$\forall t \in [0,T), \quad \frac{d^2}{dt^2}\iint_{\mathbb{R}^4\times\mathbb{R}^4} |x|^2 f(t,x)dxdv = 2\mathcal{H}(f_0). \tag{1.8}$$

In particular, given a smooth enough f_0 with nonpositive Hamiltonian $\mathcal{H}(f_0) < 0$, (1.8) implies that the quantity $\iint |x|^2 f(t,x)dxdv$ must become nonpositive in finite time and thus the solution blows up in finite time.

1.2 Variational structure and critical mass blow up solutions

Using variational tools and the Hamiltonian structure of (1.1), we exhibited in [16, 15] a sharp criterion for the existence of a global weak solution to (1.1). More precisely, let $2 < p < +\infty$ and Q_p be the polytropic stationary solution to (1.1) given by:

$$Q_p(x,v) = \begin{cases} (-1 - \frac{|v|^2}{2} - \phi_p(x))^{\frac{1}{p-1}} & \text{for } \frac{|v|^2}{2} + \phi_p(x) < -1, \\ 0 \text{ for } \frac{|v|^2}{2} + \phi_p(x) > -1 \end{cases} \tag{1.9}$$

where ϕ_p is the unique nontrivial radially symmetric nonpositive solution to:

$$\begin{cases} \Delta\phi_p - \gamma_p \max\{0, (-1-\phi_p)\}^{\frac{2p-1}{p-1}} = 0, \\ \phi_p(r) \to 0 \quad \text{as } r \to +\infty, \end{cases} \tag{1.10}$$

with $\gamma_p = 2\pi^2 \int_0^1 2t(1-t)^{\frac{1}{p-1}}dt = \frac{4\pi^2(p-1)^2}{p(2p-1)}$. We proved in [16] the following variational characterization of Q_p:

Proposition 1 (Variational characterization of Q_p, [16]) *Let $2 < p < +\infty$. Let $g \in \mathcal{E}_p$ with*

$$\mathcal{H}(g) = \mathcal{H}(Q_p) = 0 \quad \textit{and} \quad |g|_{L^1}|g|_{L^p}^{\frac{p}{p-2}} = |Q_p|_{L^1}|Q_p|_{L^p}^{\frac{p}{p-2}},$$

then there exists $\mu_0, \lambda_0, x_0 \in \mathbb{R}_+^* \times \mathbb{R}_+^* \times \mathbb{R}^4$ *such that*

$$g(x,v) = \left(\frac{\mu_0}{\lambda_0}\right)^2 Q_p\left(\frac{x+x_0}{\lambda_0}, \mu_0 v\right).$$

Moreover, we have the sharp Gagliardo-Nirenberg interpolation inequality:

$$\forall f \in \mathcal{E}_p, \quad \mathcal{H}(f) \geq ||v|^2 f|_{L^1} \left(1 - \left(\frac{|f|_{L^1}|f|_{L^p}^{\frac{p}{p-2}}}{|Q_p|_{L^1}|Q_p|_{L^p}^{\frac{p}{p-2}}}\right)^{\frac{p-2}{2(p-1)}}\right) \tag{1.11}$$

The conservation of the Hamiltonian and the L^p norm now implies:

Theorem 1 (Sharp global well posedness criterion, [16]) *Let* $2 < p < +\infty$ *and* $f_0 \in \mathcal{E}_p$ *with*

$$|f_0|_{L^1}|f_0|_{L^p}^{\frac{p}{p-2}} < |Q_p|_{L^1}|Q_p|_{L^p}^{\frac{p}{p-2}}, \tag{1.12}$$

then there exists a global weak solution $f \in L^\infty([0,+\infty), \mathcal{E}_p)$ *to (1.1).*

Under (1.12), one in fact expects the solution to asymptotically disperse, this remains to be done. Note also from (1.11) that, under condition (1.12), the Hamiltonian is nonnegative. Because the solution in this case is global, the sign of the Hamiltonian can also be deduced from (1.8).

