UNIFORMLY ACCURATE TIME-SPLITTING METHODS FOR THE SEMICLASSICAL LINEAR SCHRÖDINGER EQUATION

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Abstract. This article is devoted to the construction of numerical methods which remain insensitive to the smallness of the semiclassical parameter for the linear Schrödinger equation in the semiclassical limit. We specifically analyse the convergence behavior of the first-order splitting. Our main result is a proof of uniform accuracy. We illustrate the properties of our methods with simulations.

1. Introduction

We are concerned here with the uniformly accurate numerical approximation of the solution $\Psi^\varepsilon : \mathbb{R}_+ \times \mathbb{R}^d \to \mathbb{C}$, $d \geq 1$, of the *linear* Schrödinger equation in its semiclassical limit

$$i\varepsilon \partial_t \Psi^\varepsilon = -\frac{\varepsilon^2}{2}\Delta \Psi^\varepsilon + \mathcal{V}\Psi^\varepsilon \tag{1.1}$$

where $\mathcal{V}$ is a smooth potential which does not depend on time. The initial datum is assumed to be of the form

$$\Psi^\varepsilon(0,\cdot) = A_0(\cdot)e^{iS_0(\cdot)/\varepsilon} \quad \text{with} \quad \|A_0\|_{L^2(\mathbb{R}^d)} = 1. \tag{1.2}$$

Note that the L^2-norm, the energy and the momentum of $\Psi^\varepsilon(0,\cdot)$, namely

$$\text{Mass:} \qquad \|\Psi^\varepsilon(t,\cdot)\|^2_{L^2(\mathbb{R}^d)}, \tag{1.3}$$

$$\text{Energy:} \qquad \int_{\mathbb{R}^d} \left(\varepsilon^2|\nabla\Psi^\varepsilon(t,x)|^2 + \mathcal{V}|\Psi^\varepsilon(t,x)|^2\right) dx, \tag{1.4}$$

$$\text{Momentum:} \qquad \varepsilon \operatorname{Im} \int_{\mathbb{R}^d} \overline{\Psi^\varepsilon(t,x)}\nabla\Psi^\varepsilon(t,x)dx, \tag{1.5}$$

are all preserved by the flow of (1.1), whenever $\Psi^\varepsilon(0,\cdot) \in H^1(\mathbb{R}^d)$.

Owing to its numerous occurrences in a vast number of domains of applications in physics, equation (1.1) has been widely studied (see for instance [27, 32] and the references therein). In the semiclassical regime where the rescaled Planck constant ε is small, its asymptotic study allows for an appropriate description of the observables of Ψ^ε through the laws of hydrodynamics. We refer to [12] for a detailed presentation of the semiclassical analysis and to [25] for a review of both theoretical and numerical issues.

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Let us also mention that we do not consider the case where the initial datas are Gaussian wave packets for which efficient schemes have already been developed [22, 21, 23].

1.1. **Motivation.** Generally speaking, numerical methods for equation (1.1) exhibit an error of size $\Delta t^p/\varepsilon^r + \Delta x^q/\varepsilon^s$, where Δt and Δx are the time and space steps and p, q, r, s strictly positive numbers. For time-splitting methods for instance, the error on the wave function behaves like $\Delta x/\varepsilon + \Delta t^p/\varepsilon$ [5, 17]. Even if we content ourselves with observables[1], the error of a splitting method of Bao, Jin, and Markowich [5] grows like $\Delta x/\varepsilon + \Delta t^p$. Now, achieving a fixed accuracy for varying values of ε requires to keep both ratios $\Delta t/\varepsilon^{r/p}$ and $\Delta x/\varepsilon^{s/q}$ constant, and becomes *prohibitively costly* when $\varepsilon \to 0$. Our aim, in this article, is thus to develop *new* numerical schemes that are *Uniformly Accurate (UA)* w.r.t. ε, i.e. whose accuracy does not deteriorate for vanishing ε. In other words, schemes for which $r, s = 0$. This seems highly desirable as all available methods with the exception of [9], namely finite difference methods [1, 16, 26, 35], splitting methods [8, 17, 18, 29, 31, 34], asymptotic splitting methods [3, 4], relaxation schemes [7] and symplectic methods [33] fail to be UA.

It is the belief of the authors that, prior to the construction of UA-schemes, it is necessary to reformulate (1.1) as in [9] and we now describe how this can be done.

1.2. **Reformulation of the problem.** In the spirit of the *Wentzel-Kramers-Brillouin (WKB)* techniques, we decompose Ψ^ε as the product of a slowly varying amplitude and a fast oscillating factor[2]

$$\Psi^\varepsilon(t,\cdot) = A^\varepsilon(t,\cdot)e^{iS^\varepsilon(t,\cdot)/\varepsilon}. \tag{1.7}$$

From this point onwards, various choices are possible, depending on whether A^ε is complex or not[3]: taking $A^\varepsilon \in \mathbb{C}$ leads to the following system [12]

$$\partial_t S^\varepsilon + \frac{|\nabla S^\varepsilon|^2}{2} + \mathcal{V} = 0, \tag{1.8a}$$

$$\partial_t A^\varepsilon + \nabla S^\varepsilon \cdot \nabla A^\varepsilon + \frac{A^\varepsilon}{2}\Delta S^\varepsilon = \frac{i\varepsilon \Delta A^\varepsilon}{2} \tag{1.8b}$$

with $S^\varepsilon(0,\cdot) = S_0(\cdot)$ and $A^\varepsilon(0,\cdot) = A_0(\cdot)$. Under appropriate smoothness assumptions, $(A^\varepsilon, S^\varepsilon) \in \mathbb{C} \times \mathbb{R}$ converges when $\varepsilon \to 0$ to the solution (A^0, S^0) of

$$\partial_t S^0 + \frac{|\nabla S^0|^2}{2} + \mathcal{V} = 0, \tag{1.9a}$$

$$\partial_t A^0 + \nabla S^0 \cdot \nabla A^0 + \frac{A^0}{2}\Delta S^0 = 0. \tag{1.9b}$$

[1] These authors performed extensive numerical tests in both linear and nonlinear cases [5, 6].

[2] Considering the WKB-ansatz (1.7) transforms the invariants (1.3) into respectively

$$\|A^\varepsilon\|^2_{L^2(\mathbb{R}^d)}, \quad \int_{\mathbb{R}^d} \left(|\varepsilon\nabla A^\varepsilon + iA^\varepsilon\nabla S^\varepsilon|^2 + \mathcal{V}|A^\varepsilon|^2\right) dx \quad \text{and} \quad \mathrm{Im}\int_{\mathbb{R}^d} \overline{A^\varepsilon}\left(\varepsilon\nabla A^\varepsilon + iA^\varepsilon\nabla S^\varepsilon\right) dx. \tag{1.6}$$

[3] The Madelung transform [30] relates the semiclassical limit of (1.1) to hydrodynamic equations

$$\Psi^\varepsilon(t,\cdot) = \sqrt{\rho^\varepsilon(t,\cdot)}e^{iS^\varepsilon(t,\cdot)/\varepsilon}$$

and amounts to choosing $A^\varepsilon \in \mathbb{R}_+$. However, this formulation leads to both analytical and numerical difficulties in the presence of vacuum, *i.e.* whenever ρ^ε vanishes [14, 15].

Notice that $(\rho, v) = (|A^0|^2, \nabla S^0)$ is then solution of the Euler system

$$\partial_t v + v \cdot \nabla v + \nabla \mathcal{V} = 0, \tag{1.10a}$$

$$\partial_t \rho + \operatorname{div}(\rho v) = 0. \tag{1.10b}$$

Now, an important drawback of (1.8) stems from the formation of caustics in finite time [12]: the solution of (1.8) may indeed cease to be smooth even though Ψ^ε is globally well-defined for $\varepsilon > 0$. In order to obtain global existence for $\varepsilon > 0$, Besse, Carles and Méhats [9] suggested an alternative formulation by introducing an asymptotically-vanishing viscosity term in the eikonal equation (1.8a). Therein, system (1.8) is replaced by

$$\partial_t S^\varepsilon + \frac{|\nabla S^\varepsilon|^2}{2} + \mathcal{V} = \varepsilon^2 \Delta S^\varepsilon, \tag{1.11a}$$

$$\partial_t A^\varepsilon + \nabla S^\varepsilon \cdot \nabla A^\varepsilon + \frac{A^\varepsilon}{2} \Delta S^\varepsilon = \frac{i\varepsilon \Delta A^\varepsilon}{2} - i\varepsilon A^\varepsilon \Delta S^\varepsilon \tag{1.11b}$$

where $S^\varepsilon(0,x) = S_0(x)$, $A^\varepsilon(0,x) = A_0(x)$ and where $x \in \mathbb{R}^d$. Let us emphasize that both (1.8) and (1.11) are equivalent to (1.1) in the following sense: as long as the solution $(S^\varepsilon, A^\varepsilon)$ of (1.8) (resp. (1.11)) is smooth, the function Ψ^ε defined by (1.7) solves (1.1). The well-posedeness of (1.11) and the uniform control of the solutions with respect to ε are stated in Theorem 2.1 below.

The main advantage of the WKB reformulation (1.11) over (1.1) is apparent: the semiclassical parameter ε does not give rise to singular perturbations[4]. Hence, it constitutes a good basis for the development of UA schemes (at least prior to the appearance of caustics), as witnessed by the methods introduced later in this paper.

1.3. **Construction of the schemes.** First and only (up to our knowledge) UA schemes are based on the formulation (1.11) introduced in [9]. Nevertheless, these schemes are still subject to CFL stability conditions and are of low order in time and space. In this paper, we consider, in lieu of finite differences as in [9], time-splitting methods, for they enjoy the following favorable features:

(i) they do not suffer from stability restrictions on the time step;
(ii) they are easy to implement;
(iii) they preserve exactly the L^2-norm;
(iv) they can be adapted to semilinear Schrödinger equations;
(v) they can be composed to attain high-order of convergence in time while remaining spectrally convergent in space.

Points (iv) and (v) will be addressed in a forthcoming work using complex time steps (see [10]), while, in this paper, we introduce first and second order in time splitting-schemes and concentrate on the numerical analysis of the first-order one for the sake of clarity.

System (1.11) is split into four pieces as follows:

[4]The Cole-Hopf transformation [20, Section 4.4.1]

$$w^\varepsilon = \exp\left(-\frac{S^\varepsilon}{2\varepsilon^2}\right) - 1 \tag{1.12}$$

transforms (1.11a) into $\partial_t w^\varepsilon - \frac{\mathcal{V}}{2\varepsilon^2}(w^\varepsilon + 1) = \varepsilon^2 \Delta w^\varepsilon$ for which the regularizing effect of the viscosity term becomes arguably more apparent.

First flow: We denote φ_h^1 the approximate flow at time $h \in \mathbb{R}$ of the system

$$\partial_t S + \frac{|\nabla S|^2}{2} = 0, \tag{1.13a}$$

$$\partial_t A + \nabla S \cdot \nabla A + \frac{A}{2}\Delta S = \frac{i\Delta A}{2}. \tag{1.13b}$$

The eikonal equation (1.13a) is solved by means of the method of characteristics, while equation (1.13b) is dealt with by noticing that $w = A\exp(iS)$ satisfies the free Schrödinger equation $i\partial_t w = -\frac{1}{2}\Delta w$.

Second flow: We define φ_h^2 as the exact flow at time $h \in \mathbb{R}$ of the system

$$\partial_t S = 0, \tag{1.14a}$$

$$\partial_t A = \frac{i(\varepsilon - 1)\Delta A}{2}, \tag{1.14b}$$

which is solved in the Fourier space.

Third flow: The third flow φ_h^3 is defined as the exact flow at time $h \in \mathbb{R}$ of system

$$\partial_t S = -\mathcal{V}, \tag{1.15a}$$

$$\partial_t A = 0. \tag{1.15b}$$

Fourth flow: The fourth flow φ_h^4 is defined as the exact flow at time $h \in \mathbb{R}_+$ of

$$\partial_t S = \varepsilon^2 \Delta S, \tag{1.16a}$$

$$\partial_t A = -i\varepsilon A \Delta S. \tag{1.16b}$$

Equation (1.16a) is solved in Fourier space and the solution of (1.16b) is simply obtained through the formula $A(h,\cdot) = \exp\left(-i\varepsilon^{-1}(S(h,\cdot) - S(0,\cdot))\right) A(0,\cdot)$. Notice that φ_h^4 can thus be viewed as a regularizing flow.

The first-order scheme that we consider for (1.11) is then the concatenation of all previous flows

$$\varphi_h^1 \circ \varphi_h^2 \circ \varphi_h^3 \circ \varphi_h^4 \tag{1.17}$$

while the second-order scheme is given by

$$\varphi_{h/2}^1 \circ \varphi_{h/2}^2 \circ \varphi_{h/2}^3 \circ \varphi_h^4 \circ \varphi_{h/2}^3 \circ \varphi_{h/2}^2 \circ \varphi_{h/2}^1. \tag{1.18}$$

1.4. **Main result.** The main result of this paper is the following theorem: it states that $\varphi_h^1 \circ \varphi_h^2 \circ \varphi_h^3 \circ \varphi_h^4$ is uniformly accurate w.r.t. the semi-classical parameter ε. The proper statement of the result uses the norm $\|\cdot\|_s$ on the set $\Sigma_s = H^{s+2}(\mathbb{R}^d) \times H^s(\mathbb{R}^d)$ defined for $s \geq 0$ and $u = (S, A)$ by

$$\|u\|_s = \left(\|S\|^2_{H^{s+2}(\mathbb{R}^d)} + \|A\|^2_{H^s(\mathbb{R}^d)}\right)^{1/2}.$$

Theorem 1.1. *Let $s > d/2 + 1$, $\varepsilon_{max} > 0$, $u_0 \in \Sigma_{s+2}$ and $0 < T < T_{max}$ where*

$$T_{max} = \sup\{t > 0 : \tau \mapsto \phi_\tau^0(u_0) \in L^\infty([0,t]; \Sigma_{s+2})\}$$

and ϕ_τ^ε denotes the flow at time τ of (1.11). There exists $C > 0$ and $h_0 > 0$ such that the following error estimate holds true for any $\varepsilon \in (0, \varepsilon_{max}]$, any $h \in [0, h_0]$ and $n \in \mathbb{N}$ satisfying $nh \leq T$:

$$\|(\varphi_h^1 \circ \varphi_h^2 \circ \varphi_h^3 \circ \varphi_h^4)^n(u_0) - \phi_{nh}^\varepsilon(u_0)\|_s \leq Ch.$$

The constants C and h_0 do not depend on ε.

Remark 1.2. *The constant T_{max} appearing in Theorem 1.1 is well-defined and positive since $\tau \mapsto \phi_\tau^0$ exists locally in time (see Theorem 2.1).*

Remark 1.3. *The numerical analysis performed for the proof of Theorem 1.1 can immediately be extended after the caustics for $T_{max} \leq T$ and $\varepsilon > 0$ since the solution of (1.11) (as the one of (1.1)) are global. Nevertheless, the constants C and h_0 appearing in the result will not be independent on ε anymore. This point is illustrated in Section 4.*

Remark 1.4. *The proof of Theorem 1.1 can be adapted to any time-splitting method of order 1 obtained after permutation of the four flows in (1.17).*

Our proof is reminiscent of two previous results related to, on the one hand, splitting schemes for equations with Burgers nonlinearity [24] and on the other hand, splitting scheme for NLS in the semiclassical limit with [13]. Nonetheless, due to the finite-time existence of both exact and approximate flows, and to the peculiarity of the Lipschitz-type stability of the exact flows (see Lemma 2.3), our proof follows a different path. In particular, we lean the approximate solutions on the exact one to ensure that they do not blow up. Besides, the application of Lady Windermere's fan argument is somehow hidden in an induction procedure. Finally, let us mention that, in spite of the fact that we do not specifically address this case, it is our belief that this result can be extended to Schrödinger equations with time dependent potentials, to the second-order scheme, to the Schrödinger equation with a nonlinearity of Hartree-type and to the weakly nonlinear Schrödinger equation (see also [13, Remark 4.5]).

The paper is organized as follows. In Section 2, we first give a theorem of well-posedness of the Cauchy problem for (1.11). Then Theorem 1.1 is proved using four technical lemmas. Section 3 is devoted to the proof of these lemmas. We illustrate the properties of our methods in Section 4.

