PERVERSE NORI MOTIVES

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Abstract. Let $k = \mathbb{C}$ be the field of complex numbers (one can also choose a field of characteristic zero k with a fixed embedding of fields $\sigma : k \hookrightarrow \mathbb{C}$). Assume that K is a field. In this work, we show that the Tannakian formalism developed by M. Nori also applies to representations $\mathsf{T} : \mathcal{Q} \to \mathscr{P}$ with values in a K-linear Abelian category $\mathscr{P}$ which is Noetherian, Artinian and has finite dimensional Hom groups over K. As an application, we define a relative version, modeled after perverse sheaves, of the Abelian category of motives constructed by M. Nori over k.

1. Introduction

Let $k = \mathbb{C}$ be the field of complex numbers (one can also choose a field of characteristic zero k with a fixed embedding of fields $\sigma : k \hookrightarrow \mathbb{C}$). In this work, a k-variety will be a quasi-projective k-scheme. The reader, if he wishes, may work also with separated k-schemes of finite type. In particular, perverse motives make sense over such bases.

1.1. Using a Tannakian approach, Madhav Nori has defined a category of mixed motives $\mathrm{NMM}(k)$ over k. Though M. Nori himself did not publish his construction, several accounts of it are available in the literature (see e.g. [14, 24, 10]). Roughly speaking, $\mathrm{NMM}(k)$ is the universal Abelian category having a *faithful exact* Betti realization in the category of finite dimensional $\mathbb{Q}$-vector spaces and a relative homology theory for pairs consisting of a k-variety and a closed subscheme of it. Being a motive in $\mathrm{NMM}(k)$ is the finer structure that one can put on the relative homology of a pair of k-varieties. In particular, as the relative homology of pairs carries a (polarizable) mixed Hodge structure, the Betti realization factors through the Abelian category $\mathrm{MHS}^p_{\mathbb{Q}}$ of polarizable mixed $\mathbb{Q}$-Hodge structures i.e. one has a faithful exact functor

$$\mathrm{NMM}(k) \to \mathrm{MHS}^p_{\mathbb{Q}}.$$

1.2. Assume that K is a field. In this work, we show that the Tannakian formalism developed by M. Nori also applies to representations $\mathsf{T} : \mathcal{Q} \to \mathscr{P}$ with values in a K-linear Abelian category $\mathscr{P}$ which is Noetherian, Artinian and has finite dimensional Hom groups over K. The proof of this result is an application of Nori's statement and the characterization of categories of comodules over K-coalgebras obtained by M. Takeuchi in [21].

1.3. As an application, we develop a relative version, modeled after perverse sheaves, of the Abelian category of motives constructed by M. Nori. More precisely, for every quasi-projective k-scheme X, we construct a $\mathbb{Q}$-linear Abelian category $\mathscr{N}(X)$ of "perverse motives", with a faithful exact functor

$$\mathsf{rat}^{\mathscr{N}}_X : \mathscr{N}(X) \to \mathscr{P}(X)$$

to the category $\mathscr{P}(X)$ of perverse sheaves over X. The category $\mathscr{N}(\mathrm{Spec}(k))$ is the category $\mathrm{NMM}(k, \mathbb{Q})$ of M. Nori obtained by considering homology with rational coefficients. The Betti functor $\mathsf{rat}^{\mathscr{N}}_X$ factors through the Abelian category

$\mathrm{MHM}(X,\mathbb{Q})$ of mixed Hodge modules constructed by M. Saito in [18, 19] and this provides a faithful exact functor

$$\mathscr{N}(X)\to\mathrm{MHM}(X,\mathbb{Q}).$$

1.4. The category $\mathrm{NMM}(k)$ contains more motives, than expected a priori from its definition: (a) M. Nori constructs a realization functor (see e.g. [14] for a sketch of the construction)

$$\mathrm{DM}_{gm}(k)\to\mathrm{D}^{\mathrm{b}}(\mathrm{NMM}(k))$$

where $\mathrm{DM}_{gm}(k)$ is the triangulated category of geometric motives constructed by V. Voevodsky in [23]; (b) in [1, Theorem 3.1], D. Arapura shows that the Leray spectral sequence for Betti cohomology associated with a *projective* morphism $f: X\to Y$ of quasi-projective k-varieties is motivic in the sense of Nori i.e. is the image of a spectral sequence in the category $\mathrm{NMM}(k)$ via the Betti realization functor.

In [12] we show that, for every smooth quasi-projective k-scheme X, the derived category $\mathrm{D}^{\mathrm{b}}(\mathscr{N}(X))$ is the target of a realization functor

$$\mathbf{DA}^{\text{ét}}_{\mathsf{ct}}(X,\mathbb{Q})\to\mathrm{D}^{\mathrm{b}}(\mathscr{N}(X))$$

where $\mathbf{DA}^{\text{ét}}_{\mathsf{ct}}(X,\mathbb{Q})$ is the triangulated category of étale constructible motives with rational coefficients introduced by J. Ayoub in [3, 4]. The category $\mathbf{DA}^{\text{ét}}(X,\mathbb{Q})$ is the $\mathbb{Q}$-linear étale counterpart of the stable homotopy category of X-schemes of F. Morel and V. Voevodsky (see [13, 16, 22]). By [5, Théorème B.1] and [15, Theorem 14.30, Lemma 14.21]), it is known that the categories $\mathrm{DM}_{gm}(k,\mathbb{Q})$ and $\mathbf{DA}^{\text{ét}}_{\mathsf{ct}}(k,\mathbb{Q})$ are equivalent.

1.5. Note that a higher dimensional analog of Nori's construction, based on constructible sheaves rather than perverse sheaves, has been developed by D. Arapura in [2]. The category $\mathcal{M}(X,\mathbb{Q})$ constructed by D. Arapura in *loc.cit.* is probably related to the category $\mathscr{N}(X)$ that we construct in the present work. By analogy with perverse sheaves and mixed Hodge modules, it may be possible that $\mathcal{M}(X,\mathbb{Q})$ is equivalent to the heart of a certain t-structure on $\mathrm{D}^{\mathrm{b}}(\mathscr{N}(X))$ but we do not attempt in this work to prove such a comparison result.

2. Statement of the result and some comments

2.1. Let K be a field. Following [9, Chapitre II, §4], recall that a K-linear Abelian category $\mathscr{P}$ is said to be finite if it is Noetherian and Artinian i.e. $\mathscr{P}$ is essentially small and any object in $\mathscr{P}$ has finite length. We shall say that $\mathscr{P}$ is Hom finite if for any objects P,Q in $\mathscr{P}$ the K-vector space $\mathscr{P}(P,Q)$ is finite dimensional. The main result of this section is the following theorem:

Theorem 2.1. *Let $\mathscr{P}$ be a K-linear Abelian category which is finite and Hom finite, $\mathcal{Q}$ be a quiver*[1] *and $T:\mathcal{Q}\to\mathscr{P}$ be a representation of the quiver $\mathcal{Q}$ with values in $\mathscr{P}$. Then, there exist a K-linear Abelian category $\mathscr{A}$, a representation $R:\mathcal{Q}\to\mathscr{A}$, a K-linear faithful exact functor $F:\mathscr{A}\to\mathscr{P}$ and an invertible 2-morphism $\alpha: F\circ R\to T$ such that for every K-linear Abelian category $\mathscr{B}$, every representation $S:\mathcal{Q}\to\mathscr{B}$, every K-linear exact faithful functor $G:\mathscr{B}\to\mathscr{P}$, and every invertible 2-morphism $\beta: G\circ S\to T$ the following conditions are satisfied.*

- *There exist a K-linear functor $H:\mathscr{A}\to\mathscr{B}$ and two invertible* 2*-morphisms*

$$\gamma: H\circ R\xrightarrow{\simeq} S;\qquad \delta: G\circ H\xrightarrow{\simeq} F$$

[1] Recall that a quiver is simply a directed graph.

such that the square

$$\begin{tikzcd} G \circ H \circ R \arrow[r, "G\star\gamma"] \arrow[d, "\delta\star R"] & G \circ S \arrow[d, "\beta"] \\ F \circ R \arrow[r, "\alpha"] & T \end{tikzcd}$$

is commutative.

- *If $H' : \mathscr{A} \to \mathscr{B}$ is a K-linear functor and*

$$\gamma' : H' \circ R \xrightarrow{\simeq} S; \qquad \delta' : G \circ H' \xrightarrow{\simeq} F$$

are two invertible 2-morphisms such that the square

$$\begin{tikzcd} G \circ H' \circ R \arrow[r, "G\star\gamma'"] \arrow[d, "\delta'\star R"] & G \circ S \arrow[d, "\beta"] \\ F \circ R \arrow[r, "\alpha"] & T \end{tikzcd}$$

is commutative, then there exists a unique 2-morphism $\theta : H \to H'$ such that $\gamma' \circ (\theta \star R) = \gamma$ and $\delta' \circ (G \star \theta) = \delta$.

For $\mathscr{P} = \mathsf{mod}(K)$ the category of finite dimensional K-vector spaces, the result is due to M. Nori. A proof of Theorem 2.1 is given in Section 5. The main tool is Proposition 4.1 which allows to reduce the general case to the case considered by M. Nori.

2.2. Our main motivation behind Theorem 2.1 is to apply the Tannakian formalism developed by M. Nori to representations of quivers in categories of perverse sheaves.

Example 2.2. Let k be a field of characteristic zero with an embedding $\sigma : k \hookrightarrow \mathbb{C}$ into the field $\mathbb{C}$ of complex numbers. Given a separated k-scheme X of finite type, we consider the complex scheme X_σ obtained from X by base change along the embedding σ, and the triangulated category $\mathrm{D}^{\mathrm{b}}_{\mathrm{c}}(X, \mathbb{Q})$ of complexes, with algebraically constructible cohomology, of sheaves of $\mathbb{Q}$-vector spaces for the classical topology on the associated analytic space $X_\sigma(\mathbb{C})$.

If A, B are objects in $\mathrm{D}^{\mathrm{b}}_{\mathrm{c}}(X, \mathbb{Q})$, their Hom group is given by[2]:

$$\begin{aligned} \mathrm{Hom}(A, B) &= \mathrm{Hom}_{\mathrm{D}^{\mathrm{b}}_{\mathrm{c}}(X,\mathbb{Q})}(\mathbb{Q}_X, \mathsf{R}\mathscr{H}om(A, B)) \\ &= \mathrm{Hom}_{\mathrm{D}^{\mathrm{b}}(\mathbb{Q})}(\mathbb{Q}, \mathsf{R}\pi_*\mathsf{R}\mathscr{H}om(A, B)) \\ &= \mathsf{H}^0(\mathsf{R}\pi_*\mathsf{R}\mathscr{H}om(A, B)) \end{aligned}$$

where $\mathsf{R}\mathscr{H}om$ is the internal Hom and $\mathsf{R}\pi_*$ is the direct image along the structural morphism X/k. In particular, as $\mathsf{R}\mathscr{H}om$ and $\mathsf{R}\pi_*$ preserve constructibility, the $\mathbb{Q}$-vector space $\mathrm{Hom}(A, B)$ is finite dimensional.

Let $\mathscr{P}(X) := \mathrm{Perv}(X, \mathbb{Q})$ be the heart of $\mathrm{D}^{\mathrm{b}}_{\mathrm{c}}(X, \mathbb{Q})$ for the perverse t-structure. By [6, Théorème 4.3.1.(i)], the category $\mathscr{P}(X)$ is Noetherian and Artinian. As it is also Hom finite, it satisfies the assumption of Theorem 2.1.

[2]Note that $\mathrm{D}^{\mathrm{b}}_{\mathrm{c}}(\mathrm{Spec}(k), \mathbb{Q})$ is nothing but the triangulated category $\mathrm{D}^{\mathrm{b}}_{\mathrm{f.d.}}(\mathbb{Q})$ of complexes of $\mathbb{Q}$-vector spaces with finite dimensional cohomology, and is also equivalent to the homotopy category $\mathrm{K}^{\mathrm{b}}(\mathsf{mod}(\mathbb{Q}))$ of the Abelian category $\mathsf{mod}(\mathbb{Q})$ of finite dimensional vector spaces.

2.3. Note that the universal property of Nori's construction is unfortunately not exactly stated in the available literature [14, 24, 7] as we have stated it in Theorem 2.1. Formulations of the universal property that may be found in the literature are simplified by assuming implicitly at some point that some equivalence of categories is the identity. The statement obtained with such an assumption is not correct but its formulation is simpler and more appealing to readers wishing to avoid the cumbersome language of 2-categories. The correct formulation of the universal property is however needed here to prove Theorem 2.1 and for sake of completeness we have included the proof that Nori's category does indeed satisfy the universal property as stated in Theorem 2.1 when $\mathscr{P} = \mathsf{mod}(K)$ (see Theorem 3.2).

In the rest of this section we formulate the universal property of Theorem 2.1 in terms of 2-categories and 2-initial objects.

2.4. Let $\mathfrak{D}$ be a strict 2-category. We will denote the composition of 1-morphisms (resp. 2-morphisms) by the symbol $\circ$. Given a diagram of the form

$$\begin{tikzcd} X \arrow[r, bend left, "f"] \arrow[r, bend right, "g"'] \arrow[r, phantom, "\Downarrow\alpha"] & Y \arrow[r, bend left, "k"] \arrow[r, bend right, "h"'] \arrow[r, phantom, "\Downarrow\beta"] & Z \end{tikzcd}$$

we will denote by $\beta \star \alpha : k \circ f \to h \circ g$ the so-called horizontal composition of the 2-morphisms α and β. If $\alpha = \mathrm{id}$, we will simply write $f \star \beta = \mathrm{id} \star \beta$ and similarly if $\beta = \mathrm{id}$ we will write $k \star \alpha = \alpha \star \mathrm{id}$. Given two objects $X, Y \in \mathfrak{D}$, we denote by $\mathfrak{D}(X, Y)$ the category formed by the 1-morphisms from X to Y.

