FINITE DIMENSIONAL MOTIVES AND APPLICATIONS FOLLOWING S.-I. KIMURA, P. O'SULLIVAN AND OTHERS

by

Florian Ivorra

Abstract. — This survey paper is an expanded version of a lecture given in July 2006 at the École d'été Franco-Asiatique de géométrie algébrique et de théorie des nombres (IHÉS-Université Paris 11). It provides an overview of the notion of finite dimensionality introduced by S.-I. Kimura and P. O'Sullivan and explains some of the striking implications of this idea.

Résumé. — Ce texte reprend avec plus de détails le contenu d'un exposé donné en juillet 2006 à l'École d'été Franco-Asiatique de géométrie algébrique et de théorie des nombres (IHÉS-Université Paris 11). Il est consacré à la notion de dimension finie introduite par S.-I. Kimura et P. O'Sullivan. Son but est de présenter les principaux résultats, les grandes lignes de la théorie et d'expliquer certaines des ses applications importantes.

Contents

The author acknowledges support from the European commission through its FP6 program and its Marie Curie Intra-European Fellowship contract No. 041076. He wishes also to thank J. Riou for his careful reading of an earlier version of this survey and his detailed comments that have helped to improve the presentation.

Introduction

A. Grothendieck's idea of motives goes back to insights of A. Weil when he stated his conjectures about Zeta functions of varieties over finite fields after proving the one dimensional case. Weil's insight that it should be possible to extend to positive characteristic the powerful topological methods developed among others by S. Lefschetz and W. Hodge in the complex setting and derive from it his conjectures on Zeta functions led to the notion of Weil cohomologies and the search for such cohomologies in characteristic p. Grothendieck constructed many Weil cohomology theories related by various comparison isomorphisms, but from the beginning this work was shaped by a remark of J.-P. Serre. In the complex setting, Betti cohomology provides a Weil cohomology with coefficient field $\mathbb{Q}$, however as explained by Serre such a Weil cohomology could not exist over the finite field $\mathbb{F}_{p^2}$ [**26**, §1.7]. Indeed take a Weil cohomology theory H^* over $\mathbb{F}_{p^2}$ and denote by K its coefficient field. The H^1 of a curve of genus g being a K-vector space of dimension $2g$, the H^1 of an elliptic curve is a two dimensional K-vector space. If E is a supersingular elliptic curve, as shown by M. Deuring the endomorphism algebra $\mathrm{End}(E)_{\mathbb{Q}}$ is a quaternion division $\mathbb{Q}$-algebra which is non split exactly at p and ∞. For such an elliptic curve, functoriality provides an action of the quaternion algebra $\mathrm{End}(E)_{\mathbb{Q}} \otimes K$ over $\mathsf{H}^1(E)$ and thus a non zero morphism $\mathrm{End}(E)_{\mathbb{Q}} \otimes K \to M_2(K)$. The extension $K/\mathbb{Q}$ must therefore split the quaternion algebra $\mathrm{End}(E)_{\mathbb{Q}}$ and this prevents K to be the field $\mathbb{Q}$ of rational numbers or the field $\mathbb{Q}_p$ of p-adic numbers. This does not prevent the Weil cohomology H^* to have $\mathbb{Q}_\ell$ for a prime $\ell \neq p$ or the fraction field of the Witt ring as coefficient field and such cohomology theories were indeed developed by Grothendieck. These are the ℓ-adic cohomology theory and the crystalline cohomology theory.

To explain the comparison isomorphisms between the different available Weil cohomologies and to get a universal theory with $\mathbb{Q}$-coefficients, the idea of Grothendieck was to use instead categories built from smooth projective varieties and algebraic cycles modulo an adequate equivalence relation: the various categories of pure motives. One of the important feature of Weil cohomologies is that they are finite dimensional, more precisely they take values in categories of $\mathbb{Z}$-graded finite dimensional vector spaces. One may expect to be able to express this finiteness within the category of motives. This raises the following question: what does it mean for a motive to be finite dimensional? In [**52, 67**] S.-I. Kimura and P. O'Sullivan provide a definition of finite dimensionality for motives and conjecture that Chow motives are finite dimensional. Recall that a K-vector space V is finite dimensional if and only if some of its exterior powers vanishes and that the dimension d of V is then the smallest integer such that $\Lambda^{d+1}V = 0$. The definition of Kimura and O'Sullivan is based on this principle and works in any pseudo-abelian tensor K-linear category. It uses both symmetric powers and exterior powers in order to take into account, when applied to the category of $\mathbb{Z}$-graded finite dimensional vector spaces, the interplay between the grading and the tensor structure.

One of the striking aspect of Kimura-O'Sullivan's finite dimensionality conjecture is that it fits very well into the motivic picture and that many important results can be deduced from it. Motives of abelian type are known to be finite dimensional but in general the conjecture is largely open even for complex projective surfaces. For example a proof of the finiteness of the motive of a smooth projective surface with $p_g = 0$ would yield a proof of the conjecture due to S. Bloch that the Abel-Jacobi map of such a surface is an isomorphism.

We now go briefly through the content of this survey. The first part is devoted to the general definition and properties of finite dimensionality in pseudo-abelian rigid tensor K-linear category. The main results are the nilpotence theorem 4.29 and the structure theorem 4.31. We then state the finiteness conjecture and explain why motives of abelian type are finite dimensional. In the second part we give an idea of some important consequences namely the rationality of motivic Zeta functions and the conjecture of Bloch. We also explain some of the recent applications of finite dimensionality to motives over finite fields.

PART I

FINITENESS AND MOTIVES

In this part we consider finite dimensionality in a broad setting. We let A be a rigid tensor category such that:

- $\mathrm{End}_{\mathsf{A}}(\mathbf{1}) = K$ is a field of characteristic zero;
- A is pseudo-abelian.

The example to keep in mind is the category of pure motives modulo an adequate equivalence relation. For those categories we follow Grothendieck's convention and so the motive associated to a smooth projective variety is contravariant. The Weil cohomologies considered in this survey are the classical ones [**3**, §3.4]. If not otherwise stated all the functors considered are assumed to be K-linear and ideals to be two-sided.

1. Rigid tensor categories

1.1. Traces, Euler characteristics and ideals. —

1.1.1. Traces and Euler characteristics. — Since A is rigid every object M has a dual $M^\vee$ and we have structural maps

$$\mathbf{1} \xrightarrow{\mathrm{coev}_M} M^\vee \otimes M \qquad M \otimes M^\vee \xrightarrow{\mathrm{ev}_M} \mathbf{1}.$$

The trace of an endomorphism $f \in \mathrm{End}_{\mathsf{A}}(M)$ is the element in $K = \mathrm{End}_{\mathsf{A}}(\mathbf{1})$ given by the composition

$$\mathrm{Tr}(f) = \mathbf{1} \xrightarrow{\mathrm{coev}_M} M^\vee \otimes M \xrightarrow{1_{M^\vee}\otimes f} M^\vee \otimes M \xrightarrow{\mathsf{s}_{M^\vee,M}} M \otimes M^\vee \xrightarrow{\mathrm{ev}_M} \mathbf{1}$$

where $\mathsf{s}_{M^\vee,M}$ is the isomorphism given by the commutativity constraint which is part of the tensor structure of A. This definition involves the commutativity constraint of A and in general the trace is very sensitive to it. The Euler characteristic of an object M in A is then the scalar

$$\chi(M) = \mathrm{Tr}(1_M).$$

A $\otimes$-functor preserves duals, traces and Euler characteristics. For all objects M, N we have an isomorphism

$$\iota_{M,N} : \mathsf{A}(\mathbf{1}, M^\vee \otimes N) \to \mathsf{A}(M, N)$$

which sends a morphism $u : \mathbf{1} \to M^\vee \otimes N$ to the morphism

$$M \xrightarrow{1_M \otimes u} M \otimes M^\vee \otimes N \xrightarrow{\mathrm{ev}_M \otimes 1_N} N.$$

Its inverse sends a morphism $v : M \to N$ to the morphism

$$\mathbf{1} \xrightarrow{\mathrm{coev}_M} M^\vee \otimes M \xrightarrow{1_{M^\vee}\otimes v} M^\vee \otimes N.$$

Let $L \xrightarrow{f} M \xrightarrow{g} N$ be morphisms in A. We have then a formula relating tensor product and composition

$$g \circ f = \iota_{L,N}\Big[\big(1_{L^\vee} \otimes \mathrm{ev}_{M^\vee} \otimes 1_N\big) \circ \iota^{-1}_{M\otimes L, M\otimes N}\big((1,2) \circ (g \otimes f)\big)\Big] \tag{1}$$

where $(1,2) : N \otimes M \to M \otimes N$ is the isomorphism given by the commutativity constraint.

1.1.2. Symmetric and exterior powers. — Let M be an object of A and n be an nonnegative integer. The symmetric group $\mathfrak{S}_n$ acts on $M^{\otimes n}$ by permutation of the factors and since A is a $\mathbb{Q}$-linear pseudo-abelian category we can define the n-th symmetric power $\mathsf{S}^{\otimes n}M$ of M as the image of the projector

$$\frac{1}{n!}\sum_{\sigma\in\mathfrak{S}_n}\sigma : M^{\otimes n}\to M^{\otimes n}$$

and the n-th exterior power $\Lambda^n M$ as the image of the projector

$$\frac{1}{n!}\sum_{\sigma\in\mathfrak{S}_n}\varepsilon(\sigma)\sigma : M^{\otimes n}\to M^{\otimes n}.$$

Remark 1.1. — Let M be an object of A and $F:\mathsf{A}\to\mathsf{B}$ a $\otimes$-functor where B enjoys the same properties as A. We have then an isomorphism $F(\mathsf{S}^n M)\simeq\mathsf{S}^n F(M)$ and also an isomorphism $F(\Lambda^n M)\simeq\Lambda^n F(M)$.

Remark 1.2. — To define symmetric powers and exterior powers we do not need the category A to be rigid. So let us relax this condition and consider the category Vec_K^∞ of K-vector spaces and the category GrVec_K^∞ of $\mathbb{Z}$-graded vector spaces with the commutativity constraint defined according to Koszul's rule for signs. Then the functor « forget the grading »

$$\mathfrak{f}:\mathrm{GrVec}_K^\infty\to\mathrm{Vec}_K^\infty$$

is not a $\otimes$-functor and we see that if V is purely of odd degree then

$$\mathfrak{f}(\mathsf{S}^n V)\simeq\Lambda^n\mathfrak{f}(V)\qquad \mathfrak{f}(\Lambda^n V)\simeq\mathsf{S}^n\mathfrak{f}(V)$$

whereas for V purely of even degree

$$\mathfrak{f}(\mathsf{S}^n V)\simeq\mathsf{S}^n\mathfrak{f}(V)\qquad \mathfrak{f}(\Lambda^n V)\simeq\Lambda^n\mathfrak{f}(V).$$

1.1.3. Ideals. — Recall that an ideal in A is the data for each pair of objects (M,N) of a sub-K-vector space $\mathscr{I}(M,N)\subset\mathsf{A}(M,N)$ so that for any $f\in\mathsf{A}(M',M)$ and $g\in\mathsf{A}(N,N')$ we have

$$g\circ\mathscr{I}(M,N)\circ f\subset\mathscr{I}(M',N').$$

For each object M of A, the definition implies that $\mathscr{I}(M,M)$ is an ideal of the K-algebra $\mathsf{A}(M,M)$.

Lemma 1.3. — *Let $\mathscr{I}$ and $\mathscr{J}$ be two ideals of A. If for all object $M\in\mathsf{A}$*

$$\mathscr{I}(M,M)\subset\mathscr{J}(M,M)$$

then $\mathscr{I}\subset\mathscr{J}$.

Proof. — Let M,N be objects of A. We have a decomposition

$$\mathsf{A}(M\oplus N,M\oplus N)=\mathsf{A}(M,M)\oplus\mathsf{A}(M,N)\oplus\mathsf{A}(N,M)\oplus\mathsf{A}(N,N).$$

Looking at $\mathsf{A}(M,N)$ as a subspace of $\mathsf{A}(M\oplus N, M\oplus N)$ via this decomposition, we see that since $\mathscr{I}$ is an ideal we have

$$\mathsf{A}(M,N)\cap\mathscr{I}(M\oplus N,M\oplus N)=\mathscr{I}(M,N).$$

This formula implies the lemma. □

If $\mathscr{I}$ is an ideal in A, then it is possible to define the quotient $A/\mathscr{I}$ as the category having the same objects but whose morphisms are given by

$$\mathsf{A}/\mathscr{I}(M,N):=\mathsf{A}(M,N)/\mathscr{I}(M,N)$$

with the composition induced by the one of A. The ideal $\mathscr{I}$ is said to be monoidal if for all $f\in\mathscr{I}(M,N)$ and $g\in\mathsf{A}(P,Q)$ the morphisms $f\otimes g$ and $g\otimes f$ belong respectively to $\mathscr{I}(M\otimes P,N\otimes Q)$ and $\mathscr{I}(P\otimes M,Q\otimes N)$. It is not difficult to see that it is in fact enough that for all $f\in\mathscr{I}(M,N)$ and all object P the morphisms $f\otimes 1_P$ and $1_P\otimes f$ belong respectively to $\mathscr{I}(M\otimes P,N\otimes P)$ and $\mathscr{I}(P\otimes M,P\otimes N)$. When $\mathscr{I}$ is monoidal the quotient category $\mathsf{A}/\mathscr{I}$ inherits a tensor structure and the quotient functor is a $\otimes$-functor. Moreover if $\mathscr{I}\subsetneq\mathsf{A}$ (this condition is equivalent to $\mathscr{I}(\mathbf{1},\mathbf{1})=0$), then $\mathsf{A}/\mathscr{I}$ is also a rigid tensor category such that $\mathrm{End}_{\mathsf{A}/\mathscr{I}}(\mathbf{1})=K$, otherwise it is equivalent to 0. A priori $\mathsf{A}/\mathscr{I}$ may not be pseudo-abelian, see also remark 4.3.

1.2. Pure motives. — For a more detailed account of the definition and properties of pure motives we refer to R. Sujatha's lecture in the first volume, to [**39**] or to A. Scholl's survey article [**75**].

1.2.1. Construction. — Let R be a ring and $\sim$ be an adequate equivalence relation on algebraic cycles. We denote by $A^r_\sim(X;R)$ the group of R-linear algebraic cycles of codimension r on a smooth projective variety X modulo the relation $\sim$. One defines the category of pure motives $\mathsf{M}_\sim(F;R)$ as follows. Let

$$\mathrm{Corr}^r_\sim(X,Y;R)=\bigoplus_{i=1}^{\nu}A^{r+\dim(X_i)}_\sim(X_i\times Y;R)$$

where $X_1,\dots,X_\nu$ are the irreductible components of X, be the R-module of degree r correspondences modulo $\sim$. We have an associative composition law

$$\begin{aligned}\mathrm{Corr}^r_\sim(Y,Z;R)\otimes_R\mathrm{Corr}^s_\sim(X,Y;R)&\to\mathrm{Corr}^{r+s}_\sim(X,Z;R)\\(\beta,\alpha)\mapsto\beta\circ\alpha&=p^{XYZ}_{XZ*}(p^{XYZ*}_{YZ}\beta\frown p^{XYZ*}_{XY}\alpha)\end{aligned}$$

and so we can define $\mathsf{M}_\sim(F;R)$ as the category with triples (X,p,a) where X is a smooth projective variety, $p\in\mathrm{Corr}^0(X,X;R)$ is an idempotent and a is an integer as objects and with morphisms given by

$$\mathsf{M}_\sim(F;R)((X,p,a),(Y,q,b))=q\mathrm{Corr}^{b-a}_\sim(X,Y;R)p.$$

In the case $R = \mathbb{Z}$ we will not mention the ring in the notation. We have a monoidal functor

$$h_\sim : \mathrm{SmProj}_F^{\mathrm{op}} \to \mathsf{M}_\sim(F;R)$$

sending a variety X to $(X, 1_X, 0)$ and a morphism f to the correspondence ${}^t[\Gamma_f]$ where Γ_f denotes the graph. The n-th Tate twist $M(n)$ of a motive $M = (X, p, a)$ is defined as the motive $M(n) = (X, p, a+n)$ and every motive is a direct factor of a motive $h(X)(n)$ for some variety X and some integer n. We let $\mathbf{1}$ be the motive of $\operatorname{Spec} F$ and $\mathbb{L} = \mathbf{1}(-1)$ be the Lefchetz motive. The category $\mathsf{M}_\sim(F;R)$ is a pseudo-abelian rigid tensor category and $\mathrm{End}(\mathbf{1}) = R$.

1.2.2. Some standard conjectures. — Let X be a smooth projective variety of dimension d. Let us recall the standard conjecture $\mathsf{C}(X)$ of algebraicity of Künneth projectors.

Conjecture $\mathsf{C}(X)$ ***1.4***. — The Künneth projectors

$$\Delta_{i,2d-i} \in \mathsf{H}^{2d}(X \times X) : \mathsf{H}^*(X) \twoheadrightarrow H^i(X) \hookrightarrow H^*(X)$$

are algebraic *i.e.* belong to

$$\mathsf{M}_{\mathsf{hom}}(F;\mathbb{Q})(h(X), h(X)) \hookrightarrow \mathrm{End}(\mathsf{H}^*(\mathsf{X})).$$

In particular $\mathsf{C}(X)$ provides a canonical weight decomposition of the homological motive of X

$$h_{\mathsf{hom}}(X) = h^0_{\mathsf{hom}}(X) \oplus h^1_{\mathsf{hom}}(X) \oplus \cdots \oplus h^{2d-1}_{\mathsf{hom}}(X) \oplus h^{2d}_{\mathsf{hom}}(X)$$

where h^i_{hom} is the direct summand of the homological motive of X cut-off by the Künneth projector $\Delta_{i,2d-i}$. The sign conjecture $\mathsf{C}^+(X)$ is the weak version of $\mathsf{C}(X)$ saying that the sum of the even Künneth projectors

$$\sum_{i=0}^{d} \Delta_{2i,2d-2i} : \mathsf{H}^*(X) \twoheadrightarrow \bigoplus_{i=0}^{2d} H^i(X) \hookrightarrow H^*(X)$$

is algebraic (equivalently the sum of the odd Künneth projectors is algebraic). In particular $\mathsf{C}^+(X)$ provides a canonical decomposition

$$h_{\mathsf{hom}}(X) = h^+_{\mathsf{hom}}(X) \oplus h^-_{\mathsf{hom}}(X)$$

where h^+_{hom} (resp. h^-_{hom}) is the direct summand of the homological motive of X cut-off by the sum of the even (resp. odd) Künneth projectors. Conjecture C^+ says that we can impose a $\mathbb{Z}/2$-grading on $\mathsf{M}_{\mathsf{hom}}(F;\mathbb{Q})$ so that H^* defines a functor

$$\mathsf{H}^{\pm} : \mathsf{M}_{\mathsf{hom}}(F;\mathbb{Q}) \to \mathrm{sVec}_E$$

compatible to the $\mathbb{Z}/2$-grading. Among the standard conjecture is also the conjecture that homological and numerical equivalence agree up to torsion:

Conjecture $\mathsf{D}(X)$ ***1.5***. — For any nonnegative integer r the morphism

$$A^r_{\mathrm{hom}}(X;\mathbb{Q}) \to A^r_{\mathrm{num}}(X;\mathbb{Q})$$

is an isomorphism.

1.2.3. Jannsen's dictionary. — Although in the sequel we consider the case of an arbitrary A, it is useful to keep in mind the case where A is the category $\mathsf{M}_{\mathsf{rat}}(F;\mathbb{Q})$ of Chow motives with rational coefficients and to translate each result in that setting. For this recall that, according to [**39**], we have a 1:1 correspondence between adequate equivalence relations and proper monoidal ideals in $\mathsf{M}_{\mathsf{rat}}(F;\mathbb{Q})$. The proper monoidal ideal $\mathscr{I}_\sim$ of $\mathsf{M}_{\mathsf{rat}}(F;\mathbb{Q})$ associated to an adequate equivalence relation $\sim$ is the kernel of the functor

$$\mathsf{M}_{\mathsf{rat}}(F;\mathbb{Q}) \to \mathsf{M}_\sim(F;\mathbb{Q})$$

and we recover the category $\mathsf{M}_\sim(F;\mathbb{Q})$ from $\mathsf{M}_{\mathsf{rat}}(F;\mathbb{Q})$ by the relation

$$\mathsf{M}_\sim(F;\mathbb{Q}) = (\mathsf{M}_{\mathsf{rat}}(F;\mathbb{Q})/\mathscr{I}_\sim)^\natural$$

where $\natural$ stands for the pseudo-abelian hull. Conversely the adequate equivalence relation $\sim_{\mathscr{I}}$ associated to a proper monoidal ideal $\mathscr{I}$ is the one defined by the subgroup

$$\mathscr{I}(\mathbf{1}, h(X)(n)) \subset \mathsf{M}_{\mathsf{rat}}(F;\mathbb{Q})(\mathbf{1}, h(X)(n)) = \mathrm{CH}^n(X;\mathbb{Q}).$$

2. Tannakian categories and motives

One of the original aim of the theory of motives was to provide the foundation for a broad generalization of usual Galois theory, namely a Galois theory for families of equations. Let $\overline{F}$ be a separable closure of F and $\mathrm{FSets}^{\mathrm{cont}}(G_F)$ be the category of continuous finite G_F-sets where $G_F := \mathrm{Gal}(\overline{F}/F)$ stands for the absolute Galois group of F. If X is a finite étale F-scheme, *i.e.* the spectrum of a finite product of finite separable extensions of F, then the profinite group G_F acts continuously on the set $X(\overline{F})$. By Grothendieck's formulation of Galois theory in terms of étale coverings we have an equivalence

$$\begin{array}{ccc} \mathrm{FEt}_F & \to & \mathrm{FSets}^{\mathrm{cont}}(G_F) \\ X & \mapsto & X(\overline{F}) \end{array} \tag{2}$$

between the category FEt_F of finite étale F-schemes and the category $\mathrm{FSets}^{\mathrm{cont}}(G_F)$. Moreover if we denote by ω the functor with target the category of finite sets obtained by forgetting the action of the Galois group then one recovers G_F as the group of automorphisms of the functor ω

$$\mathrm{Gal}(\overline{F}/F) = \mathrm{Aut}(\omega).$$

Motivic Galois theory can be described as the search for higher dimensional linearized analogs of this picture. The theory of Tannakian categories provides the underlying formalism needed to deal with fibre functors and their automorphism group.