2. Preparatiory results and proof of Theorem 1.1

2.1. **Notations.** Assume that $\varepsilon \in (0, \varepsilon_{max}]$ and $s > d/2 + 1$. For the sake of simplicity, we keep the notation of all the flows independent of ε. All the constants appearing in the proof depend on $\mathcal{V}$ but not on $\varepsilon > 0$. We denote

$$\varphi_h^{ij} = \varphi_h^i \circ \varphi_h^j,\ \varphi_h^{ijk} = \varphi_h^i \circ \varphi_h^{jk},$$
$$\varphi_h^{1234} = \varphi_h^1 \circ \varphi_h^{234} = \varphi_h^1 \circ \varphi_h^2 \circ \varphi_h^3 \circ \varphi_h^4,$$

$\mathcal{N}_i$ is the possibly nonlinear operator related to φ_h^i so that

$$\partial_h \varphi_h^i = \mathcal{N}_i \varphi_h^i.$$

The quantities $\partial_h \varphi_h(u)$ and $\partial_2 \varphi_h(u)$ are the Fréchet derivatives of φ with respect to h and u. The commutator of the nonlinear operators $\mathcal{N}_i$ and $\mathcal{N}_j$ is given by

$$[\mathcal{N}_i, \mathcal{N}_j](u) = D\mathcal{N}_i(u) \cdot \mathcal{N}_j(u) - D\mathcal{N}_j(u) \cdot \mathcal{N}_i(u).$$

2.2. **Existence, uniqueness and uniform boundedness results.** The following theorem study some properties of the solutions of equations (1.11).

Theorem 2.1. *Let $\varepsilon_{max} > 0$, $s > d/2+1$ and $u_0 \in \Sigma_{s+2}$. The following two points are true.*

(i) The quantity

$$T_{max} = \sup\{t > 0 : \phi^0(u_0) \in L^\infty([0,t];\Sigma_{s+2})\} \tag{2.1}$$

is well-defined and positive.

(ii) Let $0 < T < T_{max}$. For all $\varepsilon \in [0, \varepsilon_{max}]$, there exists a unique solution

$$\phi^\varepsilon(u_0) \in C([0,T],\Sigma_{s+2})$$

of the systems of equations (1.11). Moreover, $\phi^\varepsilon(u_0)$ is bounded in

$$C([0,T],\Sigma_{s+2})$$

uniformly in $\varepsilon \in [0, \varepsilon_{max}]$.

The proof of Theorem 2.1 is given in Section 3.4.

2.3. **The main lemmas.** In this subsection, we present the main ingredients needed in the proof of Theorem 1.1. Their proof is postponed to Section 3.

Lemma 2.2. *Let $M > 0$ and $s > d/2+1$. There exist $h_1 = h_1(M) > 0$ such that for any $\varepsilon \in (0, \varepsilon_{max}]$ and any $u_0 \in \Sigma_s$ satisfying*

$$\|u_0\|_s \leq M,$$

we have that the solution $\phi_t(u_0)$ of equation (1.11) is well-defined on $[0, h_1]$ and for all $t \in [0, h_1]$

$$\|\phi_t(u_0)\|_s \leq 2M.$$

Lemma 2.3. *Let $M > 0$ and $s > d/2+1$. There exist $C_2 = C_2(M) > 0$ such that for any $\varepsilon \in (0, \varepsilon_{max}]$, any solutions $\phi_t(u_1) \in L^\infty([0,T],\Sigma_{s+1})$ and $\phi_t(u_2) \in L^\infty([0,T],\Sigma_s)$ of equation (1.11), satisfying for all $t \in [0,T]$*

$$\|\phi_t(u_1)\|_{s+1} + \|\phi_t(u_2)\|_s \leq M$$

we have

$$\|\phi_t(u_1) - \phi_t(u_2)\|_s \leq \|u_1 - u_2\|_s \exp(C_2 t).$$

Remark 2.4. *Let us insist on the fact that in Lemma 2.3, we have to control $\phi_t(u_1)$ in Σ_{s+1} and $\phi_t(u_2)$ in Σ_s to get Lipschitz-type stability in Σ_s.*

Lemma 2.5. *Let $M > 0$ and $s > d/2+1$. There exist $h_3 = h_3(M) > 0$ and $C_3 = C_3(M) > 0$ such that for any $\varepsilon \in (0, \varepsilon_{max}]$, any $u_0 \in \Sigma_s$ satisfying $\|u_0\|_s \leq M$ and any $0 \leq t \leq h_3$, we have*

(a) $\|\varphi_t^{1234}(u_0)\|_s \leq 8M$.

(b) Furthermore, if $u_0 \in \Sigma_{s+2}$, then

$$\|\varphi_t^{1234}(u_0)\|_{s+2} \leq \exp(C_3 t)\left(\|u_0\|_{s+2} + t\|\mathcal{V}\|_{H^{s+4}}\right).$$

Lemma 2.6. *Let $M > 0$ and $s > d/2+1$. There exist $h_4 = h_4(M) > 0$ and $K_4 = K_4(M) > 0$ such that for any $\varepsilon \in (0, \varepsilon_{max}]$ and any $u_0 \in \Sigma_{s+2}$ satisfying*

$$\|u_0\|_{s+2} \leq M,$$

we have for any $t \in [0, h_4]$ that

$$\|\phi_t(u_0) - \varphi_t^{1234}(u_0)\|_s \leq K_4 t^2.$$

2.4. **Proof of Theorem 1.1.** Let us denote

$$M_s^\varepsilon(T) := \sup\{\|\phi_t^\varepsilon(u_0)\|_s : \ 0 \le t \le T\}. \tag{2.2}$$

for $\varepsilon \ge 0$ and $T \ge 0$.

Let $s > d/2 + 1$, $\varepsilon \in (0, \varepsilon_{max}]$, $u_0 \in \Sigma_{s+2}$, $n \in \mathbb{N}$ and $h > 0$ be such that $nh \le T < T_{max}$ (see (2.1)). By Theorem 2.1, there exist M_s, M_{s+1} and M_{s+2} independent of $\varepsilon \in (0, \varepsilon_{max}]$ such that for all $\varepsilon \in (0, \varepsilon_{max}]$,

$$M_s^\varepsilon \le M_s, \ M_{s+1}^\varepsilon \le M_{s+1} \text{ and } M_{s+2}^\varepsilon \le M_{s+2}$$

(see (2.2)). We denote

$$\begin{aligned} &C = C_3(2M_s), \\ &c_0 = \|u_0\|_{s+2} \exp(CT) + \|\mathcal{V}\|_{H^{s+4}} e^{2TC}/C, \\ &C' = C_2(M_{s+1} + 4M_s), \\ &\widetilde{c} = K_4(c_0) a e^{C'T}/C'. \end{aligned}$$

Assume that

$$0 \le h \le \min\left(h_3(c_0), M_s/\widetilde{c}, h_1(2M_s), h_4(c_0)\right). \tag{2.3}$$

Here, h_1, C_2, h_3, h_4 and K_4 are defined in Lemmas 2.2, 2.3, 2.5 and 2.6.

We show by induction on $0 \le k \le n$ that

(i) $(\varphi_h^{1234})^k(u_0)$ is well-defined, belongs to Σ_{s+2} and

$$\|(\varphi_h^{1234})^k(u_0)\|_{s+2} \le \|u_0\|_{s+2} \exp(Ckh) + h\|\mathcal{V}\|_{H^{s+4}} \frac{e^{(k+1)hC} - e^{hC}}{e^{hC} - 1} \le c_0,$$

(ii) $\|\phi_{kh}(u_0) - (\varphi_h^{1234})^k(u_0)\|_s \le h^2 K_4(c_0) \frac{e^{C'hk}-1}{e^{C'h}-1} \le \widetilde{c}h,$

and Theorem 1.1 follows then from point (ii) with $k = n$.

The induction hypothesis is true for $k = 0$. Let us assume points (i) and (ii) true for $0 \le k \le n-1$.

Lemma 2.5, point (i) and (2.3) ensure that

$$(\varphi_h^{1234})^{k+1}(u_0)$$

is well-defined and belongs to Σ_{s+2}. By Point (ii) and (2.3), we have

$$\|(\varphi_h^{1234})^k(u_0)\|_s \le M_s + \|\phi_{kh}(u_0) - (\varphi_h^{1234})^k(u_0)\|_s \le 2M_s.$$

By Lemma 2.5 and (2.3), we have

$$\|(\varphi_h^{1234})^{k+1}\|_{s+2} \le \exp(Ch) \left(\|(\varphi_h^{1234})^k(u_0)\|_{s+2} + h\|\mathcal{V}\|_{H^{s+4}} \right)$$

and point (i) ensures that

$$\|(\varphi_h^{1234})^{k+1}\|_{s+2} \le \|u_0\|_{s+2} \exp(C(k+1)h) + h\|\mathcal{V}\|_{H^{s+4}} \frac{e^{(k+2)hC} - e^{hC}}{e^{hC} - 1} \le c_0.$$

By Lemma 2.2 and (2.3), $h' \mapsto \phi_{h'} \circ (\varphi_h^{1234})^k(u_0)$ is well-defined and satisfies for all $0 \le h' \le h$

$$\|\phi_{h'} \circ (\varphi_h^{1234})^k(u_0)\|_s \le 4M_s.$$

By Lemma 2.3, we obtain that

$$\|\phi_{h(k+1)}(u_0) - \phi_h \circ (\varphi_h^{1234})^k(u_0)\|_s \le \|\phi_{hk}(u_0) - (\varphi_h^{1234})^k(u_0)\|_s \exp(C'h).$$

By Lemma 2.6, point (i) and (2.3), we get

$$\|\phi_h \circ (\varphi_h^{1234})^k(u_0) - \varphi_h^{1234} \circ (\varphi_h^{1234})^k(u_0)\|_s \leq K_4(c_0)h^2,$$

so that

$$\|\phi_{h(k+1)}(u_0) - (\varphi_h^{1234})^{k+1}(u_0)\|_s \leq K_4(c_0)h^2 + \|\phi_{hk}(u_0) - (\varphi_h^{1234})^k(u_0)\|_s \exp(C'h).$$

By point (ii), we have then that

$$\|\phi_{h(k+1)}(u_0) - (\varphi_h^{1234})^{k+1}(u_0)\|_s \leq K_4(c_0)h^2 \left(\frac{e^{C'h(k+1)} - 1}{e^{C'h} - 1}\right).$$

Thus, points (i) and (ii) are true for $k+1$.

3. Proof of the main lemmas

3.1. **Auxiliary results.** Let us denote by $\langle \cdot, \cdot \rangle$ the L^2 scalar product, for $s > 0$

$$\Lambda^s = (1 - \Delta)^{s/2},$$

$$\Pi_1 u = S,\ \Pi_2 u = A,\ \text{for } u = \begin{pmatrix} S \\ A \end{pmatrix} \tag{3.1}$$

and

$$\begin{aligned} \langle u_1, u_2 \rangle_s &= \langle \Pi_1 u_1, \Pi_1 u_2 \rangle \\ &\quad + \left\langle \Lambda^{s+1} \nabla \Pi_1 u_1, \Lambda^{s+1} \nabla \Pi_1 u_2 \right\rangle + \operatorname{Re} \langle \Lambda^s \Pi_2 u_1, \Lambda^s \Pi_2 u_2 \rangle . \end{aligned} \tag{3.2}$$

We recall two points that will be of constant use in the following: the Sobolev space $H^s \subset L^\infty$ is an algebra for $s > d/2$ and the Kato-Ponce [28] inequality holds true:

Proposition 3.1. *Let $s_0 > d/2 + 1$. There is $c > 0$ such that for all $f \in H^{s_0}(\mathbb{R}^d)$ and $g \in H^{s_0 - 1}(\mathbb{R}^d)$*

$$\|\Lambda^{s_0}(fg) - f\Lambda^{s_0} g\|_{L^2} \leq c(\|\nabla f\|_{L^\infty} \|g\|_{H^{s_0-1}} + \|f\|_{H^{s_0}} \|g\|_{L^\infty}).$$

The following lemmas will be used several times in our proof.

Lemma 3.2. *Let $s_0 > d/2 + 1$. There is $C > 0$ such that for all v_0, v_1 and $R \in L^\infty([0, h_0], H^{s_0}(\mathbb{R}^d)^d)$ satisfying*

$$\partial_t v_0 + (v_1 \cdot \nabla) v_0 = R,$$

we have

$$\begin{aligned} \partial_t \|v_0\|_{H^{s_0}}^2 &\leq C \left(\|v_0\|_{H^{s_0}}^2 \|\nabla v_1\|_{L^\infty} + \|v_0\|_{H^{s_0}} \|v_1\|_{H^{s_0}} \|\nabla v_0\|_{L^\infty} \right) + 2 \langle \Lambda^{s_0} v_0, \Lambda^{s_0} R \rangle \\ &\leq C \|v_0\|_{H^{s_0}}^2 \|v_1\|_{H^{s_0}} + 2 \langle \Lambda^{s_0} v_0, \Lambda^{s_0} R \rangle . \end{aligned}$$

Proof. We have by integration by parts that

$$\begin{aligned} \partial_t \frac{\|v_0\|_{H^{s_0}}^2}{2} &= \langle \Lambda^{s_0} v_0, \Lambda^{s_0} \partial_t v_0 \rangle = - \langle \Lambda^{s_0} v_0, \Lambda^{s_0} (v_1 \cdot \nabla) v_0 \rangle + \langle \Lambda^{s_0} v_0, \Lambda^{s_0} R \rangle \\ &\leq \frac{1}{2} \int_{\mathbb{R}^d} |\Lambda^{s_0} v_0|^2 \operatorname{div} v_1 + \|v_0\|_{H^{s_0}} \|[\Lambda^{s_0}, (v_1 \cdot \nabla)] v_0\|_{L^2} + \langle \Lambda^{s_0} v_0, \Lambda^{s_0} R \rangle . \end{aligned}$$

Proposition 3.1 ensures that

$$\partial_t \frac{\|v_0\|_{H^{s_0}}^2}{2} \leq c \left(\|v_0\|_{H^{s_0}}^2 \|\nabla v_1\|_{L^\infty} + \|v_0\|_{H^{s_0}} \|v_1\|_{H^{s_0}} \|\nabla v_0\|_{L^\infty} \right) + \langle \Lambda^{s_0} v_0, \Lambda^{s_0} R \rangle .$$

□

Lemma 3.3. *Let $s_0 > d/2+1$. There exists $C>0$ such that for all $A \in W^{1,\infty}([0,h_0],H^{s_0}(\mathbb{R}^d))$, $v_1 \in L^\infty([0,h_0],H^{s_0+1}(\mathbb{R}^d)^d)$ and $R \in L^\infty([0,h_0],H^{s_0}(\mathbb{R}^d))$ satisfying*

$$\partial_t A + v_1\cdot\nabla A + A\frac{\operatorname{div} v_1}{2} = R,$$

we have,

$$\begin{aligned}\partial_t\|A\|_{H^{s_0}}^2 &\le C\left(\|A\|_{H^{s_0}}^2\|v_1\|_{W^{2,\infty}} + \|A\|_{H^{s_0}}\|v_1\|_{H^{s_0+1}}\|A\|_{W^{1,\infty}}\right) + 2\operatorname{Re}\langle\Lambda^{s_0}A,\Lambda^{s_0}R\rangle\\ &\le C\|A\|_{H^{s_0}}^2\|v_1\|_{H^{s_0+1}} + 2\operatorname{Re}\langle\Lambda^{s_0}A,\Lambda^{s_0}R\rangle.\end{aligned}$$

Proof. We have by integration by parts that

$$\begin{aligned}\partial_t\frac{\|A\|_{H^{s_0}}^2}{2} &= \operatorname{Re}\langle\Lambda^{s_0}A,\Lambda^{s_0}\partial_t A\rangle = -\operatorname{Re}\left\langle\Lambda^{s_0}A,\left(v_1\cdot\nabla+\frac{\operatorname{div}v_1}{2}\right)\Lambda^{s_0}A\right\rangle\\ &\quad-\operatorname{Re}\left\langle\Lambda^{s_0}A,\left[\Lambda^{s_0},\left(v_1\cdot\nabla+\frac{\operatorname{div}v_1}{2}\right)\right]A\right\rangle + \operatorname{Re}\langle\Lambda^{s_0}A,\Lambda^{s_0}R\rangle\\ &\le \|A\|_{H^{s_0}}\left\|\left[\Lambda^{s_0},\left(v_1\cdot\nabla+\frac{\operatorname{div}v_1}{2}\right)\right]A\right\|_{L^2} + \operatorname{Re}\langle\Lambda^{s_0}A,\Lambda^{s_0}R\rangle.\end{aligned}$$

Proposition 3.1 ensures that

$$\begin{aligned}\partial_t\frac{\|A\|_{H^{s_0}}^2}{2} &\le C\|A\|_{H^{s_0}}(\|\nabla v_1\|_{L^\infty}\|\nabla A\|_{H^{s_0-1}} + \|v_1\|_{H^{s_0}}\|\nabla A\|_{L^\infty})\\ &\quad + C\|A\|_{H^{s_0}}(\|\nabla(\operatorname{div}v_1)\|_{L^\infty}\|A\|_{H^{s_0-1}} + \|\operatorname{div}v_1\|_{H^{s_0}}\|A\|_{L^\infty})\\ &\quad + \operatorname{Re}\langle\Lambda^{s_0}A,\Lambda^{s_0}R\rangle\\ &\le C\left(\|A\|_{H^{s_0}}^2\|v_1\|_{W^{2,\infty}} + \|A\|_{H^{s_0}}\|v_1\|_{H^{s_0+1}}\|A\|_{W^{1,\infty}}\right) + \operatorname{Re}\langle\Lambda^{s_0}A,\Lambda^{s_0}R\rangle\\ &\le C\|A\|_{H^{s_0}}^2\|v_1\|_{H^{s_0+1}} + \operatorname{Re}\langle\Lambda^{s_0}A,\Lambda^{s_0}R\rangle.\end{aligned}$$

□

3.2. **Study of the equation** (1.11)**.** Let us prove Lemma 2.2.