Definition 2.3. Let $\mathfrak{D}$ be a strict 2-category and X be an object of $\mathfrak{D}$. Then X is said to be 2-initial if for every object Y in $\mathfrak{D}$, the category $\mathfrak{D}(X, Y)$ is non empty and every object of $\mathfrak{D}(X, Y)$ is initial.

This means that me may find at least a 1-morphism from X to Y and for every two such 1-morphisms f, g, there exists one and only one 2-morphism $\alpha : f \Rightarrow g$. Such an object X in $\mathfrak{D}$ is then unique up to an equivalence which is unique up to a unique invertible 2-morphism. Indeed if X and X' are two objects in $\mathfrak{D}$ that are 2-initial. Then there exists a 1-morphism $f : X \to X'$ and any such 1-morphism is an equivalence. Moreover given two 1-morphisms $f, g : X \to X'$, there exists one and only 2-morphism $\alpha : f \Rightarrow g$ and this 2-morphism is invertible.

Remark 2.4. If Y and Y' are equivalent in $\mathfrak{D}$, then the category $\mathfrak{D}(X, Y)$ and $\mathfrak{D}(X, Y')$ are equivalent. In particular if one of the category is non empty and all its objects are initial, the same is true for the other.

2.5. Let $\mathscr{P}$ be a K-linear Abelian category and $T : \mathcal{Q} \to \mathscr{P}$ be a representation. With this representation is associated a strict 2-category $\mathfrak{D}_T$ defined as follows.

- An object in $\mathfrak{D}_T$ is a 4-uplet $(\mathscr{A}, R, F, \alpha)$ where $\mathscr{A}$ is a K-linear abelian category, $R : \mathcal{Q} \to \mathscr{A}$ is a representation, $F : \mathscr{A} \to \mathscr{P}$ is a K-linear faithful exact functor and $\alpha : F \circ R \to T$ is an invertible 2-morphism.
- If $(\mathscr{A}, R, F, \alpha)$ and $(\mathscr{B}, S, G, \beta)$ are two objects, a 1-morphism $(\mathscr{A}, R, F, \alpha) \to (\mathscr{B}, S, G, \beta)$ is a triple (H, γ, δ) where $H : \mathscr{A} \to \mathscr{B}$ is a K-linear functor and
$$\gamma : H \circ R \xrightarrow{\simeq} S; \qquad \delta : G \circ H \xrightarrow{\simeq} F$$
are invertible 2-morphisms such that the square
$$\begin{tikzcd} G \circ H \circ R \arrow[r, "G \star \gamma"] \arrow[d, "\delta \star R"] & G \circ S \arrow[d, "\beta"] \\ F \circ R \arrow[r, "\alpha"] & T \end{tikzcd}$$

is commutative.

- If (H, γ, δ) and (H', γ', δ') are 1-morphisms between the same two objects, a 2-morphism $(H, \gamma, \delta) \Rightarrow (H', \gamma', \delta')$ is a 2-morphism[3] $\theta : H \to H'$ such that $\gamma' \circ (\theta \star R) = \gamma$ and $\delta' \circ (G \star \theta) = \delta$.

Theorem 2.1 may be reformulated as follows: if $\mathscr{P}$ is a K-linear Abelian category which is finite and Hom finite, then for every representation $T : \mathcal{Q} \to \mathscr{P}$ of a quiver $\mathcal{Q}$ the 2-category $\mathfrak{D}_T$ has a 2-initial object. It will be convenient in the sequel to use the following definition:

Definition 2.5. We say that the category $\mathscr{P}$ has the property **N** if, for every quiver $\mathcal{Q}$ and every representation $T : \mathcal{Q} \to \mathscr{P}$, the 2-category $\mathfrak{D}_T$ has a 2-initial object.

2.6. Let us fix a quiver $\mathcal{Q}$ and a representation $T : \mathcal{Q} \to \mathscr{P}$. Note that a 2-morphism θ in $\mathfrak{D}_T$ is invertible in $\mathfrak{D}_T$ if and only if it is invertible as a natural transformation. A similar result holds for 1-morphism. More precisely:

Lemma 2.6. *Let $(\mathscr{A}, R, F, \alpha)$ and $(\mathscr{B}, S, G, \beta)$ be two objects in $\mathfrak{D}_T$. A 1-morphism*

$$(H, \gamma, \delta) : (\mathscr{A}, R, F, \alpha) \to (\mathscr{B}, S, G, \beta)$$

is an equivalence in $\mathfrak{D}_T$ if and only if H is an equivalence of categories.

Proof. If (H, γ, δ) is an equivalence in $\mathfrak{D}_T$, then it follows from the definition that H is an equivalence of categories. Assume now that H is an equivalence. There exist a K-linear functor $H' : \mathscr{B} \to \mathscr{A}$, an invertible 2-morphism $\phi : H \circ H' \to \mathrm{id}_{\mathscr{B}}$ and an invertible 2-morphism $\psi : H' \circ H \to \mathrm{id}_{\mathscr{A}}$ such that $H' \star \phi = \psi \star H'$ and $H \star \psi = \phi \star H$. Consider the 2-morphisms ε, η obtained respectively by the composition of the following sequences of 2-morphisms:

$$\begin{tikzcd}
\mathcal{Q} \arrow[r, bend left=80, "G\circ H\circ H'\circ H\circ R"] \arrow[r, bend left=40, "G\circ H\circ H'\circ S"] \arrow[r, "G\circ S" description] \arrow[r, bend right=40, "T"'] \arrow[r, phantom, bend left=60, "\Downarrow G\star H\star H'\star\gamma"] \arrow[r, phantom, bend left=20, "\Downarrow G\star\phi\star S"] \arrow[r, phantom, bend right=20, "\Downarrow\beta"] & \mathscr{P}
\end{tikzcd} \quad ; \quad \begin{tikzcd}
\mathcal{Q} \arrow[r, bend left=80, "G\circ H\circ H'\circ H\circ R"] \arrow[r, bend left=40, "F\circ H'\circ H\circ R"] \arrow[r, "F\circ R" description] \arrow[r, bend right=40, "T"'] \arrow[r, phantom, bend left=60, "\Downarrow \delta\star H'\star H\star R"] \arrow[r, phantom, bend left=20, "\Downarrow F\star\psi\star R"] \arrow[r, phantom, bend right=20, "\Downarrow\alpha"] & \mathscr{P}.
\end{tikzcd}$$

We have the equality $\varepsilon = \eta$. Indeed ε precomposed with $G \star \phi^{-1} \star H \star R$ is equal to $\beta \circ (G \star \gamma)$ while η precomposed with $G \star H \star \psi^{-1} \star R$ is equal to $\alpha \circ (\delta \star R)$. The equality follows then from the equalities $\alpha \circ (\delta \star R) = \beta \circ (G \star \gamma)$ and $\phi \star H = H \star \psi$. From this one deduces that

$$\alpha \circ (F \star \gamma') = \varepsilon \circ (\delta^{-1} \star H' \star \gamma^{-1}) = \eta \circ (\delta^{-1} \star H' \star \gamma^{-1}) = \beta \circ (\delta' \star S).$$

This shows that

$$(H', \gamma', \delta') : (\mathscr{B}, S, G, \beta) \to (\mathscr{A}, R, F, \alpha)$$

is a 1-morphism in $\mathfrak{D}_T$. By definition of the composition in $\mathfrak{D}_T$,

$$\begin{aligned} (H', \gamma', \delta') \circ (H, \gamma, \delta) &= (H' \circ H, \gamma' \circ (H' \star \gamma), \delta \circ (\delta' \star H)) \\ &= (H' \circ H, \psi \star R, G \star \psi) \end{aligned}$$

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[3] i.e. a natural transformation.

Hence ψ is a 2-morphism in $\mathfrak{D}_T$. Similarly ϕ is a 2-morphism in $\mathfrak{D}_T$. As both of them are invertible, this shows that (H', γ', δ') is a 1-morphism in $\mathfrak{D}_T$ quasi-inverse to (H, γ, δ). □

3. Review of Nori's contruction

We denote by $\mathsf{mod}(K)$ the category of finite dimensional K-vector spaces. If V is a finite dimensional K-vector space, then the dual of V as a K-vector space is denoted by $V^\vee := \mathrm{Hom}_K(V, K)$. All K-coalgebras are assumed to be coassociative and counitary.

Let $\mathcal{Q}$ be a quiver and

$$T : \mathcal{Q} \to \mathsf{mod}(K)$$

be a representation of $\mathcal{Q}$ with values in the category of finite dimensional K-vector spaces $\mathsf{mod}(K)$. We will recall the unpublished construction of M. Nori. For detailed expositions of the work of M. Nori with complete proofs we refer to [14, 24, 7].

3.1. Assume that $\mathcal{Q}$ has finitely many objects. Consider the ring of left endormorphisms of T. By definition, it is the subring $\mathrm{End}_K(T)$ of

$$\prod_{q \in \mathcal{Q}} \mathrm{End}_K(T(q))$$

formed by the elements $e = (e_q)_{q \in \mathcal{Q}}$ such that for every object $p \in \mathcal{Q}$ and every morphism $m \in \mathcal{Q}(p, q)$ the square

$$\begin{tikzcd} T(p) \arrow[r, "T(m)"] \arrow[d, "e_p"] & T(q) \arrow[d, "e_q"] \\ T(p) \arrow[r, "T(m)"] & T(q) \end{tikzcd}$$

is commutative. The ring $\mathrm{End}_K(T)$ is a K-algebra which is finite dimensional as a K-vector space and its dual

$$A_T := \mathrm{End}_K(T)^\vee$$

is a K-coalgebra which is finite dimensional over K. For every object $q \in \mathcal{Q}$, the finite dimensional K-vector space $T(q)$ has a natural structure of left $\mathrm{End}_K(T)$-module via the projection

$$\mathrm{End}_K(T) \to \mathrm{End}_K(T(q))$$

and thus a structure of left A_T-comodule. This shows that the representation T may be lifted to a representation

$$R_T : \mathcal{Q} \to \mathsf{comod}(A_T)$$

by simply viewing the finite dimensional vector space $T(q)$ as a left A_T-comodule. Nori's abelian category $\mathscr{A}(T)$ is then the category $\mathsf{comod}(A_T)$ of finite dimensional A_T-comodules. The representation T is then obtained as the composition of R_T and the forgetful functor

$$F_T : \mathsf{comod}(A_T) \to \mathsf{mod}(K)$$

which is K-linear exact and faithful.

3.2. Let now $\mathcal{Q}$ be a quiver which may have infinitely many objects. Consider for every finite full sub-quiver $\mathcal{Q}' \subseteq \mathcal{Q}$, the induced representation

$$T' := T|_{\mathcal{Q}'} : \mathcal{Q}' \to \mathsf{mod}(K)$$

and the associated coalgebra $A_{T'}$. If $\mathcal{Q}''$ is a finite full subquiver of $\mathcal{Q}$ that contains $\mathcal{Q}'$, the inclusion $\mathcal{Q}' \subseteq \mathcal{Q}''$ induces by projection a morphism of K-algebras

$$\prod_{q\in\mathcal{Q}''} \mathrm{End}_K(T(q)) \to \prod_{q\in\mathcal{Q}'} \mathrm{End}_K(T(q))$$

which induces a morphism of K-algebras $\mathrm{End}_K(T|_{\mathcal{Q}''}) \to \mathrm{End}_K(T_{\mathcal{Q}'})$. This provides a morphism of K-coalgebras $A_{T|_{\mathcal{Q}'}} \to A_{T|_{\mathcal{Q}''}}$. Nori's K-coalgebra associated with the representation $T : \mathcal{Q} \to \mathsf{mod}(K)$ is then the coalgebra obtained by taking the colimit over all *finite full* sub-quivers of $\mathcal{Q}$:

$$A_T := \operatorname*{colim}_{\mathcal{Q}'\subseteq\mathcal{Q}} A_{T|_{\mathcal{Q}'}}$$

and Nori's category is then the Abelian category

$$\mathscr{A}(T) := \mathsf{comod}(A_T)$$

where $\mathsf{comod}(A_T)$ is the category of finite dimensional left comodule over A_T. For every object $q \in \mathcal{Q}$ the finite dimensional K-vector space $T(q)$ inherits a structure of left A_T-comodule and the representation T factors as a representation

$$R_T : \mathcal{Q} \to \mathsf{comod}(A_T) := \mathscr{A}(T)$$

via the forgetful functor $F_T : \mathsf{comod}(A_T) \to \mathsf{mod}(K)$ which is K-linear exact and faithful. Note that by construction $F_T \circ R_T = T$ and therefore $(\mathscr{A}(T), R_T, F_T, \alpha_T)$ where $\alpha_T = \mathrm{id}$ is an object in $\mathfrak{D}_T$.