2.1. Review of Tannakian categories. — We refer to [**10**] for a detailed exposition of the main aspects of the theory of Tannakian categories both in the neutral and non neutral cases. Recall that for A abelian, an E-valued fibre functor is an exact faithful $\otimes$-functor

$$\omega : \mathsf{A} \to \mathrm{Vec}_E$$

where E/K is an extension and Vec_E denotes the category formed by E-vector spaces of finite dimension.

Definition 2.1. — A is said to be Tannakian if it is abelian and there exists a fibre functor

$$\omega : \mathsf{A} \to \mathrm{Vec}_E$$

where E/K is an extension.

A Tannakian category A is

– neutralized if it is equipped with a K-valued fibre functor;
– neutral if there exists a K-valued fibre functor.

To a neutralized Tannakian category (A, ω) one associates an affine K-group scheme $G_{\mathsf{A},\omega}$ whose K-points are given by

$$G_{\mathsf{A},\omega}(K) = \mathrm{Aut}^{\otimes}(\omega)$$

and the original category A can be recovered entirely from this affine K-group scheme as the category of algebraic representations over K:

Theorem 2.2 **(N. Saavedra [74])**. — *Let (A, ω) be a neutralized Tannakian category. The fibre functor ω enriches into an equivalence of tensor categories*

$$\begin{tikzcd} \mathsf{A} \arrow[r, "\overline{\omega}"] \arrow[dr, "\omega"'] & \mathrm{Rep}_K(G_{\mathsf{A},\omega}) \arrow[d, "\text{forgetful}"] \\ & \mathrm{Vec}_K. \end{tikzcd}$$

2.2. Deligne's internal characterization. — In [**14**] P. Deligne has given a criterion for a category to be Tannakian that does not involve a fibre functor. This internal characterization of Tannakian categories is the main tool for proving abstractly that a category is Tannakian *i.e.* without exhibiting a specific fibre functor.

Theorem 2.3 **(Deligne [14])**. — *Assume A is abelian. The following conditions are equivalent:*

– *the category A is Tannakian;*
– *for each M in A there exists a positive integer n such that*

$$\Lambda^n M = 0;$$

– *for each M in A the Euler-Poincaré characteristic*

$$\chi(M) = \mathrm{Tr}(1_M)$$

is a nonnegative integer i.e. belongs to $\mathbb{N} \subset K$.

2.3. Are motives Tannakian ?— Ideally Grothendieck's category of pure motives $\mathsf{M}_{\mathsf{num}}(F;\mathbb{Q})$ should be Tannakian and every Weil cohomology theory H^*

$$\mathsf{H}^* : \mathsf{M}_{\mathsf{hom}}(F;\mathbb{Q}) \to \mathrm{GrVec}_E$$

with coefficients in $E/\mathbb{Q}$ should provide an E-fibre functor(1) on the category of pure motives. This raises two problems.

– For a Weil cohomology theory H^* to define a fibre functor one has to prove first that the cohomology theory is really defined over $\mathsf{M}_{\mathsf{num}}(F;\mathbb{Q})$

$$\begin{tikzcd} \mathsf{M}_{\mathsf{hom}}(F;\mathbb{Q}) \arrow[r, "\mathsf{H}^*"] \arrow[d] & \mathrm{GrVec}_E \\ \mathsf{M}_{\mathsf{num}}(F;\mathbb{Q}) \arrow[ur, dotted, "\text{need standard conjecture } \mathsf{D}"'] & \end{tikzcd}$$

and this the content of standard conjecture D.

– Even under standard conjecture D, there is little hope for $\mathsf{M}_{\mathsf{num}}(F;\mathbb{Q})$ to be Tannakian in view of Deligne's criterion. Indeed for a smooth projective variety X over $\mathbb{C}$, the Euler characteristic computed in $\mathsf{M}_{\mathsf{num}}(\mathbb{C})_{\mathbb{Q}}$ is equal to the usual Euler characteristic for Betti cohomology which may be a negative integer since for a smooth projective curve of genus g one has $\chi(X) = 2 - 2g$.

This last issue has two solutions. The more familiar one is to change the commutativity constraint in GrVec_E and so in $\mathsf{M}_{\mathsf{hom}}(F;\mathbb{Q})$ in order to view H^* as an exact $\otimes$-functor

$$\begin{tikzcd} \mathsf{M}_{\mathsf{hom}}\dot{(}F;\mathbb{Q}) \arrow[r, "\mathsf{H}^*"] \arrow[d] & \mathrm{Gr\dot{V}ec}_E \arrow[r, "\text{fibre functor}"] & \mathrm{Vec}_E \\ \mathsf{M}_{\mathsf{num}}\dot{(}F;\mathbb{Q}) \arrow[ur, dotted, "\text{need standard conjecture } D"'] & & \end{tikzcd}$$

Here if G is a $\mathbb{Z}/2$-graded tensor category, $\dot{\mathsf{G}}$ denotes the tensor category gotten from G by change of the commutativity constraint according to the Koszul rule. However this approach requires standard conjecture C or at least the sign conjecture C^+ in order to have a well defined $\mathbb{Z}/2$-grading by weight on the category of pure motives. There is another approach due to Deligne [**15**] which consists to broaden the Tannakian formalism to allow negative Euler characteristic. For this he introduces the notion

(1)One cannot expect the category of motives to be neutral in general. Indeed as shown by Serre in positive characteristic there is no Weil cohomology theory with coefficients in $\mathbb{Q}$, see the introduction for a more detailed explanation.

of super fibre functor and super Tannakian category, see also section 5 on Schur finiteness.

$$\boxed{\begin{array}{c}\text{Tannakian categories}\\ \\ \text{fibre functor}\end{array}} \overset{\text{Deligne } [\mathbf{15}]}{\rightsquigarrow} \boxed{\begin{array}{c}\text{super Tannakian categories}\\ \\ \text{super fibre functor}\end{array}}$$

Let us consider a more simple example. By Deligne's internal criterion of theorem 2.3 the categories GrVec_E of $\mathbb{Z}$-graded finite dimensional E-vector spaces and sVec_E of finite dimensional super E-vector spaces with Koszul's rule are not Tannakian. The forgetful functors

$$\mathrm{GrVec}_E \to \mathrm{Vec}_E \leftarrow \mathrm{sVec}_E$$

are exact and faithful but not compatible with the tensor product since we have put a sign in the commutativity constraint. The target of a Weil cohomology is the category of graded vector spaces with commutativity constraint given by Koszul's rule not the category of vector spaces and the reason why the category of motives $\mathsf{M}_{\mathsf{num}}(F;\mathbb{Q})$ is not Tannakian is directly linked to this sign problem in the definition of the commutativity constraint.

3. Finite dimensional objects

3.1. Three fundamental ideals. —

3.1.1. Nilideals and nilpotent ideals. — As we shall see, nilpotence theorems are one of the main consequences of finite dimensionality and one of the most useful in practice. It is appropriate here to recall some definitions. Let A be a ring,

– a nilideal of A is an ideal which contains only nilpotent elements(2);
– a nilpotent ideal of A is an ideal I such that $I^n = 0$ for some positive integer.

The second property is much stronger than the first. In-between we may introduce uniformly bounded nilideals. By definition these are the nilideals I for which there exists a positive integer n such that $a^n = 0$ for all $a \in I$. Of course nilpotent ideals are uniformly bounded nilideals, the converse is true in some cases. The following result was conjectured by M. Nagata [**66**] and then proved by G. Higman [**35**]:

Proposition 3.1. — *Let A be a $\mathbb{Q}$-algebra and I be a uniformly bounded nilideal in A. Then I is nilpotent.*

Remark 3.2. — It is possible to weaken the assumption in proposition 3.1. Indeed if each element in I has a nilpotence exponent bounded by an nonnegative integer n, the proof given in [**5**, 7.2.8] only requires $n!$ to be invertible in A and the conclusion is then that $I^{2^n-1} = 0$.

(2) An ideal generated by a nilpotent element may not be a nilideal.

We shall say that an ideal $\mathscr{I}$ of A is an nilideal when all the ideals $\mathscr{I}(M,M)$ are nilideals. Concerning the definition of a nilpotent ideal in A one has to be careful. Indeed if $\mathscr{I}$ and $\mathscr{J}$ are two ideals in A one may define the ideal $\mathscr{I}\cdot\mathscr{J}$ as follows: $(\mathscr{I}\cdot\mathscr{J})(M,N)$ is the K-vector space generated by all the morphisms $f\circ g$ where P is an object in A, $f\in\mathscr{I}(P,N)$ and $g\in\mathscr{J}(M,P)$. In general the ideal $\mathscr{I}(M,M)\cdot\mathscr{J}(M,M)$ may be strictly contained in $(\mathscr{I}\cdot\mathscr{J})(M,M)$. It may be much more stronger to require that $\mathscr{I}^n=0$ for some positive integer than to require that for each object M the ideal $\mathscr{I}(M,M)$ is nilpotent. The reasonable condition, and the one that appears throughout this survey, is the last one.

3.1.2. The radical. — In [**51**], see also [**80**, Proposition 4], it is proved that

$$\mathscr{R}_{\mathsf{A}}(M,N)=\{f\in\mathsf{A}(M,N):\ \forall g\in\mathsf{A}(N,M)\ \ 1_M-g\circ f \text{ is invertible}\}$$

defines an ideal $\mathscr{R}_{\mathsf{A}}$ in A. This is the categorical generalization, categories being viewed as rings with several objects, of the usual Jacobson radical of rings [**73**, §2.5]. This ideal may also be characterized as follows.

- The radical $\mathscr{R}_{\mathsf{A}}$ is the largest ideal $\mathscr{I}$ such that the quotient functor $\mathsf{A}\to\mathsf{A}/\mathscr{I}$ is conservative. It follows that this functor reflects also split epimorphisms and split monomorphisms.
- The radical $\mathscr{R}_{\mathsf{A}}$ is the largest ideal $\mathscr{I}$ such that

  $$\mathscr{I}(M,M)\subset\mathrm{rad}(\mathsf{A}(M,M)).$$

  Furthermore

  $$\mathscr{R}_{\mathsf{A}}(M,M)=\mathrm{rad}(\mathsf{A}(M,M)).$$

  According to lemma 1.3 those equalities characterize the radical $\mathscr{R}_{\mathsf{A}}$.

3.1.3. The numerical ideal. — Let $\mathscr{I}$ be an ideal in A. Let us first remark that $\mathscr{I}$ is monoidal if and only if we have

$$\mathscr{I}(M,N)=\iota_{M,N}(\mathscr{I}(\mathbf{1},M^\vee\otimes N)).$$

Then to any ideal $\mathscr{I}$ one can associate the monoidal ideal $\mathscr{I}^\otimes$ defined by

$$\mathscr{I}^\otimes(M,N)=\iota_{M,N}(\mathscr{I}(\mathbf{1},M^\vee\otimes N)).$$

Remark 3.3. — This ideal is equal to $\mathscr{I}$ if and only if $\mathscr{I}$ is monoidal but one has to be careful that in general it does not contain $\mathscr{I}$. For example in the category of $\mathbb{Z}$-graded finite dimensional vector spaces the ideal $\mathscr{I}$ formed by the morphisms whose even components are zero is a non zero ideal such that $\mathscr{I}^\otimes=0$.

Consider now the monoidal ideal $\mathscr{N}_{\mathsf{A}}$ defined using traces as follows:

$$\mathscr{N}_{\mathsf{A}}(M,N)=\{f\in\mathsf{A}(M,N):\ \forall g\in\mathsf{A}(N,M)\ \ \mathrm{Tr}(g\circ f)=0\}.$$

In the case of Chow motives we see that this ideal is the one defined by numerical equivalence and in general it enjoys the following properties [**5**, 7.1.4]:

Lemma 3.4. — *The numerical ideal $\mathscr{N}_{\mathsf{A}}$ is the largest proper monoidal ideal of* A *and one has the relation*

$$\mathscr{N}_{\mathsf{A}} = \mathscr{R}_{\mathsf{A}}^{\otimes}. \tag{3}$$

By the caveat given in remark 3.3 the relation (3) does not imply in general that the radical $\mathscr{R}_{\mathsf{A}}$ itself is contained in $\mathscr{N}_{\mathsf{A}}$. In case $\mathscr{R}_{\mathsf{A}}$ is a monoidal ideal then we have $\mathscr{R}_{\mathsf{A}} = \mathscr{N}_{\mathsf{A}}$ and the quotient functor

$$\mathsf{A} \to \mathsf{A}/\mathscr{N}_{\mathsf{A}}$$

is conservative. In all this survey $\overline{\mathsf{A}}$ will stand for the quotient category $\mathsf{A}/\mathscr{N}_{\mathsf{A}}$.

3.1.4. The $\otimes$-nilradical. — Let $\mathscr{I}$ be an ideal of A. The $\otimes$-radical of $\mathscr{I}$ is the monoidal ideal defined by

$$\sqrt[\otimes]{\mathscr{I}}(M,N) = \{f \in \mathsf{A}(M,N) : \exists\, n \in \mathbb{N}\ \ f^{\otimes n} \in \mathscr{I}(M^{\otimes n}, N^{\otimes n})\},$$

in particular this gives the $\otimes$-nilradical $\sqrt[\otimes]{0}$ of A. The $\otimes$-nilradical [**5**, 7.4.2] satisfies the following lemma:

Lemma 3.5. — *Let M be an object of* A*. Then the ideal $\sqrt[\otimes]{0}(M,M)$ is a nilideal and furthermore the two-sided ideal generated by any element of $\sqrt[\otimes]{0}(M,M)$ is nilpotent. In particular the nilradical $\sqrt[\otimes]{0}$ is contained in the radical $\mathscr{R}_{\mathsf{A}}$.*

Proof. — Fix a morphism $f \in \sqrt[\otimes]{0}(M,M)$ and choose n such that $f^{\otimes n} = 0$. Then by the commutativity constraint we know that for any morphism $g_1, g_2, \dots, g_n \in \mathsf{A}(M,M)$

$$g_n \otimes f \otimes g_{n-1} \otimes \cdots \otimes f \otimes g_1 = 0$$

and using induction in formula (1) we get that

$$g_n \circ f \circ g_{n-1} \circ \cdots \circ f \circ g_1 = 0$$

and the result follows. □

Remark 3.6. — The isomorphism

$$\iota_{M,N} : \mathsf{A}(\mathbf{1}, M^{\vee} \otimes N) \to \mathsf{A}(M,N)$$

preserves the $\otimes$-nilpotence. More precisely a morphism f in $\mathsf{A}(\mathbf{1}, M^{\vee} \otimes N)$ is $\otimes$-nilpotent if and only if $\iota_{M,N}(f)$ is $\otimes$-nilpotent.

3.1.5. Wedderburn categories. — If E/K is an extension, in the sequel we shall denote by A_E the category gotten from A by a naive extension of scalars

$$\mathsf{A}_E(M,N) = \mathsf{A}(M,N) \otimes_K E.$$

Definition 3.7. — A is a Wedderburn category when the following two conditions are satisfied:

– for all object M the ideal $\mathscr{R}_{\mathsf{A}}(M,M)$ is nilpotent;

– for all extension E/K the category $[(\mathsf{A}/\mathscr{R}_{\mathsf{A}})_E]^\natural$ is semi-simple(3).

Following [**5**] one may roughly describe a Wedderburn category as an extension of an absolute semi-simple category by a nilpotent radical. We have a very useful criterion for our category A to be a Wedderburn category.

Proposition 3.8. — *Assume that, for all objects M, N of A, the K-vector space*

$$\mathsf{A}(M, N)$$

is finite dimensional. Then A is a Wedderburn category.

The assumption that $\mathrm{End}_{\mathsf{A}}(\mathbf{1}) = K$ is a field of characteristic zero (in fact perfect is enough) is essential in the previous result. We refer to [**5**, 2.4.4.c] for a proof of this criterion (the converse statement is obviously false).

Remark 3.9. — Recall that a Tannakian category has finite dimensional Hom spaces as required in proposition 3.8 and so is a Wedderburn category.

3.1.6. The associated equivalence relations. — Both the numerical ideal and the $\otimes$-nilradical fit in the 1:1 correspondence between adequate equivalence relations and proper monoidal ideals in $\mathsf{M}_{\mathsf{rat}}(F;\mathbb{Q})$. More precisely we have the picture

Voevodsky's smash nilpotence relation $\otimes$-nil		$\otimes$-nilradical $\sqrt[\otimes]{0}$ of $\mathsf{M}_{\mathsf{rat}}(F;\mathbb{Q})$
	$\longleftrightarrow$	
numerical equivalence		maximal proper monoidal ideal $\mathscr{N}$ of $\mathsf{M}_{\mathsf{rat}}(F;\mathbb{Q})$.

However since the radical $\mathscr{R}$ of $\mathsf{M}_{\mathsf{rat}}(F;\mathbb{Q})$ is not a priori a monoidal ideal, it does not fit in this correspondence(4).

3.2. Kimura-O'Sullivan's definition. — Let us first go back to remark 1.2. Since we know that a vector space $V \in \mathrm{Vec}_K^\infty$ is finite dimensional if and only if it has an exterior power which vanishes, we see from the isomorphisms given in remark 1.2 that an object V in GrVec_K^∞ purely of even (resp. odd) degree is of finite dimension if and only if $\mathsf{\Lambda}^n V = 0$ (resp. $\mathsf{S}^n M = 0$) for some positive integer n. This example motivates the next definition.

Definition 3.10. — An object M in A is said to be

(3)Since we only require the pseudo-abelian hull of $(\mathsf{A}/\mathscr{R}_{\mathsf{A}})_E$ to be semi-simple, this formulation differs slightly from the original definition [**5**, Definition 2.4.1] but is equivalent to it as remarked in the proof of [**5**, Theorem A.2.10]. For more on semi-simple categories we refer to [**5**, §2.1,§A.2].

(4)As we shall see later with the structure theorem 4.31, the conjecture of Kimura-O'Sullivan implies that the radical $\mathscr{R}$ is a monoidal nilideal that satisfies $\mathscr{R} = \mathscr{N}$.

– evenly of finite dimension if

$$\mathsf{\Lambda}^n M = 0 \qquad \text{for a} \quad n > 0,$$

– oddly of finite dimension if

$$\mathsf{S}^n M = 0 \qquad \text{for a} \quad n > 0.$$

Following [**4**], for short we speak of even and odd objects. Assume M is odd and choose an integer n such that $\mathsf{S}^n M = 0$. Then, since the split epimorphism $M^{\otimes n+1} \to \mathsf{S}^{n+1} M$ factorizes through $\mathsf{S}^n M \otimes M$, we see that we also have $\mathsf{S}^{n+1} M = 0$. By induction we get that all the higher symmetric powers vanish. The same holds for the exterior powers.

Definition 3.11. — Assume M is even (resp. odd). The dimension of M denoted by $\dim(M)$ is the smallest nonnegative integer n such that

$$\mathsf{\Lambda}^{n+1}(M) = 0 \qquad (\text{resp. } \mathsf{S}^{n+1}(M) = 0).$$

Remark 3.12. — This definition of dimension is imprecise since for M both even and odd we may consider the dimension as and even object or the dimension as an odd object. In fact this does not really matter since we shall derive from the nilpotence theorem that 0 is the only object both even and odd (corollary 4.19) and its dimension is 0 either being taken as even or odd.

Definition 3.13. — An object M in A is said to be finite dimensional if it has a decomposition into a direct sum

$$M \simeq M_+ \oplus M_-$$

where M_+ is even and M_- is odd. A is said to be a Kimura-O'Sullivan category if any object of A is finite dimensional.

Remark 1.1 says that the image by a $\otimes$-functor of an even, odd or finite dimensional objects remains of the same type. The categories of finite dimensional K-vector spaces and of $\mathbb{Z}$-graded finite dimensional K-vector spaces are basic examples of Kimura-O'Sullivan categories. In the first case all objects are even, in the second case the definition of finite dimensionality given in definition 3.13 coincides with the usual one. Indeed from the isomorphisms given in remark 1.2 we see that an object V in $\mathrm{GrVec}_K^{\infty}$ is precisely of finite dimension in the usual sense if and only if it can be written as a direct sum

$$V \simeq V_+ \oplus V_-$$

where V_+ is even and V_- is odd. The category $\mathrm{Rep}_K(G)$ of algebraic representations of a affine group scheme over K is also a Kimura-O'Sullivan category in which all objects are even. More generally Tannakian categories are examples of Kimura-O'Sullivan categories.

Remark 3.14. — If M is finite dimensional, a decomposition into a direct sum

$$M \simeq M_+ \oplus M_- \qquad \text{with } M_+ \text{ even and } M_- \text{ odd}$$

is not canonical in general. We may have non zero morphisms between M_+ and M_-. Therefore decompositions into even part and odd part in a Kimura-O'Sullivan category A do not induce in general a $\mathbb{Z}/2$-grading on A. However when the $\otimes$-nilpotence result given in proposition 4.22 will be at our disposal, we shall see with proposition 4.23 that we still have some weaker uniqueness.

3.3. Basic properties. — Using the properties of exterior and symmetric powers one deduces the basic properties of even, odd and finite dimension objects.

Proposition 3.15. — *In A the following holds:*

- *even and odd objects are stable by $\oplus$, direct factor and duality $^\vee$;*
- *the $\otimes$-product of objects of same (resp. different) parity is even (resp. odd);*
- *finite dimensional objects are stable by $\oplus$, $\otimes$, and $^\vee$.*

As a consequence of the $\otimes$-nilpotence result of proposition 4.22 we shall see in proposition 4.27 that direct factors of finite dimensional object are also finite dimensional. Therefore the full subcategory of finite dimensional objects is a full thick tensor subcategory of A. The following corollary is also useful:

Corollary 3.16. — *Let M be an object of A.*

- *Assume M even. Then the $\Lambda^n M$ and the $\mathsf{S}^n M$ are even.*
- *Assume M odd. Then the $\mathsf{S}^{2n}M, \Lambda^{2n}M$ are even and the $\mathsf{S}^{2n+1}M, \Lambda^{2n+1}M$ are odd.*

4. Nilpotence theorems

4.1. Quotients by nilideals. — One of the main useful features of nilideals is the fact that they are contained in the radical $\mathscr{R}_{\mathsf{A}}$. This implies the following lemma:

Lemma 4.1. — *Let $\mathscr{I}$ be an nilideal of A. Then the quotient functor*

$$\mathsf{A} \to \mathsf{A}/\mathscr{I}$$

is conservative and in particular A does not contain non zero $\mathscr{I}$-phantom objects.