Proof. By the Cole-Hopf transform, we get that $w^\varepsilon = \exp\left(-\frac{S^\varepsilon}{2\varepsilon^2}\right)-1$ is the solution of

$$\partial_t w^\varepsilon = \varepsilon^2\Delta w^\varepsilon + \frac{\mathcal{V}}{2\varepsilon^2}(w^\varepsilon+1), \quad w^\varepsilon(0) = \exp\left(-\frac{S_0}{2\varepsilon^2}\right)-1,$$

Hence, global existence and uniqueness of the solution S^ε of (1.11a) for fixed $\varepsilon \in (0,\varepsilon_{max}]$, follows from standard semi-group theory. The function $v^\varepsilon = \nabla S^\varepsilon$ solves

$$\partial_t v^\varepsilon + (v^\varepsilon\cdot\nabla)v^\varepsilon + \nabla\mathcal{V} = \varepsilon^2\Delta v^\varepsilon.$$

Since $s > d/2$, Lemma 3.2 and an integration by parts ensure that

$$\begin{aligned}\partial_t\|v^\varepsilon\|_{H^{s+1}}^2 &\le c\|v^\varepsilon\|_{H^{s+1}}^3 + \left\langle\Lambda^{s+1}v^\varepsilon,\Lambda^{s+1}\left(-\nabla\mathcal{V}+\varepsilon^2\Delta v^\varepsilon\right)\right\rangle\\ &\le c\|v^\varepsilon\|_{H^{s+1}}^3 + \|v^\varepsilon\|_{H^{s+1}}\|\mathcal{V}\|_{H^{s+2}}.\end{aligned}$$

By (1.11a), we also have that

$$\partial_t\frac{\|S^\varepsilon\|_{L^2}^2}{2} \le \|S^\varepsilon\|_{L^2}\left(\|\mathcal{V}\|_{L^2} + \|v^\varepsilon\|_{L^4}^2/2\right)$$

so that

$$\partial_t\|S^\varepsilon\|_{H^{s+2}}^2 \le c\|\mathcal{V}\|_{H^{s+2}}\|S^\varepsilon\|_{H^{s+2}} + c\|S^\varepsilon\|_{H^{s+2}}^3.$$

The global existence and the uniqueness of a solution A^ε of equation (1.11b) follows from the fact that

$$\Psi^\varepsilon = A^\varepsilon \exp\left(iS^\varepsilon/\varepsilon\right)$$

satisfies equation (1.1). By Lemma 3.3, recalling that $s > d/2+1$, we also have

$$\partial_t \|A^\varepsilon\|_{H^s}^2 \leq c\|A^\varepsilon\|_{H^s}^2 \|S^\varepsilon\|_{H^{s+2}} + \mathrm{Re}\left\langle \Lambda^s A^\varepsilon, \Lambda^s R\right\rangle.$$

where $R = \frac{i\varepsilon\Delta A^\varepsilon}{2} - i\varepsilon A^\varepsilon \Delta S^\varepsilon$ so that an integration by parts gives us

$$\partial_t \|A^\varepsilon\|_{H^s}^2 \leq c\|A^\varepsilon\|_{H^s}^2 \|S^\varepsilon\|_{H^{s+2}}.$$

We obtain that

$$\partial_t \|\phi_t(u_0)\|_s^2 \leq c_1 \|\phi_t(u_0)\|_s \|\mathcal{V}\|_{H^{s+2}} + c_2 \|\phi_t(u_0)\|_s^3$$

and

$$\partial_t \|\phi_t(u_0)\|_s \leq c_1 \|\mathcal{V}\|_{H^{s+2}} + c_2 \|\phi_t(u_0)\|_s^2.$$

We get then that

$$\|\phi_t(u_0)\|_s \leq \sqrt{\frac{c_1\|\mathcal{V}\|_{H^{s+2}}}{c_2}} \tan\left(t\sqrt{c_1c_2\|\mathcal{V}\|_{H^{s+2}}} + \arctan\left(M\sqrt{\frac{c_2}{c_1\|\mathcal{V}\|_{H^{s+2}}}}\right)\right)$$

so that there is $h_1 = h_1(M) > 0$ such that for all $0 \leq t \leq h_1$

$$\|\phi_t(u_0)\|_s \leq 2M.$$

□

The following result will be used several times and in particular for the proof of the stability of equation (1.11) in Lemma 2.3.

Lemma 3.4. *Let $s_0 > d/2+1$. Let $u_1 = (S_1, A_1)$ be in $L^\infty([0,T], \Sigma_{s_0+1})$, $u_2 = (S_2, A_2)$, $(R_{1,S}, R_{1,A})$ and $(R_{2,S}, R_{2,A})$ be in $L^\infty([0,T], \Sigma_{s_0})$. Assume moreover that for $i = 1, 2$*

$$\partial_t S_i + \frac{|\nabla S_i|^2}{2} = R_{i,S},$$
$$\partial_t A_i + \nabla S_i \cdot \nabla A_i + A_i \frac{\Delta S_i}{2} = R_{i,A}.$$

Then, we have

$$\partial_t \|u_1 - u_2\|_{s_0}^2 \leq c\|u_1 - u_2\|_{s_0}^2 \left(\|u_1\|_{s_0+1} + \|u_2\|_{s_0}\right) + 2\left\langle u_1 - u_2, R_1 - R_2\right\rangle_{s_0}$$

where $R_i = (R_{i,S}, R_{i,A})^T$.

Proof. Let $s_0 > d/2+1$. Let us define $v_1 = \nabla S_1$, $v_2 = \nabla S_2$, $w = v_1 - v_2$, $B = A_1 - A_2$ and $u = u_1 - u_2$.

We have that

$$\begin{aligned}\partial_t w &= -(v_1 \cdot \nabla)v_1 + (v_2 \cdot \nabla)v_2 + \nabla\left(R_{1,S} - R_{2,S}\right) \\ &= -(v_2 \cdot \nabla)w - (w \cdot \nabla)v_1 + \nabla\left(R_{1,S} - R_{2,S}\right)\end{aligned}$$

and Lemma 3.2 ensures that

$$\partial_t \|w\|_{H^{s_0+1}}^2 \leq c\|w\|_{H^{s_0+1}}^2 \|v_2\|_{H^{s_0+1}} + 2\left\langle \Lambda^{s_0+1} w, \Lambda^{s_0+1} R\right\rangle.$$

where $R = -(w \cdot \nabla)v_1 + \nabla\left(R_{1,S} - R_{2,S}\right)$. We also have that

$$\|(w \cdot \nabla)v_1\|_{H^{s_0+1}} \leq c\|w\|_{H^{s_0+1}} \|v_1\|_{H^{s_0+2}}$$

and

$$\partial_t\|w\|^2_{H^{s_0+1}} \leq c\|w\|^2_{H^{s_0+1}}\left(\|S_1\|_{H^{s_0+3}}+\|S_2\|_{H^{s_0+2}}\right)+2\left\langle\Lambda^{s_0+1}w,\Lambda^{s_0+1}\nabla\left(R_{1,S}-R_{2,S}\right)\right\rangle.$$

We also have

$$\partial_t(S_1-S_2)=-\frac{1}{2}(v_1+v_2)\cdot w+(R_{1,S}-R_{2,S})$$

so that

$$\partial_t\|S_1-S_2\|^2_{L^2}\leq c\|S_1-S_2\|_{L^2}\|w\|_{L^2}\left(\|S_1\|_{W^{1,\infty}}+\|S_2\|_{W^{1,\infty}}\right)+2\left\langle S_1-S_2,R_{1,S}-R_{2,S}\right\rangle$$

and then

$$\begin{aligned}\partial_t\|S_1-S_2\|^2_{H^{s_0+2}} &\leq C\|S_1-S_2\|^2_{H^{s_0+2}}\left(\|S_1\|_{H^{s_0+3}}+\|S_2\|_{H^{s_0+2}}\right)\\&\quad+2\left\langle S_1-S_2,R_{1,S}-R_{2,S}\right\rangle+2\left\langle\Lambda^{s_0+1}\nabla\left(S_1-S_2\right),\Lambda^{s_0+1}\nabla\left(R_{1,S}-R_{2,S}\right)\right\rangle\end{aligned}$$

Let us study B, we have

$$\partial_t B+\nabla S_2\cdot\nabla B+\frac{\Delta S_2}{2}B=R$$

where

$$R=-w\cdot\nabla A_1-\frac{\operatorname{div}(w)}{2}A_1+(R_{1,A}-R_{2,A})$$

Hence, we obtain by Lemma 3.3

$$\begin{aligned}\partial_t\|B\|^2_{H^{s_0}} &\leq c\|B\|^2_{H^{s_0}}\|S_2\|_{H^{s_0+2}}+2\operatorname{Re}\left\langle\Lambda^{s_0}B,\Lambda^{s_0}R\right\rangle\\&\leq c\|B\|^2_{H^{s_0}}\|S_2\|_{H^{s_0+2}}+c\|B\|_{H^{s_0}}\|w\|_{H^{s_0+1}}\|A_1\|_{H^{s_0+1}}\\&\quad+2\operatorname{Re}\left\langle\Lambda^{s_0}B,\Lambda^{s_0}\left(R_{1,A}-R_{2,A}\right)\right\rangle\end{aligned}$$

and

$$\partial_t\|u_1-u_2\|^2_{s_0}\leq c\|u_1-u_2\|^2_{s_0}\left(\|u_1\|_{s_0+1}+\|u_2\|_{s_0}\right)+2\left\langle u_1-u_2,R_1-R_2\right\rangle_{s_0}$$

The result follows. □

Let us study now the stability of equation (1.11) and prove Lemma 2.3.

Proof. Let $s>d/2+1$ and $\varepsilon\in(0,\varepsilon_{max}]$. Let us define for $i=1,2$

$$\begin{aligned}R_{i,S}&=-\mathcal{V}+\varepsilon^2\Delta S_i,\\R_{i,A}&=i\varepsilon\frac{\Delta A_i}{2}-i\varepsilon A_i\Delta S_i.\end{aligned}$$

We apply Lemma 3.4 with $s_0=s$. We have by integrations by parts that

$$\left\langle S_1-S_2,R_{1,S}-R_{2,S}\right\rangle+\left\langle\Lambda^{s+1}\nabla\left(S_1-S_2\right),\Lambda^{s+1}\nabla\left(R_{1,S}-R_{2,S}\right)\right\rangle\leq 0,$$

and

$$\begin{aligned}\operatorname{Re}\left\langle\Lambda^s\left(A_1-A_2\right),\Lambda^s\left(R_{1,A}-R_{2,A}\right)\right\rangle &\leq c\|A_1-A_2\|^2_{H^s}\|S_1\|_{H^{s+2}}\\&\quad+c\|A_1-A_2\|_{H^s}\|S_1-S_2\|_{H^{s+2}}\|A_2\|_{H^s}.\end{aligned}$$

so that

$$\partial_t\|\phi_t(u_1)-\phi_t(u_2)\|^2_s\leq c\|u_1-u_2\|^2_s\left(\|\phi_t(u_1)\|_{s+1}+\|\phi_t(u_2)\|_s\right)$$

and the result follows. □

3.3. **Study of the numerical flow φ^{1234}.** The following lemma is inspired by the work of Holden, Lubich and Risebro [24].

Lemma 3.5. *Let $s_0 > d/2+1$ and $M > 0$. There exists $h_5 = h_5(M) > 0$ such that for any $u_0 \in \Sigma_{s_0}$ satisfying $\|u_0\|_{s_0} \leq M$ and any $0 \leq t \leq h_5$, the following two points are true.*

(i) We have that $\|\varphi_t^1(u_0)\|_{s_0} \leq 2M$.
(ii) Let $s_1 \geq s_0$. There is $C_5 = C_5(M) > 0$ such that if $u_0 \in \Sigma_{s_1}$, then

$$\|\varphi_t^1(u_0)\|_{s_1} \leq \exp(C_5 t)\, \|u_0\|_{s_1}.$$

Proof. The existence of the solution S of (1.13a) follows for instance from the method of characteristics. Lemma 3.2 ensures that for $s > d/2+1$

$$\partial_t \|\nabla S\|_{H^{s+1}}^2 \leq c\|\nabla S\|_{H^{s+1}}^2 \|\nabla(\nabla S)\|_{L^\infty} \leq C\|\nabla S\|_{H^{s+1}}^2 \|S(t)\|_{W^{2,\infty}}.$$

We also have

$$\partial_t \|S\|_{L^2}^2 \leq c\|S(t)\|_{L^2}\|\nabla S(t)\|_{L^4}^2$$

so that

$$\partial_t \|S\|_{H^{s+2}}^2 \leq C\|S(t)\|_{H^{s+2}}^2 \|S(t)\|_{W^{2,\infty}}.$$

The remaining of the proof follows exactly the same lines as the one of Lemma 2.2. By Lemma 3.3 and an integration by parts, we have

$$\begin{aligned}\partial_t \|A\|_{H^s}^2 &\leq C\left(\|A\|_{H^s}^2 \|S\|_{W^{3,\infty}} + \|A\|_{H^s}\|S\|_{H^{s+2}}\|A\|_{W^{1,\infty}}\right)\\ &\leq C\|\varphi_t^1(u_0)\|_s^2 \|\varphi_t^1(u_0)\|_{W^{3,\infty}\times W^{1,\infty}}\end{aligned}$$

and

$$\begin{aligned}\partial_t \|\varphi_t^1(u_0)\|_s^2 &\leq C\|\varphi_t^1(u_0)\|_s^2 \|\varphi_t^1(u_0)\|_{W^{3,\infty}\times W^{1,\infty}}\\ &\leq C\|\varphi_t^1(u_0)\|_s^3.\end{aligned}$$

Taking $s = s_0$, we get that

$$\|\varphi_t^1(u_0)\|_{s_0} \leq \frac{M}{1-cMt}$$

and there is $h_5 = h_5(M) > 0$ such that for all $t \in [0, h_9]$

$$\|\varphi_t^1(u_0)\|_{s_0} \leq 2M.$$

We also obtain for $s = s_1 \geq s_0 > d/2+1$ and $t \in [0, h_9]$ that

$$\begin{aligned}\partial_t \|\varphi_t^1(u_0)\|_{s_1}^2 &\leq C\|\varphi_t^1(u_0)\|_{s_1}^2 \|\varphi_t^1(u_0)\|_{s_0}\\ &\leq 2CM\|\varphi_t^1(u_0)\|_{s_1}^2.\end{aligned}$$

and the result follows from Gronwall's Lemma. □

We immediately get the following result for the second and the third flows.