Now a striking feature is that this criterion is sharp. Indeed, the ground state solution $f(t,x) = Q_p(x)$ is a global non dispersive solution to (1.1). We now apply the pseudo-conformal symmetry (1.6) to it and get the explicit finite time blow up solution

$$S_p(t,x,v) = Q_p\left(\frac{x}{1-t}, (1-t)v + x\right) \tag{1.13}$$

for which

$$|S_p(t)|_{L^1}|S_p(t)|_{L^p}^{\frac{p}{p-2}} = |Q_p|_{L^1}|Q_p|_{L^p}^{\frac{p}{p-2}}.$$

1.3 Statement of the result

Our aim in this paper is to prove the uniqueness of the critical mass finite time blow up solution to (1.1). We claim:

Theorem 2 (Uniqueness of the critical mass blow up solution) *Let* $2 < p < +\infty$. *Let* $f_0 \in \mathcal{E}_p$ *with*

$$|f_0|_{L^1}|f_0|_{L^p}^{\frac{p}{p-2}} = |Q_p|_{L^1}|Q_p|_{L^p}^{\frac{p}{p-2}}$$

and f *be a weak solution to (1.1) which blows up at* $0 < T < +\infty$, *ie:*

$$\lim_{t \to T} ||v|^2 f(t)|_{L^1} = +\infty. \tag{1.14}$$

Then we have the gain of regularity:

$$\forall t \in [0,T), \quad f(t) \in \Sigma \tag{1.15}$$

where Σ is the virial space defined by (1.5)*. In particular, the pseudo-conformal symmetrization f_T given by* (1.6) *is in the energy space:*

$$\forall \tau \in [0, +\infty), \quad f_T(\tau) \in \mathcal{E}_p.$$

Assume moreover the exact conservation of the conformal Hamiltonian defined by (1.7)*:*

$$\forall \tau \in [0, +\infty), \qquad \mathcal{H}(f_T(\tau)) = \mathcal{H}(f_T(0)). \tag{1.16}$$

Then, up to the set of symmetries of (1.1), we have

$$f(t) = S_p(t)$$

where S_p is the explicit critical mass blow up solution given by (1.13).

Remark 1 *Let us recall that the exact conservation of the pseudo-conformal Hamiltonian* (1.16) *formally holds but is known only if one assumes more regularity on the initial data, see Lions, Perthame [18].*

Similarly like in [16], the proof of Theorem 2 relies on a systematic comparison between (1.1) and the L^2 critical nonlinear Schrödinger equation

$$(NLS) \quad \begin{cases} iu_t = -\Delta u - |u|^{\frac{4}{N}} u, \quad (t,x) \in [0,T) \times \mathbb{R}^N \\ u(0,x) = u_0(x) \in H^1(\mathbb{R}^N), \quad u_0 : \mathbb{R}^N \to \mathbb{C}. \end{cases} \tag{1.17}$$

A sharp criterion for the global well-posedness of the solutions to (1.17) was first derived by Weinstein, [24], and again the existence of critical mass blow up solutions holds as a consequence of an explicit pseudo-conformal symmetry. Let us say that in the context of nonlinear dispersive equations, the existence or the non existence of critical mass blow up solutions, and then their possible classification, is in general a difficult open problem. For (1.17), the classification is a fundamental result by Merle, [21]. But for example, if we now consider (1.17) on a domain Ω in $\mathbb{R}^2$:

$$\begin{cases} iu_t = -\Delta u - |u|^2 u, \quad (t,x) \in [0,T) \times \Omega \\ u_{|\partial\Omega} = 0, \\ u(0,x) = u_0(x) \in H^1(\Omega), \quad u_0 : \mathbb{R}^2 \to \mathbb{C} \end{cases}$$

then the existence of critical mass blow up solutions is known, see Burq, Gérard, Tzvetkov [6], but their classification is a major open problem. On the other hand, if we consider the critical generalized KdV equation

$$(KdV) \quad \begin{cases} u_t + (u_x x + u^5)_x = 0, \quad (t,x) \in [0,T) \times \mathbb{R} \\ u(0,x) = u_0(x) \in H^1(\mathbb{R}), \quad u_0 : \mathbb{R} \to \mathbb{C} \end{cases}$$

which shares the same variational structure like (1.17), then there are no critical mass blow up solutions, see Martel and Merle, [19].