Furthermore the lifting of idempotents along nilideals [**73**, Corollary 1.1.28] implies the following proposition:

Proposition 4.2. — *Let $\mathscr{I}$ be an ideal of A and let M be an object in A such that $\mathscr{I}(M, M)$ is a nilideal. Then*

- *any projector of M in $\mathsf{A}/\mathscr{I}$ can be lifted to a projector of M in A;*
- *any orthogonal system of projectors of M in $\mathsf{A}/\mathscr{I}$ can be lifted to an orthogonal system of projectors of M in A.*

Remark 4.3. — If $\mathscr{I}$ is an nilideal then $\mathscr{I}(\mathbf{1},\mathbf{1}) = 0$ and so it is proper. The quotient category $\mathsf{A}/\mathscr{I}$ is thus a rigid tensor category such that $\mathrm{End}_{\mathsf{A}/\mathscr{I}}(\mathbf{1}) = K$. By proposition 4.2 we may lift also projectors from $\mathsf{A}/\mathscr{I}$ to projectors in A, therefore the category $\mathsf{A}/\mathscr{I}$ is also pseudo-abelian.

4.2. Rost's nilpotence theorem. — In [**72**, Proposition 9] M. Rost proves the following nilpotence theorem:

Theorem 4.4. — *Let E/F be a field extension and X/F be a smooth projective quadric. Then the kernel of the map*

$$\mathsf{M}_{\mathsf{rat}}(F)(h(X), h(X)) \to \mathsf{M}_{\mathsf{rat}}(E)(h(X)_E, h(X)_E)$$

is a uniformly bounded nilideal.

This result was originally used by Rost to obtain a decomposition of the motive of a Pfister quadric [**71**, Theorem 3](5) which was later used by V. Voevodsky in his proof of the Milnor conjecture [**83**, §4]. Smooth projective quadrics are particular examples of projective homogeneous varieties and Rost's nilpotence theorem for quadrics can be extended to all projective homogeneous varieties. Namely let $\mathsf{M}_{\mathsf{rat}}(F)_{\mathrm{hmg}}$ denotes the thick subcategory of $\mathsf{M}_{\mathsf{rat}}(F)$ generated by Tate twists of motives of projective homogeneous varieties then V. Chernousov, S. Gille and A. Merkurjev have proved [**12**, Theorem 8.2] the following generalization of Rost's nilpotence theorem:

Theorem 4.5. — *Let E/F be a field extension and $M \in \mathsf{M}_{\mathsf{rat}}(F)_{\mathrm{hmg}}$. The kernel of the extension of scalars*

$$-_E : \mathsf{M}_{\mathsf{rat}}(F)_{\mathrm{hmg}} \to \mathsf{M}_{\mathsf{rat}}(E) \tag{4}$$

is a uniformly bounded nilideal.

Using a transfer argument one sees that the elements in the kernel of (4) are torsion. Therefore in theorem 4.5 it is crucial to consider motives with integral coefficients and the result of Higman [**35**] does not apply in this setting even though the nilpotence exponent are uniformly bounded. One would like also to say that the scalar extension functor (4) is conservative. However since this is not a full functor, the result does not follow from lemma 4.1. Nevertheless Chernousov-Gille-Merkurjev prove [**12**, Corollary 8.4] the following weaker statement (for smooth projective quadrics this was already proved by Rost [**72**, Corollary 10]):

Proposition 4.6. — *Let E/F be a field extension and f an endomorphism of $M \in \mathsf{M}_{\mathsf{rat}}(F)_{\mathrm{hmg}}$. Then f is an isomorphism if and only if f_E is an isomorphism.*

(5) We refer also the reader to [**72**, Theorem 17] and [**48**, Proposition 5.2].

Remark 4.7. — The proofs of the nilpotence theorem 4.4 and 4.5 rely either on [**72**, Proposition 1] which was proved by Rost using his theory of cycle modules [**70**] or on the generalization of that result [**11**, Theorem 3.1] obtained by P. Brosnan using intersection theory and the refined Gysin morphisms of [**22**, Chapter 6].

4.3. Jannsen's nilpotence theorem. — Now the nilpotence result we are going to see is due to U. Jannsen [**37**, Corollary 1] and is a statement very similar to the one derived from finite dimensionality.

Theorem 4.8. — *Let X/F be a smooth projective variety and assume $\mathsf{C}^+(X)$. Then the kernel of the map*

$$\mathsf{M}_{\mathsf{hom}}(F;\mathbb{Q})(h_{\mathrm{hom}}(X), h_{\mathrm{hom}}(X)) \to \mathsf{M}_{\mathsf{num}}(F;\mathbb{Q})(\overline{h}(X), \overline{h}(X))$$

is the Jacobson radical and is a nilideal.

This result was used by T. Geisser [**24**, Theorem 2.7] to prove that Tate conjecture TC is equivalent over a finite field to the classification up to isomorphism of simple numerical motives in terms of their associated Frobenius.

4.4. The first nilpotence theorem. — One of the main properties of odd or even objects concerns their rings of endomorphisms: any endomorphism which is numerically zero is always nilpotent; moreover there exists a uniform bound for the nilpotence exponent that depends only on the dimension of the object. More precisely the theorem may be stated as follows:

Theorem 4.9. — *Let M be even or odd. Then any endomorphism $f \in \mathscr{N}_{\mathsf{A}}(M, M)$ is nilpotent. More precisely*

$$f^{\circ \dim(M)+1} = 0^{(6)}.$$

The proof of this nilpotence theorem relies on some trace computations. For computations in a slightly more general setting we refer to [**5**, §7.2]. In the sequel we shall use the following notation:

Notation 4.10. — Given a permutation $\sigma \in \mathfrak{S}_n$, we denote by Σ_σ the set of orbits of σ and by $\Sigma_{\sigma,n}$ the subset of orbits that do not contain n. The orbit of n is denoted by $\mathscr{O}_n$ and the cardinal of a finite set $\mathscr{O}$ is denoted by $|\mathscr{O}|$.

Let M be an object of A and n a nonnegative integer. Then each endomorphism f of $M^{\otimes n}$ provides an endomorphism f_n of M by taking the image of f through the

(6) Jannsen has proved [**52**, Proposition 10.1] that we have in fact $f^{\circ \dim(M)} = 0$.

map

$$\begin{tikzcd}
\mathsf{A}(M^{\otimes n}, M^{\otimes n}) \arrow[r, "\iota^{-1}_{M^{\otimes n}, M^{\otimes n}}"] & \mathsf{A}(\mathbf{1}, (M^{\otimes n})^\vee \otimes M^{\otimes n}) \arrow[d, "1_{M^\vee} \otimes \mathrm{ev}_{(M^{\otimes n-1})^\vee} \otimes 1_M"] & \\
& \mathsf{A}(\mathbf{1}, M^\vee \otimes M) \arrow[r, "\iota_{M,M}"'] & \mathsf{A}(M, M)
\end{tikzcd} \tag{5}$$

the vertical morphism being well defined via the natural isomorphism

$(M^{\otimes n})^\vee \otimes M^{\otimes n} = (M^{\otimes n-1} \otimes M)^\vee \otimes M^{\otimes n-1} \otimes M \simeq M^\vee \otimes (M^{\otimes n-1})^\vee \otimes M^{\otimes n-1} \otimes M.$

The essential point in the proof of the nilpotence theorem 4.9 is the computation of the morphism

$$(\sigma \circ f^{\otimes n})_n \qquad \sigma \in \mathfrak{S}_n \quad \text{and} \quad f \in \mathsf{A}(M, M)$$

given in proposition 4.14. We first need some reduction lemmas.

Lemma 4.11. — *Let $f \in \mathsf{A}(M^{\otimes n}, M^{\otimes n})$ be an endomorphism. Then*

$$\mathrm{Tr}(f) = \mathrm{Tr}(f_n).$$

Proof. — It follows from the commutativity of

$$\begin{tikzcd}
\mathbf{1} \arrow[d, "\iota^{-1}_{M^{\otimes n}}(f)"'] \arrow[rr, "\iota^{-1}_M(f_n)"] & & M^\vee \otimes M \\
(M^{\otimes n})^\vee \otimes M^{\otimes n} \arrow[rr, "1_{M^\vee} \otimes \mathrm{ev}_{(M^{\otimes n-1})^\vee} \otimes 1_M"'] \arrow[rrd, bend right, "\mathrm{ev}_{(M^{\otimes n})^\vee}"'] & & M^\vee \otimes M \arrow[d, "\mathrm{ev}_{M^\vee}"] \\
& & \mathbf{1}
\end{tikzcd}$$

and the definition of the trace. □

Lemma 4.12. — *Let p, q be nonnegative integers such that $n = p + q$. Then for $g \in \mathsf{A}(M^{\otimes p}, M^{\otimes p})$ and $h \in \mathsf{A}(M^{\otimes q}, M^{\otimes q})$ we have*

$$(g \otimes h)_n = \mathrm{Tr}(g) h_q.$$

Proof. — By definition of the trace, it follows from the commutativity of the diagram

$$\begin{tikzcd}
\mathbf{1} \arrow[rr, "\iota^{-1}_{M^{\otimes p}}(g) \otimes \iota^{-1}_{M^{\otimes q}}(h)"] \arrow[dr, bend right, "\iota^{-1}_{M^{\otimes n}}(g \otimes h)"'] & & (M^{\otimes p})^\vee \otimes M^{\otimes p} \otimes (M^{\otimes q})^\vee \otimes M^{\otimes q} \arrow[dl, bend left, "(6)"'] \arrow[ddl, bend left, "(\mathrm{ev}_{(M^{\otimes p})^\vee}) \otimes (1_{M^\vee} \otimes \mathrm{ev}_{(M^{\otimes q-1})^\vee} \otimes 1_M)"] \\
& (M^{\otimes n})^\vee \otimes M^{\otimes n} \arrow[d, "1_{M^\vee} \otimes \mathrm{ev}_{(M^{\otimes n-1})^\vee} \otimes 1_M"'] & \\
& M^\vee \otimes M &
\end{tikzcd}$$

where (6) is the arrow given by the commutativity constraint. □

Lemma 4.13. — *Let $f \in \mathsf{A}(M^{\otimes n}, M^{\otimes n})$ and $\beta \in \mathfrak{S}_n$ a permutation that fixes n. Then*

$$(\beta \circ f \circ \beta^{-1})_n = f_n.$$

Proof. — The result follows from the commutativity of the diagram

$$\begin{tikzcd}
\mathsf{A}(M^{\otimes n}, M^{\otimes n}) \arrow[r, "\iota^{-1}_{M^{\otimes n}, M^{\otimes n}}"] \arrow[dd, "\beta\circ - \circ\beta^{-1}"'] & \mathsf{A}(\mathbf{1}, (M^{\otimes n})^\vee \otimes M^{\otimes n}) \arrow[dr, "1_{M^\vee}\otimes \mathrm{ev}_{(M^{\otimes n-1})^\vee}\otimes 1_M"] \arrow[dd, "(\beta^\vee\otimes\beta)\circ -"'] & \\
 & & \mathsf{A}(\mathbf{1}, M^\vee \otimes M) \\
\mathsf{A}(M^{\otimes n}, M^{\otimes n}) \arrow[r, "\iota^{-1}_{M^{\otimes n}, M^{\otimes n}}"'] & \mathsf{A}(\mathbf{1}, (M^{\otimes n})^\vee \otimes M^{\otimes n}). \arrow[ur, "1_{M^\vee}\otimes \mathrm{ev}_{(M^{\otimes n-1})^\vee}\otimes 1_M"'] &
\end{tikzcd}$$

The triangle on the right is commutative indeed since we have chosen a permutation β that left n fixed. □

Let $f \in \mathsf{A}(M, M)$ be an endomorphism of an object M of A. Then the endomorphism

$$\sigma \circ f^{\otimes n} : M^{\otimes n} \to M^{\otimes n}$$

provides an endomorphism via the map (5). The next proposition computes this morphism in terms of the orbits of σ.

Proposition 4.14. — *Let $\sigma \in \mathfrak{S}_n$ be a permutation and $f \in \mathsf{A}(M, M)$ a morphism. We have the following formula*

$$(\sigma \circ f^{\otimes n})_n = \left(\prod_{\mathscr{O} \in \Sigma_{\sigma,n}} \mathrm{Tr}(f^{\circ|\mathscr{O}|}) \right) f^{\circ|\mathscr{O}_n|}.$$

Proof. — First assume that σ is the n-cycle $(1, 2, \dots, n)$. Using formula (1) and induction we have

$$(\sigma \circ f^{\otimes n})_n = f^{\circ n}$$

and thus by lemma 4.11 we have also $\mathrm{Tr}(\sigma \circ f^{\otimes n}) = \mathrm{Tr}(f^{\circ n})$. Now by lemma 4.13 we know that for a permutation $\beta \in \mathfrak{S}_n$ that fix n we have

$$(\sigma \circ f^{\otimes n})_n = (\beta \circ \sigma \circ \beta^{-1} \circ f^{\otimes n})_n.$$

Thus one may assume that σ is the product of the cycles with disjoint supports

$$\sigma_1 = (1, 2, \dots, \ell_1) \quad \sigma_2 = (\ell_1 + 1, \ell_1 + 2, \dots, \ell_1 + \ell_2) \quad \cdots$$
$$\cdots \quad \sigma_r = (n - \ell_r + 1, n - \ell_r + 2, \dots, n).$$

In that case we have

$$\sigma \circ f^{\otimes n} = (\sigma_1 \circ f^{\otimes \ell_1}) \otimes (\sigma_2 \circ f^{\otimes \ell_2}) \otimes \cdots \otimes (\sigma_r \circ f^{\otimes \ell_r})$$

and thus by lemma 4.12

$$(\sigma \circ f^{\otimes n})_n = \Big[\mathrm{Tr}(\sigma_1 \circ f^{\otimes \ell_1}) \cdots \mathrm{Tr}(\sigma_{r-1} \circ f^{\otimes \ell_{r-1}})\Big] (\sigma_r \circ f^{\otimes \ell_r})_r.$$

The proposition follows then from the case of cycles considered before. □

Remark 4.15. — By lemma 4.11 and proposition 4.14 we have

$$\mathrm{Tr}(\sigma \circ f^{\otimes n}) = \prod_{\mathscr{O} \in \Sigma_\sigma} \mathrm{Tr}(f^{\circ |\mathscr{O}|})$$

and therefore

$$\begin{aligned} \mathrm{Tr}(\mathsf{\Lambda}^n f) &= \frac{1}{n!} \sum_{\sigma \in \mathfrak{S}_n} \varepsilon(\sigma) \prod_{\mathscr{O} \in \Sigma_\sigma} \mathrm{Tr}(f^{\circ |\mathscr{O}|}), \\ \mathrm{Tr}(\mathsf{S}^n f) &= \frac{1}{n!} \sum_{\sigma \in \mathfrak{S}_n} \prod_{\mathscr{O} \in \Sigma_\sigma} \mathrm{Tr}(f^{\circ |\mathscr{O}|}). \end{aligned}$$

Taking $f = 1_M$ we derive from this the formulas computing Euler characteristics of exterior and symmetric powers

$$\chi(\mathsf{\Lambda}^n M) = \binom{\chi(M)}{n} = \frac{\chi(M)(\chi(M)-1)\cdots(\chi(M)-n+1)}{n!}, \tag{7}$$

$$\chi(\mathsf{S}^n M) = \binom{\chi(M)+n-1}{n} = \frac{\chi(M)(\chi(M)+1)\cdots(\chi(M)+n-1)}{n!}. \tag{8}$$

It is also easy to derive from the computation given in proposition 4.14 the expected positivity properties of Euler characteristic of even or odd objects.

Lemma 4.16. — *Let M be even (resp. odd). Then $\chi(M)$ is a nonnegative integer (resp. a nonpositive integer) that satisfies*

$$0 \leqslant |\chi(M)| \leqslant \dim(M).$$

Proof. — Assume M even. By formula (7) the Euler characteristic of $\mathsf{\Lambda}^n M$ is given by

$$\chi(\mathsf{\Lambda}^n M) = \binom{\chi(M)}{n} = \frac{\chi(M)(\chi(M)-1)\cdots(\chi(M)-n+1)}{n!}$$

and vanishes by assumption for $n = \dim(M)+1$. Thus $\chi(M)$ has to be a nonnegative integer $\leqslant \dim(M)$. The proof for M odd is similar using formula (8). □

We are now ready prove the first nilpotence theorem 4.9.

Proof of theorem 4.9. — Assume that $f \in \mathscr{N}_{\mathsf{A}}(M, M)$. Then by definition of the ideal $\mathscr{N}_{\mathsf{A}}$ the trace of $f^{\circ k}$ is zero for all positive integer k. Therefore given $\sigma \in \mathfrak{S}_n$ we see from proposition 4.14 that

$$(\sigma \circ f^{\otimes n})_n = \begin{cases} f^{\circ n} & \text{if } \sigma \text{ is a } n\text{-cycle} \\ 0 & \text{otherwise} \end{cases}$$

and so we have

$$(\mathsf{S}^n f)_n = \frac{1}{n} f^{\circ n} \qquad \text{and} \qquad (\mathsf{\Lambda}^n f)_n = \frac{(-1)^{n-1}}{n} f^{\circ n}.$$

The result follows now from the definition of even and odd objects. □

4.5. A structure result. — The next proposition will be essential in the sequel. The assumption that the Euler characteristic of any object of A is a nonnegative integer is truly needed.

Proposition 4.17. — *Assume that the Euler characteristic of any object of* A *is a nonnegative integer. Then the morphisms in* $\overline{\mathsf{A}}$ *are finite dimensional and for all object* M *in* A *we have*

$$\dim_K \overline{\mathsf{A}}(\overline{M}, \overline{M}) \leqslant \chi(M)^2. \tag{9}$$

Furthermore any nilideal of A *is contained in* $\mathscr{N}_{\mathsf{A}}$*, one has the inclusion* $\mathscr{R}_{\mathsf{A}} \subset \mathscr{N}_{\mathsf{A}}$ *and* $\overline{A}$ *is semi-simple.*

Proof. — It is enough to prove the inequality (9), all others Hom spaces being then also finite dimensional. For this consider r endomorphisms $f_1, \dots, f_r \in \mathsf{A}(M, M)$ which are linearly independent in $\overline{\mathsf{A}}(\overline{M}, \overline{M})$. Then the morphism

$$\mathbf{1}^r \to M^\vee \otimes M \tag{10}$$

given by the f_i is a split monomorphism. Indeed, by definition of the ideal $\mathscr{N}_{\mathsf{A}}$, the pairing

$$\overline{\mathsf{A}}(\overline{M}, \overline{M}) \otimes \overline{\mathsf{A}}(\overline{M}, \overline{M}) \to K$$

induced by the trace is non degenerate. Therefore there exist $g_1, \dots, g_r \in \mathsf{A}(M, M)$ such that $\mathrm{Tr}(g_j \circ f_i) = \delta_{ij}$. which means that

$$M^\vee \otimes M \xrightarrow{(1_{M^\vee} \otimes g_1, \cdots, 1_{M^\vee} \otimes g_r)} (M^\vee \otimes M)^r \xrightarrow{\text{com. cons.}} (M \otimes M^\vee)^r \xrightarrow{(\mathrm{ev}_M, \cdots, \mathrm{ev}_M)} \mathbf{1}^r$$

is a left inverse to the morphism (10). Therefore $\mathbf{1}^r$ splits off in $M^\vee \otimes M$ and since Euler characteristics are assumed to be nonnegative this implies $r \leqslant \chi(M)^2$. The assertion on nilideals follows from the fact [**5**, 7.3.1] that under our assumption nilpotent endomorphisms have nilpotent traces. To see that we have the inclusion $\mathscr{R}_{\mathsf{A}} \subset \mathscr{N}_{\mathsf{A}}$ it is enough to remark that the image of $\mathscr{R}_{\mathsf{A}}$ in $\overline{A}$ is contained in $\mathscr{R}_{\overline{\mathsf{A}}}$ which is a nilideal and so satisfies

$$\mathscr{R}_{\overline{\mathsf{A}}} \subset \mathscr{N}_{\overline{\mathsf{A}}} = 0.$$

The semi-simplicity result follows. □

Proposition 4.18. — *Let M be even or odd. Then $M = 0$ if and only if $\chi(M) = 0$.*

Proof. — Assume that M is even or odd. Let B be the thick full rigid subcategory of A generated by $M^\vee \otimes M$. Since $\mathscr{N}_\mathsf{B} = \mathscr{N}_\mathsf{A} \cap \mathsf{B}$, the category $\mathsf{B}/\mathscr{N}_\mathsf{B}$ is a thick full rigid subcategory of $\mathsf{A}/\mathscr{N}_\mathsf{A}$. By proposition 3.15 and lemma 4.16 every object of B is even and its Euler characteristic is a nonnegative integer. Thus proposition 4.17 implies that

$$\dim_K \left[\mathsf{B}/\mathscr{N}_\mathsf{B}(\overline{M}^\vee \otimes \overline{M}, \overline{M}^\vee \otimes \overline{M})\right] \leqslant \chi(M^\vee \otimes M)^2 = \chi(M)^4.$$

Thus if $\chi(M) = 0$ then $\overline{M}^\vee \otimes \overline{M} = 0$ in $\mathsf{A}/\mathscr{N}_\mathsf{A}$. This implies $\overline{M} = 0$ and by theorem 4.9 we have also $M = 0$. □

Corollary 4.19. — *If an object M of A is both even and odd then $M = 0$.*

Proof. — This follows from lemma 4.16 and proposition 4.18. □

Corollary 4.20. — *Assume that M is even or odd. Then $\chi(M)$ belongs to $\mathbb{Z} \subset K$ and*

$$\chi(M) = \begin{cases} \dim(M) & \text{if } M \text{ is even} \\ -\dim(M) & \text{if } M \text{ is odd.} \end{cases}$$

Proof. — Assume M even. By lemma 4.16 we have to show that $\Lambda^{\chi(M)+1}(M) = 0$. The Euler characteristic of this exterior power is given according to formula (7) by

$$\chi(\Lambda^{\chi(M)+1}(M)) = \binom{\chi(M)}{\chi(M)+1} = 0$$

and the result follows from proposition 4.18. The proof for M odd is similar using symmetric power and formula (8). □

Corollary 4.21. — *Let M, N be even objects (resp. odd objects). Then*

$$\dim(M \oplus N) = \dim(M) + \dim(N).$$

4.6. $\otimes$-Nilpotence properties. — The following $\otimes$-nilpotence theorem is one of the main tool in the proof of the second nilpotence theorem 4.29. Nice properties of finite dimensional objects are also derived from it.