Lemma 3.6. *Let $s_0 > 0$ and $M > 0$. There is $h_6 = h_6(M)$ such that for any $u_0 \in \Sigma_{s_0}$ satisfying $\|u_0\|_{s_0} \leq M$ any $0 \leq t \leq h_6$, the following two points holds true.*

(i) $\|\varphi_t^2(u_0)\|_{s_0} \leq M$ and $\|\varphi_t^3(u_0)\|_{s_0} \leq 2M$,
(ii) Let $s_1 \geq 0$. If moreover $u_0 \in \Sigma_{s_1}$, then, we have

$$\|\varphi_t^2(u_0)\|_{s_1} \leq \|u_0\|_{s_1} \text{ and } \|\varphi_t^3(u_0)\|_{s_1} \leq \|u_0\|_{s_1} + t\|\mathcal{V}\|_{H^{s_1+2}}.$$

The following lemma study the fourth flow.

Lemma 3.7. *Let $s_0 > d/2+1$ and $M>0$. There exists $h_7 = h_7(M) > 0$ such that for any $u_0 \in \Sigma_{s_0}$ satisfying $\|u_0\|_{s_0} \leq M$ and any $0 \leq t \leq h_7$, the following two points holds true.*

(i) $\|\varphi_t^4(u_0)\|_{s_0} \leq 2M$,

(ii) Let $s_1 \geq s_0$. There is $C_7 = C_7(M) > 0$ such that if $u_0 \in \Sigma_{s_1}$,

$$\|\varphi_t^4(u_0)\|_{s_1} \leq \exp(C_7 t)\,\|u_0\|_{s_1}.$$

Proof. Let $s > d/2+1$. By integration by parts, we have $\partial_t \|S(h)\|_{H^{s+2}}^2 \leq 0$ and

$$\begin{aligned}\partial_t \frac{\|A\|_{H^s}^2}{2} &= \mathrm{Re}\,\langle \Lambda^s A, \Lambda^s(-i\varepsilon A\Delta S)\rangle = \mathrm{Re}\,\langle \Lambda^s A, [\Lambda^s, -i\varepsilon\Delta S]A\rangle\\ &\leq c\|A\|_{H^s}\left(\|\nabla(\Delta S)\|_{L^\infty}\|A\|_{H^{s-1}} + \|\Delta S\|_{H^s}\|A\|_{L^\infty}\right)\\ &\leq c\|A\|_{H^s}^2\|S\|_{W^{3,\infty}} + c\|A\|_{H^s}\|S\|_{H^{s+2}}\|A\|_{L^\infty}.\end{aligned}$$

We obtain for $s = s_0$ that

$$\partial_t\|\varphi_t^4(u_0)\|_{s_0}^2 \leq c\|\varphi_t^4(u_0)\|_{s_0}^3,$$

for $s = s_1$ that

$$\partial_t\|\varphi_t^4(u_0)\|_{s_1}^2 \leq c\|\varphi_t^4(u_0)\|_{s_1}^2\|\varphi_t^4(u_0)\|_{s_0}$$

and the result follows from the arguments of the end of the proof of Lemma 3.5. □

Taking $s_0 = s$ and $s_1 = s_0+2$, we immediately get Lemma 2.5 combining Lemmas 3.5, 3.6 and 3.7.

3.4. **Proof of Theorem 2.1.** Let $M > 0$. Lemma 2.2 ensures that there is $h_1 = h_1(M) > 0$ such that for any $\varepsilon \in (0, \varepsilon_{max}]$ and any $u_0 \in \Sigma_{s+2}$ satisfying $\|u_0\|_{s+2} \leq M$, the solutions $t \mapsto \phi_t^\varepsilon(u_0)$ of equation (1.11) are well-defined in $L^\infty([0,h_1], \Sigma_{s+2})$ and uniformly bounded with respect to ε.

Let $\varepsilon, \varepsilon' \in (0, \varepsilon_{max}]$, $u_0, u_0' \in \Sigma_{s+2}$ such that $\|u_0\|_{s+2} \leq M$ and $\|u_0'\|_{s+2} \leq M$. We define $(S^\varepsilon, A^\varepsilon)^T = \phi^\varepsilon(u_0)$, $(S^{\varepsilon'}, A^{\varepsilon'})^T = \phi^{\varepsilon'}(u_0')$ and

$$\begin{aligned}&R_{1,S} = -\mathcal{V} + \varepsilon^2\Delta S^\varepsilon, \quad R_{2,S} = -\mathcal{V} + \varepsilon'^2\Delta S^{\varepsilon'},\\ &R_{1,A} = i\varepsilon\frac{\Delta A^\varepsilon}{2} - i\varepsilon A^\varepsilon\Delta S^\varepsilon, \quad R_{2,A} = i\varepsilon'\frac{\Delta A^{\varepsilon'}}{2} - i\varepsilon' A^{\varepsilon'}\Delta S^{\varepsilon'}.\end{aligned}$$

We apply Lemma 3.4 with $s_0 = s$, $u_1 = \phi^\varepsilon(u_0)$ and $u_2 = \phi^{\varepsilon'}(u_0)$. We have by integrations by parts that

$$\begin{aligned}&\left\langle S^\varepsilon - S^{\varepsilon'}, R_{1,S} - R_{2,S}\right\rangle + \left\langle \Lambda^{s+1}\nabla\left(S^\varepsilon - S^{\varepsilon'}\right), \Lambda^{s+1}\nabla\left(R_{1,S} - R_{2,S}\right)\right\rangle\\ &\quad\leq c|\varepsilon-\varepsilon'|\|S^\varepsilon - S^{\varepsilon'}\|_{H^{s+2}}\|S^\varepsilon\|_{H^{s+4}},\end{aligned}$$

and

$$\begin{aligned}\mathrm{Re}\left\langle \Lambda^s\left(A^\varepsilon - A^{\varepsilon'}\right), \Lambda^s\left(R_{1,A} - R_{2,A}\right)\right\rangle &\leq c|\varepsilon-\varepsilon'|\|A^\varepsilon - A^{\varepsilon'}\|_{H^s}\|A^\varepsilon\|_{H^{s+2}}\\ &+ c\|A^\varepsilon - A^{\varepsilon'}\|_{H^s}^2\|S^{\varepsilon'}\|_{H^{s+2}} + c\|A^\varepsilon\|_{H^s}\|A^\varepsilon - A^{\varepsilon'}\|_{H^s}\|S^\varepsilon - S^{\varepsilon'}\|_{H^{s+2}}\\ &+ c|\varepsilon-\varepsilon'|\|A^\varepsilon - A^{\varepsilon'}\|_{H^s}\|A^\varepsilon\|_{H^s}\|S^\varepsilon\|_{H^{s+2}}.\end{aligned}$$

so that

$$\begin{aligned}\partial_t\|\phi_t^\varepsilon(u_0) - \phi_t^{\varepsilon'}(u_0')\|_s^2 &\leq c\|\phi_t^\varepsilon(u_0) - \phi_t^{\varepsilon'}(u_0')\|_s^2\left(\|\phi_t^\varepsilon(u_0)\|_{s+1} + \|\phi_t^{\varepsilon'}(u_0')\|_s\right)\\ &+ c|\varepsilon-\varepsilon'|\|\phi_t^\varepsilon(u_0) - \phi_t^{\varepsilon'}(u_0')\|_s(\|\phi_t^\varepsilon(u_0)\|_{s+2} + \|\phi_t^\varepsilon(u_0)\|_s^2).\end{aligned}$$

Gronwall's Lemma ensures that for all $t \in [0, h_1]$

$$\|\phi_t^\varepsilon(u_0) - \phi_t^{\varepsilon'}(u_0')\|_s \leq C\left(\|u_0 - u_0'\|_s + |\varepsilon - \varepsilon'|\right) \tag{3.3}$$

where

$$C = C(\|\phi^\varepsilon(u_0)\|_{L^\infty([0,h_1],\Sigma_{s+2})}, \|\phi^{\varepsilon'}(u_0')\|_{L^\infty([0,h_1],\Sigma_s)}) > 0.$$

Thus, $(\phi_t^\varepsilon(u_0))_{t\in[0,h_1]}$ is a Cauchy sequence of ε of $L^\infty([0,h_1],\Sigma_s)$. The limit $\phi^0(u_0)$ is solution of (1.11) with $\varepsilon = 0$. Uniqueness follows from (3.3). We get immediately that Lemma 2.2 is also true for $\varepsilon = 0$ and $\phi^0(u_0) \in L^\infty([0,h_1],\Sigma_{s+2})$.

Let

$$T_{max} = \sup\{t > 0 : \phi^0(u_0) \in L^\infty([0,t];\Sigma_{s+2})\} > 0,$$

then, for any $0 < T < T_{max}$, $\phi^0(u_0) \in L^\infty([0,T];\Sigma_{s+2})$. Let us define $\widetilde{T} = h_1(2M_s^0)$ (see Lemma 2.2 and (2.2)), $C = C(M_{s+2}^0, 2M_s^0)$ (see inequality (3.3)) and N the smallest $n \in \mathbb{N}$ such that

$$n\widetilde{T} \geq T.$$

Let $\varepsilon_0 > 0$ be such that $\varepsilon_0 \sum_{j=1}^N C^j \leq M_s^0$ and $\varepsilon \in (0, \varepsilon_0]$. By inequality (3.3) and Lemma 2.2, we obtain by induction on $0 \leq k \leq N$ that

$$\|\phi_t^\varepsilon(u_0)\|_s \leq \|\phi_t^0(u_0)\|_s + \|\phi_t^0(u_0) - \phi_t^\varepsilon(u_0)\|_s \leq M_s^0 + \varepsilon \sum_{j=1}^k C^j \leq M_s^0 \leq 2M_s^0$$

for all $t \in [0, k\widetilde{T}]$. Thus, $\phi^\varepsilon(u_0)$ is well-defined on $[0,T]$, belongs to $L^\infty([0,T];\Sigma_s)$ and

$$\|\phi^\varepsilon(u_0)\|_{L^\infty([0,T];\Sigma_s)} \leq 2M_s^0. \tag{3.4}$$

Following the arguments of the proofs of Lemmas 3.5, 3.6 and 3.7, we obtain that

$$\partial_t \|\phi^\varepsilon(u_0)\|_{s+2}^2 \leq c\|\phi^\varepsilon(u_0)\|_{s+2}^2 \|\phi^\varepsilon(u_0)\|_s + \|\phi^\varepsilon(u_0)\|_{s+2}\|\mathcal{V}\|_{H^{s+4}}.$$

Gronwall's lemma ensures that there is $\widetilde{C} = \widetilde{C}(M_0^s) > 0$ independent of ε such that

$$\|\phi_t^\varepsilon(u_0)\|_{s+2} \leq \exp(t\widetilde{C})\left(t\|\mathcal{V}\|_{H^{s+4}} + \|u_0\|_{s+2}\right)$$

for all $t \in [0,T]$. Moreover, $\phi^\varepsilon(u_0)$ is well-defined in $L^\infty([0,T],\Sigma_{s+2})$ for any $\varepsilon \in (0, \varepsilon_{max}]$. Then, the same arguments ensure that $\varepsilon \in (0,\varepsilon_{max}] \mapsto \phi^\varepsilon(u_0)$ is continuous in $L^\infty([0,T],\Sigma_s)$ so that $(\phi^\varepsilon(u_0))_{\varepsilon\in[0,\varepsilon_{max}]}$ is uniformly bounded in $L^\infty([0,T],\Sigma_{s+2})$ and the result follows.

3.5. **The local error estimates.** The proof of Lemma 2.6 given in this section is inspired by [2] where the two flows case is treated. The local error of scheme (1.17) is defined by

$$\mathscr{R}(h,u) = \varphi_h^{1234}(u) - \phi_h(u).$$

3.5.1. *Main lemmas.* Let us give the main ingredients that will be used in the proof of Lemma 2.6. The balls in Σ_{s_0} are denoted by

$$B_{s_0}(M) = \{u \in \Sigma_{s_0} : \|u\|_{s_0} \leq M\} \tag{3.5}$$

for $s_0 \geq 0$ and $M > 0$. The strategy to get estimates on $\mathscr{R}(h,u)$ is to differentiate $\mathscr{R}$ with respect to h. Hence, we will be in need of the following lemma whose proof is postponed to Appendix A.

Lemma 3.8. *Let $s > d/2 + 1$ and $M > 0$. There exists $h_8 = h_8(M) > 0$ such that the following two points hold true.*

(i) Let $s_1 \geq s$. The functions

$$(h,u) \in [0,h_8] \times (B_s(M) \cap \Sigma_{s_1+3}) \mapsto \varphi_h^1(u) \in \Sigma_{s_1},$$
$$(h,u) \in [0,h_8] \times (B_s(M) \cap \Sigma_{s_1+2}) \mapsto \varphi_h^2(u) \in \Sigma_{s_1},$$
$$(h,u) \in [0,h_8] \times (B_s(M) \cap \Sigma_{s_1}) \mapsto \varphi_h^3(u) \in \Sigma_{s_1},$$

are C^1-applications.

(ii) Let $s_2 \geq s$ and $M_2 > 0$. There exists $C_8 = C_8(M, M_2) > 0$ such that for any $u \in B_s(M) \cap B_{s_2+1}(M_2)$, $h \in [0,h_8]$ and any $u_0 \in \Sigma_{s_2}$, we have

$$\|\partial_2 \varphi_h^1(u) \cdot u_0\|_{s_2} \leq \exp(C_8 h)\, \|u_0\|_{s_2},$$
$$\|\partial_2 \varphi_h^2(u) \cdot u_0\|_{s_2} \leq \exp(C_8 h)\, \|u_0\|_{s_2},$$
$$\|\partial_2 \varphi_h^3(u) \cdot u_0\|_{s_2} \leq \exp(C_8 h)\, \|u_0\|_{s_2},$$

and

$$|\left\langle u_0, D\mathcal{N}_1(\varphi_h^1(u)) \cdot u_0 \right\rangle_{s_2}| \leq C_8 \|u_0\|_{s_2}^2,$$
$$|\left\langle u_0, D\mathcal{N}_2(\varphi_h^2(u)) \cdot u_0 \right\rangle_{s_2}| \leq C_8 \|u_0\|_{s_2}^2,$$
$$|\left\langle u_0, D\mathcal{N}_3(\varphi_h^3(u)) \cdot u_0 \right\rangle_{s_2}| \leq C_8 \|u_0\|_{s_2}^2,$$

where $\langle \cdot, \cdot \rangle_{s_0}$ is defined in (3.2) and $B_{s_0}(M)$ in (3.5).

The following lemma ensures that the object studied in the proof of Lemma 2.6 are well-defined.

Lemma 3.9. *Let $s > d/2 + 1$ and $M > 0$. There is $h_9 = h_9(M) > 0$ such that the following three points are true. Let $u \in \Sigma_{s+7}$ such that $\|u\|_{s+2} \leq M$.*

(i) We have for all $h \in [0,h_9]$,

$$\varphi_h^{1234}(u),\ \varphi_h^{234}(u),\ \varphi_h^{34}(u) \text{ and } \varphi_h^4(u)$$

are well-defined, belong to $L^\infty([0,h_9], \Sigma_{s+7})$ and satisfy

$$\max\left(\|\varphi_h^4(u)\|_{s+2}, \|\varphi_h^{34}(u)\|_{s+2}, \|\varphi_h^{234}(u)\|_{s+2}\right) \leq 4M.$$

(ii) The application $h \in [0,h_9] \mapsto \mathscr{R}(h,u) \in \Sigma_s$ is differentiable,

$$\partial_h \mathscr{R}(h,u) = \sum_{k=1}^{4} \mathcal{N}_k(\varphi_h^{1234}(u)) - \mathcal{N}_k(\phi_h(u)) + \mathscr{S}(h,u),$$
$$\mathscr{R}(0,u) = 0.$$

where

$$\begin{aligned}\mathscr{S}(h,u) = {} & (\chi_{12} + \chi_{13} + \chi_{14})\,(h, \varphi_h^{234}(u)) \\ & + \partial_2 \varphi^1(h, \varphi_h^{234}(u)) \cdot (\chi_{23} + \chi_{24})(h, \varphi_h^{34}(u)) \\ & + \partial_2 \varphi^1(h, \varphi_h^{234}(u)) \cdot \partial_2 \varphi^2(h, \varphi_h^{34}(u)) \cdot \chi_{34}(h, \varphi_h^4(u))\end{aligned}$$

and $\chi_{ij}(h,v) = \partial_2 \varphi_h^i(v) \cdot \mathcal{N}_j(v) - \mathcal{N}_j(\varphi_h^i(v))$ (see [2, Section 3]).