More generally, Theorem 2 lies in the framework of obtaining a *dynamical classification of the stationary polytropic solution among the solutions to the Hamiltonian system.* This kind of issues has turned out to be fundamental for the description of the asymptotic dynamics of (NLS) and (KdV) type equations both in the subcritical and critical cases, see

for example Martel and Merle [20], Merle and Raphaël [23]. Let us stress that there is in this context a major difference between (NLS) or (KdV) type systems and the gravitational (VP) system. The variational study of the first class of problem as pursued by Berestycki, Lions [4], Weinstein [24], Kwong [14], exhibits *a unique ground state solitary wave up to the set of H^1 symmetries which is isolated among the set of stationary solutions* -up to a finite number of parameters related to the symmetries of the equation-. This is in deep contrast with the (VP) system where both in dimensions 3 and 4, *any smooth enough convex functional j generates a solitary wave*, see [16], [3]. The polytropic steady state Q_p corresponds to $j(f) = f^p$, see [16] for a further discussion. Applying the pseudo-conformal symmetry to these solutions yields explicit finite time blow up solutions with a blow up profile arbitrarily close but not equal to Q_p, and thus the asymptotic dynamics of (1.1) are much richer than the one of (NLS) or (KdV) type of systems. In this context, it is remarkable that Merle's proof of the classification of the critical mass blow up solution for (1.17) may still be adapted for (1.1) and the very key here is the existence in both cases of a pseudo-conformal symmetry.

Acknowledgment: We acknowledge support by the European IHP network HYKE, by the ACINIM 176-2004 funded by the French ministry of research as well as the ACI Jeunes chercheurs no. JC1035. Pierre Raphaël would like to thank the laboratoire MIP, Université Paul Sabatier, Toulouse, France, where part of this work was done.

2 Proof of Theorem 2

The proof of Theorem 2 is an adaptation of the original proof by Merle for the L^2 critical (NLS), [21], and incorporates the technical simplifications further obtained by Merle [22], Antonini [1], Banica [2], Hmidi and Keraani [12]. Let us briefly recall the strategy.

Let $f(t,x)$ be a critical mass finite time blow up solution. We may assume up to a fixed rescaling that $|f|_{L^1} = |Q_p|_{L^1}$. The variational characterization of the ground state Q_p, Proposition 1, together with the conservation of the Hamiltonian and the L^p norm imply that the solution forms a Dirac mass in L^1 as $t \to T$ at some point $x(T)$:

$$\rho_f(t, \cdot + x(t)) = \int_{\mathbb{R}^4} f(t, \cdot + x(t), v)\, dv \rightharpoonup |Q|_{L^1}\, \delta_{x=x(T)} \quad \text{as } t \to T.$$

The fact that the point up point $x(t)$ remains bounded in space is an issue we will have to deal with. Now observe that the mass that is put into blow up equals the total and conserved amount of L^1 mass of the initial data from the critical mass assumption, and in this sense the solution is non dispersive. The key is now to prove that even though there is a priori an infinite speed of propagation of mass due to the blow up solution -the support in v is unbounded-, the fact that there is no mass away from the blow up point at time T implies that there was "no mass" far away at time zero. This is measured in term of the weighted norm:

$$|x|^2 f(0) \in L^1 \quad \text{and} \quad ||x - x(T)|^2 f(t)|_{L^1} \to 0 \quad \text{as} \quad t \to T.$$

This is the gain of regularity (1.15) which is the heart of the proof. Now a global obstructive argument based on the pseudo-conformal symmetry and the variational characterization of Q_p allows one to conclude.

Proof. Let $f_0 \in \mathcal{E}_p$ such that the corresponding weak solution $f(t)$ to (1.1) blows at time $0 < T < +\infty$ and satisfies the hypothesis of Theorem 2. Up to a fixed rescaling of the initial data and using (1.2), we may without loss of generality assume:

$$\forall t \in [0,T), \quad |f(t)|_{L^1} = |Q_p|_{L^1} \quad \text{and} \quad |f(t)|_{L^p} = |Q_p|_{L^p}\,. \tag{2.18}$$

Step 1. The solution is non dispersive.