Proposition 4.22. — *Assume that M, N are objects of A of different parities. Then any morphism $f \in \mathsf{A}(M, N)$ is $\otimes$-nilpotent. Furthermore*

$$f^{\otimes \dim(M)\dim(N)+1} = 0.$$

Proof. — By proposition 3.15, our assumption implies that $M^\vee \otimes N$ is odd of dimension less than $\dim(M)\dim(N)$. Since $\mathbf{1}$ is even of dimension 1 we may assume using remark 3.6 that $M = \mathbf{1}$ and N is odd. In that case the morphism

$$f^{\otimes n} : \mathbf{1}^{\otimes n} \to N^{\otimes n}$$

is $\mathfrak{S}_n$-equivariant and we have a factorization

$$\begin{tikzcd} \mathbf{1} \arrow[r, dotted] \arrow[rr, bend left, "f^{\otimes n}"] & \mathsf{S}^n N \arrow[r] & N^{\otimes n}. \end{tikzcd}$$

The result follows. □

Proposition 4.23. — *Let M be finite dimensional and*

$$M \simeq M_+ \oplus M_- \qquad M \simeq M'_+ \oplus M'_-$$

be two decompositions where M_+, M'_+ are even and M_-, M'_- are odd. Then M_+ and M'_+ are isomorphic and similarly M_- and M'_- are isomorphic.

Proof. — [**52**, Proposition 6.3] Let p be the projector of M defined by the direct summand M_+ and p' the one defined by the direct summand M'_+. Then $1_M - p'$ is the projector onto M'_- and the composition

$$p - p' \circ p = (1_M - p') \circ p$$

is $\otimes$-nilpotent by proposition 4.22. Lemma 3.5 implies that there exists an $n > 0$ such that

$$(p - p' \circ p)^{\circ n} = 0. \tag{11}$$

By expanding the left hand side of (11) we get a relation $p = h \circ p' \circ p$ for some morphism $h \in \mathsf{A}(M, M)$. Since p is a projector, we have the following commutative diagram

$$\begin{tikzcd} M \arrow[rr, "p"] \arrow[rrrrrr, bend left, "p"] \arrow[dr, dotted] & & M \arrow[rr, "p'"] \arrow[dr, dotted] & & M \arrow[rr, "h"] \arrow[dr, dotted] & & M \\ & M_+ \arrow[ur] \arrow[rr, dotted] \arrow[rrrr, bend right, "1_{M_+}"'] & & M'_+ \arrow[ur] \arrow[rr, dotted] & & M_+. \arrow[ur] & \end{tikzcd}$$

Therefore M_+ is a direct summand of M'_+ and there is an object N such that M'_+ is isomorphic to $M_+ \oplus N$. By corollary 4.21 we have $\dim(M_+) \leqslant \dim(M'_+)$ and since the other inequality may be proved similarly we have in fact $\dim(M_+) = \dim(M'_+)$. Then according to corollary 4.21 the dimension of N is zero and the result follows from theorem 4.20. □

Remark 4.24. — The decomposition into even and odd parts of a finite dimensional object is unique up to isomorphism but it may not be unique up to a unique isomorphism. In other words there is no natural decomposition into even and odd parts. However if $\sqrt[\otimes]{0} = 0$ then the decomposition is unique up to a unique isomorphism.

Proposition 4.25. — *Let* A *be a Kimura-O'Sullivan category such that* $\sqrt[\otimes]{0} = 0$. *Then there exists a commutativity constraint* s *on* A *such that the category* A_s *obtained from* A *by change of commutativity constraint is a Kimura-O'Sullivan category in which all objects are even.*

Proof. — Let us denote by s_A the original commutativity constraint of A. Our assumption on the $\otimes$-nilradical implies that in A the decomposition in even and odd parts is unique. Let M be an object of A and $p_{M,+}$ the projector on the even part. Then one may consider the endomorphism

$$e_M = 2p_{M,+} - 1$$

and it is easy to see that the isomorphism $\mathsf{s}_{M,N} : M \otimes N \to N \otimes M$ given by

$$\mathsf{s}_{M,N} = \mathsf{s}_{\mathsf{A}M,N} \circ (e_M \otimes e_N)$$

defines a new commutativity constraint on A such that $\chi_\mathsf{s}(M) = \dim(M)$. For this commutativity constraint all objects are even. □

The uniqueness up to isomorphism of the odd and even part in any decomposition of a finite dimensional object proved in proposition 4.23 shows that the following definition of the dimension is not ambiguous.

Definition 4.26. — Let M be a finite dimensional object. Then the dimension of M is the nonnegative integer

$$\dim(M) = \dim(M_+) \oplus \dim(M_-)$$

where M_+ and M_- are even and odd objects such that we have a decomposition into a direct sum $M \simeq M_+ \oplus M_-$.

The next result is proposition 6.9 of [**52**].

Proposition 4.27. — *A direct factor of a finite dimensional object is also finite dimensional.*

Proof. — By assumption M has a decomposition $M \simeq M_+ \oplus M_-$ with M_+ even and M_- odd. It is enough to show that N has a decomposition $N \simeq N_+ \oplus N_-$ such that N_+ is a direct factor of M_+ and N_- is a direct factor of M_-. Let p_+ and p_- be the projectors of M defined by the direct factors M_+ and M_-. Since N is a direct factor of M we have morphisms

$$\begin{tikzcd} N \arrow[r, "\iota"'] \arrow[rr, bend left, "1_N"] & M \arrow[r, "\pi"'] & N. \end{tikzcd}$$

Consider the endomorphisms of N

$$r_+ = \pi \circ p_+ \circ \iota \qquad r_- = \pi \circ p_- \circ \iota$$

and remark that they commute one with the other. We know from proposition 4.22 that the morphism $r_+ \circ r_-$ is nilpotent. So we may choose a positive integer n such that

$$(r_- \circ r_+)^{\circ n} = 0$$

and let q_+ be the endomorphism of N given by

$$q_+ = \left(1_N - r_-^{\circ n}\right)^{\circ n} = \left(\sum_{k=1}^{n} (-1)^k r_+^{\circ k}\right)^{\circ n} = r_+^{\circ n} \circ \left(\sum_{k=1}^{n} (-1)^k r_+^{\circ k-1}\right)^{\circ n}.$$

Since $r_-^{\circ n} \circ r_+^{\circ n} = (r_- \circ r_+)^{\circ n} = 0$ we have

$$q_+ \circ q_+ = \left(1_N - r_-^{\circ n}\right)^{\circ n} \circ r_+^{\circ n} \circ \left(\sum_{k=1}^{n} (-1)^k r_+^{\circ k-1}\right)^{\circ n} = q_+$$

and thus q_+ is a projector of N and induces a decomposition $N \simeq N_+ \oplus N_-$. By construction q_+ may be written as a polynomial in r_+ with no constant term

$$q_+ = \sum_{s=1}^{\nu} a_s r_+^{\circ s}$$

and so by definition of r_+ we have the identity

$$q_+ \circ \pi \circ p_+ \circ \left(\iota \circ \sum_{s=1}^{\nu} a_s r_+^{\circ s-1}\right) = q_+ \circ r_+ \circ \left(\sum_{s=1}^{\nu} a_s r_+^{\circ s-1}\right) = q_+ \circ q_+ = q_+$$

which proves that N_+ is a direct factor of M_+. Similarly we prove that N_- is a direct factor of M_- and the result follows. □

This implies the following useful result:

Corollary 4.28. — *A is a Kimura-O'Sullivan category if and only if it is generated by a family of finite dimensional objects.*

4.7. The second nilpotence theorem. — The next result is proposition 7.5 of [**52**] and is crucial in all application of finite dimensionality.

Theorem 4.29. — *Let M be finite dimensional. Then $\mathscr{N}_{\mathsf{A}}(M, M)$ is a nilpotent ideal and $\overline{\mathsf{A}}(\overline{M}, \overline{M})$ is a semi-simple K-algebra of finite dimension.*

Proof. — By the result of Higman [**35**] recalled in proposition 3.1, an ideal I in a $\mathbb{Q}$-algebra is nilpotent if and only if there exists a positive integer n such that $a^n = 0$ for all $a \in I$. Therefore it is enough to show that for $f \in \mathscr{N}_{\mathsf{A}}(M, M)$ we have $f^{\circ n} = 0$ for some positive integer n that depends only on M. We may assume that M is the direct sum of M_+ even and M_- odd, and write the morphism f as a sum

$$f = f_+ + f_- + f_\pm$$

where f_+ preserves the even summand, f_- preserves the odd one and $f_\pm$ is the sum of the two antidiagonal morphisms. In that case the morphism $f^{\circ n}$ is a sum of monomial terms

$$\underbrace{f_{\varepsilon_1}^{k_1} \circ f_\pm \circ f_{\varepsilon_2}^{k_2} \circ \cdots \circ f_\pm \circ f_{\varepsilon_r}^{k_r}}_{f_\pm \text{ appears } k_\pm \text{ times}} \tag{12}$$

where $\varepsilon_i \in \{-1, +1\}$ and $k_\pm, k_1, \ldots, k_r$ are nonnegative integers such that

$$k_\pm + k_1 + \cdots + k_r = n. \tag{13}$$

On the other hand we know that (12) vanishes as soon as one of the following condition is fulfilled:

- one of the k_i is larger than a constant that depends only on M (this is implied by theorem 4.9);
- the integer $k_\pm$ is larger than a constant that depends only on M (this is implied by proposition 4.22 and lemma 3.5).

Now we see from relation (13) that there exists a constant that depends only on M such that for n larger than this constant, one of the two conditions above has to be fulfilled for each monomial term in the expansion of $f^{\circ n}$. We have shown that $\mathscr{N}_{\mathsf{A}}(M, M)$ is a nilpotent ideal. To get the semi-simplicity statement it is enough to apply proposition 4.25, lemma 4.16 and then proposition 4.17. □

Corollary 4.30. — *Finite dimensional objects are not phantom objects. In particular a Kimura-O'Sullivan category does not contain phantom objects.*

The following is [**5**, 9.2.2]:

Theorem 4.31. — *A Kimura-O'Sullivan category is a Wedderburn category such that*

$$\mathscr{R}_{\mathsf{A}} = \mathscr{N}_{\mathsf{A}}$$

and $\overline{\mathsf{A}}$ is a semi-simple abelian category that can be made Tannakian after a change of the commutativity constraint.

Proof. — Assume that A is a Kimura-O'Sullivan category. The nilpotence theorem 4.29 implies that $\overline{\mathsf{A}}$ is semi-simple and that $\mathscr{N}_{\mathsf{A}}$ is a nilideal. Since any nilideal is contained in $\mathscr{R}_{\mathsf{A}}$ one has the inclusion $\mathscr{N}_{\mathsf{A}} \subset \mathscr{R}_{\mathsf{A}}$. For the converse inclusion it is enough to remark that the image of $\mathscr{R}_{\mathsf{A}}$ in $\overline{\mathsf{A}}$ in contained in $\mathscr{R}_{\overline{\mathsf{A}}}$ which is zero by semi-simplicity. Being pseudo-abelian and semi-simple, the category $\overline{\mathsf{A}}$ is also abelian. The category $\overline{\mathsf{A}}$ is a Kimura-O'Sullivan category such that $\sqrt[\otimes]{0}_{\overline{\mathsf{A}}} = 0$ and by proposition 4.25 there exists a commutativity constraint s on it such that all object of $\overline{\mathsf{A}}_{\mathsf{s}}$ are even. Therefore for all object M in A the Euler characteristic $\chi_{\mathsf{s}}(M)$ is a nonnegative integer. This implies by proposition 4.17 that the Hom spaces in $\overline{\mathsf{A}}$ are finite dimensional and thus by proposition 3.8 that $\overline{\mathsf{A}}$ is a Wedderburn category. Since

$\mathscr{R}_{\mathsf{A}} = \mathscr{N}_{\mathsf{A}}$, we get that A is also a Wedderburn category and to complete the proof it is enough to apply proposition 4.25 and Deligne's criterion 2.3. □

5. Schur finiteness

5.1. Schur functors. — Let M be an object in A. If V is a finite dimensional K-vector space, we define the objects $V \otimes M$ and $\mathscr{H}om(V, M)$ of A by

$$\mathrm{Hom}_{\mathsf{A}}(V \otimes M, -) = \mathrm{Hom}_K(V, \mathrm{Hom}_{\mathsf{A}}(M, -)),$$
$$\mathrm{Hom}_{\mathsf{A}}(-, \mathscr{H}om(V, M)) = \mathrm{Hom}_{\mathsf{A}}(V \otimes -, M).$$

Let G be a finite group, assume that V is a linear representation of G and that G acts on M. Then we get an action of G on $\mathscr{H}om(V, M)$ and since A is pseudo-abelian we can consider

$$\mathscr{H}om_G(V, M) := \mathrm{Im}\left[\frac{1}{|G|}\sum_{g \in G} g : \mathscr{H}om(V, M) \to \mathscr{H}om(V, M)\right]$$

which is a direct factor in $\mathscr{H}om(V, M)$. Assume that all irreductible representations of G over $\overline{K}$ are already defined over K, and choose a representative V_λ of each isomorphism classes of irreductible representations of G, then the regular representation decomposes into a direct sum of the V_λ each one appearing with multiplicity $\dim(V_\lambda)$

$$K[G] = \bigoplus_\lambda V_\lambda^{\dim(V_\lambda)}.$$

This says that the morphism

$$\bigoplus_\lambda V_\lambda \otimes \mathscr{H}om_G(V_\lambda, M) \to M$$

is an isomorphism. Recall that an isomorphim class of irreductible representations of the permutation group $\mathfrak{S}_n$ can be identified to a partition of n. Since the tensor power $M^{\otimes n}$ has a natural action of $\mathfrak{S}_n$, we get in particular a decomposition

$$\bigoplus_\lambda V_\lambda \otimes \mathscr{H}om_{\mathfrak{S}_n}(V_\lambda, M^{\otimes n}) \simeq M^{\otimes n}$$

over the partitions λ of n. The functor

$$M \mapsto \mathsf{S}_\lambda(M) := \mathscr{H}om_{\mathfrak{S}_n}(V_\lambda, M^{\otimes n})$$

is called the Schur functor associated to the partition λ. For more details on Schur functors we refer to [**15**, §1].

Example 5.1. — Symmetric powers and exteriors power are special cases of Schur functors. The symmetric power S^n is the Schur functor associated to the partition $\lambda = (n, 0, \ldots, 0, 0, \ldots)$ and the exterior power Λ^n is the Schur functor associated to the partition $\lambda = (1, 1, \ldots, 1, 0, \ldots)$.

Let μ, ν be partitions of p and q and λ be a partition of $n = p + q$. In the sequel we let $[V_\lambda : V_\mu \otimes V_\nu]$ be the Littlewood-Richardson's coefficient *i.e.* the multiplicity of the irreductible representation $V_\mu \otimes V_\nu$ of $\mathfrak{S}_p \times \mathfrak{S}_q$ in the restriction of V_λ to this subgroup. Similarly if λ, μ, ν are partitions of n, we denote by $[V_\mu \otimes V_\nu : V_\lambda]$ the multiplicity of the irreductible representation V_λ into $V_\mu \otimes V_\nu$. The next proposition sums up the main properties of Schur functors.

Proposition 5.2. — *Schur functors enjoy the following properties:*

– *let μ, ν be partitions of p, q*

$$\mathsf{S}_\mu(M) \otimes \mathsf{S}_\nu(M) \simeq \oplus_\lambda \mathsf{S}_\lambda(M)^{[V_\lambda : V_\mu \otimes V_\nu]}$$

with sum over λ partition of $n = p + q$;

– *let λ be a partition of n*

$$\mathsf{S}_\lambda(M \oplus N) \simeq \oplus_{\mu,\nu} \left(\mathsf{S}_\mu(M) \otimes \mathsf{S}_\nu(N)\right)^{[V_\lambda : V_\mu \otimes V_\nu]}$$

with sum over partitions μ, ν of p, q such that $p + q = n$;

– *let λ be a partition of n*

$$\mathsf{S}_\lambda(M \otimes N) \simeq \oplus_{\mu,\nu} \left(\mathsf{S}_\mu(M) \otimes \mathsf{S}_\nu(N)\right)^{[V_\mu \otimes V_\nu : V_\lambda]}$$

with sum over μ, ν partition of n.

For any object M we have also an isomorphism $\mathsf{S}_\lambda(M^\vee) = \mathsf{S}_\lambda(M)^\vee$. Moreover if we have an inclusion $D_\lambda \subset D_\mu$ of the corresponding Young diagrams then the vanishing of $\mathsf{S}_\lambda(M)$ implies the vanishing of $\mathsf{S}_\mu(M)$.

5.2. Schur finiteness. — Finite dimensional objects have a lot of good stability properties, however in triangulated categories they do not have in general the two out of three properties. Schur finiteness is the generalization of finite dimensionality obtained by taking into account not only symmetric or exterior powers but also all Schur functors. This notion has nicer properties in the triangulated setting and is also related to the Tannakian formalism.

Definition 5.3. — An object M in A is said to be Schur finite if

$$\mathsf{S}_\lambda M = 0$$

for some Schur functor S_λ.

Schur finite objects enjoy the same stability properties as finite dimensional objects, namely they are stable by $\oplus$, tensor product, direct factor and duality $^\vee$. Let us denote by $DM_{gm}(F; \mathbb{Q})$ the triangulated category of geometrical motives with rational coefficients constructed by Voevodsky in [**82**].

***Proposition 5.4* (Mazza [60], Guletskii [29]).** — *In a distinguished triangle of $DM_{gm}(F;\mathbb{Q})$*

$$M' \to M \to M'' \xrightarrow{+1}$$

if two objects are Schur finite then so is the third.

Remark 5.5. — In fact we have a more precise result. Indeed in [**60**, Lemma 3.6] it is proved that if $M' \to M \to M'' \xrightarrow{+1}$ is a a distinguished triangle of $DM_{gm}(F;\mathbb{Q})$ such that $\mathsf{S}_\lambda(M' \oplus M'') \simeq 0$, then one has also $\mathsf{S}_\lambda(M) \simeq 0$.

Any finite dimensional object is also Schur finite. However in [**60**] an example due to O'Sullivan of a Schur finite object in $DM_{gm}(F;\mathbb{Q})$ which is not finite dimensional is provided for some field F. The field F in the example is obtained as the function field of a smooth projective surface X over an algebraically closed field, and the motive is obtained by removing a finite set of rational points in X_F, see [**60**, Corollary 5.20]. O'Sullivan's example proves also that the two out of three property does not hold for finite dimensional objects. Nevertheless another interesting application of proposition 5.4 obtained via remark 5.5 is the following:

***Corollary 5.6* (Mazza [60], Guletskii [29]).** — *If M' and M'' are even (resp. odd) objects and*

$$M' \to M \to M'' \xrightarrow{+1}$$

is a distinguished triangle in $DM_{gm}(F;\mathbb{Q})$, then M is even (resp. odd).

Remark 5.7. — Let λ be a partition of an integer n. When dealing with Schur functors in $DM_{gm}(F;\mathbb{Q})$, it is useful to keep also in mind the effect of the shift functor $-[1]$ on S_λ. The two are related by a natural isomorphism $\mathsf{S}_\lambda(M[1]) \simeq \mathsf{S}_{\lambda^t}(M)[n]$, where λ^t denotes the transpose of the partition λ. In particular we have natural isomorphisms

$$\mathsf{S}^n(M[1]) \simeq \Lambda^n(M)[n] \qquad \text{and} \qquad \Lambda^n(M[1]) \simeq \mathsf{S}^n(M)[n]$$

in $DM_{gm}(F;\mathbb{Q})$.

One may also prove that the motive of a curve in $DM_{-}(F;\mathbb{Q})$ is always finite dimensional by reduction to the case of a smooth projective curve.

5.3. Super Tannakian categories. — Assume A is abelian. An E-valued super fibre functor [**15**] is an exact faithful $\otimes$-functor

$$\omega : \mathsf{A} \to \mathrm{sVec}_E$$

where E/K is an extension and sVec_E denotes the category formed by super E-vector spaces of finite dimension.

Definition 5.8. — A is said to be super Tannakian if it is abelian and there exists a super fibre functor

$$\omega : \mathsf{A} \to \mathrm{sVec}_E$$

where E/K is an extension.

Schur finiteness provides an internal description of super Tannakian categories parallel to the one we have for Tannakian categories with theorem 2.3.

***Theorem 5.9* (Deligne [15]).** — *Assume* A *is abelian. The following assertions are equivalent:*

– A *is a super Tannakian category;*
– *every object of* A *is Schur finite.*

In particular an abelian Kimura-O'Sullivan category is super Tannakian. However theorem 5.9 does not provide any specific super fibre functor. To sum up the link between finite dimensionality, Schur finiteness and the Tannakian formalism, we have the following picture:

$$\begin{tikzcd}
\boxed{\begin{array}{c}\mathsf{A}\text{ is abelian}\\ \text{and all its objets}\\ \text{are Schur finite}\end{array}} \arrow[r, Leftrightarrow, "\text{theorem 5.9}"] & \boxed{\begin{array}{c}\mathsf{A}\text{ is a super Tannakian}\\ \text{category}\end{array}} \\
\boxed{\begin{array}{c}\mathsf{A}\text{ is an abelian}\\ \text{Kimura-O'Sullivan}\\ \text{category}\end{array}} \arrow[u, Rightarrow] & \\
\boxed{\begin{array}{c}\mathsf{A}\text{ is abelian}\\ \text{and all its objets}\\ \text{are even}\end{array}} \arrow[u, Rightarrow] \arrow[r, Leftrightarrow, "\text{theorem 2.3}"] & \boxed{\begin{array}{c}\mathsf{A}\text{ is a Tannakian}\\ \text{category}\end{array}} \arrow[uu, Rightarrow]
\end{tikzcd}$$

6. The Kimura-O'Sullivan's conjecture

6.1. Statement of the conjecture. — Kimura [**52**] and O'Sullivan have made the following conjecture which has deep implication in the motivic world:

***Conjecture 6.1* (Kimura, O'Sullivan).** — Any Chow motive $M \in \mathsf{M}_{\mathsf{rat}}(F;\mathbb{Q})$ is finite dimensional in other words $\mathsf{M}_{\mathsf{rat}}(F;\mathbb{Q})$ is a Kimura-O'Sullivan category.

As we have seen before this implies:

– Jannsen's theorem: $\mathsf{M}_{\mathsf{num}}(F;\mathbb{Q})$ is abelian semi-simple;

– the functor
$$\mathsf{M}_{\mathsf{rat}}(F;\mathbb{Q}) \to \mathsf{M}_{\mathsf{num}}(F;\mathbb{Q})$$
is conservative;
– projectors and orthogonal systems of projectors lift;
– endomorphisms which are numerically zero are rationally nilpotent.

6.2. Motives of abelian type. — Unfortunately conjecture 6.1 is unknown outside the world of curves. Even for surfaces finite dimensionality would have tremendous implications as we shall see with the motivic reformulation of the Bloch conjecture in theorem 9.5. Let us introduce the category of motives of abelian type.