(iii) Let $v \in \Sigma_{s+7}$. We have,

$$\partial_h \chi_{ij}(h,v) = D\mathcal{N}_i(\varphi_h^i(v)) \cdot \chi_{ij}(h,v) + [\mathcal{N}_i, \mathcal{N}_j](\varphi_h^i(v))$$
$$\chi_{ij}(0,v) = 0.$$

The following lemma gives bounds on the commutators.

Lemma 3.10. *Let $s > d/2 + 1$. There is $C > 0$ such that for any $u \in \Sigma_{s+2}$ and any $1 \leq i < j \leq 4$, we have*

$$\|[\mathcal{N}_i, \mathcal{N}_j](u)\|_s \leq C\|u\|_{s+2}^2(1 + \|u\|_{s+2}).$$

C does not depend on $\varepsilon \in (0, \varepsilon_{max}]$.

3.5.2. *Proof of Lemma 2.6.* Let $s > d/2 + 1$ and $M > 0$. Let us define $h_4 = h_4(M) = h_9(M)$. Assume for the moment that $u \in \Sigma_{s+7}$ and $\|u\|_{s+2} \leq M$. By Lemmas 3.8, 3.9, 3.10 and Gronwall's Lemma, there is $C = C(M) > 0$ such that for any $h \in [0, h_4]$

$$\begin{aligned}
&\|\chi_{12}(h, \varphi_h^{234}(u))\|_s + \|\chi_{13}(h, \varphi_h^{234}(u))\|_s + \|\chi_{14}(h, \varphi_h^{234}(u))\|_s \leq Ch,\\
&\|\chi_{23}(h, \varphi_h^{34}(u))\|_s + \|\chi_{24}(h, \varphi_h^{34}(u))\|_s \leq Ch\\
&\|\chi_{34}(h, \varphi_h^{4}(u))\|_s \leq Ch.
\end{aligned}$$

Using again Lemmas 3.8 and 3.9, we obtain that

$$\|\mathscr{S}(h, u)\|_s \leq Ch.$$

Let us define

$$\begin{aligned}
R_{1,S} &= -\mathcal{V} + \varepsilon^2 \Delta\Pi_1 \phi_h(u),\\
R_{2,S} &= -\mathcal{V} + \varepsilon^2 \Delta\Pi_1 \varphi_h^{1234}(u) + \Pi_1 \mathscr{S}(h, u),\\
R_{1,A} &= \frac{i\varepsilon \Delta\Pi_2 \phi_h(u)}{2} - i\varepsilon \frac{\Delta\Pi_1 \phi_h(u)}{2} \Pi_2 \phi_h(u),\\
R_{2,A} &= \frac{i\varepsilon \Delta\Pi_2 \varphi_h^{1234}(u)}{2} - i\varepsilon \frac{\Delta\Pi_1 \varphi_h^{1234}(u)}{2} \Pi_2 \varphi_h^{1234}(u) + \Pi_2 \mathscr{S}(h, u)
\end{aligned}$$

where Π_1 and Π_2 are defined in (3.1). Then, Lemma 3.4 ensures that

$$\partial_t \|\varphi_h^{1234}(u) - \phi_h(u)\|_s^2 \leq C\|\varphi_h^{1234}(u) - \phi_h(u)\|_s^2 + C\|\varphi_h^{1234}(u) - \phi_h(u)\|_s \|\mathscr{S}(h, u)\|_s.$$

Gronwall's lemma ensures that there is $K_4 = K_4(M)$ such that

$$\|\varphi_h^{1234}(u) - \phi_h(u)\|_s^2 \leq K_4 h^2.$$

Let us insist on the fact that K_4 and h_4 only depend on M. Hence, using the fact that for all $h \in [0, h_4]$, the applications

$$u \in \Sigma_{s+2} \mapsto \phi_h(u) \in \Sigma_s$$

and

$$u \in \Sigma_{s+2} \mapsto \varphi_h^{1234}(u) \in \Sigma_s$$

are continuous (see Lemma 2.3 and the proof of Lemma 3.8), we get that

$$\|\varphi_h^{1234}(u) - \phi_h(u)\|_s^2 \leq K_4 h^2.$$

holds true for any $u \in \Sigma_{s+2}$ such that $\|u\|_{s+2} \leq M/2$ and the result follows.

3.5.3. *Proof of Lemma 3.9.* Let $u \in \Sigma_{s+7}$ such that $\|u\|_{s+2} \leq M$. Let us define

$$0 < h_9 = h_9(M) := \min\left(h_5(4M), h_6(2M), h_7(M), h_8(4M)\right), \tag{3.6}$$

where h_5, h_6, h_7 and h_8 are defined by Lemmas 3.5, 3.6, 3.7 and 3.8. Using these lemmas, we get that for all $h \in [0, h_9]$,

$$\varphi_h^{1234}(u),\ \varphi_h^{234}(u),\ \varphi_h^{34}(u) \text{ and } \varphi_h^{4}(u)$$

are well-defined, belong to $L^\infty([0,h_9], \Sigma_{s+7})$ and satisfy

$$\max\left(\|\varphi_h^4(u)\|_{s+2}, \|\varphi_h^{34}(u)\|_{s+2}, \|\varphi_h^{234}(u)\|_{s+2}\right) \leq 4M.$$

Define for $i = 1,2,3,4$, $h \geq 0$ and $u_0 \in \Sigma_{s+2}$, the applications

$$\vartheta^i(h,u_0) = (h, \varphi_h^i(u_0))^T \text{ and } \Xi(h,u_0) = u_0.$$

By Lemma 3.8, we obtain that

$$h \in [0,h_9] \mapsto \varphi_h^{1234}(u) \in B_s(8M)$$

is a C^1-application since $\varphi_h^{1234}(u) = \Xi \circ \vartheta^1 \circ \vartheta^2 \circ \vartheta^3 \circ \vartheta^4(h,u)$. We have that

$$\begin{aligned}
\partial_h \varphi_h^{1234}(u) \quad &= \mathcal{N}_1 \varphi_h^{1234}(u) + \partial_2 \varphi_h^1(\varphi_h^{234}(u)) \cdot \mathcal{N}_2 \varphi_h^{234}(u) \\
&+ \partial_2 \varphi_h^1(\varphi_h^{234}(u)) \cdot \partial_2 \varphi_h^2(\varphi_h^{34}(u)) \cdot \mathcal{N}_3 \varphi_h^{34}(u) \\
&+ \partial_2 \varphi_h^1(\varphi_h^{234}(u)) \cdot \partial_2 \varphi_h^2(\varphi_h^{34}(u)) \cdot \partial_2 \varphi_h^3(\varphi_h^4(u)) \cdot \mathcal{N}_4 \varphi_h^4(u),
\end{aligned}$$

so that

$$\begin{aligned}
&\partial_h \varphi_h^{1234}(u) \\
&= \mathcal{N}_1 \varphi_h^{1234}(u) + \mathcal{N}_2 \varphi_h^{1234}(u) + \mathcal{N}_3 \varphi_h^{1234}(u) + \mathcal{N}_4 \varphi_h^{1234}(u) \\
&\quad + \chi_{12}(h, \varphi_h^{234}(u)) + \chi_{13}(h, \varphi_h^{234}(u)) + \chi_{14}(h, \varphi_h^{234}(u)) \\
&\quad + \partial_2 \varphi_h^1(\varphi_h^{234}(u)) \cdot \left(\chi_{23}(h, \varphi_h^{34}(u)) + \chi_{24}(h, \varphi_h^{34}(u))\right) \\
&\quad + \partial_2 \varphi_h^1(\varphi_h^{234}(u)) \cdot \partial_2 \varphi_h^2(\varphi_h^{34}(u)) \cdot \chi_{34}(h, \varphi_h^4(u)).
\end{aligned}$$

Let us show the last point. We have for $u_0 \in \Sigma_{s+7}$ that

$$\partial_h\left(\partial_2 \varphi_h^i(v) \cdot u_0\right) = D\mathcal{N}_i(\varphi_h^i(v)) \cdot \left(\partial_2 \varphi_h^i(v) \cdot u_0\right),$$

so that

$$\begin{aligned}
\partial_h \chi_{ij}(h,v) &= D\mathcal{N}_i(\varphi_h^i(v)) \cdot \partial_2 \varphi_h^i(v) \cdot \mathcal{N}_j(v) - D\mathcal{N}_j(\varphi_h^i(v)) \cdot \partial_h \varphi_h^i(v) \\
&= D\mathcal{N}_i(\varphi_h^i(v)) \cdot \chi_{ij}(h,v) + [\mathcal{N}_i, \mathcal{N}_j](\varphi_h^i(v)).
\end{aligned}$$

3.5.4. *Proof of Lemma 3.10.* Let us consider

$$u = \begin{pmatrix} S \\ A \end{pmatrix} \text{ and } u_0 = \begin{pmatrix} S_0 \\ A_0 \end{pmatrix}.$$

We have

$$\begin{aligned}
D\mathcal{N}_1(u) \cdot u_0 &= \begin{pmatrix} -\nabla S \cdot \nabla S_0 \\ -\nabla S \cdot \nabla A_0 - A_0 \frac{\Delta S}{2} - \nabla S_0 \cdot \nabla A - A \frac{\Delta S_0}{2} + i \frac{\Delta A_0}{2} \end{pmatrix}, \\
D\mathcal{N}_2(u) \cdot u_0 &= \mathcal{N}_2 u_0 = \begin{pmatrix} 0 \\ i(\varepsilon - 1)\frac{\Delta A_0}{2} \end{pmatrix}, \\
D\mathcal{N}_3(u) \cdot u_0 &= 0, \\
D\mathcal{N}_4(u) \cdot u_0 &= \begin{pmatrix} \varepsilon^2 \Delta S_0 \\ -i\varepsilon\left(A_0 \Delta S + A \Delta S_0\right) \end{pmatrix},
\end{aligned}$$

so that, $[\mathcal{N}_1, \mathcal{N}_3](u) = 0$, $[\mathcal{N}_2, \mathcal{N}_3](u) = 0$, $[\mathcal{N}_3, \mathcal{N}_4](u) = 0$ and

$$\begin{aligned}[\mathcal{N}_1, \mathcal{N}_2](u) &= D\mathcal{N}_1(u) \cdot \mathcal{N}_2(u) - D\mathcal{N}_2(u) \cdot \mathcal{N}_1(u) \\ &= \frac{i(\varepsilon - 1)}{2} \begin{pmatrix} 0 \\ \nabla \Delta S \cdot \nabla A + A \frac{\Delta^2 S}{2} + 2 \sum_{k=1}^d \nabla \partial_k S \cdot \nabla \partial_k A + \partial_k A \frac{\Delta \partial_k S}{2} \end{pmatrix}.\end{aligned}$$

We obtain

$$\|[\mathcal{N}_1, \mathcal{N}_2](u)\|_s \quad \leq C \|u\|_{s+2}^2.$$

We also have

$$\begin{aligned}[\mathcal{N}_1, \mathcal{N}_4](u) &= D\mathcal{N}_1(u) \cdot \mathcal{N}_4(u) - D\mathcal{N}_4(u) \cdot \mathcal{N}_1(u) \\ &= \begin{pmatrix} \varepsilon^2 \sum_{k=1}^d \nabla \partial_k S \cdot \nabla \partial_k S \\ (\varepsilon - \varepsilon^2) \left(\nabla \Delta S \cdot \nabla A + A \frac{\Delta^2 S}{2} \right) - i \varepsilon A \sum_{k=1}^d \nabla \partial_k S \cdot \nabla \partial_k S \end{pmatrix},\end{aligned}$$

and

$$\|[\mathcal{N}_1, \mathcal{N}_4](u)\|_s \leq \varepsilon C \|u\|_{s+2}^2 \left(1 + \|u\|_{s+2}\right).$$

We also get

$$\begin{aligned}[\mathcal{N}_2, \mathcal{N}_4](u) &= D\mathcal{N}_2(u) \cdot \mathcal{N}_4(u) - D\mathcal{N}_4(u) \cdot \mathcal{N}_2(u) \\ &= \frac{\varepsilon(\varepsilon - 1)}{2} \begin{pmatrix} 0 \\ A \Delta^2 S + 2 \nabla A \cdot \nabla \Delta S \end{pmatrix},\end{aligned}$$

so that

$$\|[\mathcal{N}_2, \mathcal{N}_4](u)\|_s \leq \varepsilon C \|u\|_{s+2}^2 \ ,$$

and the result follows.

4. **Numerical experiments**

In this part, we illustrate the behavior of the schemes (1.17) and (1.18) introduced in Section 1.3. We restrict ourselves to the one-dimensional periodic setting in which the equations studied remain unchanged and a Fourier spectral discretization can be used. Note that eikonal equation (1.13a) is solved using the method of characteristics and an interpolation method based on a direct discrete Fourier series evaluation. Many other methods are available to solve this equation. Let us mention in particular [11, 19] where these questions are discussed in the context of advection equations.

We consider the following initial data:

$$\begin{aligned} &A_0(x) = \sin(x), \quad S_0(x) = \sin(x)/2, \\ &\Psi^\varepsilon(0, \cdot) = A_0(\cdot) e^{i S_0(\cdot)/\varepsilon}, \end{aligned} \tag{4.1}$$

and the potential

$$\mathcal{V}(x) = \frac{\sin(x)}{1 + \cos(x)^2}$$

where $x \in \mathbb{T} = \mathbb{R}/2\pi\mathbb{Z}$, for which caustics appear numerically at time $T_c = 0.8$. In our simulations, the semiclassical parameter ε varies from 1 to 2^{-10}.

The numerical solutions $(S^\varepsilon, A^\varepsilon)$, resp. Ψ^ε, are compared to corresponding reference solutions $(S^\varepsilon_{ref}, A^\varepsilon_{ref})$, resp. Ψ^ε_{ref}, which, in the absence of analytical solutions, are respectively obtained thanks to our second order splitting method (1.18) and thanks to a splitting scheme of order 4 for (1.1) (see [36]), with very small time and

space steps. More precisely, to compute $(S^\varepsilon_{ref}, A^\varepsilon_{ref})$, we have taken $N_x = 2^8$ and $h = 2^{-13}T_f$, and to compute Ψ^ε_{ref}, in order to fit with the constraints on the time step and on the space step

$$h \ll \varepsilon \text{ and } \Delta x \ll \varepsilon,$$

the space interval $[0, 2\pi]$ is discretized with $N_x = 2^{12}$ points and the time step is $h = 2^{-13}T_f$.

The various errors that are represented in the figures below are defined as follows:

$$err_{\rho^\varepsilon}(T) = \frac{\|\rho^\varepsilon_{ref}(T) - \rho^\varepsilon(T)\|_{L^1}}{\|\rho^\varepsilon_{ref}(T)\|_{L^1}}, \quad err_{\Psi^\varepsilon}(T) = \frac{\|\Psi^\varepsilon_{ref}(T) - \Psi^\varepsilon(T)\|_{L^2}}{\|\psi^\varepsilon_{ref}(T)\|_{L^2}},$$

and

$$err_{(S^\varepsilon, A^\varepsilon)}(T) = \left(\frac{\|S^\varepsilon_{ref}(T) - S^\varepsilon(T)\|^2_{L^2} + \|A^\varepsilon_{ref}(T) - A^\varepsilon(T)\|^2_{L^2}}{\|S^\varepsilon_{ref}(T)\|^2_{L^2} + \|A^\varepsilon_{ref}(T)\|^2_{L^2}} \right)^{1/2},$$

where

$$\|u\|_{L^1} = \Delta x \sum_{k=0}^{N_x-1} |u_k|, \quad \|u\|_{L^2} = \sqrt{\Delta x \sum_{k=0}^{N_x-1} |u_k|^2},$$

with $\rho^\varepsilon_{ref}(T) = |\Psi^\varepsilon_{ref}(T)|^2$ and $\rho^\varepsilon(T) = |A^\varepsilon(T)|^2$.

We first study qualitatively the dynamics, in order to guess what is the time of appearance of the caustics. Figures 1a and 1b represent the density $|A^\varepsilon|^2$ and the phase S^ε at times $T_f = 0$, 0.3, 0.6, 0.8, 1 for $\varepsilon = 2^{-4}$. The caustics appear around $t = 0.8$. At time $t = 1$, oscillations at other scales than those of the phase can be observed in $|A^\varepsilon|^2$ whereas S^ε ceases to be smooth. These figures are obtained by using our scheme (1.18) with $N_x = 2^8$ and $N_t = T_f/h = 2^9$.