Let

$$h(t,x) = f\left(t, \lambda(t)x, \frac{v}{\lambda(t)}\right) \quad \text{with} \quad \lambda(t) = \left(\frac{||v|^2 Q_p|_{L^1}}{||v|^2 f(t)|_{L^1}}\right)^{1/2},$$

then from (2.18):

$$|h(t)|_{L^1} = |Q_p|_{L^1}, \quad |h(t)|_{L^p} = |Q_p|_{L^p}, \quad ||v|^2 h(t)|_{L^1} = ||v|^2 Q_p|_{L^1}. \tag{2.19}$$

Moreover, from (1.4),

$$\mathcal{H}(h(t)) = \lambda^2(t)\mathcal{H}(f(t)) \leq \lambda^2(t)\mathcal{H}(f(0)),$$

and thus the blow up assumption on $f(t)$, ie $\lambda(t) \to 0$, implies:

$$\limsup_{t\to T} \mathcal{H}(h(t)) \leq 0. \tag{2.20}$$

From standard concentration techniques as introduced by Lions, [17], and the variational characterization of Q_p, see explicitly Proposition 5.1 in [16], (2.19) and (2.20) imply the existence of $y(t) \in \mathbb{R}^N$ such that

$$h(t, x + y(t), v) \to Q_p \quad \text{in} \quad \mathcal{E}_p \quad \text{as} \quad t \to T.$$

In particular, this shows that $f(t)$ does not disperse and accumulates all its L^1 mass at blow up time:

$$\rho_f(t, \cdot + x(t)) = \int_{\mathbb{R}^4} f(t, \cdot + x(t), v)\, dv \rightharpoonup |Q|_{L^1}\, \delta_{x=0} \quad \text{as } t \to T, \tag{2.21}$$

where $x(t) = \lambda(t)y(t)$.

Step 2. Refined Cauchy-Schwarz inequality.

We claim the following refined Cauchy-Schwarz inequality which is crucial for the control of the flux type of terms: $\forall g \in \mathcal{E}_p$ with $|g|_{L^1}\, |g|_{L^p}^{\frac{p}{p-2}} \leq |Q_p|_{L^1}\, |Q_p|_{L^p}^{\frac{p}{p-2}}$, $\forall \phi(x) \in C^\infty(\mathbb{R}^4)$,

$$\left| \iint_{\mathbb{R}^4\times\mathbb{R}^4} \nabla_x \phi \cdot v g(x,v)\, dx\, dv \right| \leq (\mathcal{H}(g))^{1/2} \left(\iint_{\mathbb{R}^4\times\mathbb{R}^4} |\nabla_x \phi|^2\, g(x,v)\, dx\, dv \right)^{1/2}. \tag{2.22}$$

Indeed, let $a \in \mathbb{R}$ and $g_a(x,v) = g(x, v - a\nabla\phi)$, then $|g_a|_{L^1}\, |g_a|_{L^p}^{\frac{p}{p-2}} \leq |Q_p|_{L^1}\, |Q_p|_{L^p}^{\frac{p}{p-2}}$ implies $\mathcal{H}(g_a) \geq 0$ from (1.11). Now $\mathcal{H}(g_a)$ is a second order polynomial in a which discriminant must be nonpositive, this is (2.22).

Step 3. Control of the concentration point.

We now claim that the concentration point $x(t)$ does not run to infinity as $t \to T$:

$$\exists R_0 > 0 \ \text{ such that } \ \forall t \in [0,T), \quad |x(t)| \leq R_0. \tag{2.23}$$

Indeed, this follows from (2.21) and the fact that f is L^1 compact at blow up time:

$$\forall \varepsilon > 0, \quad \exists A > 0 \quad \text{such that } \forall t \in [0,T), \quad \int_{|x|>A} \rho_f(t,x)dx < \varepsilon, \tag{2.24}$$

Proof of (2.24): Let $\phi(x)$ be a bounded $C^\infty(\mathbb{R}^4)$ function. Multiplying (1.1) by ϕ and integrating on $\mathbb{R}^4 \times \mathbb{R}^4$ gives:

$$\frac{d}{dt}\int_{\mathbb{R}^4} \phi(x)\rho_f(t,x)dx = \iint_{\mathbb{R}^4\times\mathbb{R}^4} \nabla_x\phi \cdot v f(t,x,v)\, dx\, dv\,. \tag{2.25}$$

Remark that this calculation can be justified when f is a weak solution of (1.1) by a standard truncation argument (i.e. by replacing ϕ by a regularization of $\phi\, 1\!\!1_{|x|<R}\, 1\!\!1_{|v|<R}$ then passing to the limit as $R \to +\infty$).