Definition 6.2. — Let $\mathsf{M}_{\mathsf{rat}}(F;\mathbb{Q})^{\mathrm{ab}}$ be the smallest thick strictly full rigid tensor subcategory of $\mathsf{M}_{\mathsf{rat}}(F;\mathbb{Q})$ that contains Artin motives and motives of abelian varieties.

Recall that the category $\mathsf{AM}(F;\mathbb{Q})$ of Artin motives is the pseudo-abelian hull of the essential image of the restriction of
$$h : \mathrm{SmProj}_F^{\mathrm{op}} \to \mathsf{M}_{\mathsf{rat}}(F;\mathbb{Q})$$
to the category of smooth projective of dimension 0. Artin motives are all even and so finite dimensional. This can be seen from the fact that $\mathsf{AM}(F;\mathbb{Q})$ is tensor equivalent to the category $\mathrm{Rep}_{\mathbb{Q}}^{\mathrm{cont}}(G_F)$ of continuous finite dimensional $\mathbb{Q}$-representations of the absolute Galois group. Recall that smooth projective varieties of dimension 0 are finite étale schemes *i.e.* spectrum of finite product of finite separable extensions of F. For two such varieties X and Y, a correspondence from X to Y is simply a $\mathbb{Q}$-linear combination of connected components of $X \times Y$ and thus the space of correspondences may be identified to the $\mathbb{Q}$-vector space
$$\left[\mathbb{Q}^{X(\overline{F})\times Y(\overline{F})}\right]^{G_F} \simeq \mathrm{End}_{G_F}(\mathbb{Q}^{X(\overline{F})}, \mathbb{Q}^{Y(\overline{F})})$$
of $\mathbb{Q}$-valued functions over $X(\overline{F}) \times Y(\overline{F})$ left invariant by Galois. From this we obtain a fully faithful $\otimes$-functor
$$\mathsf{AM}(F;\mathbb{Q}) \to \mathrm{Rep}_{\mathbb{Q}}^{\mathrm{cont}}(G_F) \tag{14}$$
which is in fact an equivalence. This tensor equivalence may be seen as a linearized version of Grothendieck's formulation of Galois theory. To be more precise we have a commutative square
$$\begin{tikzcd} \mathrm{FEt}_F^{\mathrm{op}} \arrow[r, "(2)"] \arrow[d, "h"] & \mathrm{FSets}^{\mathrm{cont}}(G_F)^{\mathrm{op}} \arrow[d] \\ \mathsf{AM}(F;\mathbb{Q}) \arrow[r, "(14)"] & \mathrm{Rep}_{\mathbb{Q}}^{\mathrm{cont}}(G_F) \end{tikzcd}$$

in which the vertical arrow on the right is obtained as the composition of the functor induced by transposition and the functor which sends a continuous finite $\mathrm{Gal}(\overline{F}/F)$-set X to the continuous finite dimensional $\mathbb{Q}$-representation $\mathbb{Q}^X$ where the action is given by

$$(g\phi)(x) = \phi(g^{-1}x)$$

for $\phi \in \mathbb{Q}^X$, $x \in X$ and g in the Galois group. Motives of abelian type provide the best known examples for which conjecture 6.1 is known.

Theorem 6.3. — *Every motive in* $\mathsf{M}_{\mathsf{rat}}(F;\mathbb{Q})^{\mathrm{ab}}$ *is finite dimensional.*

Using corollary 4.28 and the above discussion for Artin motives it is enough to prove that motives of abelian varieties are finite dimensional.

Sketch of proof. — Let A be an abelian variety of dimension d. Then the works of A.M. Shermenev [**77**] and C. Deninger-J. Murre [**19**] prove that there exists a unique decomposition

$$h(A) = \bigoplus_{i=0}^{2d} h^i(A) \tag{15}$$

such that for all integer n we have $[\times n] = n^i$ on $h^i(A)$. Furthermore the map

$$h(A)^{\otimes n} = h(A \times \cdots \times A) \xrightarrow{\Delta_A} h(A)$$

induces an isomorphism for all i

$$\mathsf{S}^i\Big[h^1(A)\Big] \xrightarrow{\sim} h^i(A).$$

In particular $h^1(A)$ is odd of dimension $2d$ and so $h(A)$ is of finite dimension by corollary 3.16. $\square$

Remark 6.4. — The decomposition (15) of the motive of an abelian variety is canonical and is a Chow-Künneth decomposition in the sense of 7.1. In particular the works of Shermenev [**77**] and Deninger-Murre [**19**] provide a canonical Chow-Künneth decomposition for abelian varieties. Such decompositions are conjectured to exist for any smooth projective variety but there is no reason for them to be unique in general.

7. Chow-Künneth decomposition

7.1. Murre's conjecture. — Let X be a smooth projective variety of dimension d. Standard conjecture $\mathsf{C}(X)$ says that the Künneth projectors

$$\Delta_{i,2d-i} \in \mathsf{H}^{2d}(X \times X) : \mathsf{H}^*(X) \twoheadrightarrow H^i(X) \hookrightarrow H^*(X)$$

are algebraic *i.e.* belong to

$$\mathsf{M}_{\mathsf{hom}}(F;\mathbb{Q})(h(X), h(X)) \hookrightarrow \mathrm{End}(\mathsf{H}^*(\mathsf{X})).$$

In particular $\mathsf{C}(X)$ provides a canonical weight decomposition

$$h_{\mathsf{hom}}(X) = h^0_{\mathsf{hom}}(X) \oplus h^1_{\mathsf{hom}}(X) \oplus \cdots \oplus h^{2d-1}_{\mathsf{hom}}(X) \oplus h^{2d}_{\mathsf{hom}}(X)$$

where h^i_{hom} is the direct summand of the homological motive of X cut-off by the Künneth projector $\Delta_{i,2d-i}$. Let us assume furthermore that the Chow motive of X is finite dimensional. Then the kernel of

$$\mathsf{M}_{\mathsf{rat}}(F;\mathbb{Q})(h(X),h(X)) \to \mathsf{M}_{\mathsf{hom}}(F;\mathbb{Q})(h_{\mathsf{hom}}(X),h_{\mathsf{hom}}(X))$$

is a nilideal and the Künneth system of orthogonal projectors lifts to the category of Chow motives. Thus under a finite dimensionality assumption, we see that X admits a Chow-Künneth decomposition:

***Definition 7.1* (Murre [64]).** — A smooth projective variety X of dimension d is said to have a Chow-Künneth decomposition when there exist orthogonal projectors of sum 1

$$\Pi_i \in \mathsf{M}_{\mathsf{rat}}(F;\mathbb{Q})(h(X),h(X)) \qquad 0 \leqslant i \leqslant 2d$$

such that for any Weil cohomology theory H^*

$$\mathsf{H}^*(\Pi_i) = \Delta_{i,2d-i}.$$

A decomposition is said to be self dual when $\Pi_{2d-i} = \Pi_i^t$.

Thus finite dimensionality and standard conjecture $\mathsf{C}(X)$ imply an older conjecture made by Murre in [**64**]:

***Conjecture 7.2* (Murre [64]).** — Any smooth projective variety X has a Chow-Künneth decomposition.

***Remark 7.3*.** — This implies that the conjectural homological weight decomposition lifts

$$h(X) = h^0(X) \oplus h^1(X) \oplus \cdots \oplus h^{2d-1}(X) \oplus h^{2d}(X)$$

in $\mathsf{M}_{\mathsf{rat}}(F;\mathbb{Q})$. However in general neither the projectors Π_i nor the decomposition are unique.

In fact Murre conjectures much more in [**64**] and as we shall see in section 8 the strong form of Murre's conjecture, namely conjecture 8.2, along with the standard conjecture D implies the Kimura-O'Sullivan conjecture: see theorem 8.4. As for most of the cohomology theories, the deepest part of a decomposition is the half-dimensional part and we shall have a striking example of this important phenomenon with the Bloch conjecture in section 9.

7.2. Some known cases. — The conjecture is known to be true in some important cases which are low dimensional cases. For curves one has always a decomposition

$$h(X) = h^0(X) \oplus h^1(X) \oplus h^2(X)$$

and moreover if X is irreductible and has a rational point then $h^0(X) = \mathbf{1}$ and $h^2(X) = \mathbb{L}$. The half-dimensional part corresponds to Jacobians. More precisely for two smooth projective curves X, X' one has an isomorphism

$$\mathsf{M}_{\mathsf{rat}}(F;\mathbb{Q})(h^1(X), h^1(X')) = \mathsf{VarAb}_F(\mathrm{Jac}_X, \mathrm{Jac}_{X'})_{\mathbb{Q}}$$

where VarAb_F denotes the category of abelian varieties over F. For a surface the work of Murre [**63**] provides a decomposition

$$h(X) = h^0(X) \oplus h^1(X) \oplus h^2(X) \oplus h^3(X) \oplus h^4(X)$$

and again if X is irreductible and has a rational point then $h^0(X) = \mathbf{1}$ and $h^4(X) = \mathbb{L}^{\otimes 2}$. The h^1 part corresponds to Picard abelian variety through the isomorphism

$$\mathsf{M}_{\mathsf{rat}}(F;\mathbb{Q})(h^1(X), h^1(X')) = \mathsf{VarAb}_F(\mathrm{Pic}^0_X, \mathrm{Pic}^0_{X'})_{\mathbb{Q}}$$

and the h^3 part corresponds to Albanese abelian variety (dual to Picard) through the isomorphism

$$\mathsf{M}_{\mathsf{rat}}(F;\mathbb{Q})(h^3(X), h^3(X')) = \mathsf{VarAb}_F(\mathrm{Alb}_{X'}, \mathrm{Alb}_X)_{\mathbb{Q}}.$$

Already for smooth projective surfaces the h^2 carries remarkable information since the half-dimensional part is strongly related to the Bloch conjecture. For abelian varieties the conjecture is a consequence of the works of Shermenev [**77**] and Deninger-Murre [**19**] as we have seen in subsection 6.2.

8. The link with BBM's conjecture

In order to state Bloch-Beilinson's conjecture or Murre's conjecture, we assume in the sequel that the standard conjecture C is fulfilled. As explained by A. Beilinson in [**6**], the most fundamental conjecture about the motivic world, namely the existence of a suitable abelian tensor category of mixed motives inside the category of triangulated motives, provides very precise information on the structure of Chow groups of smooth projective varieties through a remarkable filtration. Let us recall very briefly how this filtration follows from the motivic formalism, we refer to [**38**, §4] for a more thorough account. Let $\mathscr{D}(F;\mathbb{Q})$ be the triangulated category of mixed motives over F with rational coefficients(7) and $\mathscr{M}\mathscr{M}(F;\mathbb{Q})$ be the abelian category of mixed motives. In

(7)Recall that V. Voevodsky [**82**], M. Levine [**57**] and M. Hanamura [**31, 33, 32**] have independently constructed such a category of triangulated motives satisfying property (B). The existence of a motivic t-structure is still however a wide open question. Nevertheless in [**56**] Levine has proved that for a field satisfying Beilinson-Soulé's vanishing conjecture (by A. Borel's work [**9**] number fields are among them) there exists a motivic t-structure on the triangulated subcategory of mixed Tate motives.

this conjectural picture one should have the following commutative square

$$\begin{tikzcd}
\mathrm{SmProj}_F^{\mathrm{op}} \arrow[r, "\text{embedd.}"] \arrow[d, "h"] & \mathrm{Sm}_F^{\mathrm{op}} \arrow[d, "\mathfrak{R}"] \\
\mathsf{M}_{\mathsf{rat}}(F;\mathbb{Q}) \arrow[r, "\text{embedd.}"] \arrow[d] & \mathscr{D}(F;\mathbb{Q}) \arrow[d, "\oplus_i {}^{\mathfrak{m}}H^i"] \\
\mathsf{M}_{\mathsf{num}}(F;\mathbb{Q}) \arrow[r, "\text{embedd.}"] & \mathscr{M}\mathscr{M}(F)_{\mathbb{Q}}
\end{tikzcd}$$

where ${}^{\mathfrak{m}}H^i$ are the cohomological functors associated to the motivic t-structure and the functor $\mathfrak{R}$ should enjoy the following properties:

L. (see [**17**]) for every smooth projective variety X one has a (non canonical) isomorphism
$$\mathfrak{R}(X) = \bigoplus_{i=0}^{\infty} {}^{\mathfrak{m}}H^i\mathfrak{R}(X)[-i];$$

B. the motivic cohomology groups of a smooth variety X are canonically isomorphic to Bloch's higher Chow groups
$$\mathrm{CH}^p(X,q)\otimes\mathbb{Q} = \mathrm{Hom}_{\mathscr{D}(F;\mathbb{Q})}(\mathbf{1},\mathfrak{R}(X)(p)[2p-q]).$$

Beilinson's filtration is then the filtration gotten from the spectral sequence
$$E_2^{p,q}(X) = \mathrm{Hom}_{\mathscr{D}(F;\mathbb{Q})}(\mathbf{1},{}^{\mathfrak{m}}H^q\mathfrak{R}(X)(r)[p]) \Longrightarrow \mathrm{Hom}_{\mathscr{D}(F;\mathbb{Q})}(\mathbf{1},\mathfrak{R}(X)(r)[p+q])$$
on its limit term which, according to (B), is precisely the Chow group
$$\mathrm{CH}^r(X;\mathbb{Q}) = \mathrm{Hom}_{\mathscr{D}(F;\mathbb{Q})}(\mathbf{1},\mathfrak{R}(X)(r)[2r]).$$
The condition (L) implies the degeneracy at E_2 of the above spectral sequence and so the graded pieces of Beilinson's conjectural filtration are given by
$$\mathrm{Gr}^{\nu}\mathrm{CH}^r(X;\mathbb{Q}) = E_2^{\nu,r-2\nu}(X) = \mathrm{Hom}_{\mathscr{D}(F;\mathbb{Q})}(\mathbf{1},h^{2r-\nu}(X)(r)[\nu]).$$
The idea of such a filtration can be traced back to works of Bloch [**8**]. More precisely Bloch-Beilinson conjecture may be stated as follows:

Conjecture 8.1 **(Bloch-Beilinson)**. — For all smooth projective varieties there exists a filtration
$$F^{\nu}\mathrm{CH}^r(X;\mathbb{Q}) \subset \mathrm{CH}^r(X;\mathbb{Q})$$
such that:

BB_1. one has $F^0\mathrm{CH}^r(X;\mathbb{Q}) = \mathrm{CH}^r(X;\mathbb{Q})$ and
$$F^1\mathrm{CH}^r(X;\mathbb{Q}) = \left|\begin{array}{l}\text{elements in } \mathrm{CH}^r(X;\mathbb{Q}) \\ \text{such that} \\ \alpha =_{\mathrm{hom}} 0;\end{array}\right.$$

BB_2. the filtration is stable by pull-back, push-forward and[8]

$$F^\nu \mathrm{CH}^r(X;\mathbb{Q}) \cdot F^\mu \mathrm{CH}^s(X;\mathbb{Q}) \subset F^{\nu+\mu}\mathrm{CH}^{r+s}(X;\mathbb{Q});$$

BB_3. the action of the Künneth projectors is given by:

$$\mathrm{Gr}^\nu \mathrm{CH}^r(X;\mathbb{Q}) \xrightarrow{\pi^i} \mathrm{Gr}^\nu \mathrm{CH}^r(X;\mathbb{Q}) = \begin{cases} 1 & \text{if } i = 2r-\nu \\ 0 & \text{otherwise;} \end{cases}$$

BB_4. the filtration is separated:

$$F^\nu \mathrm{CH}^r(X;\mathbb{Q}) = 0 \quad \text{for} \quad \nu >> 0.$$

In [**64**] Murre was also led to conjecture the existence of a remarkable filtration on the Chow groups of a smooth projective variety given in terms of a Chow-Künneth decomposition (Murre's conjecture is therefore a strengthening of conjecture 7.2).

***Conjecture 8.2* (Murre [64])**. — Every smooth projective variety X has a Chow-Künneth decomposition Π_i. Furthermore:

M_1. the action of Π_i on $\mathrm{CH}^r(X;\mathbb{Q})$ is zero for $i > 2r$;

M_2. the filtration

$$F^\nu \mathrm{CH}^r(X;\mathbb{Q}) = \cap_{i>2r-\nu} \ker(\Pi^i)$$

does not depend on the Chow-Künneth decomposition;

M_3. one has $F^0\mathrm{CH}^r(X;\mathbb{Q}) = \mathrm{CH}^r(X;\mathbb{Q})$ and

$$F^1\mathrm{CH}^r(X;\mathbb{Q}) = \left|\begin{array}{l} \text{elements in } \mathrm{CH}^r(X;\mathbb{Q}) \\ \text{such that} \\ \alpha =_{\mathrm{hom}} 0. \end{array}\right.$$

The formulation in terms of a Chow-Künneth decomposition has the advantage over conjecture 8.1 to be meaningful for a single smooth projective variety whereas in order to state axiom BB_2 the filtration has to be defined over all smooth projective varieties. As proved by Jannsen in [**38**] the above conjectures are equivalent:

***Theorem 8.3* (Jannsen [38])**. — *Murre's conjecture 8.2 is equivalent to Bloch-Beilinson's conjecture 8.1 and the two filtrations are the same.*

In the sequel we denote simply by BBM those equivalent conjectures. The link with finite dimensionality is given by the next result which is due to O'Sullivan. We refer to [**3**, 11.5.3.1] for a proof.

***Theorem 8.4* (O'Sullivan)**. — *The following two conjectures*

- *standard conjecture* D,
- BBM *conjecture*

implies the Kimura-O'Sullivan's conjecture.

[8] Consequently homological correspondences, in particular Künneth projectors, act on the graded pieces.

PART II
APPLICATIONS

9. The Bloch conjecture

We assume in this section that F is the field of complex numbers $\mathbb{C}$. Let $(V_{\mathbb{Z}}, F^*V_{\mathbb{C}})$ be a pure Hodge structure of weight $2k-1$. We have a decomposition

$$V_{\mathbb{C}} = F^kV_{\mathbb{C}} \oplus \overline{F^kV_{\mathbb{C}}}$$

and this implies that $V_{\mathbb{R}} \cap F^kV_{\mathbb{C}} = 0$. So the morphism $V_{\mathbb{R}} \to V_{\mathbb{C}}/F^kV_{\mathbb{C}}$ is an isomorphism of $\mathbb{R}$-vector spaces. The image of $V_{\mathbb{Z}}$ in $V_{\mathbb{R}}$ is a lattice and so provides a lattice in $V_{\mathbb{C}}/F^kV_{\mathbb{C}}$. The complex torus quotient of $V_{\mathbb{C}}/F^kV_{\mathbb{C}}$ by this lattice

$$J(V_{\mathbb{Z}}, F^*V_{\mathbb{C}}) := V_{\mathbb{C}}/(F^kV_{\mathbb{C}} \oplus V_{\mathbb{Z}}/\mathrm{torsion})$$

is called the intermediate Jacobian of the pure Hodge structure. Now if X denotes a smooth connected projective variety of dimension d, then $H^{2k-1}(X, \mathbb{Z})$ is the underlying abelian group of a pure Hodge structure of weight $2k-1$ and the associated complex torus

$$J^{2k-1}(X) = H^{2k-1}(X, \mathbb{C})/(F^kH^{2k-1}(X, \mathbb{C}) \oplus H^{2k-1}(X, \mathbb{Z})/\mathrm{torsion})$$

is the k-th intermediate Jacobian of X. By Poincaré duality we have a square

$$\begin{array}{ccc} H^{2k-1}(X,\mathbb{Z})/\mathrm{torsion} & \simeq & H_{2d-2k+1}(X,\mathbb{Z})/\mathrm{torsion} \\ \downarrow & & \downarrow \\ H^{2k-1}(X,\mathbb{C}) & \simeq & H^{2d-2k+1}(X,\mathbb{C})^{\vee} \end{array}$$

where the last map is given by the integration of forms along chains. The horizontal isomorphism identifies $F^kH^{2k-1}(X, \mathbb{C})$ with $F^{d-k+1}H^{2d-2k+1}(X, \mathbb{C})^{\perp}$ and so the k-th intermediate Jacobian of X may also be described as the complex torus

$$J^{2k-1}(X) = F^{d-k+1}H^{2d-2k+1}(X, \mathbb{C})^{\vee}/(H_{2d-2k+1}(X, \mathbb{Z})/\mathrm{torsion}).$$

For $k = d$, the complex torus $\mathrm{Alb}(X) := J^{2d-1}(X)$ is an abelian variety called the Albanese variety. We have by definition

$$\mathrm{Alb}(X) = F^1H^1(X, \mathbb{C})^{\vee}/(H_1(X, \mathbb{Z})/\mathrm{torsion})$$

and since $F^1H^1(X, \mathbb{C}) = H^0(X; \Omega^1_X)$ we have finally the following description of the Albanese variety

$$\mathrm{Alb}(X) = H^0(X, \Omega^1_X)^{\vee}/(H_1(X, \mathbb{Z})/\mathrm{torsion})$$

where $(H_1(X, \mathbb{Z})/\mathrm{torsion})$ is seen as a lattice in $H^0(X, \Omega^1_X)^{\vee}$ through the integration of forms. For more on intermediate Jacobians we refer to [**84**, §12.1]. Now choose a base point a in X and let x be another point. By integration of forms along a path from a to x, we get for each path a element in $H^0(X, \Omega^1_X)^\vee$. All those linear forms for different paths have the same image in the quotient $\mathrm{Alb}(X)$ and we get this way a morphism

$$\mathrm{alb}_{X,a} : X \to \mathrm{Alb}(X)$$

which depends on a only up to a translation. This morphism extends linearly to a morphism $Z_0(X) \to \mathrm{Alb}(X)$ where $Z_0(X)$ is the group of 0-cycles. The restriction of this morphism to the subgroup $Z_0(X)_0$ of 0-cycles of degree 0 does not depend any more on the base point a and passes to rational equivalence. We obtain then the Abel-Jacobi map:

$$\mathrm{AJ}_X : \mathrm{CH}_0(X)_0 \to \mathrm{Alb}(X).$$

9.1. The refined Chow-Künneth decomposition of a surface. — Let X be a smooth projective surface. The Neron-Severi group of the surface X is the finitely generated abelian group given by 1-codimensional algebraic cycles modulo algebraic equivalence:

$$NS(X) := A^1_{\mathrm{alg}}(X) = \left| \begin{array}{c} \text{1-codimensional algebraic cycles} \\ \text{modulo} \\ \text{algebraic equivalence.} \end{array} \right.$$

As proved by T. Matsusaka [**59**, Theorem 4] this group is also up to torsion the group of 1-codimension algebraic cycles modulo numerical equivalence

$$NS(X)_{\mathbb{Q}} = A^1_{\mathrm{alg}}(X; \mathbb{Q}) = A^1_{\mathrm{num}}(X; \mathbb{Q}).$$

Thus the intersection pairing

$$NS(X)_{\mathbb{Q}} \otimes NS(X)_{\mathbb{Q}} \xrightarrow{\frown} A^2_{\mathrm{num}}(X; \mathbb{Q}) \xrightarrow{\deg} \mathbb{Q}$$

is non degenerate and compatible with the pairing on a Weil cohomology H^*

$$\mathsf{H}^2(X) \otimes \mathsf{H}^2(X) \xrightarrow{\smile} \mathsf{H}^4(X) \xrightarrow{\mathrm{Tr}} K.$$

This breaks the H^2 in two pieces: an algebraic piece given by the finitely generated Neron-Severi group and a remaining piece called the transcendental part. We have thus a decomposition

$$\mathsf{H}^2(X) = NS(X)_K \oplus \mathsf{H}^2_{\mathrm{tr}}(X) \dashleftarrow \left| \begin{array}{c} \text{transcendental part of} \\ \text{the cohomology} \end{array} \right.$$

where $\mathsf{H}^2_{\mathrm{tr}}(X)$ is called the transcendental part. This decomposition is motivic:

Proposition 9.1 **(Kahn-Murre-Pedrini [45]).** — *In a Chow-Künneth decomposition of X there exists a canonical decomposition*

$$h^2(X) = h^2_{\mathrm{alg}}(X) \oplus \mathfrak{t}^2(X)$$

that induces the previous decomposition of a Weil cohomology H^*:

$$\mathsf{H}^*(h^2_{\mathrm{alg}}(X)) = NS(X)_K \qquad \mathsf{H}^*(\mathfrak{t}^2(X)) = \mathsf{H}^2_{\mathrm{tr}}(X).$$

One of the essential result proved in [**45**] is the explicit computation of the Chow groups of each part. Indeed the Neron-Severi group is recovered up to torsion by the only non vanishing Chow group of the algebraic part while the computation of the Chow group of the transcendental part is related to the kernel of the Abel-Jacobi map. Namely the computation gives the following results:

– the Chow groups of the algebraic part are given by
$$\mathrm{CH}^i(h^2_{\mathrm{alg}}(X)) = \begin{cases} NS(X)_{\mathbb{Q}} & \text{if } i=1 \\ 0 & \text{otherwise;} \end{cases}$$
– the Chow groups of the transcendental part are given by
$$\mathrm{CH}^i(\mathfrak{t}^2(X)) = \begin{cases} T(X)_{\mathbb{Q}} & \text{if } i=2 \\ 0 & \text{otherwise} \end{cases}$$
where $T(X)$ is the kernel of the Abel-Jacobi map.