Let us now focus on the experiments performed with our first and second-order methods at time $T_f = 0.2$ before the caustics. We start with the first-order scheme (1.17). Figures 2 and 3 represent the errors on ρ^ε and $(S^\varepsilon, A^\varepsilon)$ *w.r.t.* the time step h for a fixed $N_x = 2^7$. Figures 4 and 5 represent the errors *w.r.t.* Δx for fixed $N_t = h/T_f = 2^{13}$. All these figures illustrate the fact that our scheme is UA with respect to ε, for the quadratic observables as well as for the whole unknown $(S^\varepsilon, A^\varepsilon)$ itself. Figures 2 and 3 show that (1.17) is uniformly of order 1 in time, whereas Figures 4 and 5 show that the convergence is uniformly spectral in space.

Figures 6 to 9 illustrate the behavior of our second-order scheme (1.18) at $T_f = 0.2$: here again, it appears that, before the caustics, our method is UA with an order 2 in time and with spectral in space accuracy.

Finally, let us explore the behavior of the splitting methods after caustics, by observing the error on the density ρ^ε. Figures 10 and 11 present the same simulations as Figures 2 and 4, except that the final time is now $T_f = 1$, *i.e.* we illustrate the behaviors of scheme (1.18) after the caustics. In that case, it appears that our methods are not UA, neither in h, nor in Δx, with respect to ε. Notice that, although it is not UA any longer, our scheme (1.18) still has second-order accuracy in time and spectral accuracy in space (with ε-dependent constants). Recall that the same scheme written on (1.8) would not be usable in the same situation, since S^ε ceases to be regular for $\varepsilon > 0$, after the formation of caustics.

Appendix A. Proof of Lemma 3.8

A.1. **Study of the differentiability of φ^1.** The proof of this lemma is divided in several steps. Let us fix $s > d/2 + 1$ and $M > 0$.

A.1.1. *Notations.* For any Banach spaces E and F, we denote $\mathscr{L}(E, F)$ the set of continuous linear maps between E and F endowed with the norm

$$\|l\|_{\mathscr{L}(E,F)} = \sup\{\|l(x)\|_F,\ x \in E,\ \|x\|_E \leq 1\}$$

where $\|\cdot\|_E$ and $\|\cdot\|_F$ are the norms of E and F.

Let us define for $u_0 = (S_0, A_0)$

$$\Theta_h^1 \cdot u_0 = \begin{pmatrix} \widetilde{S}_h^1 \\ \widetilde{A}_h^1 \end{pmatrix}$$

the solution of

$$\begin{aligned} \partial_h \Theta_h^1 &= D\mathcal{N}_1(\varphi_h^1(u)) \cdot \Theta_h^1 \\ \Theta_0^1 \cdot u_0 &= u_0. \end{aligned}$$

We denote $\Gamma_h^1 = \varphi_h^1(u + u_0) - \varphi_h^1(u) - \Theta_h^1 \cdot u_0$,

$$\varphi_h^1(u) = \begin{pmatrix} S_h^1 \\ A_h^1 \end{pmatrix},\ \varphi_h^1(u + u_0) = \begin{pmatrix} S_h^1 \\ \underline{A_h^1} \end{pmatrix},$$

$v_h^1 = \nabla S_h^1$, $\underline{v}_h^1 = \nabla \underline{S}_h^1$, $\widetilde{v}_h^1 = \nabla \widetilde{S}_h^1$, $\omega_h^1 = \underline{S}_h^1 - S_h^1 - \widetilde{S}_h^1$ and $B_h^1 = \underline{A}_h^1 - A_h^1 - \widetilde{A}_h^1$.

A.1.2. *Definition of h_8.* Lemma 3.5 ensures that for any $u \in B_s(2M)$, we have for $h \in [0, h_5(2M)]$ that

$$\|\varphi_h^1(u)\|_s \leq 4M. \tag{A.1}$$

We denote $h_8(M) = h_5(2M)$.

Let $s' \geq s$. If moreover, $u \in \Sigma_{s'}$, then we have

$$\|\varphi_h^1(u)\|_{s'} \leq \exp\left(C_5(2M)h\right) \|u\|_{s'}. \tag{A.2}$$

A.1.3. *Continuity of φ^1.* Let $s' \geq s$, $M' > 0$ and $u_1, u_2 \in B_s(M) \cap B_{s'+1}(M')$.

By (A.1) and (A.2), we obtain that $\varphi_h^1(u_1)$ and $\varphi_h^1(u_2)$ are well-defined on $[0, h_8]$ and satisfy

$$\|\varphi_h^1(u_1)\|_{s'+1} + \|\varphi_h^1(u_2)\|_{s'+1} \leq 2\exp\left(C_5(2M)h_8\right) M'$$

for all $h \in [0, h_8]$. By Lemma 3.4 and an integration by parts, we get that there exists $C = C(M, M') > 0$ such that for all $h \in [0, h_8]$

$$\|\varphi_h^1(u_1) - \varphi_h^1(u_2)\|_{s'} \leq C\|u_1 - u_2\|_{s'+1}. \tag{A.3}$$

Moreover, for fixed $u \in B_s(M) \cap B_{s'+1}(M')$, Lemma 3.5 ensures that $h \in [0, h_8] \mapsto \varphi_h^1(u) \in \Sigma_{s'}$ is continuous so that

$$(h, u) \in [0, h_8] \times \Sigma_{s'+1} \mapsto \varphi_h^1(u) \in \Sigma_{s'} \tag{A.4}$$

is also continuous.

A.1.4. *Well-posedness, continuity and estimates on the norm for* Θ_h^1. Let $s_2 \geq s$, $M_2 > 0$, $u \in B_s(M) \cap B_{s_2+1}(M_2)$ and $u_0 \in \Sigma_{s_2}$. We recall that the function $\Theta_h^1 \cdot u_0 =: (\widetilde{S}_h^1, \widetilde{A}_h^1)^T$ satisfies

$$\partial_h \widetilde{S}_h^1 + \nabla S_h^1 \cdot \nabla \widetilde{S}_h^1 = 0$$

$$\partial_h \widetilde{A}_h^1 + \nabla S_h^1 \cdot \nabla \widetilde{A}_h^1 + \frac{\widetilde{A}_h^1}{2} \Delta S_h^1 = -\nabla \widetilde{S}_h^1 \cdot \nabla A_h^1 - \frac{A_h^1}{2} \Delta \widetilde{S}_h^1 + \frac{i}{2} \Delta \widetilde{A}_h^1$$

and $\Theta_0^1 \cdot u_0 = u_0$. The existence and uniqueness of $\widetilde{S}_h^1$ follows for instance from the method of characteristics. We have

$$\partial_h \widetilde{v}_h^1 + \left(v_h^1 \cdot \nabla\right) \widetilde{v}_h^1 = -\left(\widetilde{v}_h^1 \cdot \nabla\right) v_h^1$$

and Lemma 3.2 with $R = -\left(\widetilde{v}_h^1 \cdot \nabla\right) v_h^1$ gives us that

$$\partial_h \|\widetilde{v}_h^1\|_{H^{s_2+1}}^2 \leq C \|\widetilde{v}_h^1\|_{H^{s_2+1}}^2 \|S_h^1\|_{H^{s_2+3}} \leq C \|\widetilde{v}_h^1\|_{H^{s_2+1}}^2 \|\varphi_h^1(u)\|_{s_2+1}.$$

We also have

$$\partial_h \|\widetilde{S}_h^1\|_{L^2}^2 \leq C \|\widetilde{S}_h^1\|_{L^2} \|\widetilde{S}_h^1\|_{H^1} \|S_h^1\|_{W^{1,\infty}}$$

so that

$$\partial_h \|\widetilde{S}_h^1\|_{H^{s_2+2}}^2 \leq C \|\widetilde{S}_h^1\|_{H^{s_2+2}}^2 \|\varphi_h^1(u)\|_{s_2+1}.$$

The existence and uniqueness of $\widetilde{A}_h^1$ follows from the fact that $\widetilde{w}_h^1 = \widetilde{A}_h^1 \exp\left(iS_h^1\right)$ satisfies

$$i\partial_h \widetilde{w}_h^1 = -\frac{\Delta}{2} \widetilde{w}_h^1 - \left(\nabla \widetilde{S}_h^1 \cdot \nabla A_h^1 + \frac{A_h^1}{2} \Delta \widetilde{S}_h^1\right) \exp\left(iS_h^1\right).$$

Lemma 3.3 with $R = -\nabla \widetilde{S}_h^1 \cdot \nabla A_h^1 - \frac{A_h^1}{2} \Delta \widetilde{S}_h^1 + \frac{i}{2} \Delta \widetilde{A}_h^1$ ensures that

$$\partial_t \|\widetilde{A}_h^1\|_{H^{s_2}}^2 \leq C \|\Theta_h^1 \cdot u_0\|_{s_2}^2 \|\varphi_h^1(u)\|_{s_2+1}$$

so that

$$\partial_t \|\Theta_h^1 \cdot u_0\|_{s_2}^2 \leq C \|\Theta_h^1 \cdot u_0\|_{s_2}^2 \|\varphi_h^1(u)\|_{s_2+1}.$$

By (A.2) and Gronwall's Lemma, there is $C_8 = C_8(M, M_2) > 0$ such that for any $h \in [0, h_8]$,

$$\|\Theta_h^1 \cdot u_0\|_{s_2} \leq \exp\left(C_8 h\right) \|u_0\|_{s_2}. \tag{A.5}$$

Using directly the integrations by parts of the proof of Lemmas 3.2 and 3.3, we obtain actually that

$$|\left\langle u_0, D\mathcal{N}_1(\varphi_h^1(u)) \cdot u_0 \right\rangle_{s_2}| \leq C_8 \|u_0\|_{s_2}^2,$$

for all $u_0 \in \Sigma_{s_2}$.

A.1.5. *Differentiability of* φ^1. By Lemma 3.5 and equations (1.13), the application

$$h \in [0, h_8] \mapsto \varphi_h^1(u) \in \Sigma_{s_1}$$

is differentiable in h for any $u \in B_s(M) \cap \Sigma_{s_1+2}$.

Let us prove that φ_h^1 is differentiable in u and that Θ_h^1 is its derivative.

Let $M_1 > 0$ and $u, u_0 \in B_s(M) \cap B_{s_1+2}(M_1)$. We have that $u, u+u_0 \in B_s(2M) \cap B_{s_1+2}(2M_1)$. By (A.1) and (A.2), we obtain that for all $h \in [0, h_8]$,

$$\|\varphi_h^1(u)\|_{s_1+2} + \|\varphi_h^1(u+u_0)\|_{s_1+2} \leq 4 \exp\left(C_5(2M)h\right) M_1.$$

We have

$$\partial_h \nabla \omega_h^1 = -(v_h^1 + \widetilde{v}_h^1) \cdot \nabla(\nabla \omega_h^1) - \left(\nabla \omega_h^1 \cdot \nabla\right) \underline{v_h^1} - \left(\widetilde{v}_h^1 \cdot \nabla\right) \widetilde{v}_h^1.$$

By Lemma 3.2, we obtain taking $v_1 = v_h^1 + \widetilde{v}_h^1$ and

$$R = -\left(\nabla\omega_h^1 \cdot \nabla\right) \underline{v_h^1} - \left(\widetilde{v}_h^1 \cdot \nabla\right) \widetilde{v}_h^1$$

that

$$\begin{aligned}\partial_t \|\nabla\omega_h^1\|_{H^{s_1+1}}^2 \leq C\|\nabla\omega_h^1\|_{H^{s_1+1}}^2 \left(\|v_h^1\|_{H^{s_1+1}} + \|\widetilde{v}_h^1\|_{H^{s_1+1}} + \|\underline{v}_h^1\|_{H^{s_1+2}}\right) \\ + C\|\nabla\omega_h^1\|_{H^{s_1+1}} \|\widetilde{v}_h^1\|_{H^{s_1+2}}^2.\end{aligned}$$

Moreover, we have

$$\partial_h \omega_h^1 = -\frac{1}{2}\left(\nabla\omega_h^1 \cdot (\widetilde{v}_h^1 + v_h^1) + \underline{v}_h^1 \cdot \nabla\omega_h^1 + |\widetilde{v}_h^1|^2\right)$$

so that

$$\begin{aligned}\partial_h \|\omega_h^1\|_{H^{s_1+2}}^2 \leq \|\Theta_h^1 \cdot u_0\|_{s_1+1}^4 \\ + C\|\omega_h^1\|_{H^{s_1+2}}^2 \left(1 + \|\varphi_h^1(u)\|_{s_1} + \|\varphi_h^1(u+u_0)\|_{s_1+1} + \|\Theta_h^1 \cdot u_0\|_{s_1}\right)\end{aligned}$$

We also have

$$\begin{aligned}\partial_h B_h^1 = -\nabla \underline{S}_h^1 \cdot \nabla B_h^1 - B_h^1 \frac{\Delta \underline{S}_h^1}{2} - \nabla\omega_h^1 \cdot \nabla(A_h^1 + \widetilde{A}_h^1) - (A_h^1 + \widetilde{A}_h^1)\frac{\Delta\omega_h^1}{2} \\ + i\frac{\Delta B_h^1}{2} - \nabla \widetilde{S}_h^1 \cdot \nabla \widetilde{A}_h^1 - \widetilde{A}_h^1 \frac{\Delta \widetilde{S}_h^1}{2}\end{aligned}$$

and Lemma 3.3 ensures taking

$$R = -\nabla\omega_h^1 \cdot \nabla(A_h^1 + \widetilde{A}_h^1) - (A_h^1 + \widetilde{A}_h^1)\frac{\Delta\omega_h^1}{2} + i\frac{\Delta B_h^1}{2} - \nabla \widetilde{S}_h^1 \cdot \nabla \widetilde{A}_h^1 - \widetilde{A}_h^1 \frac{\Delta \widetilde{S}_h^1}{2}$$

that,

$$\begin{aligned}\partial_t \|B_h^1\|_{H^{s_1}}^2 \leq \|\Theta_h^1 \cdot u_0\|_{s_1+1}^4 \\ + C\|\Gamma_h^1\|_{s_1}^2 \left(1 + \|\varphi_h^1(u+u_0)\|_{s_1} + \|\varphi_h^1(u)\|_{s_1+1} + \|\Theta_h^1 \cdot u_0\|_{s_1+1}\right)\end{aligned}$$

and

$$\begin{aligned}\partial_h \|\Gamma_h^1\|_{s_1}^2 \leq \|\Theta_h^1 \cdot u_0\|_{s_1+1}^4 \\ + C\|\Gamma_h^1\|_{s_1}^2 \left(1 + \|\varphi_h^1(u+u_0)\|_{s_1+1} + \|\varphi_h^1(u)\|_{s_1+1} + \|\Theta_h^1 \cdot u_0\|_{s_1+1}\right).\end{aligned}$$

By (A.5) with $s_2 = s_1 + 1$ and Gronwall's Lemma, we get that there exists $C = C(M, M_1) > 0$ such that for all $h \in [h, h_8]$,

$$\|\Gamma_h^1\|_{s_1} \leq C\|u_0\|_{s_1+1}^2 \leq C\|u_0\|_{s_1+2}^2$$

We proved that for any $h \in [0, h_8]$

$$\varphi_h^1 : B_s(M) \cap \Sigma_{s_1+2} \to \Sigma_{s_1}$$

is differentiable in $B_s(M) \cap \Sigma_{s_1+2}$.

A.1.6. *Proof of point* (i). Let us prove that the application

$$(h, u) \in [0, h_8] \times (B_s(M) \cap \Sigma_{s_1+4}) \mapsto \varphi_h^1(u) \in \Sigma_{s_1}$$

is a C^1-function.