We now apply (2.25) with $\phi(x) = \chi_A(x) = \chi(\frac{x}{A})$ for some smooth radially symmetric cut off function $\chi(r) = 0$ for $r \leq 1/2$, $\chi(r) = 1$ for $r \geq 1$, and estimate the right hand-side using (2.22) and (1.4) to conclude:

$$\begin{aligned} &\left|\frac{d}{dt}\int_{\mathbb{R}^4} \chi_A(x)\,\rho_f(t,x)\,dx\right| \leq (\mathcal{H}(f(t)))^{1/2}\left(\int_{\mathbb{R}^4} |\nabla_x\chi_A|^2\,\rho_f(t,x)\,dx\right)^{1/2} \\ \leq\ & \frac{C}{A}\,(\mathcal{H}(f_0))^{1/2}\,|f(t)|_{L^1}^{\frac{1}{2}} = \frac{C}{A}\,(\mathcal{H}(f_0))^{1/2}\,|f_0|_{L^1}^{1/2}. \end{aligned}$$

Integrating this in time between 0 and t yields: $\forall t \in [0,T)$,

$$\int_{\mathbb{R}^4} \chi_A(x)\,\rho_f(t,x)\,dx \leq \frac{C(f_0)T}{A} + \iint_{\mathbb{R}^4\times\mathbb{R}^4} \chi_A(x)\,f_0(x,v)\,dx\,dv \leq \varepsilon$$

for A large enough, and (2.24) is proved.

Step 4. Dispersive control in the virial space Σ.

From (2.23), there exists a point $x_T \in \mathbb{R}^4$ and a sequence $t_n \to T$ such that

$$x(t_n) \to x_T \qquad \text{as} \quad n \to +\infty. \tag{2.26}$$

Consider now a nonnegative radial cut-off function $\psi(r)$ such that $\psi(r) = r^2$ for $r \leq 1$, $\psi(r) = 6$ for $r \geq 2$, and $(\psi'(r))^2 \leq C\psi(r)$. Consider then (2.25) with

$$\phi(x) = \psi_A(x) := A^2\,\psi\left(\frac{|x - x_T|}{A}\right), \quad A \geq 1.$$

The inequality

$$|\nabla_x\psi_A(x)|^2 \leq C\,\psi_A(x)$$

together with (2.22) and (1.4) yields:

$$\left|\frac{d}{dt}\int_{\mathbb{R}^4} \psi_A(x)\,\rho_f(t,x)\,dx\right| \leq C\,\mathcal{H}(f_0))^{1/2}\left(\int_{\mathbb{R}^4} \psi_A(x)\,\rho_f(t,x)\,dx\right)^{1/2},$$

and thus:

$$\left|\frac{d}{dt}\left(\int_{\mathbb{R}^4}\psi_A(x)\,\rho_f(t,x)\,dx\right)^{1/2}\right| \leq C\,\mathcal{H}(f_0))^{1/2}.$$

An integration between t and t_n gives

$$\left|\int_{\mathbb{R}^4}\psi_A(x)\,\rho_f(t,x)\,dx\right|^{1/2} \leq \left|\int_{\mathbb{R}^4}\psi_A(x)\,\rho_f(t_n,x)\,dx\right|^{1/2} + C\,\mathcal{H}(f_0))^{1/2}|t_n - t|.$$

Now (2.21), (2.24) and (2.26) imply:

$$\int_{\mathbb{R}^4}\psi_A(x)\,\rho_f(t_n,x)\,dx \to 0 \quad \text{as } n \to +\infty,$$

hence:

$$\forall A \geq 1, \quad \forall t \in [0,T), \quad \left|\int_{\mathbb{R}^4}\psi_A(x)\rho_f(t,x)\,dx\right|^{1/2} \leq C(f_0)\mathcal{H}(f_0))^{1/2}(T-t),$$

where the constant $C(f_0)$ is independent of A. Letting $A \to +\infty$, we conclude that

$$|x|^2\rho_f \in L^\infty((0,T), L^1(\mathbb{R}^4))$$

and

$$\int_{\mathbb{R}^4}|x - x_T|^2\rho_f(t,x)\,dx \to 0 \quad \text{as} \quad t \to T. \tag{2.27}$$

Step 5. Conclusion.