Remark 9.2. — Since the algebraic part $h^2_{\mathrm{alg}}(X)$ is a twisted Artin motive [**45**, Proposition 14.2.3], one sees that the only part of the Chow motive $h(X)$ of a surface that may fail to be of finite dimension is its transcendental part $\mathfrak{t}^2(X)$. As usual the most interesting phenomena occur in the middle part of the cohomology and for a surface this is precisely in the transcendental part of the h^2.

9.2. Motivic reformulation of the Bloch conjecture. — Let $X/\mathbb{C}$ be a smooth projective surface. Recall that the geometric genus of X is the number
$$p_g(X) = \dim H^2(X, \mathcal{O}_X).$$
Then one has the following conjecture:

Bloch conjecture 9.3. — Let $X/\mathbb{C}$ be smooth projective surface with $p_g(X) = 0$. The kernel $T(X)$ of the Abel-Jacobi map
$$\mathrm{AJ}_X : \mathrm{CH}_0(X)_0 \to \mathrm{Alb}(X)$$
vanishes.

Remark 9.4. — As shown by D. Mumford if $p_g(X) \neq 0$ the kernel of the Abel-Jacobi map is huge and thus far from being zero.

The following is the motivic reformulation of the Bloch conjecture:

Theorem 9.5 **(Guletskii-Pedrini [28], Kahn-Murre-Pedrini [45])**
Let $X/\mathbb{C}$ be a smooth projective surface. The following are equivalent:

1. $p_g(X) = 0$ *and* $h(X)$ *is of finite dimension.*
2. *The transcendental part vanishes:* $\mathfrak{t}^2(X) = 0$.
3. *Bloch conjecture holds:* $T(X) = 0$.

Sketch of proof. — $(1 \Rightarrow 2)$ Assume that $p_g(X) = 0$ and that the Chow motive $h(X)$ is finite dimensional. Then for a Weil cohomology H^* one has

$$\mathsf{H}^2_{\mathrm{tr}}(X) = \mathsf{H}^*(\mathfrak{t}^2(X)) = 0.$$

Hence $\mathfrak{t}^2(X)$ is homologically (and so numerically) phantom and an application of the nilpotence theorem via corollary 4.30 implies that $\mathfrak{t}^2(X) = 0$.

$(2 \Rightarrow 3)$ Assume that $\mathfrak{t}^2(X) = 0$. The explicit computation of the Chow groups of the transcendental part gives

$$\mathrm{CH}^2(\mathfrak{t}^2(X)) = T(X)_{\mathbb{Q}} = 0$$

and the result follows then from A. Roĭtman's theorem [**69**] which assures that the torsion part of the kernel is zero.

$(3 \Rightarrow 1)$ Now assume that $T(X) = 0$. Choose a finitely generated extension $k/\mathbb{Q}$ and $\mathcal{X}/k$ a smooth projective surface such that

$$\mathcal{X} \times_k \mathbb{C} = X.$$

Then $k(\mathcal{X}) \hookrightarrow \mathbb{C}$ and using a transfer argument one shows that

$$T(\mathcal{X}_{k(\mathcal{X})})_{\mathbb{Q}} \subset T(X)_{\mathbb{Q}} = 0.$$

A result of [**45**] gives

$$\mathsf{M}_{\mathsf{rat}}(F;\mathbb{Q})(\mathfrak{t}^2(\mathcal{X}), \mathfrak{t}^2(\mathcal{X})) = T(\mathcal{X}_{k(\mathcal{X})})_{\mathbb{Q}} = 0$$

and so $\mathfrak{t}^2(\mathcal{X}) = 0$. This implies $\mathfrak{t}^2(X) = 0$ and proves that $p_g(X) = 0$ and that $h(X)$ is finite dimensional. □

10. Motives over finite fields

In the sequel $\mathbb{F}_q$ denotes a fixed finite field. In this section we explain the relation highlighted in [**42**] between the finite dimensionality conjecture and two other deep conjectures: the strong form of the Tate conjecture and the Beilinson conjecture. It is appropriate to recall here the statement of these two conjectures. We let $X/\mathbb{F}_q$ be a smooth projective variety.

- The strong Tate conjecture predicts that the order of the pole of the Hasse-Weil zeta function of X at an nonnegative integer r is the dimension of the $\mathbb{Q}$-vector space of codimension r algebraic cycles with $\mathbb{Q}$-coefficients modulo numerical equivalence:
$$\mathrm{ord}_{s=r}\zeta(X,s) = -\dim_{\mathbb{Q}} A^r_{\mathrm{num}}(X;\mathbb{Q}).$$
- Beilinson's conjecture predicts that numerical and rational equivalences coincide up to torsion *i.e.* the morphism
$$\mathrm{CH}^r(X;\mathbb{Q}) \to A^r_{\mathrm{num}}(X;\mathbb{Q})$$
is an isomorphism for any nonnegative integer r. This is a strengthening of the standard conjecture $\mathsf{D}(X)$.

In [**78**] C. Soulé was able to prove both of them for varieties of abelian type and dimension at most three. In [**24**, Theorem 3.3] Geisser shows that the conjunction of the Tate conjecture and the Beilinson conjecture (called the Tate-Beilinson conjecture in [**44**]) implies Parshin's conjecture that the higher K-groups $K_a(X)$ for $a > 0$ are all torsion. B. Kahn has given a reformulation of the Tate-Beilinson conjecture and has shown that it implies many other conjectures. We refer to [**44**, §4.7.5 to §4.7.8] for a detailed account of the consequences of the Beilinson-Tate conjecture.

Our main purpose is to explain how the nilpotence theorem is used by Kahn in [**42**] to prove Beilinson's conjecture for a smooth projective variety $X/\mathbb{F}_q$ of abelian type for which the Tate conjecture is known. This result extends Soulé's pioneer work [**78**] and provides new families of smooth projective varieties over $\mathbb{F}_q$ satisfying Beilinson's conjecture such as for example products of elliptic curves (the Tate conjecture is known for them by [**79**]). In particular the proof of [**42**] allows to revisit Geisser's result [**24**, Theorem 3.3][(9)].

10.1. The Tate conjecture. — We refer to [**81**] and [**61**, §8] for a more detailed account of the Tate conjecture. Let F be a field finitely generated over its prime subfield and $\overline{F}$ a separable closure of F. We denote by ℓ a prime number different from the characteristic of F and we let X/F be a smooth projective variety. The geometric ℓ-adic cohomology group $H^{2r}_{\text{ét}}(\overline{X}, \mathbb{Q}_\ell(r))$ is a finite dimensional $\mathbb{Q}_\ell$-vector space with a continuous action of the profinite group $G_F := \mathrm{Gal}(\overline{F}/F)$. The image of the ℓ-adic cycle class map $\mathrm{CH}^r(X) \to H^{2r}_{\text{ét}}(\overline{X}, \mathbb{Q}_\ell(r))$ is contained in the subspace of points left fixed under the action of the Galois group. Since the $\mathbb{Q}_\ell$-vector space spanned by the image of the ℓ-adic cycle class map is isomorphic to the subspace $A^r_{\text{hom}}(X; \mathbb{Q}_\ell)$ of cycles with $\mathbb{Q}_\ell$-coefficients modulo homological equivalence, the cycle class map gives an injection

$$A^r_{\text{hom}}(X; \mathbb{Q}_\ell) \hookrightarrow H^{2r}_{\text{ét}}(\overline{X}, \mathbb{Q}_\ell(r))^{G_F}.$$

The cohomological Tate conjecture asserts that the subspace of cohomology classes left fixed under Galois is precisely the $\mathbb{Q}_\ell$-vector space $A^r_{\text{hom}}(X; \mathbb{Q}_\ell)$ spanned by the image of the ℓ-adic cycle class map.

Tate conjecture $\mathsf{CTC}(X)$ ***10.1***. — The geometric ℓ-adic cycle class map

$$\mathrm{CH}^r(X; \mathbb{Q}_\ell) \to H^{2r}(\overline{X}, \mathbb{Q}_\ell(r))^{G_k} \tag{16}$$

is a surjective morphism for any nonnegative integer r.

This conjecture is widely opened and known in some scattered cases. Even for abelian varieties the result is not known in general. It has been proved for the h^1 part

(9) The proofs in [**42**] and [**24**] are applications of a method introduced by Soulé [**78**]. They both rely on a nilpotence result, namely Jannsen's theorem 4.8 in [**24**] and Kimura's theorem 4.29 in [**42**].

of an abelian variety[(10)] in the work of Tate and Zarhin for positive characteristic and in the work of Faltings for characteristic zero. Over a finite field more is known:

– in [**78**] Soulé proves the conjecture for varieties of abelian type and dimension at most three[(11)];
– in [**79**] M. Spiess proves the conjecture for product of elliptic curves.

We consider now only the case of a finite field. The conjecture $\mathsf{CTC}(X)$ is then a weak form of the conjecture that relates the order of the pole of the Hasse-Weil Zeta function of X to the dimension of the space of codimension r cycles with $\mathbb{Q}$-coefficients modulo numerical equivalence. As shown in [**81**, Theorem 2.9] the conjunction of $\mathsf{CTC}(X)$ and $\mathsf{D}(X)$ is equivalent to the strong Tate conjecture:

Tate conjecture $\mathsf{TC}(X)$ ***10.2***. — We have

$$\mathrm{ord}_{s=r}\zeta(X,s) = -\dim_{\mathbb{Q}} A^r_{\mathrm{num}}(X;\mathbb{Q})$$

for all nonnegative integers r.

The relation between the weak and the strong form of the Tate conjecture is also given by the partial semi-simplicity conjecture.

Conjecture 10.3. — 1 is not a multiple root of the minimal polynomial of Frobenius acting on $H^{2r}_{\text{ét}}(\overline{X},\mathbb{Q}_\ell(r))$ for any nonnegative integer r.

As shown in [**81**, Theorem 2.9] the strong form $\mathsf{TC}(X)$ is equivalent to the conjunction of the weaker form $\mathsf{CTC}(X)$ and the partial semi-simplicity conjecture 10.3. Since the later is known for varieties of abelian type (see *e.g.* [**42**, Lemma 1.9]) the conjecture $\mathsf{TC}(X)$ is equivalent to its weaker form $\mathsf{CTC}(X)$ for those varieties.

10.2. Motives over finite fields. — Four results give very powerful tools to study motives over finite fields:

– the proof of the Weil conjecture by Deligne (see also [**50**]);
– Jannsen's semi-simplicity theorem;
– the nilpotence theorem;
– the existence of the Frobenius automorphism.

For more about the interplay between motives over finite fields and Frobenius we refer to [**62**]. In [**78**] Soulé introduces a very efficient weight argument[(12)] to prove the vanishing of some $\mathbb{Q}$-vector space H attached to a motive M. This method is latter used by Geisser [**24**, Theorem 3.3] to obtain a conditional result on Parshin's conjecture, and roughly speaking works as follows:

(10) Both sides of (16) are meaningful for Chow motives, so it makes sense to state the cohomological Tate conjecture in that setting.
(11) This statement is actually a slight strengthening of the original result proved by Soulé, see [**44**, Example 75.1,Theorem 82].
(12) This argument is explained in the introduction of [**78**].

– first you have to find a constant $\lambda \in \mathbb{Q}$ such that Frobenius acts on H as multiplication by λ;
– then you have to find a polynomial $P \in \mathbb{Q}[t]$ such that the endomorphism $P(F_M)$ acts by zero on H;
– at last you must prove that λ is not a root of P.

In practice the constant λ is related to the weight of H and the polynomial P is some power (produced if needed by the use of the nilpotence theorem) of the characteristic or minimal polynomial associated to the action of Frobenius on some Weil cohomology group, K-theory group or motivic cohomology group. The deepest part is then to prove that the constant by which Frobenius acts is not a root of this polynomial for this is achieved through either the Weil conjecture or the Tate conjecture.

For the application of this argument to the Beilinson conjecture in [**42**], the Tate conjecture is used via proposition 10.6. To state this result we need to recall some facts about Frobenius of numerical motives.

Lemma 10.4. — *Let M be a simple numerical motive. The algebra $\mathbb{Q}[F_M]$ generated by F_M in $\mathrm{End}_{\mathrm{num}}(M)$ is a finite extension of $\mathbb{Q}$.*

Proof. — It is enough to remark that the algebra generated by Frobenius $\mathbb{Q}[F_M]$ is a commutative subalgebra of $\mathrm{End}_{\mathrm{num}}(M)$ which is a finite dimensional $\mathbb{Q}$-division algebra by Jannsen's semi-simplicity theorem. □

By lemma 10.4 the Frobenius automorphism F_M of a simple numerical motive defines an algebraic number up to conjugacy by the absolute Galois group $G_{\mathbb{Q}} := \mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$ of $\mathbb{Q}$. In the sequel we denote by

$$[F_M] \in G_{\mathbb{Q}} \backslash \overline{\mathbb{Q}}$$

the conjugacy class of F_M. In [**62**] Milne has studied the properties of the category of numerical motives $\mathsf{M}_{\mathsf{num}}(\mathbb{F}_q)_{\mathbb{Q}}$ over $\mathbb{F}_q$ under Tate conjecture TC. He shows then that simple numerical motives are entirely known through their associated Frobenius isomorphism. Let $\Sigma(\mathsf{M}_{\mathsf{num}}(\mathbb{F}_q)_{\mathbb{Q}})$ be the set of isomorphism classes of simple numerical motives. Recall that an algebraic number α is said to be a Weil q-number of weight m if

1. for every embedding $\rho : \mathbb{Q}[\alpha] \hookrightarrow \mathbb{C}$ we have $|\rho(\alpha)| = q^{m/2}$,
2. for some nonnegative integer n the number $q^n\alpha$ is an algebraic integer,

and let $G_{\mathbb{Q}} \backslash \mathscr{W}(q)$ be the conjugacy classes of Weil q-numbers under the action of the absolute Galois group $G_{\mathbb{Q}}$ of $\mathbb{Q}$. Then the Honda-Tate flavoured[13] result proved in [**62**, Proposition 2.6] is the following statement:

---

[13] The proof that the map of proposition 10.5 is onto uses indeed the classification up to isogenies of simple abelian varieties over $\mathbb{F}_q$ in terms of their associated Weil q-number obtained by Honda and Tate [**36**]. Furthermore the arguments in [**62**] shows that $\mathsf{M}_{\mathsf{num}}(\mathbb{F}_q)_{\mathbb{Q}}$ is generated as a Tannakian category by the full tensor subcategory $\mathsf{M}_{\mathsf{num}}(\mathbb{F}_q)_{\mathbb{Q}}^{\mathrm{ab}}$ of motives of abelian type.

Proposition 10.5. — *Assume Tate's conjecture* TC *i.e. that* $\mathsf{TC}(X)$ *holds for any smooth projective variety* $X/\mathbb{F}_q$. *The map*

$$\begin{array}{ccc} \Sigma(\mathsf{M}_{\mathsf{num}}(\mathbb{F}_q)_{\mathbb{Q}}) & \longrightarrow & G_{\mathbb{Q}}\backslash\mathscr{W}(q) \\ M & \mapsto & [F_M] \end{array}$$

is a bijection.

In [**24**, Theorem 2.7] Geisser has proved the converse statement by showing that proposition 10.5 is in fact equivalent to Tate's conjecture TC given in 10.2. In [**62**] the strong form of Tate's conjecture is assumed once for all to be true for all smooth projective varieties. However if one does not want to do so, and has a closer look at the proof of proposition 2.6 in *loc.cit.* we see that the following result holds:

Proposition 10.6. — *Let* $X/\mathbb{F}_q$ *be a smooth projective variety satisfying* $\mathsf{TC}(X)$ *and* r *an integer. Then for each simple factor* M *of* $\overline{h}(X)$ *one has* $[F_M] = q^r$ *if and only if* M *is isomorphic to* $\mathbb{L}^r$.

10.3. Beilinson conjecture. — As a consequence of his conjecture on the existence of a very special filtration on Chow groups [**6**], Beilinson was led over finite fields to the conjecture that up to torsion there exists only one adequate equivalence relation on algebraic cycles. Let $X/\mathbb{F}_q$ be a smooth projective variety:

Beilinson conjecture $\mathsf{Be}(X)$ ***10.7***. — Rational and numerical equivalences agree on X up to torsion, in other words the maps

$$\mathrm{CH}^r(X;\mathbb{Q}) \to A^r_{\mathrm{num}}(X;\mathbb{Q})$$

are isomorphisms for all nonnegative integers r.

That numerical and homological equivalences agree is of course the content of the standard conjecture $\mathsf{D}(X)$. What is so specific to finite fields and also needs to kill torsion is the statement that rational and homological equivalence should also agree. This strong strengthening of the standard conjecture follows from the fact that finite fields offer, as far as the BBM conjecture is concerned, the simplest picture possible. Indeed we have the following application of Soulé's argument (see [**39**, Theorem 4.16]):

Theorem 10.8. — *The BBM filtration over* $\mathbb{F}_q$ *is trivial i.e. for any smooth projective* $X/\mathbb{F}_q$ *and all nonnegative integers* r *we have*

$$F^{\nu}_{\mathrm{BBM}}\mathrm{CH}^r(X;\mathbb{Q}) = F^{1}_{\mathrm{BBM}}\mathrm{CH}^r(X;\mathbb{Q}) = 0$$

for any integer $\nu \geqslant 1$.

Proof. — Fix a Weil cohomology theory H^* with coefficients in K and consider the characteristic polynomial $P_r \in K[T]$ of the action of the Frobenius F_X on the r-th cohomology group $\mathsf{H}^r(X)$:

$$P_r = \det(F_X - T\mathrm{id} \mid \mathsf{H}^r(X)).$$

Using the Weil conjecture and the result of Katz and Messing [**50**, Theorem 2], this polynomial is independent of the chosen Weil cohomology and its roots are Weil q-numbers of weight r. By Cayley-Hamilton's theorem $P_r(F_X) = 0$, and so for a positive integer ν we have $P_{2r-\nu}(F_X) = 0$ as endomorphisms of $\mathsf{H}^{2r-\nu}(X)$. The properties of the BBM filtration imply then that $P_{2r-\nu}(F_X)$ acts as zero on $\mathrm{Gr}^{\nu}_{\mathrm{BBM}}\mathrm{CH}^r(X;\mathbb{Q})$. On the other hand Soulé [**78**, Proposition 2] has proved that on $\mathrm{CH}^r(X;\mathbb{Q})$ the Frobenius F_X is the multiplication by q^r. Therefore $P_{2r-\nu}(F_X)$ is the multiplication by $P_{2r-\nu}(q^r)$ on the ν-th graded piece of the BBM filtration and it is enough to check that $P_{2r-\nu}(q^r)$ is non zero. This follows from the Weil conjecture since q^r is not a Weil q-number of weight $2r-\nu$ and so cannot be a root of $P_{2r-\nu}$. □

As the first step of the BBM filtration is given by elements $\alpha \in \mathrm{CH}^r(X;\mathbb{Q})$ that are homologically zero the above theorem proves that over a finite field the BBM conjecture implies that homological equivalence coincides with rational equivalence up to torsion.

The main theorem of [**42**] allows to deduce the Beilinson conjecture from the Tate conjecture for smooth projective varieties of abelian type. Before this result the only non trivial known cases were proved in all codimension by Soulé in [**78**] for smooth projective varieties of abelian type and dimension at most three and the conjecture was also known for 0-cycles by the work of K. Kato and S. Saito [**49**].

10.4. Beilinson conjecture and finite dimensionality. — We now turn to the interplay between Tate conjecture and finite dimensionality, namely the proof that the conjunction of Tate's conjecture $\mathsf{TC}(X)$ and Kimura-O'Sullivan's conjecture $\mathsf{KS}(X)$ implies Beilinson's conjecture $\mathsf{Be}(X)$ and Parshin's vanishing conjecture [**24, 42**]. By theorem 6.3 we know that Chow motives of smooth projective varieties of abelian type are finite dimensional, therefore in practice the above implication allows to deduce the Beilinson conjecture from the known cases of the Tate conjecture. Recall first Parshin's vanishing conjecture:

Parshin conjecture $\mathsf{Pa}(X)$ ***10.9***. — Let $X/\mathbb{F}_q$ be a smooth projective variety and $a > 0$ an integer. Then $K_a(X)_{\mathbb{Q}} = 0$.