Using equations (1.13) and (A.4), we get that

$$(h, u) \in [0, h_8] \times (B_s(M) \cap \Sigma_{s_1+3}) \mapsto \varphi_h^1(u) \in \Sigma_{s_1+2}$$

is continuous so that the partial derivative

$$(h, u) \in [0, h_8] \times (B_s(M) \cap \Sigma_{s_1+3}) \mapsto \partial_h \varphi_h^1(u) = \mathcal{N}_1 \varphi_h^1(u) \in \Sigma_{s_1}$$

is also continuous. Let us study the continuity of

$$(h,u)\mapsto \partial_2\varphi_h^1(u).$$

Let $u_1,u_2\in B_s(M)\cap B_{s_1+2}(M_1)$. We denote $\varphi_h^1(u_i)=(S_h^{1,i},A_h^{1,i})$ and $\partial_2\varphi_h^1(u_i)\cdot u_0=(\widetilde{S}_h^{1,i},\widetilde{A}_h^{1,i})$ for $i=1,2$. We have

$$\partial_h\left(\widetilde{S}_h^{1,1}-\widetilde{S}_h^{1,2}\right)+\nabla S_h^{1,1}\cdot\nabla\left(\widetilde{S}_h^{1,1}-\widetilde{S}_h^{1,2}\right)=-\nabla\left(S_h^{1,1}-S_h^{1,2}\right)\cdot\nabla\widetilde{S}_h^{1,2}$$

so that

$$\begin{aligned}&\partial_h\left(\nabla\widetilde{S}_h^{1,1}-\nabla\widetilde{S}_h^{1,2}\right)+\left(\nabla S_h^{1,1}\cdot\nabla\right)\left(\nabla\widetilde{S}_h^{1,1}-\nabla\widetilde{S}_h^{1,2}\right)\\&\quad=-\left(\left(\nabla\widetilde{S}_h^{1,1}-\nabla\widetilde{S}_h^{1,2}\right)\cdot\nabla\right)\left(\nabla S_h^{1,1}+\widetilde{S}_h^{1,2}\right)\\&\qquad-\left(\nabla\widetilde{S}_h^{1,2}\cdot\nabla\right)\left(\nabla S_h^{1,1}-\nabla S_h^{1,2}\right).\end{aligned}$$

By Lemma 3.2 with $v_1=\nabla S_h^{1,1}$ and

$$\begin{aligned}R=&-\left(\left(\nabla\widetilde{S}_h^{1,1}-\nabla\widetilde{S}_h^{1,2}\right)\cdot\nabla\right)\left(\nabla S_h^{1,1}+\widetilde{S}_h^{1,2}\right)\\&-\left(\nabla\widetilde{S}_h^{1,2}\cdot\nabla\right)\left(\nabla S_h^{1,1}-\nabla S_h^{1,2}\right),\end{aligned}$$

which satisfies

$$\begin{aligned}\|R\|_{H^{s_1+1}}\le&\, C\|\widetilde{S}_h^{1,1}-\widetilde{S}_h^{1,2}\|_{H^{s_1+2}}\left(\|S_h^{1,1}\|_{H^{s_1+3}}+\|\widetilde{S}_h^{1,2}\|_{H^{s_1+3}}\right)\\&+C\|S_h^{1,1}-S_h^{1,2}\|_{H^{s_1+3}}\|\widetilde{S}_h^{1,2}\|_{H^{s_1+2}}\end{aligned}$$

we obtain that

$$\begin{aligned}\partial_t\|\nabla\widetilde{S}_h^{1,1}-\nabla\widetilde{S}_h^{1,2}\|_{H^{s_1+1}}^2\le&\, C\|\nabla\widetilde{S}_h^{1,1}-\nabla\widetilde{S}_h^{1,2}\|_{H^{s_1+1}}^2\left(\|S_h^{1,1}\|_{H^{s_1+3}}+\|\widetilde{S}_h^{1,2}\|_{H^{s_1+3}}\right)\\&+C\|\nabla\widetilde{S}_h^{1,1}-\nabla\widetilde{S}_h^{1,2}\|_{H^{s_1+1}}\|S_h^{1,1}-S_h^{1,2}\|_{H^{s_1+3}}\|\widetilde{S}_h^{1,2}\|_{H^{s_1+2}}.\end{aligned}$$

Moreover, we have

$$\begin{aligned}&\partial_t\|\widetilde{S}_h^{1,1}-\widetilde{S}_h^{1,2}\|_{L^2}^2\\&\le C\|\widetilde{S}_h^{1,1}-\widetilde{S}_h^{1,2}\|_{L^2}\left(\|\widetilde{S}_h^{1,1}-\widetilde{S}_h^{1,2}\|_{H^1}\|S_h^{1,1}\|_{W^{1,\infty}}+\|S_h^{1,1}-S_h^{1,2}\|_{H^1}\|\widetilde{S}_h^{1,2}\|_{W^{1,\infty}}\right)\end{aligned}$$

so that

$$\begin{aligned}\partial_t\|\widetilde{S}_h^{1,1}-\widetilde{S}_h^{1,2}\|_{H^{s_1+2}}^2\le&\, C\|\widetilde{S}_h^{1,1}-\widetilde{S}_h^{1,2}\|_{H^{s_1+2}}^2\left(\|S_h^{1,1}\|_{H^{s_1+3}}+\|\widetilde{S}_h^{1,2}\|_{H^{s_1+3}}\right)\\&+C\|\widetilde{S}_h^{1,1}-\widetilde{S}_h^{1,2}\|_{H^{s_1+2}}\|S_h^{1,1}-S_h^{1,2}\|_{H^{s_1+3}}\|\widetilde{S}_h^{1,2}\|_{H^{s_1+2}}.\end{aligned}$$

We also have

$$\begin{aligned}&\partial_h\left(\widetilde{A}_h^{1,1}-\widetilde{A}_h^{1,2}\right)+\nabla S_h^{1,1}\cdot\nabla\left(\widetilde{A}_h^{1,1}-\widetilde{A}_h^{1,2}\right)+\left(\widetilde{A}_h^{1,1}-\widetilde{A}_h^{1,2}\right)\frac{\Delta S_h^{1,1}}{2}\\&\quad=-\nabla\left(S_h^{1,1}-S_h^{1,2}\right)\cdot\nabla\widetilde{A}_h^{1,2}-\frac{\widetilde{A}_h^{1,2}}{2}\Delta\left(S_h^{1,1}-S_h^{1,2}\right)\\&\qquad-\nabla\left(\widetilde{S}_h^{1,1}-\widetilde{S}_h^{1,2}\right)\cdot\nabla A_h^{1,1}-\frac{A_h^{1,1}}{2}\Delta\left(\widetilde{S}_h^{1,1}-\widetilde{S}_h^{1,2}\right)\\&\qquad-\nabla\widetilde{S}_h^{1,2}\cdot\nabla\left(A_h^{1,1}-A_h^{1,2}\right)-\frac{\left(A_h^{1,1}-A_h^{1,2}\right)}{2}\Delta\widetilde{S}_h^{1,2}+\frac{i}{2}\Delta\left(\widetilde{A}_h^{1,1}-\widetilde{A}_h^{1,2}\right).\end{aligned}$$

Using Lemma 3.3 with $v_1 = \nabla S_h^{1,1}$ and

$$\begin{aligned} R = &-\nabla\left(S_h^{1,1} - S_h^{1,2}\right)\cdot\nabla\widetilde{A}_h^{1,2} - \frac{\widetilde{A}_h^{1,2}}{2}\Delta\left(S_h^{1,1} - S_h^{1,2}\right) \\ &- \nabla\left(\widetilde{S}_h^{1,1} - \widetilde{S}_h^{1,2}\right)\cdot\nabla A_h^{1,1} - \frac{A_h^{1,1}}{2}\Delta\left(\widetilde{S}_h^{1,1} - \widetilde{S}_h^{1,2}\right) \\ &- \nabla\widetilde{S}_h^{1,2}\cdot\nabla\left(A_h^{1,1} - A_h^{1,2}\right) - \frac{\left(A_h^{1,1} - A_h^{1,2}\right)}{2}\Delta\widetilde{S}_h^{1,2} + \frac{i}{2}\Delta\left(\widetilde{A}_h^{1,1} - \widetilde{A}_h^{1,2}\right). \end{aligned}$$

which satisfies

$$\begin{aligned} &\mathrm{Re}\left\langle \Lambda^{s_1}\left(\widetilde{A}_h^{1,1} - \widetilde{A}_h^{1,2}\right), \Lambda^{s_1} R\right\rangle \\ &\quad\le C\|\widetilde{A}_h^{1,1} - \widetilde{A}_h^{1,2}\|_{H^{s_1}}\|S_h^{1,1} - S_h^{1,2}\|_{H^{s_1+2}}\|\widetilde{A}_h^{1,2}\|_{H^{s_1+1}} \\ &\qquad + C\|\widetilde{A}_h^{1,1} - \widetilde{A}_h^{1,2}\|_{H^{s_1}}\|\widetilde{S}_h^{1,1} - \widetilde{S}_h^{1,2}\|_{H^{s_1+2}}\|A_h^{1,1}\|_{H^{s_1+1}} \\ &\qquad + C\|\widetilde{A}_h^{1,1} - \widetilde{A}_h^{1,2}\|_{H^{s_1}}\|A_h^{1,1} - A_h^{1,2}\|_{H^{s_1+1}}\|\widetilde{S}_h^{1,2}\|_{H^{s_1+2}} \end{aligned}$$

we obtain that

$$\begin{aligned} &\partial_h\|\left(\partial_2\varphi_h^1(u_1) - \partial_2\varphi_h^1(u_2)\right)\cdot u_0\|_{s_1}^2 \\ &\quad\le C\|\left(\partial_2\varphi_h^1(u_1) - \partial_2\varphi_h^1(u_2)\right)\cdot u_0\|_{s_1}^2\left(\|\varphi_h^1(u_1)\|_{s_1+1} + \|\partial_2\varphi_h^1(u_2)\cdot u_0\|_{s_1+1}\right) \\ &\quad + C\|\left(\partial_2\varphi_h^1(u_1) - \partial_2\varphi_h^1(u_2)\right)\cdot u_0\|_{s_1}\|\varphi_h^1(u_1) - \varphi_h^1(u_2)\|_{s_1+1}\|\partial_2\varphi_h^1(u_2)\cdot u_0\|_{s_1+1}. \end{aligned}$$

Let us recall that $u_1, u_2 \in B_s(M)\cap B_{s_1+2}(M_1)$. By (A.2), (A.3) and (A.5) with $s_2 = s_1$ and Gronwall's Lemma, we get that for all $h\in[0,h_8]$,

$$u \in B_s(M)\cap\Sigma_{s_1+2} \mapsto \partial_2\varphi_h^1(u) \in \mathscr{L}(\Sigma_{s_1+2}, \Sigma_{s_1})$$

is continuous. Hence, we obtain that

$$(h,u)\in[0,h_8]\times(B_s(M)\cap\Sigma_{s_1+3}) \mapsto (\partial_2\varphi_h^1(u), \partial_h\varphi_h^1(u)) \in \mathscr{L}(\Sigma_{s_1+3},\Sigma_{s_1})\times\Sigma_{s_1}$$

is continuous and the result follows.

A.2. **Study of the differentiability of φ^2 and φ^3.** Let $u, u_0 \in \Sigma_s$. Since $\mathcal{N}_2$ is linear, we have that

$$\Theta_h^2\cdot u_0 = \varphi_h^2(u_0),$$

φ_h^2 is differentiable on Σ_s and for any $h\ge 0$,

$$\begin{aligned} &\|\partial_2\varphi_h^2(u)\cdot u_0\|_s = \|\varphi_h^2(u_0)\|_s = \|u_0\|_s, \\ &|\left\langle u_0, D\mathcal{N}_2(\varphi_h^2(u))\cdot u_0\right\rangle_s| \le C\|u_0\|_s^2. \end{aligned}$$

and the result follows. We easily prove that φ_i^3 is differentiable, that for any $h\ge 0$, $\Theta_h^3\cdot u_0 = u_0$, that

$$\|\Theta_h^3\cdot u_0\|_s = \|\Theta_0^3\cdot u_0\|_s = \|u_0\|_s.$$

and

$$|\left\langle\chi, \left(D\mathcal{N}_3(\varphi_h^3(u))\cdot\chi\right)\right\rangle_s| \le C\|\chi\|_s^2,$$

for all $\chi\in\Sigma_s$.

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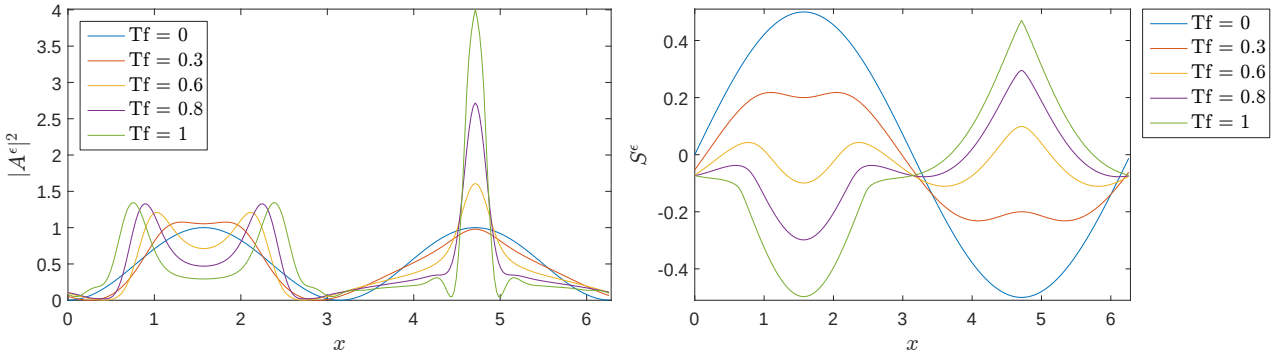
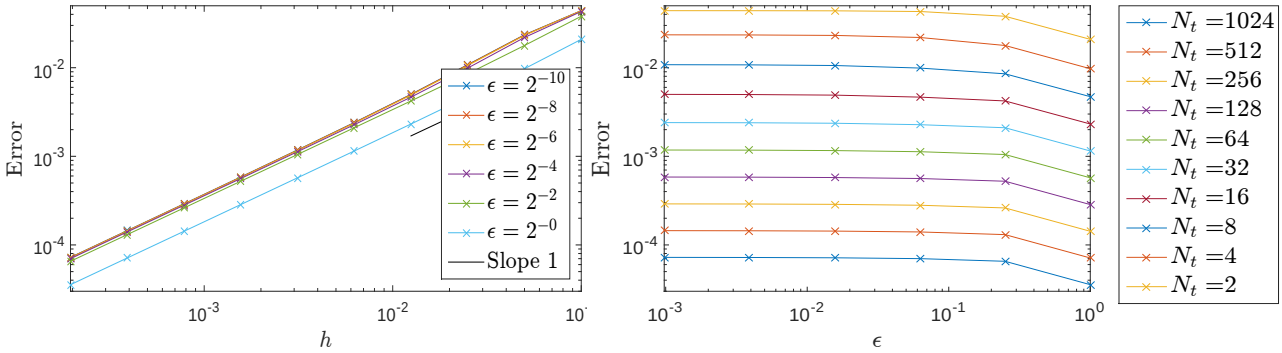


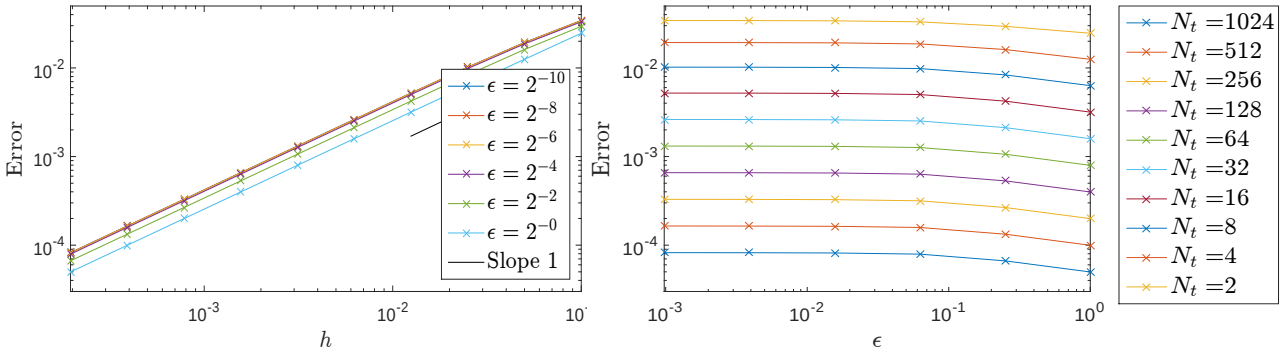
(a) Evolution of the density $|A^\varepsilon|^2$ for $\varepsilon = 2^{-4}$. (b) Evolution of the phase S^ε for $\varepsilon = 2^{-4}$.