3.3. Now let us consider the functoriality of this construction. Let $T_1 : \mathcal{Q}_1 \to \mathsf{mod}(K)$ and $T_2 : \mathcal{Q}_2 \to \mathsf{mod}(K)$ be two representations of quivers. Let $F : \mathcal{Q}_1 \to \mathcal{Q}_2$ be a morphism of quivers, and $\alpha : T_2 \circ F \to T_1$ be an invertible 2-morphism. Assume first that $\mathcal{Q}_1$ and $\mathcal{Q}_2$ have finitely many objects. Consider the morphism of rings:

$$\prod_{q_2\in\mathcal{Q}_2} \mathrm{End}_K(T_2(q_2)) \xrightarrow{\Pi_\alpha} \prod_{q_1\in\mathcal{Q}_1} \mathrm{End}_K(T_1(q_1)) \tag{1}$$

where the map Π_α is defined for every $e = (e_{q_2})_{q_2\in\mathcal{Q}_2}$ by

$$\Pi_\alpha(e) = (\alpha \cdot e_{F(q_1)} \cdot \alpha^{-1})_{q_1\in\mathcal{Q}_1}.$$

If for every object $p_2 \in \mathcal{Q}_2$ and every morphism $m_2 \in \mathcal{Q}_2(p_2, q_2)$ the square

$$\begin{tikzcd} T_2(p_2) \arrow[r, "T_2(m_2)"] \arrow[d, "e_{p_2}"] & T_2(q_2) \arrow[d, "e_{q_2}"] \\ T_2(p_2) \arrow[r, "T_2(m_2)"] & T_2(q_2) \end{tikzcd}$$

is commutative, then in particular for every object $p_1 \in \mathcal{Q}_1$ and every morphism $m_1 \in \mathcal{Q}_1(p_1, q_1)$ the square

$$\begin{tikzcd} T_1(q_1) \arrow[r, "T_1(m_1)"] \arrow[d, "\Pi_\alpha(e)_{p_1}"] & T_1(q_1) \arrow[d, "\Pi_\alpha(e)_{q_1}"] \\ T_1(p_1) \arrow[r, "T_1(m_1)"] & T_1(q_1) \end{tikzcd}$$

is commutative. This shows that the morphism of rings (1) induces a morphism of K-algebras $\mathrm{End}_K(T_2) \to \mathrm{End}_K(T_1)$ and thus a morphism of K-coalgebras: $A(F)$:

$A_{T_1} \to A_{T_2}$. If $\mathcal{Q}_1$ and $\mathcal{Q}_2$ are not finite, then for any finite sub-quivers $\mathcal{Q}'_1 \subseteq \mathcal{Q}_1$ and $\mathcal{Q}'_2 \subseteq \mathcal{Q}_2$ such that $F(\mathcal{Q}'_1) \subseteq \mathcal{Q}'_2$ one has a morphism of K-coalgebras

$$A_{T_1|_{\mathcal{Q}'_1}} \to A_{T_2|_{\mathcal{Q}'_2}}.$$

By taking the colimit one obtains a morphism of K-coalgebras

$$A(F) : A_{T_1} \to A_{T_2}.$$

This morphism induces by restriction of scalars a K-linear exact functor

$$\mathscr{A}(F) : \mathscr{A}(T_1) := \mathsf{comod}(A_{T_1}) \to \mathscr{A}(T_2) := \mathsf{comod}(A_{T_2}).$$

The invertible 2-morphism α lifts and provides an invertible 2-morphism $\alpha : R_{T_2} \circ F \Rightarrow \mathscr{A}(F) \circ R_{T_1}$ i.e.

$$\begin{tikzcd} \mathcal{Q}_1 \arrow[r, "F"] \arrow[d, "R_{T_1}"'] \arrow[dr, phantom, "\not\Downarrow_{\alpha}"] & \mathcal{Q}_2 \arrow[d, "R_{T_2}"] \\ \mathscr{A}(T_1) \arrow[r, "\mathscr{A}(F)"] & \mathscr{A}(T_2). \end{tikzcd}$$

3.4. Let us recall the main result related to these categories. For a complete proof we refer to [24, Satz 3.28] or [11, Theorem 6.1.19].

Theorem 3.1 (M. Nori)**.** *Let $\mathscr{A}$ be a K-linear Abelian category and $\varrho : \mathscr{A} \to$ $\mathsf{mod}(K)$ be an exact faithful functor. Then the representation*

$$R_\varrho : \mathscr{A} \to \mathscr{A}(\varrho)$$

is a K-linear functor and an equivalence of categories.

As a corollary, one obtains the following result:

Theorem 3.2 (M. Nori)**.** *The K-linear Abelian category* $\mathsf{mod}(K)$ *of finite dimensional K-vector spaces satisfies the property* **N**.

Proof. Let $\mathcal{Q}$ be a quiver and $T : \mathcal{Q} \to \mathsf{mod}(K)$ be a representation of $\mathcal{Q}$ in $\mathsf{mod}(K)$. The construction of M. Nori recalled in §3.2, provides a K-linear Abelian category $\mathscr{A}(T)$, a representation $R_T : \mathcal{Q} \to \mathscr{A}(T)$ and a K-linear faithful exact functor $F_T : \mathscr{A}(T) \to \mathsf{mod}(K)$ such that $F_T \circ R_T = T$. It is enough to check that the object

$$(\mathscr{A}(T), R_T, F_T, \alpha_T)$$

where $\alpha_T = \mathrm{id}$, is 2-initial in the strict 2-category $\mathfrak{D}_T$ (see Definition 2.3). Let $\mathscr{B}$ be a K-linear Abelian category, $S : \mathcal{Q} \to \mathscr{B}$ be a representation, $G : \mathscr{B} \to \mathsf{mod}(K)$ be a K-linear exact faithful functor and $\beta : G \circ S \to T$ be an invertible 2-morphism.

By Theorem 3.1, the representation $R_G : \mathscr{B} \to \mathscr{A}(G)$ is a K-linear exact functor and an equivalence of categories such that

$$\begin{tikzcd} \mathscr{B} \arrow[r, "R_G"] \arrow[dr, "G"'] & \mathscr{A}(G) = \mathsf{comod}(A_G) \arrow[d, "F_G"] \\ & \mathsf{mod}(K) \end{tikzcd}$$

is commutative, where F_G is the forgetful functor. Note that

$$(R_G, \mathrm{id}, \mathrm{id}) : (\mathscr{B}, S, G, \beta) \to (\mathscr{A}(G), R_G \circ S, F_G, \beta)$$

is a 1-morphism in $\mathfrak{D}_T$. Since R_G is an equivalence, Lemma 2.6 assures that the objects $(\mathscr{B}, S, G, \beta)$ and $(\mathscr{A}(G), R_G \circ S, F_G, \beta)$ are then equivalent in $\mathfrak{D}_T$. It is therefore enough, by Remark 2.4, to prove that the category

$$\mathfrak{D}_T((\mathscr{A}(T), R_T, F_T, \alpha_T), (\mathscr{A}(G), R_G \circ S, F_G, \beta))$$

is non empty and that all its objects are initial. By functoriality of NORI's construction as recalled in §3.3, there is a morphism of K-coalgebras $A_T \to A_G$ such that the functor

$$H : \mathscr{A}(T) := \mathsf{comod}(A_T) \to \mathsf{comod}(A_G) =: \mathscr{A}(G)$$

obtained by restriction of scalars, fits into a commutative diagram

$$\begin{tikzcd}
\mathcal{Q} \arrow[r, "R_T"] \arrow[dd, "S"'] & \mathscr{A}(T) \arrow[dr, "F_T"] \arrow[dd, "H"] & \\
\arrow[r, phantom, "\overset{\beta}{\nearrow}"] & & \mathsf{mod}(K) \\
\mathscr{B} \arrow[r, "R_G"'] & \mathscr{A}(G). \arrow[ur, "F_G"'] &
\end{tikzcd}$$

Hence $(H, \beta^{-1}, \mathrm{id})$ is a 1-morphism from $(\mathscr{A}(T), R_T, F_T, \alpha_T)$ to $(\mathscr{A}(G), R_G \circ S, F_G, \beta)$. Let $H' : \mathscr{A}(T) \to \mathscr{A}(G)$ be a K-linear functor and

$$\gamma' : H' \circ R_T \xrightarrow{\simeq} R_G \circ S; \qquad \delta' : F_G \circ H' \xrightarrow{\simeq} F_T$$

be two invertible 2-morphisms such that the square

$$\begin{tikzcd}
F_G \circ H' \circ R_T \arrow[r, "F_G \star \gamma'"] \arrow[d, "\delta' \star R_T"] & F_G \circ R_G \circ S \arrow[d, "\beta"] \\
F_T \circ R_T \arrow[r, equal] & T
\end{tikzcd} \tag{2}$$

is commutative. Let $\theta : H \to H'$ be a 2-morphism such that $\gamma' \circ (\theta \star R_T) = \beta^{-1}$ and $\delta' \circ (F_G \star \theta) = \mathrm{id}$. Then, for every $V \in \mathsf{comod}(A_T)$, the morphism $\theta_V : H(V) \to H'(V)$ is equal to $(\delta'_V)^{-1}$ as a morphism of K-vector spaces. Hence it is unique. The existence follows from the fact that the commutativity of (2) implies that every V, of the form $V = R_T(q)$ for some $q \in \mathcal{Q}$, the morphism $(\delta'_V)^{-1}$ is a morphism of A-comodules[4]. As they generate $\mathscr{A}(T) = \mathsf{comod}(A_T)$ as an Abelian category, the morphism $\theta_V := (\delta'_V)^{-1}$ is a morphism of A-comodules for any V. The commutativity of (2) assures that $\gamma' \circ (\theta \star R_T) = \beta^{-1}$. As $\delta' \circ (F_G \star \theta) = \mathrm{id}$ by definition, this concludes. □

4. Finite and Hom finite K-linear Abelian categories

If A is a K-coalgebra, we denote by $\mathsf{coMod}(A)$ the category of left A-comodules and by $\mathsf{comod}(A)$ the full subcategory of finite dimensional left A-comodules. To obtain Theorem 2.1 from Theorem 3.2, we will need that the K-linear Abelian categories which are finite and Hom finite are precisely the categories of finite dimensional comodules over some K-coalgebra. More precisely:

Proposition 4.1. *Let $\mathscr{P}$ be a K-linear Abelian category. If $\mathscr{P}$ is finite and Hom finite, then there exist a K-coalgebra A and a K-linear equivalence of categories between $\mathscr{P}$ and $\mathsf{comod}(A)$.*

Note that this characterization is simply a finite dimensional variant of the result proved by M. TAKEUCHI in [21], and that we actually deduce it from TAKEUCHI's result. As a preliminary we need the following basic lemma:

Lemma 4.2. *Let A be K-coalgebra.*

[4]The functoriality assures that β lifts as a morphism of A-comodules.

(1) *The category* $\mathsf{comod}(A)$ *is the full subcategory of* $\mathsf{coMod}(A)$ *formed by the Noetherian objects of* $\mathsf{coMod}(A)$*.*
(2) *The category* $\mathsf{comod}(A)$ *is a finite and Hom finite K-linear abelian category.*

Proof. The set of all sub-comodules of a comodule V that are finite dimensional over K is partially ordered, filtrant, and has union V (see e.g. [8, Chapter II, Proposition 2.3]). Hence if V is Noetherian it must be finite dimensional over K. The reciprocal follows from the fact that the forgetful functor $\mathsf{coMod}(A) \to \mathsf{Mod}(K)$ is K-linear exact and faithful. This implies also that every object in $\mathsf{comod}(A)$ is Artinian. □

Let us now prove Proposition 4.1.

Proof of Proposition 4.1. By [9, Chapitre II, Théorème 1], we may find a locally Noetherian K-linear Abelian category $\mathscr{L}$ and an exact[5] K-linear functor $\mathscr{P} \to \mathscr{L}$ which induces an equivalence between $\mathscr{P}$ and the full subcategory of $\mathscr{L}$ formed by the Noetherian objects of $\mathscr{L}$. Since $\mathscr{P}$ is Artinian and Hom finite, the category $\mathscr{L}$ is locally finite and, for every objects of finite length $X, Y \in \mathscr{L}$, the K-vector space $\mathscr{L}(X, Y)$ is finite dimensional. We may thus apply [21, 5.1 Theorem, 5.8], to find a K-coalgebra A and a K-linear exact functor

$$\mathscr{L} \to \mathsf{coMod}(A)$$

which is an equivalence between $\mathscr{L}$ and the category $\mathsf{coMod}(A)$ of A-comodules. The result follows then from Lemma 4.2. □

Note that Proposition 4.1 has the following corollary:

Corollary 4.3. *Let $\mathscr{P}$ be a K-linear Abelian category. The following conditions are equivalent:*

(1) *$\mathscr{P}$ is finite and Hom finite;*
(2) *there exist a K-coalgebra A and a K-linear exact equivalence of categories between $\mathscr{P}$ and the category* $\mathsf{comod}(A)$*;*
(3) *there exists a K-linear exact and faithful functor $\omega : \mathscr{P} \to \mathsf{mod}(K)$.*

Proof. The equivalence of conditions 1, 2 and 3 is a consequence of Lemma 4.2, Proposition 4.1 and Theorem 3.1. □

5. Proof of Theorem 2.1

We start with the following observation:

Lemma 5.1. *If $\mathscr{P}_1$ be a K-linear Abelian category that satisfies the property* **N**, *then every K-linear Abelian category $\mathscr{P}_2$ such that there exists a K-linear exact faithful functor $\omega : \mathscr{P}_2 \to \mathscr{P}_1$ satisfies also the property* **N**.