The conclusion now follows from a global rigidity property which is a consequence of the pseudo-conformal symmetry (1.6). Indeed, from $f(t) \in \Sigma$,

$$g(\tau,x,v) = f\left(\frac{T\tau}{T+\tau}, \frac{Tx}{T+\tau} + x_T, \frac{(T+\tau)v - x}{T}\right)$$

is a weak solution in $\mathcal{E}_p$ of the Vlasov-Poisson system (1.1) defined for $\tau \in [0,\infty)$ with

$$|g(0)|_{L^1} = |Q_p|_{L^1} \quad \text{and} \quad |g(0)|_{L^p} = |Q_p|_{L^p}. \tag{2.28}$$

Remark that, in view of (1.7), we have

$$g(\tau,x,v) = f_T\left(\tau, x + \frac{T+\tau}{T}x_T, v + \frac{x_T}{T}\right)$$

so that the Hamiltonian of g can be deduced from the one of f_T defined by (1.6) thanks to

$$\mathcal{H}(g(\tau)) = \mathcal{H}(f_T(\tau)) - 2\frac{x_T}{T}\cdot\iint_{\mathbb{R}^4\times\mathbb{R}^4} v f_T(\tau,x,v)dxdv + \frac{|x_T|^2}{T^2}\iint_{\mathbb{R}^4\times\mathbb{R}^4} f_T(\tau,x,v)\,dx\,dv.$$

Observe now from the conservation of the L^1 norm (1.2), the momentum (1.3) and the assumption (1.16) that this implies the conservation of the Hamiltonian of $g(\tau)$:

$$\forall \tau \in [0,+\infty), \qquad \mathcal{H}(g(\tau)) = \mathcal{H}(g(0)). \tag{2.29}$$

We now claim:

$$\mathcal{H}(g(\tau)) \to 0 \quad \text{as} \quad \tau \to +\infty. \tag{2.30}$$

Indeed, a direct calculation gives with $t = \frac{T\tau}{T+\tau}$:

$$\begin{aligned}\mathcal{H}(g(\tau)) \quad = \quad & \frac{1}{T^2} \iint_{\mathbb{R}^4\times\mathbb{R}^4} |x - x_T|^2 \, f(t,x,v) \, dx \, dv \\ & + \frac{2}{T+\tau} \iint_{\mathbb{R}^4\times\mathbb{R}^4} (x - x_T) \cdot v f(t,x,v) \, dx \, dv \\ & + \left(\frac{T}{T+\tau}\right)^2 \mathcal{H}(f(t)).\end{aligned}$$

The first term and the third term are controlled using respectively (2.27) and (1.4). For the second term, we use (2.22) with $\phi(x) = \frac{1}{2}(x - x_T)^2$ to derive:

$$\begin{aligned}\left| \iint_{\mathbb{R}^4\times\mathbb{R}^4} (x - x_T) \cdot v f(t,x,v) \, dx \, dv \right| \quad & \leq \mathcal{H}(f(t))^{1/2} \left(\iint_{\mathbb{R}^4\times\mathbb{R}^4} |x - x_T|^2 \, f(t,x,v) \, dx \, dv \right)^{1/2} \\ & \to 0 \quad \text{as} \quad \tau \to +\infty\end{aligned}$$

where we used (1.4) and (2.27) in the last step. This concludes the proof of (2.30).

From (2.29) and (2.30), we conclude:

$$\mathcal{H}(g(0)) = 0.$$

Together with (2.28) and the variational characterization of the ground state given by Proposition 1, this implies

$$g(0) = Q_p \,,$$

up to symmetries, and concludes the proof of Theorem 2. □

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