FIGURE 1. Evolution of the density and of the phase



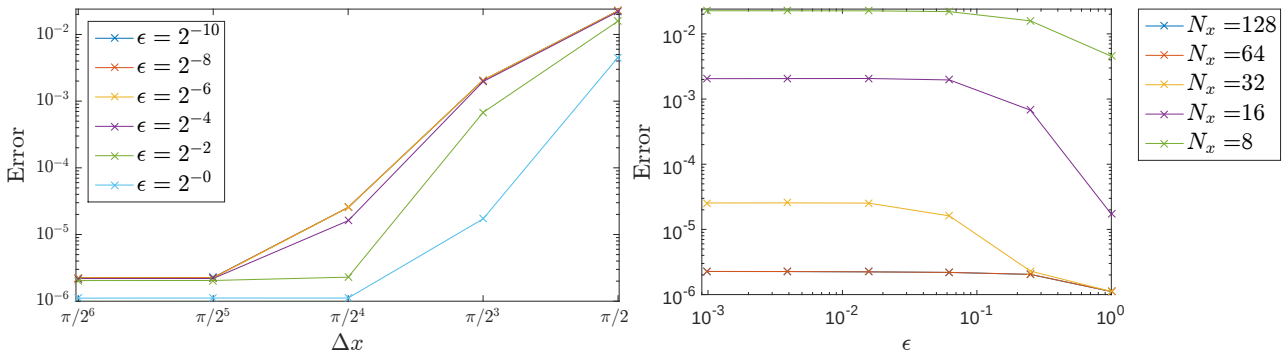
(a) $err_{\rho^\varepsilon}(T_f = 0.2)$ w.r.t h, $N_x = 2^8$ (b) $err_{\rho^\varepsilon}(T_f = 0.2)$ w.r.t ε, $N_x = 2^8$

FIGURE 2. Error on the density ρ^ε for the splitting scheme (1.17) of order 1 before the caustics: dependence on ε and on h.



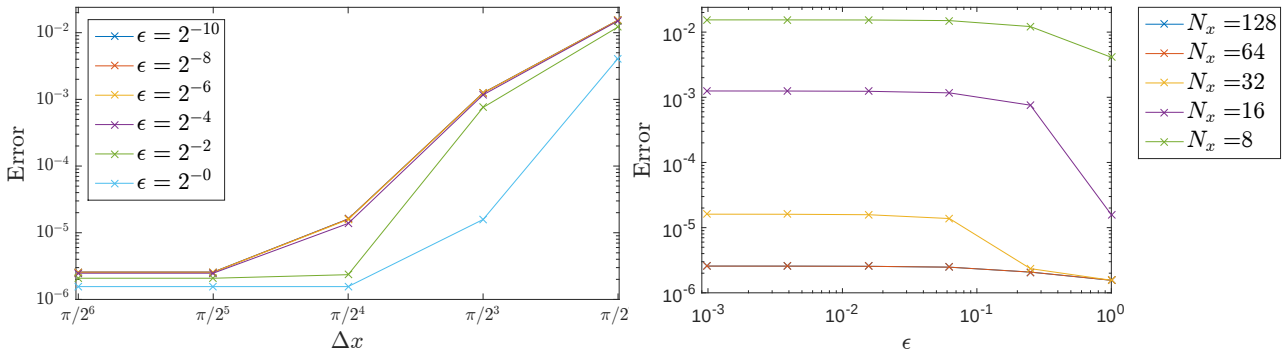
(a) $err_{(S^\varepsilon, A^\varepsilon)}(T_f = 0.2)$ w.r.t h, $N_x = 2^8$ (b) $err_{(S^\varepsilon, A^\varepsilon)}(T_f = 0.2)$ w.r.t ε, $N_x = 2^8$

FIGURE 3. Error on $(S^\varepsilon, A^\varepsilon)$ for the splitting scheme (1.17) of order 1 before the caustics: dependence on ε and on h.



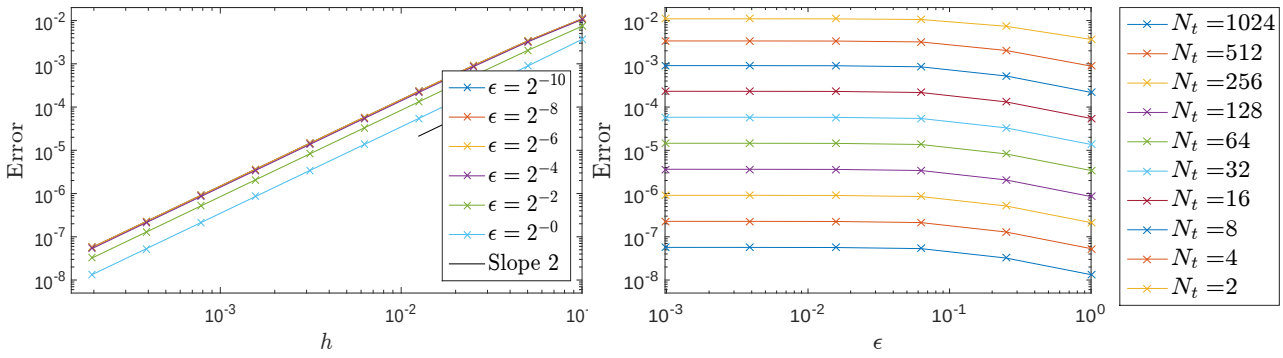
(a) $err_{\rho^\varepsilon}(T_f = 0.2)$ w.r.t Δx, $N_t = 2^{15}$ (b) $err_{\rho^\varepsilon}(T_f = 0.2)$ w.r.t ε, $N_t = 2^{15}$

FIGURE 4. Error on the density ρ^ε for the splitting scheme (1.17) of order 1 before the caustics: dependence on ε and on Δx.



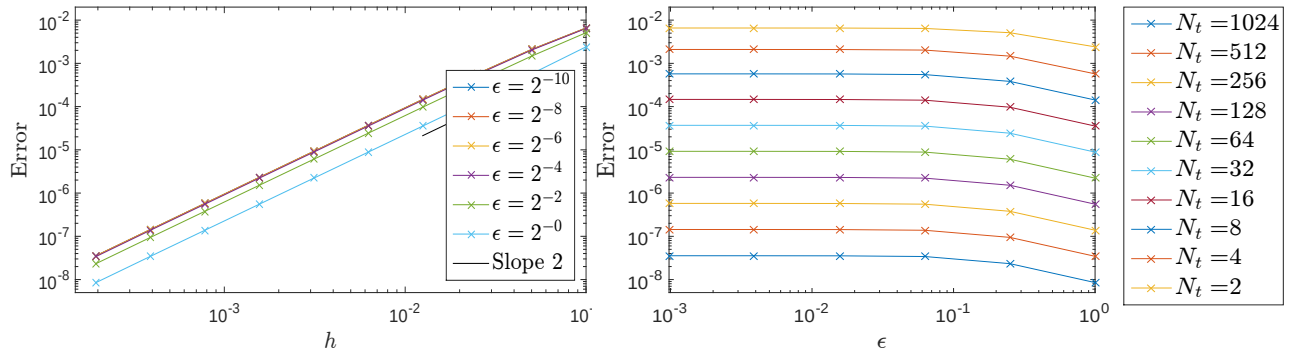
(a) $err_{(S^\varepsilon,A^\varepsilon)}(T_f = 0.2)$ w.r.t Δx, $N_t = 2^{15}$ (b) $err_{(S^\varepsilon,A^\varepsilon)}(T_f = 0.2)$ w.r.t ε, $N_t = 2^{15}$

FIGURE 5. Error on $(S^\varepsilon, A^\varepsilon)$ for the splitting scheme (1.17) of order 1 before the caustics: dependence on ε and on Δx.



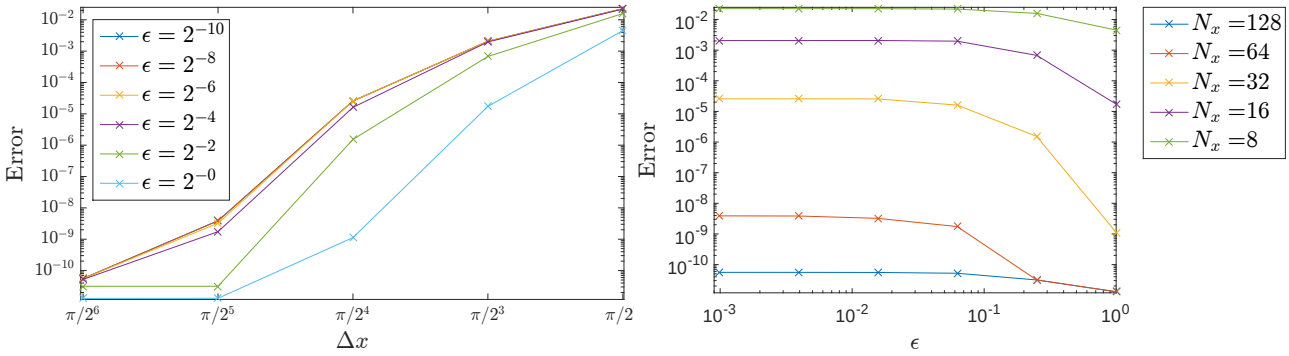
(a) $err_{\rho^\varepsilon}(T_f = 0.2)$ w.r.t h, $N_x = 2^8$ (b) $err_{\rho^\varepsilon}(T_f = 0.2)$ w.r.t ε, $N_x = 2^8$

FIGURE 6. Error on the density ρ^ε for the splitting scheme (1.18) of order 2 before the caustics: dependence on ε and on h.



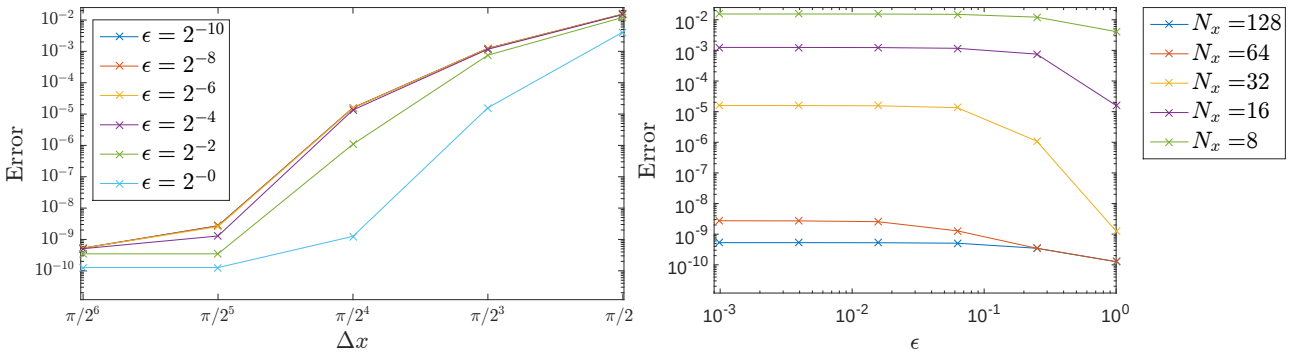
(a) $err_{(S^\varepsilon,A^\varepsilon)}(T_f = 0.2)$ w.r.t h, $N_x = 2^8$ (b) $err_{(S^\varepsilon,A^\varepsilon)}(T_f = 0.2)$ w.r.t ε, $N_x = 2^8$

FIGURE 7. Error on $(S^\varepsilon, A^\varepsilon)$ for the splitting scheme (1.18) of order 2 before the caustics: dependence on ε and on h.



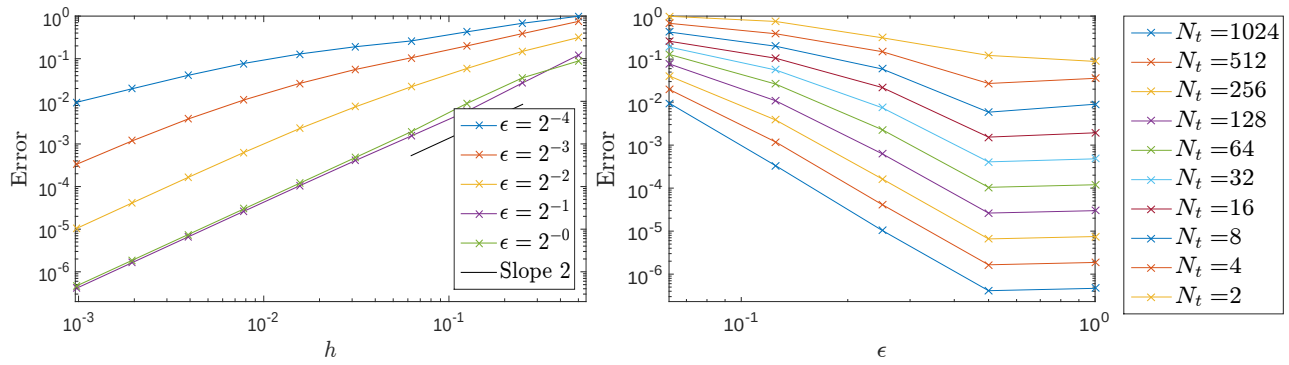
(a) $err_{\rho^\varepsilon}(T_f = 0.2)$ w.r.t Δx, $N_t = 2^{15}$ (b) $err_{\rho^\varepsilon}(T_f = 0.2)$ w.r.t ε, $N_t = 2^{15}$

FIGURE 8. Error on ρ^ε for the splitting scheme (1.18) of order 2 before the caustics: dependence on ε and on Δx.



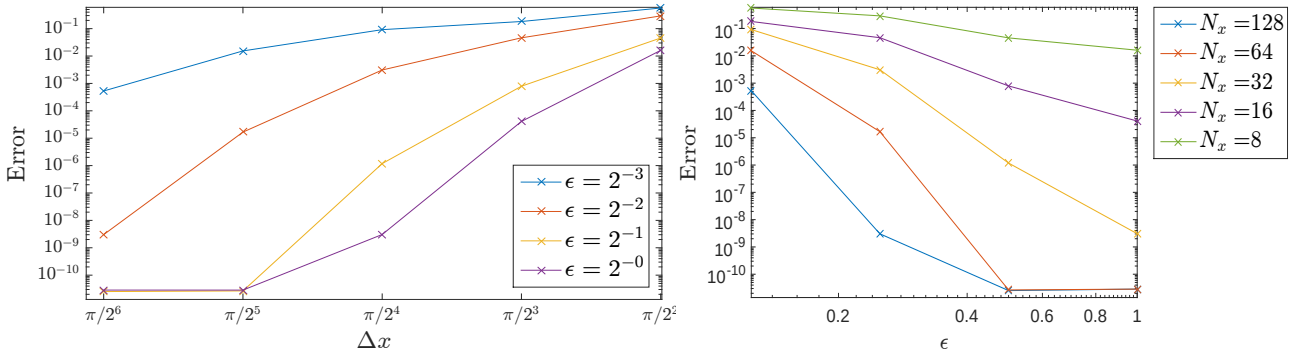
(a) $err_{(S^\varepsilon,A^\varepsilon)}(T_f = 0.2)$ w.r.t Δx, $N_t = 2^{15}$ (b) $err_{(S^\varepsilon,A^\varepsilon)}(T_f = 0.2)$ w.r.t ε, $N_t = 2^{15}$

FIGURE 9. Error on $(S^\varepsilon, A^\varepsilon)$ for the splitting scheme (1.18) of order 2 before the caustics: dependence on ε and on Δx.



(a) $err_{\rho^\varepsilon}(T_f = 1)$ w.r.t h, $N_x = 2^8$ (b) $err_{\rho^\varepsilon}(T_f = 1)$ w.r.t ε, $N_x = 2^8$

FIGURE 10. Error on ρ^ε for the splitting scheme (1.18) of order 2 after the caustics, dependence on ε and on h.



(a) $err_{\rho^\varepsilon}(T_f = 1)$ w.r.t Δx, $N_t = 2^{17}$ (b) $err_{\rho^\varepsilon}(T_f = 1)$ w.r.t ε, $N_t = 2^{17}$

FIGURE 11. Error on ρ^ε for the splitting scheme (1.18) of order 2 after the caustics, dependence on ε and on Δx.