Proof. Let $T_2 : \mathcal{Q} \to \mathscr{P}_2$ be a representation of the quiver $\mathcal{Q}$. We may apply the property **N** to the representation $T_1 := \omega \circ T_2 : \mathcal{Q} \to \mathscr{P}_1$, and find a K-linear Abelian category $\mathscr{A}_1$, a representation $R_1 : \mathcal{Q} \to \mathscr{A}_1$, a faithful exact functor $F_1 : \mathscr{A}_1 \to \mathscr{P}_1$, and an invertible 2-morphism $\alpha_1 : F_1 \circ R_1 \to T_1$ such that $(\mathscr{A}_1, R_1, F_1, \alpha_1)$ is 2-initial in the category $\mathfrak{D}_{T_1}$. We set

$$R_2 := R_1 \qquad \mathscr{A}_2 := \mathscr{A}_1.$$

The representation $T_2 : \mathcal{Q} \to \mathscr{P}_2$ and the K-linear exact faithful functor $\omega : \mathscr{P}_2 \to \mathscr{P}_1$, provide an object $(\mathscr{P}_2, T_2, \omega, \mathrm{id})$ in $\mathfrak{D}_{T_1}$. Since $(\mathscr{A}_1, R_1, F_1, \alpha_1)$ is 2-initial in

[5]The exactness is proved in [9, Chapitre II, Proposition 6]

$\mathfrak{D}_{T_1}$, we obtain the existence of a K-linear functor $F_2 : \mathscr{A}_2 \to \mathscr{P}_2$ and two invertible 2-morphisms

$$\alpha_2 : F_2 \circ R_2 \xrightarrow{\simeq} T_2; \qquad \delta : \omega \circ F_2 \xrightarrow{\simeq} F_1$$

such that the square

$$\begin{tikzcd} \omega \circ F_2 \circ R_2 \arrow[r, "\omega\star\alpha_2"] \arrow[d, "\delta\star R_2"] & \omega \circ T_2 \arrow[d, equal] \\ F_1 \circ R_1 \arrow[r, "\alpha_1"] & T_1 \end{tikzcd} \tag{3}$$

is commutative. It is enough to check that the object $(\mathscr{A}_2, R_2, F_2, \alpha_2)$ of $\mathfrak{D}_{T_2}$ is 2-initial in $\mathfrak{D}_{T_2}$. For this let $\mathscr{B}_2$ be a K-linear Abelian category, $S_2 : \mathscr{Q} \to \mathscr{B}_2$ be a representation, $G_2 : \mathscr{B}_2 \to \mathscr{P}_2$ be a K-linear faithful exact functor, and $\beta_2 : G_2 \circ S_2 \to T_2$ be an invertible 2-morphism. Let $\mathscr{B}_1 := \mathscr{B}_2$, $S_1 := S_2$, $G_1 := \omega \circ G_2$ and $\beta_1 := \omega \star \beta_2$. Since $(\mathscr{A}_1, R_1, F_1, \alpha_1)$ is 2-initial in $\mathfrak{D}_{T_1}$, and $(\mathscr{B}_1, S_1, G_1, \beta_1)$ is an object in $\mathfrak{D}_{T_1}$, we obtain a K-linear functor $H_1 : \mathscr{A}_1 \to \mathscr{B}_1$ and two invertible 2-morphisms

$$\gamma_1 : H_1 \circ R_1 \to S_1 \qquad \delta_1 : G_1 \circ H_1 \to F_1$$

such that the square

$$\begin{tikzcd} G_1 \circ H_1 \circ R_1 = \omega \circ G_2 \circ H_1 \circ R_1 \arrow[r, "G_1\star\gamma_1=\omega\star G_2\star\gamma_1"] \arrow[d, "\delta_1\star R_1"] & G_1 \circ S_1 = \omega \star G_2 \circ S_2 \arrow[d, "\beta_1=\omega\star\beta_2"] \\ F_1 \circ R_1 \arrow[r, "\alpha_1"] & T_1 = \omega \circ T_2 \end{tikzcd} \tag{4}$$

is commutative. Let us set

$$H_2 := H_1 \qquad \gamma_2 := \gamma_1.$$

Using the squares (3) and (4), we may apply again the fact that $(\mathscr{A}_1, R_1, F_1, \alpha_1)$ is 2-initial in $\mathfrak{D}_{T_1}$ to the 1-morphisms of $\mathfrak{D}_{T_1}$

$$(G_2 \circ H_2, \beta_2 \circ (G_2 \circ \gamma_1), \delta_1) \qquad (F_2, \alpha_2, \mathrm{id})$$

to get an invertible 2-morphism $\delta_2 : G_2 \circ H_2 \to F_2$ such that $\omega \star \delta_2 = \delta_1$ and such that the square

$$\begin{tikzcd} G_2 \circ H_2 \circ R_2 \arrow[r, "G_2\star\gamma_2"] \arrow[d, "\delta_2\star R_2"] & G_2 \circ S_2 \arrow[d, "\beta_2"] \\ F_2 \circ R_2 \arrow[r, "\alpha_2"] & T_2 \end{tikzcd}$$

is commutative. Now let $H'_2 : \mathscr{A}_2 \to \mathscr{B}_2$ be a K-linear functor and

$$\gamma'_2 : H'_2 \circ R_2 \to S_2 \qquad \delta'_2 : G_2 \circ H_2 \to F_2$$

be two invertible 2-morphisms such that the square

$$\begin{tikzcd} G_2 \circ H_2 \circ R_2 \arrow[r, "G_2\star\gamma'_2"] \arrow[d, "\delta'_2\star R_2"] & G_2 \circ S_2 \arrow[d, "\beta_2"] \\ F_2 \circ R_2 \arrow[r, "\alpha_2"] & T_2 \end{tikzcd}$$

is commutative. By composing with ω, since $\omega \star \beta_2 = \beta_1$, $\alpha_1 = \omega \star \alpha_2$, $G_1 = \omega \circ G_2$ and $F_1 = \omega \circ F_2$, we obtain the existence of a unique 2-morphism $\theta : H_2 \to H'_2$ such that

$$\gamma'_2 \circ (\theta \star R_2) = \gamma_2 \qquad \omega \star (\delta'_2 \circ (G_2 \star \theta)) = \delta_1.$$

Since $\delta_1 = \omega \star \delta_2$ and ω is faithful, we obtain $\delta_2' \circ (G_2 \star \theta) = \delta_2$ as desired. Note that θ is the unique 2-morphism $\theta : H_2 \to H_2'$ such that $\gamma_2' \circ (\theta \star R_2) = \gamma_2$ and $\delta_2' \circ (G_2 \star \theta) = \delta_2$ since such a 2-morphism satisfies also

$$\gamma_2' \circ (\theta \star R_2) = \gamma_2 \qquad \omega \star (\delta_2' \circ (G_2 \star \theta)) = \omega \star \delta_2 = \delta_1.$$

This relation ensures the uniqueness since $(\mathscr{A}_1, R_1, F_1, \alpha_1)$ is 2-initial in $\mathfrak{D}_{T_1}$. This shows the lemma. □

Proof of Theorem 2.1. Let us fix a K-linear Abelian category $\mathscr{P}$ which is finite and Hom finite. By Corollary 4.3, there exists a K-linear faithful exact functor

$$\omega : \mathscr{P} \to \mathsf{mod}(K).$$

By applying Theorem 3.2 and Lemma 5.1, we obtain that $\mathscr{P}$ satisfies also the property $\mathbf{N}$. This concludes the proof. □

6. Remarks on functoriality

In this section, we draw some simple remarks concerning the functoriality of the previous construction.

6.1. Let $\mathscr{A}$, $\mathscr{B}$ and $\mathscr{C}$ be K-linear Abelian categories and $\mathsf{F} : \mathscr{A} \to \mathscr{C}$ and $\mathsf{G} : \mathscr{B} \to \mathscr{C}$ be K-linear exact functors. We denote by $\mathscr{A} \times_{\mathscr{C}} \mathscr{B}$ the category obtained by glueing $\mathscr{A}$ and $\mathscr{B}$ over $\mathscr{C}$. An object in this category is a 5-uplet (A, B, C, α, β) where $A \in \mathscr{A}$, $B \in \mathscr{B}$, $C \in \mathscr{C}$ and $\alpha : \mathsf{F}(A) \to C$, $\beta : \mathsf{G}(B) \to C$ are isomorphisms in $\mathscr{C}$. A morphism

$$(A, B, C, \alpha, \beta) \to (A', B', C', \alpha', \beta')$$

is a triple (a, b, c) where $a \in \mathscr{A}(A, A')$, $b \in \mathscr{B}(B, B')$, $c \in \mathscr{C}(c, c')$ are such that $\alpha' \circ \mathsf{F}(a) = c \circ \alpha$ and $\beta' \circ \mathsf{G}(b) = c \circ \beta$.

This category is K-linear and Abelian with kernel and cokernel computed componentwise, and there are projection functors

$$\Pi_1 : \mathscr{A} \times_{\mathscr{C}} \mathscr{B} \to \mathscr{A} \qquad \Pi_2 : \mathscr{A} \times_{\mathscr{C}} \mathscr{B} \to \mathscr{B}$$

that are K-linear and exact.

Remark 6.1. The comparison isomorphisms $\alpha^{-1} \circ \beta$, provide a canonical isomorphism of functors $\mathsf{G} \circ \Pi_2 \to \mathsf{F} \circ \Pi_1$.

The following obvious remark will be useful:

Remark 6.2. By construction if F is exact and faithful, then so is the projection functor Π_2. Indeed assume that $\Pi_2(a, b, c) = b = 0$. Then

$$c = \beta' \circ \mathsf{G}(b) \circ \beta^{-1} = 0$$

which implies that $\mathsf{F}(a) = \alpha'^{-1} \circ c \circ \alpha = 0$. As F is faithful, one has also $a = 0$.

6.2. Let $\mathcal{Q}$ be a quiver, $\mathscr{P}_1$, $\mathscr{P}_2$ be K-linear Abelian categories that are finite and Hom finite and

$$\mathsf{T}_1 : \mathcal{Q} \to \mathscr{P}_1, \qquad \mathsf{T}_2 : \mathcal{Q} \to \mathscr{P}_2$$

be representation of quivers. Let $(\mathscr{A}_1, \mathsf{F}_1, \mathsf{R}_1, \alpha_1)$ and $(\mathscr{A}_2, \mathsf{F}_2, \mathsf{R}_2, \alpha_2)$ be 4-uplets obtained by applying Theorem 2.1 to the representations T_1 and T_2 respectively.

Proposition 6.3. *Let (Φ, ϕ) be a pair where $\Phi : \mathscr{P}_1 \to \mathscr{P}_2$ is an* exact *K-linear functor and $\phi : \Phi \circ \mathsf{T}_1 \to \mathsf{T}_2$ is an isomorphism of representations. There exist an* *exact functor* $\Psi : \mathscr{A}_1 \to \mathscr{A}_2$*, an invertible natural transformation* $\rho : \Phi \circ \mathsf{F}_1 \to \mathsf{F}_2 \circ \Psi$*, and an isomorphism of representations* $\varrho : \Psi \circ \mathsf{R}_1 \to \mathsf{R}_2$ *such that*

$$\begin{tikzcd}
\Phi \circ \mathsf{F}_1 \circ \mathsf{R}_1 \arrow[r, "\Phi\star\alpha_1"] \arrow[dd, "\rho\star\mathsf{R}_1"'] & \Phi \circ \mathsf{T}_1 \arrow[dr, "\phi"] & \\
 & & \mathsf{T}_2 \\
\mathsf{F}_2 \circ \Psi \circ \mathsf{R}_1 \arrow[r, "\mathsf{F}_2\star\varrho"'] & \mathsf{F}_2 \circ \mathsf{R}_2 \arrow[ur, "\alpha_2"'] &
\end{tikzcd}$$

is commutative.

Remark 6.4. Note that in Proposition 6.3 the functor $\Phi : \mathscr{P}_1 \to \mathscr{P}_2$ is not assumed to be faithful. If Φ is faithful, then the theorem follows from Lemma 5.1 and the functor $\Psi : \mathscr{A}_1 \to \mathscr{A}_2$ is an equivalence. However, for applications to functoriality of the categories of perverse motives introduced in Section 7, it is necessary to consider the more general case where Φ is only K-linear and exact (e.g., if $f : Y \to X$ is an étale morphism, the pullback functor $f^* : \mathscr{P}(Y) \to \mathscr{P}(X)$ between the categories of perverse sheaves is not faithful in general).