We are now ready to state the main theorem of [**42**].

Theorem 10.10. — *Let $X/\mathbb{F}_q$ be a smooth projective variety. Then the conjunction of Tate's conjecture $\mathsf{TC}(X)$ and Kimura-O'Sullivan's conjecture $\mathsf{KS}(X)$ implies Beilinson's conjecture $\mathsf{Be}(X)$ and Parshin's vanishing conjecture $\mathsf{Pa}(X)$.*

Proof. — Fix a nonnegative integer r. Using Jannsen's semi-simplicity theorem the numerical motive $\overline{h}(X)$ decomposes into a direct sum of simple motives

$$\overline{h}(X) = N_1 \oplus \cdots \oplus N_s. \tag{17}$$

By assumption $h(X)$ is finite dimensional, therefore the nilpotence theorem allows to lift the decomposition (17) to a decomposition of finite dimensional Chow motives

$$h(X) = M_1 \oplus \cdots \oplus M_s.$$

such that $\overline{M_i}$ is a simple numerical motive. Since we have

$$\mathrm{CH}^r(X;\mathbb{Q}) = \mathrm{Hom}_{\mathrm{rat}}(\mathbb{L}^r, h(X)) = \bigoplus_{i=1}^{s} \mathrm{Hom}_{\mathrm{rat}}(\mathbb{L}^r, M_i)$$

as well as the analogous decomposition for numerical equivalence, it is enough to prove that the maps

$$\mathrm{Hom}_{\mathrm{rat}}(\mathbb{L}^r, M_i) \to \mathrm{Hom}_{\mathrm{num}}(\mathbb{L}^r, \overline{M_i})$$

are isomorphisms. We may assume that M_i is not isomorphic to $\mathbb{L}^r$. Let $\Pi_i \in \mathbb{Q}[T]$ be the minimal polynomial of the Frobenius automorphism $F_{\overline{M_i}}$. Since $\mathrm{End}_{\mathrm{num}}(\overline{M_i})$ is a finite dimensional $\mathbb{Q}$-division algebra, this is an irreductible polynomial and using once again the nilpotence theorem, Cayley-Hamilton's theorem assures that there exists a nonnegative integer N such that

$$\Pi_i(F_{M_i})^N = 0$$

in $\mathrm{End}_{\mathrm{rat}}(M_i)$. Let $\alpha : \mathbb{L}^r \to M_i$ be a morphism. Since $F_{M_i} \circ \alpha = q^r\alpha$, we have

$$\Pi_i(q^r)(\alpha) = \Pi_i(F_{M_i})^N(\alpha) = 0$$

and it remains to check[14] that $\Pi_i(q^r) \neq 0$. Since we have assumed that M_i is non isomorphic to $\mathbb{L}^r$, the nilpotence theorem implies that $\overline{M_i}$ is also non isomorphic to $\mathbb{L}^r$. By proposition 10.6 we have therefore $F_{\overline{M_i}} \neq q^r$ and thus the irreductible polynomial Π_i is not equal to $T - q^r$. The proof of the other implication is similar using the eigenspaces for the Adams operations on K-theory [**24**, Theorem 3.3]. □

Then we have the following corollary:

Corollary 10.11. — *Let $X/\mathbb{F}_q$ be a smooth projective variety of abelian type such that conjecture* $\mathsf{CTC}(X)$ *holds. Then Beilinson conjecture* $\mathsf{Be}(X)$ *and Parshin conjecture* $\mathsf{Pa}(X)$ *are also true.*

Proof. — It is enough to see that for a smooth projective variety of abelian type $X/\mathbb{F}_q$ the cohomological Tate conjecture implies the full Tate conjecture. □

[14] Until now we have only used the finite dimensionality assumption. The assumption that $\mathsf{TC}(X)$ holds is only used via proposition 10.6.

11. Motivic Zeta function

11.1. Kapranov's definition. — The idea of a motivic Zeta function can be traced back to insights of Grothendieck [**13**, Grothendieck 24/09/1964] among those where the existence of a motivic Euler characteristic with compact support which allows to attach to each variety a virtual Chow motive [**13**, Grothendieck 16/08/1964]. Building upon the case of the Hasse-Weil series, M. Kapranov defines in [**47**] a motivic Zeta function for each quasi-projective variety X. This is a power series with coefficients in a K-group of isomorphism classes of varieties over F modulo the cutting and pasting relation. The construction of a motivic Euler characteristic with compact support was achieved by H. Gillet and C. Soulé in [**25**] and is stated in terms of the above mentioned K-group: see theorem 11.14.

More precisely consider the following K-group of isomorphism classes of varieties over F modulo the cutting and pasting relation:

$$K_0(\mathrm{Var}_F) = \left| \begin{array}{l} \text{Free abelian group} \\ \text{on isomorphism classes} \\ \text{of varieties with relation} \\ [X] = [Z] + [X \setminus Z] \\ \text{for } Z \text{ closed subvariety.} \end{array} \right.$$

Let E be a rank n vector bundle over X and denote by $\mathbb{L}$ the class of $\mathbb{A}^1$ in our K-group. Then we have $[E] = \mathbb{L}^n \cdot [X]$ as well as a projective bundle formula

$$[\mathbb{P}(E)] = [1 + \mathbb{L} + \cdots + \mathbb{L}^{n-1}] \cdot [X]$$

and thus $\mathbb{L}$ is somehow analogous to the Lefschetz motive in the usual category of Chow motives.

Remark 11.1. — In case the field F is perfect we get exactly the same ring if in the definition we restrict ourselves to smooth varieties. In characteristic zero we may even find a presentation by smooth projective varieties as we shall see in theorem 11.7.

If A is a ring we denote by $1 + A[\![t]\!]^+$ the multiplicative group of formal power series with constant coefficient equal to 1. Following Kapranov [**47**] one can associate to a quasi-projective variety X a motivic Zeta function given by the formal power series

$$Z(X,t) = \sum_{i=0}^{\infty} [\mathrm{Sym}^n X] t^n \in 1 + K_0(\mathrm{Var}_F)[\![t]\!]^+ \tag{18}$$

with coefficients in $K_0(\mathrm{Var}_F)$. Given a ring A one can define an A-valued additive invariant of algebraic varieties as a map of ring[(15)]

$$K_0(\mathrm{Var}_F) \xrightarrow{\mu} A.$$

[(15)] These maps of rings are also called motivic measures in the works of Kapranov [**47**] and M. Larsen-V. Lunts [**54, 55**]. We follow here the terminology of [**18**], the motivic measure being the measure defined on a reasonable class of subsets of the arc space of X in motivic integration theory.

Then for any such additive invariant of algebraic varieties, the motivic zeta function specializes to a power series with coefficients in the ring A

$$Z_\mu(X,t) = \sum_{i=0}^{\infty} \mu([\mathrm{Sym}^n X])t^n \in 1 + A[\![t]\!]^+.$$

Example 11.2. — Euler characteristics with compact support associated with usual cohomology theories provide such additive invariant of algebraic varieties. For example if $F = \mathbb{C}$ then Betti cohomology defines an additive invariant

$$\mu_{\mathrm{Betti}}([X]) = \sum_i (-1)^i \dim H^i_c(X(\mathbb{C}), \mathbb{C}).$$

Similarly one can consider ℓ-adic cohomology with compact support for a prime ℓ not equal to the characteristic of F. In case $F = \mathbb{C}$ Deligne's mixed Hodge theory [**16**] provides also an additive invariant. Indeed each $H^i_c(X(\mathbb{C}), \mathbb{C})$ carries a natural mixed Hodge structure (see e.g. [**21**, 3.7.14]) and we get a additive invariant by taking the Hodge polynomial

$$H(X)(u,v) = \sum_{p,q} \left[\sum_{i \geqslant 0} (-1)^i h^{p,q}(H^i_c(X(\mathbb{C}), \mathbb{C})) \right] u^p v^q$$

where $h^{p,q}(H^i_c(X(\mathbb{C}), \mathbb{C}))$ is the (p,q)-Hodge number of the mixed Hodge structure carried by $H^i_c(X(\mathbb{C}), \mathbb{C})$. However very important additive invariants also arise by simpler means. Indeed if $F = \mathbb{F}_q$ then counting the number of rational points over $\mathbb{F}_q$ of a variety X is an additive invariant

$$\mu_{\mathrm{HW}}(X) = |X(\mathbb{F}_q)|$$

and the associated Zeta function coincides with the usual Hasse-Weil series of X which was proved to be rational by B. Dwork [**20**].

In [**47**] Kapranov raised the question of the existence of a motivic proof of Dwork's theorem. This amounts to ask whether the rationality of the Hasse-Weil Zeta series lifts to the Grothendieck ring $K_0(\mathrm{Var}_F)$ and for this one has to look at rationality for power series in ring that may fail to be integral domains[16]. In this context there are several possible definitions which may not be equivalent and we refer to [**55**, §2] for a review of them. To be precise in the sequel a formal power series in $1 + A[\![t]\!]^+$ will be said to be rational if it may be written as the quotient of two polynomials with constant coefficient equal to 1, in particular it is then rational in the sense of [**55**, Definition 2.1]. The « naive » rationality conjecture may then be stated as follows:

Rationality conjecture 11.3. — The motivic Zeta series (18) of a quasi-projective variety X is rational.

[16] Indeed B. Poonen has proved in [**68**] that $K_0(\mathrm{Var}_F)$ may not a domain (see remark 11.13).

Any proof of conjecture 11.3 would also provide a proof of the rationality of any Zeta series gotten from some additive invariant such as the Hasse-Weil series. Kapranov managed in his original article to give some pieces of evidence in direction of this conjecture.

Theorem 11.4 **(Kapranov [47]).** — *Let X be smooth irreductible projective curve with a line bundle of degree 1. Then for any additive invariant*

$$K_0(\mathrm{Var}_F) \xrightarrow{\mu} A$$

such that A is a field and $\mu(\mathbb{L}) \neq 0$, the Zeta series $Z_\mu(X,t)$ is rational.

However in [**55**, Theorem 7.6] Larsen-Lunts proved that conjecture 11.3 cannot be true as stated by exhibiting some counter example:[17]

Theorem 11.5. — *Let $X/\mathbb{C}$ be a complex smooth projective surface with Kodaira dimension $\geqslant 0$. Then there exist a field H and an additive invariant*

$$K_0(\mathrm{Var}_F) \xrightarrow{\mu} H$$

such that $\mu(\mathbb{L}) = 0$ and the Zeta series $Z_\mu(X,t)$ is not rational. In particular the Zeta series (18) is not rational.

Although theorem 11.5 proves that conjecture 11.3 cannot be true, there are still hopes for a motivic proof of Dwork's theorem. Indeed if one looks at the theorems 11.4 and 11.5 one sees that the additive invariants considered in those statements behave orthogonally with respect to the class of the affine line $\mathbb{L}$. This is where birational geometry appears and provides a better understanding of the kind of rationality results we might still expect to be true.

11.2. Birational invariants. — We will assume throughout this subsection that F is a field of characteristic zero. As we have said before the class of the affine line $\mathbb{L}$ is very similar to the Lefschetz motive and indeed Gillet-Soulé motivic Euler characteristic with compact support sends $\mathbb{L}$ to the Lefschetz motive. Recall that in the case of motives, the Lefschetz motive

$$\mathbb{L} \in \mathsf{M}^{\mathrm{eff}}_{\mathsf{rat}}(F;\mathbb{Q})$$

in the category of effective Chow motives has two different fates:

– one can invert $\mathbb{L}$ in order to get the category of (non effective) Chow motives

$$\mathsf{M}_{\mathsf{rat}}(F;\mathbb{Q}) = \mathsf{M}^{\mathrm{eff}}_{\mathsf{rat}}(F;\mathbb{Q})[\mathbb{L}^{-1}];$$

[17] We refer to [**55**] to some further developments in this direction. Note that the original formulation of Kapranov [**47**, Remark 1.3.5.b] is not as precise as the statement of conjecture 11.3.

– one can also impose the relation $\mathbb{L} = 0$ and this leads to birational Chow motives[(18)] and birational geometry.

As we shall see the same holds true for additive invariant. A central result in what follows is the following result known as weak factorization theorem:

Theorem 11.6. — *(Włodarczyk [**85**], Abramovich and al. [**2**]) Let $\phi : X \dashrightarrow Y$ be a birational map between smooth proper connected varieties and $U \subset X$ an open subset where ϕ is an isomorphism. Then ϕ can be factored into a sequence of blow-ups and blow-downs with smooth centers disjoint from U : there exists a sequence of birational maps*

$$X = X_1 \overset{\phi_1}{\dashrightarrow} X_2 \overset{\phi_2}{\dashrightarrow} \cdots \overset{\phi_{r-1}}{\dashrightarrow} X_{r-1} \overset{\phi_r}{\dashrightarrow} X_r = Y$$

with $\phi = \phi_r \circ \phi_{r-1} \circ \cdots \circ \phi_1$ such that each ϕ_i is an isomorphism over U and either ϕ_i or ϕ_i^{-1} is a blow-up with smooth center disjoint from U.

Using the weak factorization theorem and resolution of singularities, F. Heinloth provides in [**7**] a simpler description of the Grothendieck group of varieties having a strong birational flavor. More precisely let $K_0^{\mathrm{bl}}(\mathrm{Var}_F)$ denotes the free abelian group on isomorphism classes of smooth projective varieties modulo the blow-up relation

$$[\mathrm{Bl}_Y X] - [E] = [X] - [Y]$$

where X is smooth projective, Y is a smooth closed subvariety of X, $\mathrm{Bl}_Y X$ is the blow-up of Y in X and E is the exceptional divisor of this blow-up. Then we have the following result:

***Theorem 11.7* (Bittner [7]).** — *Then the canonical morphism*

$$K_0^{\mathrm{bl}}(\mathrm{Var}_F) \to K_0(\mathrm{Var}_F)$$

is an isomorphism.

Idea of the proof. — The theorem is proved by constructing an explicit inverse to the natural morphism. For this take a smooth connected variety[(19)] X of dimension d and a smooth compactification $X \to \overline{X}$ such that $D = \overline{X} \setminus X$ is a normal crossing divisor. Let $D^{(k)}$ be the disjoint union of the the k-fold intersections of the irreductible components of D and consider the element

$$\sum_{k=0}^{d} (-1)^k [D^{(k)}]$$

in $K_0^{\mathrm{bl}}(\mathrm{Var}_F)$. It is then enough to show that this element does not depend on the chosen compactification and that this definition satisfies the cut and paste relation.

(18) We refer to [**46**] for a precise construction of birational motives either in Chow theory or in Voevodsky's triangulated setting.

(19) Using smooth stratifications and the cut and paste relation one sees that $K_0(\mathrm{Var}_F)$ may also be presented by smooth connected varieties.

The weak factorization theorem is used to prove that the independence of the compactification by reducing the proof to the case where one compactification is a blow-up of the other one which can be handle out explicitly using the blow-up relation. □

Recall now that an additive invariant μ is said to be birational (stably birational) when $\mu(X) = \mu(X')$ for X and X' birational (stably birational[20]) smooth proper connected varieties.

Remark 11.8. — For a birational additive invariant μ we have

$$\mu(\mathbb{P}^n) = 1 \qquad \mu(\mathbb{L}^n) = 0$$

for any positive integer n. Indeed it is enough to show that $\mu(\mathbb{P}^1) = 1$ since $\mathbb{P}^n$ and $(\mathbb{P}^1)^n$ are birational and $[\mathbb{P}^1] = 1 + \mathbb{L}$. For this consider the blow-up B of a point in a smooth proper connected surface X. Then the class of the exceptional divisor E satisfies $[E] = [\mathbb{P}^1]$ and using birational invariance and the cut and paste relation we deduce that

$$\begin{aligned}\mu(X) = \mu(B) &= \mu(E) + \mu(X) - 1 \\ &= \mu(\mathbb{P}^1) + \mu(X) - 1.\end{aligned}$$

This implies the desired relation $\mu(\mathbb{P}^1) = 1$.

Consider the free abelian group $\mathbb{Z}[SB]$ on stable birational equivalence classes of smooth proper connected varieties over F. Then a stable birational invariant on algebraic varieties may be seen as a map of ring

$$\mu : \mathbb{Z}[SB] \to A$$

where A is a ring. As proved by Larsen-Lunts in [**54**] stable birational invariants are always additive *i.e.* satisfy the paste un cup relation. This is an easy corollary of theorem 11.7 which was not yet available when [**54**] was written. The original proof in [**54**] although it does not rely on theorem 11.7 uses the same kind of ideas and the weak factorization theorem too.

***Corollary 11.9* (Larsen-Lunts [54])**. — *There exists a natural morphism of rings*

$$\mu_{SB} : K_0(\mathrm{Var}_F) \to \mathbb{Z}[SB] \tag{19}$$

such that $\mu_{SB}(X) = [X]$ for a smooth projective variety X.

---

[20] Recall that X and X' are stably birational if there exist $k, l \geqslant 0$ such that $X \times \mathbb{P}^k$ and $Y \times \mathbb{P}^l$ are birational. When $\mathrm{Char}(F) = 0$ any connected variety is birational to a smooth proper (even projective) connected variety, thus it is then enough to define a birational invariant on smooth proper (even projective) varieties.

Proof. — Consider a smooth projective variety X and the blow-up $\mathrm{Bl}_Y X$ with smooth center Y. Then $\mathrm{Bl}_Y X$ and X are birational and the exceptional divisor E is stably birational to Y thus the blow-up relation

$$[\mathrm{Bl}_Y X] - [E] = [X] - [Y]$$

is satisfied in $\mathbb{Z}[SB]$ and the result follows from theorem 11.7. □

***Proposition 11.10* (Larsen-Lunts [54]).** — *The kernel of the map (19) is the principal ideal* $(\mathbb{L})$ *generated by the class of the affine line* $\mathbb{L}$ *and therefore*

$$K_0(\mathrm{Var}_F)/(\mathbb{L}) = \mathbb{Z}[SB].$$

Proof. — Since μ_{SB} is stably birational, we must have $\mu_{SB}(\mathbb{P}^n) = 1$ and therefore $\mu_{SB}(\mathbb{L}) = 0$. Now take an element in $\ker \mu_{SB}$ and write it as a linear combination

$$[X_1] + \cdots + [X_r] - [Y_1] - \cdots - [Y_s]$$

where X_i and Y_j are smooth proper connected varieties. Then

$$[X_1]_{SB} + \cdots + [X_r]_{SB} - [Y_1]_{SB} - \cdots - [Y_s]_{SB} = 0$$

in $\mathbb{Z}[BS]$ and so $r = s$ and after renumbering X_i and Y_i may be assume to be stably birational. It is enough to prove that $[X_i] - [Y_i]$ belongs to $(\mathbb{L})$. Since

$$[X \times \mathbb{P}^k] = [X] + \mathbb{L} \cdot (1 + \mathbb{L} + \cdots + \mathbb{L}^{k-1}) \cdot [X]$$

we may even assume that X_i and Y_i are birational. Now using the weak factorization theorem we may as well assume that X_i is a blow-up of Y_i with smooth connected center Z. If E is the exceptional divisor of this blow-up we have then

$$\begin{aligned}[X_i] - [Y_i] = [E] - [Z] &= (1 + \mathbb{L} \cdots + \mathbb{L}^{c-1}) \cdot [Z] - [Z] \\ &= (\mathbb{L} \cdots + \mathbb{L}^{c-1}) \cdot [Z]\end{aligned}$$

where c is the codimension of Z in Y. The proposition follows. □

***Corollary 11.11* (Larsen-Lunts [54]).** — *For any additive invariant*

$$\mu : K_0(\mathrm{Var}_F) \to A,$$

the following properties are equivalent:

1. $\mu(\mathbb{L}) = 0$*;*
2. μ *is birational;*
3. μ *is stably birational.*

Proof. — By remark 11.8 we already know that for a birational invariant one has $\mu(\mathbb{L}) = 0$. Conversely assuming $\mu(\mathbb{L}) = 0$ we have by proposition 11.10 a natural factorization

$$\begin{tikzcd} K_0(\mathrm{Var}_F) \arrow[r, "\mu_{SB}"] \arrow[dr, "\mu"'] & \mathbb{Z}[SB] \arrow[d, dotted] \\ & A \end{tikzcd}$$

and thus μ is stably invariant. □

11.3. Rationality conjectures. — Let us go back to theorems 11.4 and 11.5. One sees that the additive invariants considered in Kapranov's theorem 11.4 are the non birational ones whereas Larsen-Lunts counterexample is provided by a birational invariant. Counting rational points of varieties over finite fields is surely not a birational invariant and thus there is still hope for a motivic proof of Dwork's theorem. In views of Larsen-Lunts' results one must deal only with non birational invariant as far as rationality is concerned. Let us introduce the naive motivic ring of varieties as the localization

$$\mathcal{M}_F = K_0(\mathrm{Var}_F)[\mathbb{L}^{-1}]$$

and denote by $[X]_{\mathcal{M}}$ the class of a varieties in $\mathcal{M}_F$. This construction is pretty similar to the one of Chow motives from effective Chow motives and one may still believe in the following conjecture:

Rationality conjecture 11.12. — Let X/F a variety. Then the Zeta function given by the formal power series

$$Z_{\mathcal{M}}(X,t) = \sum_{i=0}^{\infty} [\mathrm{Sym}^n X]_{\mathcal{M}} t^n \in 1 + \mathcal{M}_F[\![t]\!]^+$$

with coefficients in $\mathcal{M}_F[\![t]\!]$ is rational.

Remark 11.13. — It is strongly believed that the localization map $K_0(\mathrm{Var}_F) \to \mathcal{M}_F$ is not injective. Poonen has proved in [**68**] that for a field of characteristic zero $K_0(\mathrm{Var}_F)$ was not a domain. However his proof uses a birational invariant and so provides no information about zero divisors in $\mathcal{M}_F$.

11.4. Gillet-Soulé's invariant. — As in the previous subsection we still assume F to be a field of characteristic zero. The motivic Euler characteristic with compact support was obtained by Gillet-Soulé as a special consequence of the constructions in [**25**]. Another proof taking into account the equivariant case[21] was given by F. Guillén and V. Navarro Aznar in [**27**]. This result is also a straightforward corollary of theorem 11.7.

[21] The equivariant case is needed to get the formula (20) that involves symmetric powers.