Proof. Consider the glued category $\mathscr{A} := \mathscr{A}_2 \times_{\mathscr{P}_2} \mathscr{P}_1$ with respect to the exact functors $\mathsf{F}_2 : \mathscr{A}_2 \to \mathscr{P}_2$ and $\Phi : \mathscr{P}_1 \to \mathscr{P}_2$, and denote by

$$\Pi_1 : \mathscr{A} \to \mathscr{A}_2, \qquad \Pi_2 : \mathscr{A} \to \mathscr{P}_1$$

the (exact) projection functors. Since F_2 is exact and faithful, so is the functor Π_2 (see Remark 6.2). Moreover we have a representation of quivers $\mathsf{T} : \mathcal{Q} \to \mathscr{A}$:

$$\begin{aligned}
p \in \mathcal{Q} \quad &\mapsto \quad \mathsf{T}(p) := (\mathsf{R}_2(p), \mathsf{T}_1(p), \mathsf{T}_2(p), \alpha_2(p), \phi(p));\\
m \in \mathcal{Q}(p,q) \quad &\mapsto \quad \mathsf{T}(m) := (\mathsf{R}_2(m), \mathsf{T}_1(m), \mathsf{T}_2(m)).
\end{aligned}$$

Note that $\Pi_2 \circ \mathsf{T} = \mathsf{T}_1$ and $\Pi_1 \circ \mathsf{T} = \mathsf{R}_2$ by definition. Apply then Theorem 2.1, with respect to the representation T_1, to the 4-uplet $(\mathscr{A}, \Pi_2, \mathsf{T}, \mathrm{id})$. It yields a triple $(\Upsilon, \gamma, \delta)$ such that

$$\begin{tikzcd}
\Pi_2 \circ \Upsilon \circ \mathsf{R}_1 \arrow[r, "\Pi_2\star\gamma"] \arrow[d, "\delta\star\mathsf{R}_1"] & \Pi_2 \circ \mathsf{T} \arrow[d, equal] \\
\mathsf{F}_1 \circ \mathsf{R}_1 \arrow[r, "\alpha_1"] & \mathsf{T}_1
\end{tikzcd}$$

is commutative, where $\Upsilon : \mathscr{A}_1 \to \mathscr{A}$ is an exact faithful functor, $\gamma : \Upsilon \circ \mathsf{R}_1 \to \mathsf{T}$ is an isomorphism of representations of quivers and $\delta : \Pi_2 \circ \Upsilon \to \mathsf{F}_1$ is an isomorphism of functors. Now define

$$\Psi := \Pi_1 \circ \Upsilon, \quad \varrho := \Pi_1 \star \gamma.$$

There is a canonical isomorphism of functors $\varepsilon : \Phi \circ \Pi_2 \to \mathsf{F}_2 \circ \Pi_1$, such that $\varepsilon \star \mathsf{T} = \alpha_2^{-1} \circ \phi$. Define $\rho : \Phi \circ \mathsf{F}_1 \to \mathsf{F}_2 \circ \Psi$ to be the composition

$$\Phi \circ \mathsf{F}_1 \xrightarrow{\Phi\star\delta^{-1}} \Phi \circ \Pi_2 \circ \Upsilon \xrightarrow{\varepsilon\star\Upsilon} \mathsf{F}_2 \circ \Pi_1 \circ \Upsilon =: \mathsf{F}_2 \circ \Psi.$$

One has then the commutative diagram

$$\begin{tikzcd}
 & \Phi \circ \mathsf{F}_1 \circ \mathsf{R}_1 \arrow[r, "\Phi\star\alpha_1"] \arrow[d, "\Phi\star\delta^{-1}\star\mathsf{R}_1"'] \arrow[ddl, bend right, "\rho\star\mathsf{R}_1"'] & \Phi \circ \mathsf{T}_1 \arrow[d, equal] \\
 & \Phi \circ \Pi_2 \circ \Upsilon \circ \mathsf{R}_1 \arrow[r, "(\Phi\circ\Pi_2)\star\gamma"'] \arrow[d, "\varepsilon\star(\Psi\circ\mathsf{R}_1)"'] & \Phi \circ \Pi_2 \circ \mathsf{T} \arrow[d, "\varepsilon\star\mathsf{T}=\alpha_2^{-1}\circ\phi"] \\
\mathsf{F}_2 \circ \Psi \circ \mathsf{R}_1 \arrow[r, equal] & \mathsf{F}_2 \circ \Pi_1 \circ \Upsilon \circ \mathsf{R}_1 \arrow[r, "\mathsf{F}_2\star\varrho=(\mathsf{F}_2\circ\Pi_1)\star\gamma"'] & \mathsf{F}_2 \circ \Pi_1 \circ \mathsf{T}
\end{tikzcd}$$

as desired. □

6.3. We have the following uniqueness statement.

Proposition 6.5. *Let (Φ_1, ϕ_1) and (Φ_2, ϕ_2) be pairs where $\Phi_1, \Phi_2 : \mathscr{P}_1 \to \mathscr{P}_2$ are exact functors and $\phi_1 : \Phi_1 \circ \mathsf{T}_1 \to \mathsf{T}_2$, $\phi_2 : \Phi_2 \circ \mathsf{T}_1 \to \mathsf{T}_2$ are isomorphisms of representations. Let $\theta : \Phi_1 \to \Phi_2$ be a natural transformation such that*

$$\phi_2 \circ (\theta \star \mathsf{T}_1) = \phi_1.$$

Let $(\Psi_1, \rho_1, \varrho_1)$, $(\Psi_2, \rho_2, \varrho_2)$ be triples where $\Psi_1, \Psi_2 : \mathscr{A}_1 \to \mathscr{A}_2$ are exact functors, $\rho_1 : \Phi_1 \circ \mathsf{F}_1 \to \mathsf{F}_2 \circ \Psi_1$, $\rho_2 : \Phi_2 \circ \mathsf{F}_1 \to \mathsf{F}_2 \circ \Psi_2$ are invertible natural transformations, $\varrho_1 : \Psi_1 \circ \mathsf{R}_1 \to \mathsf{R}_2$, $\varrho_2 : \Psi_2 \circ \mathsf{R}_1 \to \mathsf{R}_2$ are isomorphisms of representations such that

$$\begin{tikzcd}
\Phi_1 \circ \mathsf{F}_1 \circ \mathsf{R}_1 \arrow[r, "\Phi_1\star\alpha_1"] \arrow[dd, "\rho_1\star\mathsf{R}_1"'] & \Phi_1 \circ \mathsf{T}_1 \arrow[dr, "\phi_1"] & \\
 & & \mathsf{T}_2 \\
\mathsf{F}_2 \circ \Psi_1 \circ \mathsf{R}_1 \arrow[r, "\mathsf{F}_2\star\varrho_1"'] & \mathsf{F}_2 \circ \mathsf{R}_2 \arrow[ur, "\alpha_2"'] &
\end{tikzcd}
\qquad
\begin{tikzcd}
\Phi_2 \circ \mathsf{F}_1 \circ \mathsf{R}_1 \arrow[r, "\Phi_2\star\alpha_1"] \arrow[dd, "\rho_2\star\mathsf{R}_1"'] & \Phi_2 \circ \mathsf{T}_1 \arrow[dr, "\phi_2"] & \\
 & & \mathsf{T}_2 \\
\mathsf{F}_2 \circ \Psi_2 \circ \mathsf{R}_1 \arrow[r, "\mathsf{F}_2\star\varrho_2"'] & \mathsf{F}_2 \circ \mathsf{R}_2 \arrow[ur, "\alpha_2"'] &
\end{tikzcd}$$

are commutative. Then there exists one and only one natural transformation $\vartheta : \Psi_1 \to \Psi_2$ such that

$$\begin{tikzcd}
\Phi_1 \circ \mathsf{F}_1 \arrow[r, "\rho_1"] \arrow[d, "\theta\star\mathsf{F}_1"'] & \mathsf{F}_2 \circ \Psi_1 \arrow[d, "\mathsf{F}_2\star\vartheta"] \\
\Phi_2 \circ \mathsf{F}_1 \arrow[r, "\rho_2"'] & \mathsf{F}_2 \circ \Psi_2
\end{tikzcd}
\qquad
\begin{tikzcd}
\Psi_1 \circ \mathsf{R}_1 \arrow[d, "\vartheta\circ\mathsf{R}_1"'] \arrow[dr, "\varrho_1"] & \\
\Psi_2 \circ \mathsf{R}_1 \arrow[r, "\varrho_2"'] & \mathsf{R}_2
\end{tikzcd}$$

are commutative.

6.4. There is also a variant of Proposition 6.3 where one starts with two quivers $\mathscr{Q}_1$, $\mathscr{Q}_2$, a morphism of quivers $\mathsf{Q} : \mathscr{Q}_1 \to \mathscr{Q}_2$ and two representations

$$\mathsf{T}_1 : \mathscr{Q}_1 \to \mathscr{P}_1, \qquad \mathsf{T}_2 : \mathscr{Q}_2 \to \mathscr{P}_2.$$

Let $(\mathscr{A}_1, \mathsf{F}_1, \mathsf{R}_1, \alpha_1)$ and $(\mathscr{A}_2, \mathsf{F}_2, \mathsf{R}_2, \alpha_2)$ be 4-uplets obtained by applying Theorem 2.1 to the representations T_1 and T_2 respectively.

Proposition 6.6. *Let (Φ, ϕ) be a pair where $\Phi : \mathscr{P}_1 \to \mathscr{P}_2$ is an* exact *K-linear functor and $\phi : \Phi \circ \mathsf{T}_1 \to \mathsf{T}_2 \circ \mathsf{Q}$ is an isomorphism of representations. There exist an exact functor $\Psi : \mathscr{A}_1 \to \mathscr{A}_2$, an invertible natural transformation $\rho : \Phi \circ \mathsf{F}_1 \to \mathsf{F}_2 \circ \Psi$,* *and an isomorphism of representations* $\varrho : \Psi \circ \mathsf{R}_1 \to \mathsf{R}_2 \circ \mathsf{Q}$ *such that*

$$\begin{tikzcd}
\Phi \circ \mathsf{F}_1 \circ \mathsf{R}_1 \arrow[r, "\Phi\star\alpha_1"] \arrow[dd, "\rho\star\mathsf{R}_1"'] & \Phi \circ \mathsf{T}_1 \arrow[dr, "\phi"] & \\
& & \mathsf{T}_2 \circ \mathsf{Q} \\
\mathsf{F}_2 \circ \Psi \circ \mathsf{R}_1 \arrow[r, "\mathsf{F}_2\star\varrho"'] & \mathsf{F}_2 \circ \mathsf{R}_2 \circ \mathsf{Q} \arrow[ur, "\alpha_2\star\mathsf{Q}"'] &
\end{tikzcd} \tag{5}$$

is commutative.

Proof. The proof reduces to Proposition 6.3. Indeed consider the representations $\mathsf{T}_1 : \mathscr{Q}_1 \to \mathscr{P}_1$ and $\mathsf{T}'_2 := \mathsf{T}_2 \circ \mathsf{Q} : \mathscr{Q}_1 \to \mathscr{P}_2$ of the quiver $\mathscr{Q}_1$. Let $(\mathscr{A}'_2, \mathsf{F}'_2, \mathsf{R}'_2, \alpha'_2)$ be a 4-uplet obtained by applying Theorem 2.1 to the representations T'_2. By Proposition 6.3 applied to the pair (Φ, ϕ), there exist an exact functor $\Psi' : \mathscr{A}_1 \to \mathscr{A}'_2$, an invertible natural transformation $\rho' : \Phi \circ \mathsf{F}_1 \to \mathsf{F}'_2 \circ \Psi'$, and an isomorphism of representations $\varrho' : \Psi' \circ \mathsf{R}_1 \to \mathsf{R}'_2$ such that

$$\begin{tikzcd}
\Phi \circ \mathsf{F}_1 \circ \mathsf{R}_1 \arrow[r, "\Phi\star\alpha_1"] \arrow[dd, "\rho'\star\mathsf{R}_1"'] & \Phi \circ \mathsf{T}_1 \arrow[dr, "\phi"] & \\
& & \mathsf{T}'_2 \\
\mathsf{F}'_2 \circ \Psi' \circ \mathsf{R}_1 \arrow[r, "\mathsf{F}'_2\star\varrho'"'] & \mathsf{F}_2 \circ \mathsf{R}'_2 \arrow[ur, "\alpha'_2"'] &
\end{tikzcd} \tag{6}$$

is commutative. Now apply Theorem 2.1, with respect to T'_2, to the 4-uplet $(\mathscr{A}_2, \mathsf{F}_2, \mathsf{R}_2 \circ \mathsf{Q}, \alpha_2)$. We obtain a triple $(\Upsilon, \gamma, \delta)$ such that

$$\begin{tikzcd}
\mathsf{F}_2 \circ \Upsilon \circ \mathsf{R}'_2 \arrow[r, "\mathsf{F}_2\star\gamma"] \arrow[d, "\delta\star\mathsf{R}'_2"] & \mathsf{F}_2 \circ \mathsf{R}_2 \circ \mathsf{Q} \arrow[d, "\alpha_2"] \\
\mathsf{F}'_2 \circ \mathsf{R}'_2 \arrow[r, "\alpha'_2"] & \mathsf{T}'_2 := \mathsf{T}_2 \circ \mathsf{Q}
\end{tikzcd} \tag{7}$$

is commutative, where $\Upsilon : \mathscr{A}'_2 \to \mathscr{A}_2$ is an exact faithful functor, $\gamma : \Upsilon \circ \mathsf{R}'_2 \to \mathsf{R}_2 \circ \mathsf{Q}$ is an isomorphism of representations of quivers and $\delta : \mathsf{F}_2 \circ \Upsilon \to \mathsf{F}'_2$ is an isomorphism of functors. Now let $\Psi = \Upsilon \circ \Psi'$, define the natural transformation $\rho : \Phi \circ \mathsf{F}_1 \to \mathsf{F}_2 \circ \Psi$ as the composition:

$$\Phi \circ \mathsf{F}_1 \xrightarrow{\rho'} \mathsf{F}'_2 \circ \Psi' \xrightarrow{\delta^{-1}\star\Psi'} \mathsf{F}_2 \circ \Upsilon \circ \Psi' = \mathsf{F}_2 \circ \Psi$$

and $\varrho : \Psi \circ \mathsf{R}_1 \to \mathsf{R}_2 \circ \mathsf{Q}$ as the composition

$$\Psi \circ \mathsf{R}_1 = \Upsilon \circ \Psi' \circ \mathsf{R}_1 \xrightarrow{\Upsilon\star\varrho'} \Upsilon \circ \mathsf{R}'_2 \xrightarrow{\gamma} \mathsf{R}_2 \circ \mathsf{Q}.$$

The commutativity of (5) is then an immediate consequence of the commutativity of the diagrams (6) and (7). ☐

The uniqueness statement in Proposition 6.5 admits then the following variant:

Proposition 6.7. *Let* (Φ_1, ϕ_1) *and* (Φ_2, ϕ_2) *be pairs where* $\Phi_1, \Phi_2 : \mathscr{P}_1 \to \mathscr{P}_2$ *are exact functors and* $\phi_1 : \Phi_1 \circ \mathsf{T}_1 \to \mathsf{T}_2 \circ \mathsf{Q}$, $\phi_2 : \Phi_2 \circ \mathsf{T}_1 \to \mathsf{T}_2 \circ \mathsf{Q}$ *are isomorphisms of representations. Let* $\theta : \Phi_1 \to \Phi_2$ *be a natural transformation such that*

$$\phi_2 \circ (\theta \star \mathsf{T}_1) = \phi_1.$$

Let $(\Psi_1, \rho_1, \varrho_1)$, $(\Psi_2, \rho_2, \varrho_2)$ *be triples where* $\Psi_1, \Psi_2 : \mathscr{A}_1 \to \mathscr{A}_2$ *are exact functors,* $\rho_1 : \Phi_1 \circ \mathsf{F}_1 \to \mathsf{F}_2 \circ \Psi_1$, $\rho_2 : \Phi_2 \circ \mathsf{F}_1 \to \mathsf{F}_2 \circ \Psi_2$ *are invertible natural transformations,* $\varrho_1 : \Psi_1 \circ \mathsf{R}_1 \to \mathsf{R}_2 \circ \mathsf{Q}$, $\varrho_2 : \Psi_2 \circ \mathsf{R}_1 \to \mathsf{R}_2 \circ \mathsf{Q}$ *are isomorphisms of representations such that*

$$\begin{tikzcd}
\Phi_1 \circ \mathsf{F}_1 \circ \mathsf{R}_1 \arrow[r, "\Phi_1\star\alpha_1"] \arrow[dd, "\rho_1\star\mathsf{R}_1"'] & \Phi_1 \circ \mathsf{T}_1 \arrow[dr, "\phi_1"] & \\
 & & \mathsf{T}_2 \circ \mathsf{Q} \\
\mathsf{F}_2 \circ \Psi_1 \circ \mathsf{R}_1 \arrow[r, "\mathsf{F}_2\star\varrho_1"'] & \mathsf{F}_2 \circ \mathsf{R}_2 \circ \mathsf{Q} \arrow[ur, "\alpha_2\star\mathsf{Q}"'] &
\end{tikzcd}
\qquad
\begin{tikzcd}
\Phi_2 \circ \mathsf{F}_1 \circ \mathsf{R}_1 \arrow[r, "\Phi_2\star\alpha_1"] \arrow[dd, "\rho_2\star\mathsf{R}_1"'] & \Phi_2 \circ \mathsf{T}_1 \arrow[dr, "\phi_2"] & \\
 & & \mathsf{T}_2 \circ \mathsf{Q} \\
\mathsf{F}_2 \circ \Psi_2 \circ \mathsf{R}_1 \arrow[r, "\mathsf{F}_2\star\varrho_2"'] & \mathsf{F}_2 \circ \mathsf{R}_2 \circ \mathsf{Q} \arrow[ur, "\alpha_2\star\mathsf{Q}"'] &
\end{tikzcd}$$

are commutative. Then there exists one and only one natural transformation $\vartheta : \Psi_1 \to \Psi_2$ *such that*

$$\begin{tikzcd}
\Phi_1 \circ \mathsf{F}_1 \arrow[r, "\rho_1"] \arrow[d, "\theta\star\mathsf{F}_1"'] & \mathsf{F}_2 \circ \Psi_1 \arrow[d, "\mathsf{F}_2\star\vartheta"] \\
\Phi_2 \circ \mathsf{F}_1 \arrow[r, "\rho_2"'] & \mathsf{F}_2 \circ \Psi_2
\end{tikzcd}
\qquad
\begin{tikzcd}
\Psi_1 \circ \mathsf{R}_1 \arrow[d, "\vartheta\circ\mathsf{R}_1"'] \arrow[dr, "\varrho_1"] & \\
\Psi_2 \circ \mathsf{R}_1 \arrow[r, "\varrho_2"'] & \mathsf{R}_2 \circ \mathsf{Q}
\end{tikzcd}$$

are commutative.

6.5. Consider two finite and Hom finite K-linear Abelian categories $\mathscr{P}$ and $\mathscr{Q}$. There is finally a useful consequence of Proposition 6.6 where one starts with a quiver $\mathcal{Q}$, a representation of quivers

$$\mathsf{T} : \mathcal{Q} \to \mathscr{P}$$

a K-linear Abelian category $\mathscr{B}$, a representation $\mathsf{S} : \mathcal{Q} \to \mathscr{B}$ and a faithful exact functor $\mathsf{G} : \mathscr{B} \to \mathscr{Q}$. We consider a 4-uplet $(\mathscr{A}, \mathsf{F}, \mathsf{R}, \alpha)$ obtained by applying Theorem 2.1 to the representation T.

Proposition 6.8. *Let* (Φ, ϕ) *be a pair where* $\Phi : \mathscr{P} \to \mathscr{Q}$ *is an* exact *K-linear functor and* $\phi : \Phi \circ \mathsf{T} \to \mathsf{G} \circ \mathsf{S}$ *is an isomorphism of representations. There exist an exact functor* $\Psi : \mathscr{A} \to \mathscr{B}$*, an invertible natural transformation* $\rho : \Phi \circ \mathsf{F} \to \mathsf{G} \circ \Psi$*, and an isomorphism of representations* $\varrho : \Psi \circ \mathsf{R} \to \mathsf{S}$ *such that*

$$\begin{tikzcd}
\Phi \circ \mathsf{F} \circ \mathsf{R} \arrow[r, "\Phi\star\alpha"] \arrow[d, "\rho\star\mathsf{R}"'] & \Phi \circ \mathsf{T} \arrow[d, "\phi"] \\
\mathsf{G} \circ \Psi \circ \mathsf{R} \arrow[r, "\mathsf{G}\star\varrho"'] & \mathsf{G} \circ \mathsf{S}
\end{tikzcd}$$

is commutative.

Proof. Let $\mathcal{Q}_1 := \mathcal{Q}$, $\mathscr{P}_1 := \mathscr{P}$ and $\mathsf{T}_1 := \mathsf{T}$. Now consider the category $\mathscr{B}$ as a quiver $\mathcal{Q}_2 := \mathscr{B}$, and set $\mathscr{P}_2 := \mathscr{Q}$, $\mathsf{T}_2 := \mathsf{G}$ and $\mathsf{Q} := \mathsf{S}$. The the 4-uplet $(\mathscr{B}, \mathsf{G}, \mathrm{id}, \mathrm{id})$ satisfies the universal property for the representation T_2. Apply Proposition 6.6. □

The uniqueness statement in Proposition 6.7 implies then the following:

Proposition 6.9. *Let* (Φ_1, ϕ_1) *and* (Φ_2, ϕ_2) *be pairs where* $\Phi_1, \Phi_2 : \mathscr{P} \to \mathscr{Q}$ *are exact functors and* $\phi_1 : \Phi_1 \circ \mathsf{T} \to \mathsf{G} \circ \mathsf{S}$, $\phi_2 : \Phi_2 \circ \mathsf{T} \to \mathsf{G} \circ \mathsf{S}$ *are isomorphims of* *representations. Let* $\theta : \Phi_1 \to \Phi_2$ *be a natural transformation such that*

$$\phi_2 \circ (\theta \star \mathsf{T}) = \phi_1.$$

Let $(\Psi_1, \rho_1, \varrho_1)$, $(\Psi_2, \rho_2, \varrho_2)$ *be triples where* $\Psi_1, \Psi_2 : \mathscr{A} \to \mathscr{B}$ *are exact functors,* $\Phi_1 \circ \mathsf{F} \to \mathsf{G} \circ \Psi_1$, $\rho_2 : \Phi_2 \circ \mathsf{F} \to \mathsf{G} \circ \Psi_2$ *are invertible natural transformations,* $\varrho_1 : \Psi_1 \circ \mathsf{R} \to \mathsf{S}$, $\varrho_2 : \Psi_2 \circ \mathsf{R} \to \mathsf{S}$ *are isomorphisms of representations such that*

$$\begin{tikzcd}
\Phi_1 \circ \mathsf{F} \circ \mathsf{R} \arrow[r, "\Phi_1 \star \alpha"] \arrow[d, "\rho_1 \star \mathsf{R}"'] & \Phi_1 \circ \mathsf{T} \arrow[d, "\phi_1"] \\
\mathsf{G} \circ \Psi_1 \circ \mathsf{R} \arrow[r, "\mathsf{G} \star \varrho_1"'] & \mathsf{G} \circ \mathsf{S}
\end{tikzcd}
\qquad
\begin{tikzcd}
\Phi_2 \circ \mathsf{F} \circ \mathsf{R} \arrow[r, "\Phi_2 \star \alpha"] \arrow[d, "\rho_2 \star \mathsf{R}"'] & \Phi_2 \circ \mathsf{T} \arrow[d, "\phi_2"] \\
\mathsf{G} \circ \Psi_2 \circ \mathsf{R} \arrow[r, "\mathsf{G} \star \varrho_2"'] & \mathsf{G} \circ \mathsf{S}
\end{tikzcd}$$

are commutative. Then there exists one and only one natural transformation $\vartheta : \Psi_1 \to \Psi_2$ *such that*

$$\begin{tikzcd}
\Phi_1 \circ \mathsf{F} \arrow[r, "\rho_1"] \arrow[d, "\theta \star \mathsf{F}"'] & \mathsf{G} \circ \Psi_1 \arrow[d, "\mathsf{F}_2 \star \vartheta"] \\
\Phi_2 \circ \mathsf{F} \arrow[r, "\rho_2"'] & \mathsf{G} \circ \Psi_2
\end{tikzcd}
\qquad
\begin{tikzcd}
\Psi_1 \circ \mathsf{R} \arrow[d, "\vartheta \circ \mathsf{R}"'] \arrow[dr, "\varrho_1"] & \\
\Psi_2 \circ \mathsf{R} \arrow[r, "\varrho_2"'] & \mathsf{G} \circ \mathsf{S}
\end{tikzcd}$$

are commutative.

7. Abelian categories of Perverse motives

Recall that $k = \mathbb{C}$ is the field of complex numbers (one can also choose a field of characteristic zero k with a fixed embedding of fields $\sigma : k \hookrightarrow \mathbb{C}$) and that by a k-variety we mean a quasi-projective k-scheme. The reader, if he wishes, may work also with separated k-schemes of finite type. In particular, perverse motives make sense over such bases.

In this section we use Theorem 2.1 to construct a perverse analog over k-varieties of the category $\mathrm{NMM}(k, \mathbb{Q})$.

7.1. Let X be a k-variety. We denote by $\mathscr{M}(X)$ one of the following Abelian categories: (a) the category of perverse sheaves $\mathscr{P}(X)$; (b) the category of perverse sheaves of geometric origins $\mathscr{P}(X)^{\mathrm{go}}$ introduced in [6, 6.2.4]; (c) the category of mixed Hodge modules $\mathrm{MHM}(X, \mathbb{Q})$; (d) the category $\mathrm{MHM}(X, \mathbb{Q})^{\mathrm{go}}$ of mixed Hodge modules of geometric origin (see [20, (2.6) Définition]). Recall that the derived categories $\mathrm{D^b}(\mathscr{M}(X))$, as X runs over k-varieties, are endowed with a six functors formalism

$$\begin{tikzcd}
\mathrm{D^b}(\mathscr{M}(X)) \arrow[r, shift right, "f_*^{\mathscr{M}}"'] & \mathrm{D^b}(\mathscr{M}(Y)) \arrow[l, shift right, "f^*_{\mathscr{M}}"'] \arrow[r, shift left, "f_!^{\mathscr{M}}"] & \mathrm{D^b}(\mathscr{M}(X)). \arrow[l, shift left, "f^!_{\mathscr{M}}"]
\end{tikzcd}$$

We denote by

$$\mathsf{H}^i_{\mathscr{M}} : \mathrm{D^b}(\mathscr{M}(X)) \to \mathscr{M}(X) \quad i \in \mathbb{Z}$$

the cohomological functor associated with the usual t-structure.

7.2. A relative X-triple is a triple $(Y \xrightarrow{a} X, Z, i)$ where Y is a k-variety, $a : Y \to X$ is a morphism of k-varieties, Z is a closed subscheme of Y and $i \in \mathbb{Z}$ is an integer.

Definition 7.1. Let $(Y \xrightarrow{a} X, Z, i)$ be a relative X-triple. We set

$$\mathsf{T}^{\mathscr{M}}_X(Y \xrightarrow{a} X, Z, i) := \mathsf{H}^{-i}_{\mathscr{M}}(a_!^{\mathscr{M}}(u_*^{\mathscr{M}} u^*_{\mathscr{M}} a^!_{\mathscr{M}}(\mathbb{Q}^{\mathscr{M}}_X)))$$

where $u : U \hookrightarrow Y$ is the open immersion of the complement of Z in Y.

Note that by definition $\mathsf{T}_X^{\mathscr{M}}(Y \xrightarrow{a} X, Z, i)$ is an object in $\mathscr{M}(X)$ which depends only on the reduced structure of Y and Z. If there is no confusion, we will also use the notation (Y, Z, i) to denote a relative X-triple and write $\mathsf{T}_X^{\mathscr{M}}(Y, Z, i)$ instead of $\mathsf{T}_X^{\mathscr{M}}(Y \xrightarrow{a} X, Z, i)$.

7.3. Using the formalism of the six operations, it is easy to construct two different sorts of morphisms between these objects in $\mathscr{M}(X)$: the functoriality morphisms and the boundary morphisms.

Let (Y_1, Z_1, i) and (Y_2, Z_2, i) be relative X-triples. Assume that $f : Y_2 \to Y_1$ is a morphism of X-schemes, such that $f(Z_2) \subseteq Z_1$. The functoriality morphisms are maps in $\mathscr{M}(X)$

$$f_{\mathsf{T}}^{\mathscr{M}} : \mathsf{T}_X^{\mathscr{M}}(Y_2, Z_2, i) \to \mathsf{T}_X^{\mathscr{M}}(Y_1, Z_1, i) \tag{8}$$

such that if (Y_3, Z_3, i) is a relative X-triple, and $g : Y_3 \to Y_2$ is a morphism of X-schemes such that $g(Z_3) \subseteq Z_2$, then

$$f_{\mathsf{T}}^{\mathscr{M}} \circ g_{\mathsf{T}}^{\mathscr{M}} = (fg)_{\mathsf{T}}^{\mathscr{M}}.$$

The morphism (8) is obtained as follows. Consider the commutative diagram

$$\begin{tikzcd}
f^{-1}(U_1) \arrow[r, "u"] \arrow[d, "f"'] \arrow[dr, phantom, "\square"] & U_2 \arrow[r, "u_2"] & Y_2 \arrow[d, "f"] \arrow[dr, "a_2"] & \\
U_1 \arrow[rr, "u_1"] & & Y_1 \arrow[r, "a_1"'] & X
\end{tikzcd}$$

in which U_1 (resp. U_2) is the open complement of Z_1 (resp. Z_2) and all arrows are the canonical morphisms. Using Smooth Base Change and adjunction, we have a morphism in $\mathrm{D^b}(\mathscr{M}(Y_1))$

$$\begin{aligned}
f_!^{\mathscr{M}}(u_2)_*^{\mathscr{M}}(u_2)^*_{\mathscr{M}}(a_2)^!_{\mathscr{M}} \to {} & f_!^{\mathscr{M}}(u_2)_*^{\mathscr{M}} u_*^{\mathscr{M}} u^*_{\mathscr{M}}(u_2)^*_{\mathscr{M}}(a_2)^!_{\mathscr{M}} \\
&\| \\
& f_!^{\mathscr{M}}(u_2)_*^{\mathscr{M}} u_*^{\mathscr{M}} f^!_{\mathscr{M}}(u_1)^*_{\mathscr{M}}(a_1)^!_{\mathscr{M}} \\
&\| \\
& f_!^{\mathscr{M}} f^!_{\mathscr{M}}(u_1)_*^{\mathscr{M}}(u_1)^*_{\mathscr{M}}(a_1)^!_{\mathscr{M}} \longrightarrow (u_1)_*^{\mathscr{M}}(u_1)^*_{\mathscr{M}}(a_1)^!_{\mathscr{M}}.
\end{aligned}$$

Applying successively $(a_1)_!^{\mathscr{M}}$ and the cohomological functor $\mathsf{H}^{-i}_{\mathscr{M}}$ to this morphism, we obtain the morphism (8) in $\mathscr{M}(X)$.

Let (Y, Z, i) be a relative X-triple, and $W \subseteq Z$ be a closed subset. Then the boundary morphism is the map in $\mathscr{M}(X)$

$$\mathsf{T}_X^{\mathscr{M}}(Y, Z, i) \to \mathsf{T}_X^{\mathscr{M}}(Z, W, i-1) \tag{9}$$

defined as follows. Consider the commutative diagram

$$\begin{tikzcd}
U := Y \setminus Z \arrow[r, "j"] \arrow[rr, bend left, "u"] & Y \setminus W \arrow[r, "v_Y"] \arrow[dr, phantom, "\square"] & Y \arrow[r, "a"] & X \\
& V := Z \setminus W \arrow[u, "z_V"] \arrow[r, "v"] & Z \arrow[u, "z"'] \arrow[ur, "b"'] &
\end{tikzcd}$$

where v, v_Y, j are the open immersions, z the closed immersion and a, b the structural morphisms. The localization triangle in $\mathrm{D^b}(\mathscr{M}(Y \setminus W))$

$$(z_V)_!^{\mathscr{M}}(z_V)^!_{\mathscr{M}} \to \mathrm{id} \to j_*^{\mathscr{M}} j^*_{\mathscr{M}} \xrightarrow{+1},$$

applied to $(v_Y)^!_{\mathscr{M}} a^!_{\mathscr{M}}(\mathbb{Q}^{\mathscr{M}}_X)$, provides a morphism

$$j^{\mathscr{M}}_* u^*_{\mathscr{M}} a^!_{\mathscr{M}}(\mathbb{Q}^{\mathscr{M}}_X) \to (z_V)^{\mathscr{M}}_! v^*_{\mathscr{M}} b^!_{\mathscr{M}}(\mathbb{Q}^{\mathscr{M}}_X)[1].$$

As z and z_V are closed immersions, applying $(v_Y)^{\mathscr{M}}_*$, yields a morphism

$$u^{\mathscr{M}}_* u^*_{\mathscr{M}} a^!_{\mathscr{M}}(\mathbb{Q}^{\mathscr{M}}_X) \to z^{\mathscr{M}}_! v_* v^*_{\mathscr{M}} b^!_{\mathscr{M}}(\mathbb{Q}^{\mathscr{M}}_X)[1]$$

Applying $a^{\mathscr{M}}_!$, the cohomological functor $\mathsf{H}^{-i}_{\mathscr{M}}$ yields the boundary map (9).

7.4. Let $\mathscr{D}^{\mathrm{Nori}}_X$ be the quiver with a vertex for every relative X-triple (Y, Z, i) and the following edges.

- Let (Y_1, Z_1, i) and (Y_2, Z_2, i) be relative X-triples. Then every morphism of X-schemes $f : Y_2 \to Y_1$ such that $f(Z_2) \subseteq Z_1$ defines and edge
$$(Y_2, Z_2, i) \to (Y_1, Z_1, i). \tag{10}$$
- For every relative X-triple (Y, Z, i) and every closed subscheme $W \subseteq Z$, we have an edge
$$(Y, Z, i) \to (Z, W, i-1). \tag{11}$$

The quiver $\mathscr{D}^{\mathrm{Nori}}_X$ admits then a representation

$$\mathsf{T}^{\mathscr{M}}_X : \mathscr{D}^{\mathrm{Nori}}_X \to \mathscr{M}(X)$$

where the edges (10) and (11) are maps respectively to the morphisms (8) and (9). As the functor $\mathsf{rat}^{\mathscr{H}}_X : \mathrm{MHM}(X, \mathbb{Q}) \to \mathscr{P}(X)$ is compatible with the formalism of cohomological operations, there is a canonical isomorphism of representations of quivers

$$\alpha^{\mathscr{H}}_X : \mathsf{rat}^{\mathscr{H}}_X \circ \mathsf{T}^{\mathscr{H}}_X \to \mathsf{T}^{\mathscr{P}}_X.$$

7.5. Let X be a k-variety. Since the Abelian category of perverse sheaves $\mathscr{P}(X)$ is Noetherian, Artinian and has finite dimensional Homs (see Example 2.2), we may apply Theorem 2.1, to the representation

$$\mathsf{T}^{\mathscr{P}}_X : \mathscr{D}^{\mathrm{Nori}}_X \to \mathscr{P}(X)$$

to obtain an Abelian category $\mathscr{N}^{\mathrm{eff}}(X)$ together with a faithful exact functor $\mathsf{rat}^{\mathscr{N}}_X$, a representation $\mathsf{T}^{\mathscr{N}}_X$:

$$\mathsf{rat}^{\mathscr{N}}_X : \mathscr{N}^{\mathrm{eff}}(X) \to \mathscr{P}(X) \qquad \mathsf{T}^{\mathscr{N}}_X : \mathscr{D}^{\mathrm{Nori}}_X \to \mathscr{N}^{\mathrm{eff}}(X)$$

and an isomorphism of representations of quivers $\alpha^{\mathrm{Nori}}_X : \mathsf{rat}^{\mathscr{N}}_X \circ \mathsf{T}^{\mathscr{N}}_X \to \mathsf{T}^{\mathscr{P}}_X$ such that the following theorem holds:

Theorem 7.2. *For every k-variety X, the 4-uplet*

$$(\mathscr{N}^{\mathrm{eff}}(X), \mathsf{rat}^{\mathscr{N}}_X, \mathsf{T}^{\mathscr{N}}_X, \alpha^{\mathscr{N}}_X)$$

has the following properties.

- *For every $(\mathscr{A}, \mathsf{B}, \mathsf{T}, \alpha)$ where $\mathsf{B} : \mathscr{A} \to \mathscr{P}(X)$ is a faithful exact functor, $\mathsf{T} : \mathscr{D}^{\mathrm{Nori}}_X \to \mathscr{A}$ is a representation, and $\alpha : \mathsf{B} \circ \mathsf{T} \to \mathsf{T}^{\mathscr{P}}_X$ is an isomorphism of representations of quivers, there exists a triple $(\mathrm{R}, \rho, \varrho)$ where*
$$\mathsf{R} : \mathscr{N}^{\mathrm{eff}}(X) \to \mathscr{A}$$
is a faithful exact functor, $\rho : \mathsf{B} \circ \mathsf{R} \to \mathsf{rat}^{\mathscr{N}}_X$ is an invertible natural transformation and $\varrho : \mathsf{R} \circ \mathsf{T}^{\mathscr{N}}_X \to \mathsf{T}$ is an isomorphism of quiver representations such that
$$\begin{tikzcd} \mathsf{B} \circ \mathsf{R} \circ \mathsf{T}^{\mathscr{N}}_X \arrow[r, "\mathsf{B}\circ\varrho"] \arrow[d, "\rho\circ\mathsf{T}^{\mathscr{N}}_X"] & \mathsf{B} \circ \mathsf{T} \arrow[d, "\alpha"] \\ \mathsf{rat}^{\mathscr{N}}_X \circ \mathsf{T}^{\mathscr{N}}_X \arrow[r, "\alpha^{\mathscr{N}}_X"] & \mathsf{T}^{\mathscr{P}}_X \end{tikzcd}$$

is commutative.

- *If* $(\mathsf{R}', \varrho', \rho')$ *is another such triple, then there exists one and only one natural transformation* $\theta : \mathsf{R} \to \mathsf{R}'$ *such that*

$$\rho' \circ (\mathsf{B} \circ \theta) = \rho \qquad \varrho' \circ (\theta \circ \mathsf{T}_X^{\mathscr{N}}) = \varrho.$$

Remark 7.3. The category $\mathscr{N}^{\mathrm{eff}}(\mathrm{Spec}(k))$ is nothing but the category of effective (homological) motives $\mathrm{EHM}(k)$ constructed by M. Nori (see [17] or [14, Definition 3.13]).

By applying Theorem 7.2 to the 4-uplet

$$(\mathrm{MHM}(X, \mathbb{Q}), \mathsf{rat}_X^{\mathscr{H}}, \mathsf{T}_X^{\mathscr{H}}, \alpha_X^{\mathscr{H}})$$

where $\mathsf{rat}_X^{\mathscr{H}} : \mathrm{MHM}(X, \mathbb{Q}) \to \mathscr{P}(X)$ is the forgetful functor, one obtains a faithful exact functor

$$\mathsf{R}_X^{\mathscr{H}} : \mathscr{N}^{\mathrm{eff}}(X) \to \mathrm{MHM}(X, \mathbb{Q})$$

which takes its values in the full Abelian subcategory formed by the mixed Hodge modules of geometric origin.

7.6. We now want to apply Proposition 6.6 to construct the categories of perverse motives out of the categories of effective perverse motives.

Let $\mathscr{A}$ be a K-linear Abelian category and $\mathsf{L} : \mathscr{A} \to \mathscr{A}$ be a K-linear exact functor. Let us construct a category $\mathscr{A}[\mathsf{L}^{-1}]$ as follows. An object in $\mathscr{A}[\mathsf{L}^{-1}]$ is a pair (A, n) where A is an object in $\mathscr{A}$ and $n \in \mathbb{Z}$ is an integer. Morphisms in $\mathscr{A}[\mathsf{L}^{-1}]$ are given by

$$\mathrm{Hom}_{\mathscr{A}[\mathsf{L}^{-1}]}((A, n), (B, m)) = \operatorname*{colim}_i \mathrm{Hom}_{\mathscr{A}}(\mathsf{L}^{\circ i+n}(A), \mathsf{L}^{\circ i+m}(B)),$$

where the colimit is taken over all integers $i \in \mathbb{N}$ such that $i+n \geqslant 0$ and $i+m \geqslant 0$. There is a functor

$$\mathscr{A} \to \mathscr{A}[\mathsf{L}^{-1}]$$
$$A \mapsto (A, 0).$$

Lemma 7.4. *The category* $\mathscr{A}[\mathsf{L}^{-1}]$ *is* K*-linear Abelian and the functor* $\mathscr{A} \to \mathscr{A}[\mathsf{L}^{-1}]$ *is* K*-linear and exact. If* L *is faithful (resp. full) then* $\mathscr{A} \to \mathscr{A}[\mathsf{L}^{-1}]$ *is a faithful (resp. full) functor.*

Proof. We only sketch the proof as the details are straightforward. The category $\mathscr{A}[\mathsf{L}^{-1}]$ is K-linear and admits finite direct sums given by

$$(A, n) \oplus (B, m) = (\mathsf{L}^{i+n}(A) \oplus \mathsf{L}^{\circ i+m}(B), -i)$$

where $i \in \mathbb{N}$ is some integer such that $i+n \geqslant 0$ and $i+m \geqslant 0$. Let $\alpha : (A, n) \to (B, m)$ be a morphism in $\mathscr{A}[\mathsf{L}^{-1}]$. Let $i \in \mathbb{N}$ an integer such that $i+n \geqslant 0$ and $i+m \geqslant 0$ and $a : \mathsf{L}^{\circ i+n}(A) \to \mathsf{L}^{\circ i+m}(B)$ a morphism in $\mathscr{A}$ that lifts α. The maps $\mathrm{Ker}\, a \to \mathsf{L}^{\circ i+n}$ and $\mathsf{L}^{i+m}(B) \to \mathrm{Coker}\, a$ in $\mathscr{A}$ define maps

$$(\mathrm{Ker}\, a, -i) \to (A, n) \qquad (B, m) \to (\mathrm{Coker}\, a, -i)$$

in $\mathscr{A}[\mathsf{L}^{-1}]$. Using that L is exact, it is easy to check that these maps are respectively a kernel and a cokernel of α in $\mathscr{A}[\mathsf{L}^{-1}]$. Since $\mathscr{A}$ is an Abelian category, the canonical map

$$(A, n)/\operatorname{Ker} \alpha = (\mathsf{L}^{\circ i+n}(A)/\operatorname{Ker} a, -i) \to (\mathrm{Im}\, a, -i) = \mathrm{Im}\, \alpha$$

is an isomorphism (here $\mathrm{Im}\, \alpha$ denotes, as usual, the kernel of the cokernel map). □

Remark 7.5. Let $A \in \mathscr{A}$, and $n \in \mathbb{Z}$, $r \in \mathbb{N}$ be integers. For every integer $i \in \mathbb{N}$ such that $i+n-r \geqslant 0$, the identity of $\mathsf{L}^{\circ i+n}(A) = \mathsf{L}^{\circ i+n-r}(\mathsf{L}^{\circ r}(A))$ induces an isomorphism between (A, n) and $(\mathsf{L}^{\circ r}(A), n-r)$ in $\mathscr{A}[\mathsf{L}^{-1}]$. In particular (A, r) is isomorphic to $(\mathsf{L}^{\circ r}(A), 0)$ in $\mathscr{A}[\mathsf{L}^{-1}]$.

Recall that by definition (we use the convention of §7.1)

$$\mathbb{Q}_k^{\mathscr{M}}(1) = \mathsf{T}_k^{\mathscr{M}}(\mathbf{G}_{m,k}, \{1\}, 1).$$

The Tate twist of an object $A \in \mathscr{M}(X)$ is then defined by $A(1) := A \boxtimes \mathbb{Q}_k^{\mathscr{M}}(1)$ where $\boxtimes : \mathscr{M}(X) \times \mathscr{M}(k) \to \mathscr{M}(X)$ is the external product.

Let X be a quasi-projective k-scheme. Consider the quivers $\mathscr{Q}_1 := \mathscr{Q}_2 = \mathscr{D}_X^{\mathrm{Nori}}$ and the representations

$$\mathsf{T}_1 := \mathsf{T}_2 := \mathsf{T}_X^{\mathscr{M}}.$$

Consider the morphism of quivers $\mathsf{Q} : \mathscr{Q}_1 \to \mathscr{Q}_2$.

$$(Y, Z, i) \mapsto (\mathbf{G}_{m,Y}, \mathbf{G}_{m,Z} \cup Y, i+1)$$

where Y is embedded in $\mathbf{G}_{m,Y}$ via the unit section. Denote by $\Phi_X^{\mathscr{M}} : \mathscr{M}(X) \to \mathscr{M}(X)$ the functor which maps K to its twist $K(1)$.

Lemma 7.6. *There are canonical isomorphisms of representations of quivers*

$$\phi_X^{\mathscr{M}} : \Phi_X^{\mathscr{M}} \circ \mathsf{T}_X^{\mathscr{M}} \to \mathsf{T}_X^{\mathscr{M}} \circ \mathsf{Q}$$

which are compatible via the functor $\mathsf{rat}_X^{\mathscr{H}}$.

Proof. Let $z : Z \hookrightarrow Y$ be a closed immersion, and $u : U \hookrightarrow Y$ be its open complement. We denote by $\pi : \mathbf{G}_{m,k} \to \mathrm{Spec}(k)$ the projection and $v : \mathbf{G}_{m,k} \setminus \{1\} \hookrightarrow \mathbf{G}_{m,k}$ the open immersion. The open immersion in $\mathbf{G}_{m,Y}$ of the complement of $\mathbf{G}_{m,Z} \cup Y$ is

$$u \times_k v : U \times_k (\mathbf{G}_{m,k} \setminus \{1\}) \hookrightarrow \mathbf{G}_{m,Y}.$$

Let $\sigma : \mathrm{Spec}(k) \to \mathbf{G}_{m,k}$ be the unit section. The triangle $\sigma_!^{\mathscr{M}} \sigma_{\mathscr{M}}^! \to \mathrm{Id} \to v_*^{\mathscr{M}} v_{\mathscr{M}}^! \xrightarrow{+1}$ yields the distinguished triangle

$$\mathbb{Q}_k^{\mathscr{M}} \to \pi_!^{\mathscr{M}} \pi_{\mathscr{M}}^!(\mathbb{Q}_k^{\mathscr{M}}) \to \pi_!^{\mathscr{M}} v_*^{\mathscr{M}} v_{\mathscr{M}}^! \pi_{\mathscr{M}}^!(\mathbb{Q}_k^{\mathscr{M}}) \xrightarrow{+1}.$$

Via the canonical isomorphism $\pi_!^{\mathscr{M}} \pi_{\mathscr{M}}^!(\mathbb{Q}_k^{\mathscr{M}}) = \mathbb{Q}_k^{\mathscr{M}} \oplus \mathbb{Q}_k^{\mathscr{M}}(1)[1]$, one gets

$$\pi_!^{\mathscr{M}} v_*^{\mathscr{M}} v_{\mathscr{M}}^* \pi_{\mathscr{M}}^!(\mathbb{Q}_k^{\mathscr{M}}) = \mathbb{Q}_k^{\mathscr{M}}(1)[1].$$

We have thus an isomorphism

$$(u \times_k v)_{\mathscr{M}}^!(a \times_k \pi)_{\mathscr{M}}^!(\mathbb{Q}_X^{\mathscr{M}}) = u_{\mathscr{M}}^! a_{\mathscr{M}}^!(\mathbb{Q}_X^{\mathscr{M}}) \boxtimes v^! \pi_{\mathscr{M}}^!(\mathbb{Q}_k^{\mathscr{M}}).$$

The object $(a \times_k \pi)_!^{\mathscr{M}}(u \times_k v)_*^{\mathscr{M}}(u \times_k v)_{\mathscr{M}}^!(a \times_k \pi)_{\mathscr{M}}^!(\mathbb{Q}_X^{\mathscr{M}})$ of $\mathrm{D}^{\mathrm{b}}(\mathscr{P}(X))$ is therefore isomorphic to

$$(a_!^{\mathscr{M}} u_*^{\mathscr{M}} u_{\mathscr{M}}^! a_{\mathscr{M}}^!(\mathbb{Q}_X^{\mathscr{M}})) \boxtimes (\pi_!^{\mathscr{M}} v_*^{\mathscr{M}} v_{\mathscr{M}}^! \pi_{\mathscr{M}}^!(\mathbb{Q}_k^{\mathscr{M}}) = a_!^{\mathscr{M}} u_*^{\mathscr{M}} u_{\mathscr{M}}^! a_{\mathscr{M}}^!(\mathbb{Q}_X^{\mathscr{M}})(1)[1].$$

Hence for every integer $i \in \mathbb{Z}$, we have a canonical isomorphism

$$\mathsf{T}_X^{\mathscr{M}}(\mathbf{G}_{m,Y}, \mathbf{G}_{m,Z} \cup Y, i+1) = \mathsf{T}_X^{\mathscr{M}}(Y, Z, i)(1).$$

The right hand side is $\mathsf{T}_X^{\mathscr{M}}(\mathsf{Q}(Y, Z, i))$ while the right hand side is $\Phi_X^{\mathscr{M}}(\mathsf{T}_X^{\mathscr{M}}(X, Z, i))$. One checks easily that this defines an isomorphism of quivers as desired and this concludes the proof. □

Note that $\Phi := \Phi_X^{\mathscr{P}}$ is an exact functor. Hence we may apply Proposition 6.6 which yields an exact functor $\mathsf{L} : \mathscr{N}^{\mathrm{eff}}(X) \to \mathscr{N}^{\mathrm{eff}}(X)$, an invertible natural transformation $\rho : \Phi \circ \mathsf{rat}_X^{\mathscr{N}} \to \mathsf{rat}_X^{\mathscr{N}} \circ \mathsf{L}$, and an isomorphism of representations of quivers

$\varrho : \mathsf{L} \circ \mathsf{T}_X^{\mathscr{N}} \to \mathsf{T}_X^{\mathscr{N}} \circ \mathsf{Q}$ such that

$$\begin{tikzcd}
\Phi \circ \mathsf{rat}_X^{\mathscr{N}} \circ \mathsf{T}_X^{\mathscr{N}} \arrow[r, "\Phi\star\alpha_X^{\mathscr{N}}"] \arrow[dd, "\rho\star\mathsf{T}_X^{\mathscr{N}}"'] & \Phi \circ \mathsf{T}_X^{\mathscr{N}} \arrow[dr, "\phi_X^{\mathscr{P}}"] & \\
 & & \mathsf{T}_X^{\mathrm{Nori}} \circ \mathsf{Q} \\
\mathsf{rat}_X^{\mathscr{N}} \circ \mathsf{L} \circ \mathsf{T}_X^{\mathscr{N}} \arrow[r, "\mathsf{rat}_X^{\mathscr{N}}\star\varrho"'] & \mathsf{rat}_X^{\mathscr{N}} \circ \mathsf{T}_X^{\mathscr{N}} \circ \mathsf{Q} \arrow[ur, "\alpha_X^{\mathscr{N}}\star\mathsf{Q}"'] &
\end{tikzcd}$$

is commutative.

The category of perverse motives $\mathscr{N}(X)$ is then defined by

$$\mathscr{N}(X) := \mathscr{N}^{\mathrm{eff}}(X)[\mathsf{L}^{-1}].$$

Since the functor L is faithful, the canonical exact functor

$$\mathscr{N}^{\mathrm{eff}}(X) \to \mathscr{N}(X)$$
$$A \mapsto (A, 0).$$

is also faithful. It is easy to see that the functors on the category of perverse effective motives $\mathsf{rat}_X^{\mathscr{N}}$ and $\mathsf{R}_X^{\mathscr{H}}$ extend to compatible faithful exact functors

$$\mathsf{rat}_X^{\mathscr{N}} : \mathscr{N}(X) \to \mathscr{P}(X) \qquad \mathsf{R}_X^{\mathscr{H}} : \mathscr{N}(X) \to \mathrm{MHM}(X, \mathbb{Q}).$$

Remark 7.7. Note that by Proposition 6.7 the Abelian category $\mathscr{N}(X)$ does not depend up to an exact equivalence of categories on the choice of the functor L (and all other functors satisfying the above conditions are isomorphic).

Remark 7.8. The categories $\mathscr{N}^{\mathrm{eff}}(X)$ and $\mathscr{N}(X)$ are the Abelian categories of perverse motives used in [12]. Note that instead of using the representation $\mathsf{T}_X^{\mathscr{P}}$, we may also define Abelian categories of perverse motives out of the representation

$$\mathscr{D}_X^{\mathrm{Nori}} \to \mathscr{P}(X)$$
$$(Y \xrightarrow{a} X, Z, i) \mapsto \mathsf{H}_{\mathscr{P}}^{-i}(a_!^{\mathscr{P}} u_*^{\mathscr{P}} \mathrm{K}_U^{\mathscr{P}})$$

where $u : U \hookrightarrow Y$ is the open immersion of the complement of Z in Y and $\mathrm{K}_U^{\mathscr{P}} := \pi^!_{\mathscr{P}} \mathbb{Q}_k^{\mathscr{P}}$ is the dualizing complex of U (here $\pi : U \to \mathrm{Spec}(k)$ is the structural morphism). For functoriality reasons, it might be simpler to work with the categories of motives defined using dualizing complexes. If X is smooth, it is easy to see that the two representations yield equivalent categories. This should also be true over singular schemes.

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