Theorem 11.14 **(Gillet-Soulé [25], Guillén-Navarro Aznar [27], Bittner [7])**
There exists a unique morphism of rings

$$\mu_{\mathrm{GS}} : K_0(\mathrm{Var}_F) \to K_0(\mathsf{M}_{\mathsf{rat}}(F;\mathbb{Q}))$$

such that in $K_0(\mathsf{M}_{\mathsf{rat}}(F;\mathbb{Q}))$

$$\mu_{\mathrm{GS}}([X]) = [h(X)]$$

for X *smooth projective. Furthermore*

$$\mu_{\mathrm{GS}}([\mathrm{Sym}^n X]) = [\mathsf{S}^n h(X)]. \tag{20}$$

The additive invariant μ_{GS} is far from being birational and, since it sends the class of the affine line to the class of the Lefschetz motive which is invertible, it even factors through the naive motivic ring of varieties

$$\mu_{\mathrm{GS}} : \mathcal{M}_F \to K_0(\mathsf{M}_{\mathsf{rat}}(F;\mathbb{Q}))$$

Therefore as a special case of conjecture 11.12 one may believe in the following weaker statement:

Rationality conjecture 11.15. — Let X/F a variety. Then the Zeta function given by the formal power series

$$Z_{\mathsf{M}}(X,t) = \sum_{i=0}^{\infty} [\mathsf{S}^n h(X)] t^n \in 1 + K_0(\mathsf{M}_{\mathsf{rat}}(F;\mathbb{Q}))[\![t]\!]^+$$

with coefficients in $K_0(\mathsf{M}_{\mathsf{rat}}(F;\mathbb{Q}))[\![t]\!]$ is rational.

11.5. Rationality and finiteness. — As we shall see now the weaker rationality conjecture 11.15 is a consequence of finite dimensionality.

Theorem 11.16 **(André [3], Heinloth [34])**. — *Assume* $M \in \mathsf{A}$ *is finite dimensional. Then the Zeta series*

$$Z_{\mathsf{A}}(M,t) = \sum_{i=0}^{\infty} [\mathsf{S}^n M] t^n$$

is rational.

Recall that if A is a ring $1 + A[\![t]\!]^+$ denotes the multiplicative group of formal power series with constant coefficient equal to 1. Let λ^n and σ^n be the operations on $K_0(\mathsf{A})$ induced by exterior powers and symmetric powers

$$\lambda^n[M] = [\Lambda^n M] \qquad \sigma^n[M] = [\mathsf{S}^n M].$$

To an element a in $K_0(\mathsf{A})$ we can associate the formal powers series

$$\lambda_t(a) = \sum_{i=0}^{\infty} \lambda^n(a) t^n \qquad \sigma_t(a) = \sum_{i=0}^{\infty} \sigma^n(a) t^n$$

and we obtain this way two morphisms of groups[22]

$$\lambda_t : K_0(\mathsf{A}) \to 1 + K_0(\mathsf{A})[\![t]\!]^+ \qquad \sigma_t : K_0(\mathsf{A}) \to 1 + K_0(\mathsf{A})[\![t]\!]^+$$

such that $\lambda_t(a) = 1 + at + \cdots$ and $\sigma_t(a) = 1 + at + \cdots$. The rationality of Zeta series is a consequence of the following formula that relates symmetric powers operations and exterior powers operations.

Proposition 11.17. — *Let $M \in \mathsf{A}$ be a finite dimensional object. Then we have the relation*

$$\sigma_t(M) = \lambda_{-t}(M)^{-1}.$$

In other words the following relation holds

$$\sum_{n=0}^{\infty} [\mathsf{S}^n M] t^n = \frac{1}{\displaystyle\sum_{n=0}^{\infty} [\Lambda^n M](-t)^n}. \tag{21}$$

Let us first assume that proposition 11.17 is true and show how to deduce from it the rationality theorem.

Proof of theorem 11.16. — Assume

$$M \simeq M_+ \oplus M_- \qquad M_+ \text{ even and } M_- \text{ odd}$$

Then the Zeta series of M factors in terms of the Zeta series of the odd and even parts

$$Z_{\mathsf{M}}(M, t) = Z_{\mathsf{M}}(M_+, t) \cdot Z_{\mathsf{M}}(M_-, t).$$

Therefore one may assume that M is either even or odd. If M is odd then $Z_{\mathsf{M}}(M, t)$ is a polynomial and if M is even then the result follows from formula (21). $\square$

Recall that if A is a ring and a is any element of A then we have the relation

$$\frac{1}{1 - at} = 1 + at + a^2 t^2 + \cdots + a^n t^n + \cdots$$

in the ring of formal power series $A[\![t]\!]$. As we shall see now by dévissage, the formula (21) of proposition 11.17 is a consequence of this identity. To perform the reduction steps we need some remarks.

Remark 11.18. — Let M and N two objects of A. Since λ_t and σ_t are morphisms of abelian groups we have

$$\lambda_t([M \oplus N]) = \lambda_t([M] + [N]) = \lambda_t([M])\lambda_t([N])$$
$$\sigma_t([M \oplus N]) = \sigma_t([M] + [N]) = \sigma_t([M])\sigma_t([N])$$

and so formula (21) is true for $M \oplus N$ as soon as it is true for M and N.

[22] These morphisms define on $K_0(\mathsf{A})$ two pre-λ-ring structures in the sense of [**1**, V, Definition 2.1].

Remark 11.19. — Let B and C be two additive categories. The morphism on K_0

$$K_0(\mathsf{B}) \to K_0(\mathsf{C})$$

induced by an essentially surjective, conservative and full additive functor $F : \mathsf{B} \to \mathsf{C}$ is an isomorphism. When B is abelian an ambiguity appears, we can look at the Grothendieck group of B as an additive or as an abelian category, the second one being only a quotient of the first one. Indeed if L denotes the free abelian group on isomorphism classes of objects of B, the Grothendieck group of B as an additive category is the quotient of L obtained by imposing the relations $[M \oplus N] = [M] + [N]$ whereas its Grothendieck group as an abelian category is the quotient of L obtained by imposing the relations $[M] = [M'] + [M'']$ for any exact sequence

$$0 \to M' \to M \to M'' \to 0.$$

For a semi-simple abelian category the two groups are the same.

Proof of proposition 11.17. — First let us assume that A is the abelian category $\mathrm{Rep}_K(G)$ of algebraic representations of an affine group scheme G over K and that M is a one dimensional representation. Since $\Lambda^n M = 0$ for $n \geqslant 2$ and $\mathsf{S}^n M = M^{\otimes n}$ for all nonnegative n, we get the relations in $K_0(\mathsf{A})$

$$[\mathsf{S}^n M] = [M]^n \quad \text{and} \quad [\Lambda^n M] = \begin{cases} 0 & n \geqslant 2 \\ [M] & n = 1 \\ 1 & n = 0 \end{cases}$$

and our relation (21) is simply

$$\frac{1}{1 - [M]t} = 1 + [M]t + [M]^2 t^2 + \cdots + [M]^n t^n + \cdots .$$

The equality is then also true for a finite direct sum of one dimensional representations. Now take M to be any algebraic representation $G \to \mathrm{GL}_{n,K}$. Using the ring morphism

$$K_0(\mathrm{Rep}_K(\mathrm{GL}_{n,K})) \to K_0(\mathrm{Rep}_K(G))$$

induced by the restriction $\otimes$-functor $\mathrm{Rep}_K(G) \to \mathrm{Rep}_K(\mathrm{GL}_{n,K})$ we may assume that $G = \mathrm{GL}_{n,K}$ and that M is the tautological representation $\mathrm{GL}_{n,K} \to \mathrm{GL}_{n,K}$. Now we know by [**76**, Theorem 4] that the restriction morphism

$$K_0(\mathrm{Rep}_K(\mathrm{GL}_{n,K})) \to K_0(\mathrm{Rep}_K(\mathbb{G}_{m,K}^n))$$

is injective, so we may assume that $G = \mathbb{G}_{m,K}^n$ and that M is the representation $\mathbb{G}_{m,K}^n \to \mathrm{GL}_{n,K}$ induced by inclusion. This representation is a finite direct sum of one dimensional representations and so satisfies the formula.

Let us now go back to the general case. We may assume that M is even and replace A by the thick strictly full rigid tensor subcategory generated by M. The category has then a $\otimes$-generator and all Euler characteristic are nonnegative now. By theorem

4.31 the full and essentially surjective functor $\mathsf{A} \to \overline{\mathsf{A}}$ is conservative and so by remark 11.19 the morphism

$$K_0(\mathsf{A}) \to K_0(\overline{A})$$

is an isomorphism. So we may assume that $\mathsf{A} = \overline{\mathsf{A}}$ is absolute semi-simple abelian category in which Euler characteristics are nonnegative. By Deligne's criterion A is then an absolute semi-simple Tannakian category. Let E/K be a field extension over which a fibre functor is defined. Since A is absolute semi-simple, the faithful functor $\mathsf{A} \to \mathsf{A}_E$ induces an injection

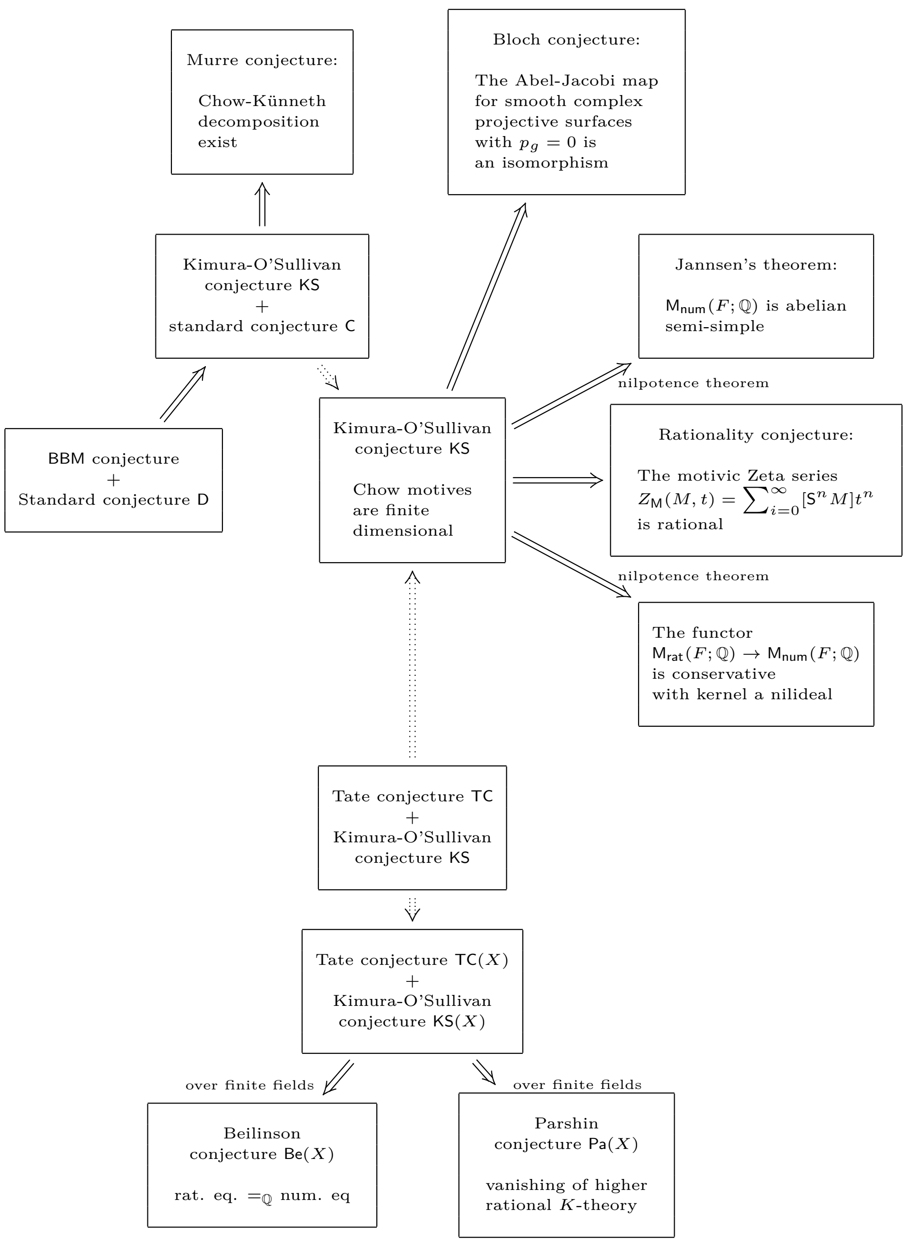
$$K_0(\mathsf{A}) \to K_0(\mathsf{A}_E).$$

We may finally assume that A is a semi-simple neutral Tannakian category which has a $\otimes$-generator. By Tannaka theory A is equivalent to the category $\mathrm{Rep}_K(G)$ of algebraic representations of an affine reductive K-group scheme and we are reduced to the case considered above. □

A different proof of this proposition is given in [**34**, §4.2]. Some developments related to functional equations may be found in [**34, 41**].

A
Leitfaden

We give below a synoptic view of the main implications discussed in this survey, the obvious one are represented by dotted arrows. Remark that Jannsen's theorem is an unconditional statement.



References

[1] *Théorie des intersections et théorème de Riemann-Roch*, Springer-Verlag, Berlin, 1971, Séminaire de Géométrie Algébrique du Bois-Marie 1966–1967 (SGA 6), Dirigé par P. Berthelot, A. Grothendieck et L. Illusie. Avec la collaboration de D. Ferrand, J. P. Jouanolou, O. Jussila, S. Kleiman, M. Raynaud et J. P. Serre, Lecture Notes in Mathematics, Vol. 225.

[2] Dan Abramovich, Kalle Karu, Kenji Matsuki, and Jarosław Włodarczyk, *Torification and factorization of birational maps*, J. Amer. Math. Soc. **15** (2002), no. 3, 531–572 (electronic).

[3] Yves André, *Une introduction aux motifs (motifs purs, motifs mixtes, périodes)*, Panoramas et Synthèses [Panoramas and Syntheses], vol. 17, Société Mathématique de France, Paris, 2004.

[4] ______, *Motifs de dimension finie (d'après S.-I. Kimura, P. O'Sullivan…)*, Astérisque (2005), no. 299, Exp. No. 929, viii, 115–145, Séminaire Bourbaki. Vol. 2003/2004.

[5] Yves André, Bruno Kahn, and Peter O'Sullivan, *Nilpotence, radicaux et structures monoïdales*, Rend. Sem. Mat. Univ. Padova **108** (2002), 107–291.

[6] A. A. Beĭlinson, *Height pairing between algebraic cycles*, K-theory, arithmetic and geometry (Moscow, 1984–1986), Lecture Notes in Math., vol. 1289, Springer, Berlin, 1987, pp. 1–25.

[7] Franziska Bittner, *The universal Euler characteristic for varieties of characteristic zero*, Compos. Math. **140** (2004), no. 4, 1011–1032.

[8] Spencer Bloch, *Some elementary theorems about algebraic cycles on Abelian varieties*, Invent. Math. **37** (1976), no. 3, 215–228.

[9] Armand Borel, *Stable real cohomology of arithmetic groups*, Ann. Sci. École Norm. Sup. (4) **7** (1974), 235–272 (1975).

[10] Lawrence Breen, *Tannakian categories*, Motives (Seattle, WA, 1991), Proc. Sympos. Pure Math., vol. 55, Amer. Math. Soc., Providence, RI, 1994, pp. 337–376.

[11] Patrick Brosnan, *A short proof of Rost nilpotence via refined correspondences*, Doc. Math. **8** (2003), 69–78 (electronic).

[12] Vladimir Chernousov, Stefan Gille, and Alexander Merkurjev, *Motivic decomposition of isotropic projective homogeneous varieties*, Duke Math. J. **126** (2005), no. 1, 137–159.

[13] Pierre Colmez and Jean-Pierre Serre (eds.), *Correspondance Grothendieck-Serre*, Documents Mathématiques (Paris) [Mathematical Documents (Paris)], 2, Société Mathématique de France, Paris, 2001.

[14] P. Deligne, *Catégories tannakiennes*, The Grothendieck Festschrift, Vol. II, Progr. Math., vol. 87, Birkhäuser Boston, Boston, MA, 1990, pp. 111–195.

[15] ______, *Catégories tensorielles*, Mosc. Math. J. **2** (2002), no. 2, 227–248, Dedicated to Yuri I. Manin on the occasion of his 65th birthday.

[16] Pierre Deligne, *Théorie de Hodge. III*, Inst. Hautes Études Sci. Publ. Math. (1974), no. 44, 5–77.

[17] ______, *Décompositions dans la catégorie dérivée*, Motives (Seattle, WA, 1991), Proc. Sympos. Pure Math., vol. 55, Amer. Math. Soc., Providence, RI, 1994, pp. 115–128.

[18] J. Denef and F. Loeser, *One some rational generating series*, Preprint (2005), http://www.dma.ens.fr/~loeser.

[19] Christopher Deninger and Jacob Murre, *Motivic decomposition of abelian schemes and the Fourier transform*, J. Reine Angew. Math. **422** (1991), 201–219.

[20] Bernard Dwork, *On the rationality of the zeta function of an algebraic variety*, Amer. J. Math. **82** (1960), 631–648.

[21] Fouad El Zein, *Introduction à la théorie de Hodge mixte*, Actualités Mathématiques. [Current Mathematical Topics], Hermann, Paris, 1991.

[22] William Fulton, *Intersection theory*, second ed., Ergebnisse der Mathematik und ihrer Grenzgebiete. 3. Folge. A Series of Modern Surveys in Mathematics [Results in Mathematics and Related Areas. 3rd Series. A Series of Modern Surveys in Mathematics], vol. 2, Springer-Verlag, Berlin, 1998.

[23] William Fulton and Joe Harris, *Representation theory*, Graduate Texts in Mathematics, vol. 129, Springer-Verlag, New York, 1991, A first course, Readings in Mathematics.

[24] Thomas Geisser, *Tate's conjecture, algebraic cycles and rational K-theory in characteristic p*, K-Theory **13** (1998), no. 2, 109–122.

[25] H. Gillet and C. Soulé, *Descent, motives and K-theory*, J. Reine Angew. Math. **478** (1996), 127–176.

[26] A. Grothendieck, *Crystals and the de Rham cohomology of schemes*, Dix Exposés sur la Cohomologie des Schémas, North-Holland, Amsterdam, 1968, pp. 306–358.

[27] Francisco Guillén and Vicente Navarro Aznar, *Un critère d'extension des foncteurs définis sur les schémas lisses*, Publ. Math. Inst. Hautes Études Sci. (2002), no. 95, 1–91.

[28] V. Guletskiĭ and C. Pedrini, *Finite-dimensional motives and the conjectures of Beilinson and Murre*, K-Theory **30** (2003), no. 3, 243–263, Special issue in honor of Hyman Bass on his seventieth birthday. Part III.

[29] Vladimir Guletskiĭ, *Finite-dimensional objects in distinguished triangles*, J. Number Theory **119** (2006), no. 1, 99–127.

[30] Vladimir Guletskiĭ and Claudio Pedrini, *The Chow motive of the Godeaux surface*, Algebraic geometry, de Gruyter, Berlin, 2002, pp. 179–195.

[31] Masaki Hanamura, *Mixed motives and algebraic cycles. I*, Math. Res. Lett. **2** (1995), no. 6, 811–821.

[32] ———, *Mixed motives and algebraic cycles. III*, Math. Res. Lett. **6** (1999), no. 1, 61–82.

[33] ———, *Mixed motives and algebraic cycles. II*, Invent. Math. **158** (2004), no. 1, 105–179.

[34] Franziska Heinloth, *A note on functional equations for zeta functions with values in Chow motives*, Ann. Inst. Fourier (Grenoble) **57** (2007), no. 6, 1927–1945.

[35] Graham Higman, *On a conjecture of Nagata*, Proc. Cambridge Philos. Soc. **52** (1956), 1–4.

[36] Taira Honda, *Isogeny classes of abelian varieties over finite fields*, J. Math. Soc. Japan **20** (1968), 83–95.

[37] Uwe Jannsen, *Motives, numerical equivalence, and semi-simplicity*, Invent. Math. **107** (1992), no. 3, 447–452.

[38] ———, *Motivic sheaves and filtrations on Chow groups*, Motives (Seattle, WA, 1991), Proc. Sympos. Pure Math., vol. 55, Amer. Math. Soc., Providence, RI, 1994, pp. 245–302.

[39] ———, *Equivalence relations on algebraic cycles*, The arithmetic and geometry of algebraic cycles (Banff, AB, 1998), NATO Sci. Ser. C Math. Phys. Sci., vol. 548, Kluwer Acad. Publ., Dordrecht, 2000, pp. 225–260.

[40] André Joyal and Ross Street, *Braided tensor categories*, Adv. Math. **102** (1993), no. 1, 20–78.

[41] Bruno Kahn, *Zeta functions and motives*, to appear in *Pure and Applied Mathematics Quarterly* (2009).

[42] ______, *Équivalences rationnelle et numérique sur certaines variétés de type abélien sur un corps fini*, Ann. Sci. École Norm. Sup. (4) **36** (2003), no. 6, 977–1002 (2004).

[43] ______, *Formes quadratiques et cycles algébriques (d'après Rost, Voevodsky, Vishik, Karpenko)*, (2004), Exp. No. 941, Séminaire Bourbaki. Vol. 2004/2005.

[44] ______, *Algebraic K-theory, algebraic cycles and arithmetic geometry*, Handbook of K-theory. Vol. 1, 2, Springer, Berlin, 2005, pp. 351–428.

[45] Bruno Kahn, Jacob P. Murre, and Claudio Pedrini, *On the transcendental part of the motive of a surface*, Algebraic cycles and motives. Vol. 2, London Math. Soc. Lecture Note Ser., vol. 344, Cambridge Univ. Press, Cambridge, 2007, pp. 143–202. MR MR2187153

[46] Bruno Kahn and Ramdorai Sujatha, *Birational motives I (preliminary version)*, Preprint (2002), Ktheory/0596.

[47] Mikhail Kapranov, *The elliptic curve in the S-duality theory and Eisenstein series for Kac-Moody groups*, Preprint (2000), ArXiv:math.AG/0001005 v2.

[48] Nikita A. Karpenko, *Characterization of minimal Pfister neighbors via Rost projectors*, J. Pure Appl. Algebra **160** (2001), no. 2-3, 195–227.

[49] Kazuya Kato and Shuji Saito, *Unramified class field theory of arithmetical surfaces*, Ann. of Math. (2) **118** (1983), no. 2, 241–275.

[50] Nicholas M. Katz and William Messing, *Some consequences of the Riemann hypothesis for varieties over finite fields*, Invent. Math. **23** (1974), 73–77.

[51] G. M. Kelly, *On the radical of a category*, J. Austral. Math. Soc. **4** (1964), 299–307.

[52] Shun-Ichi Kimura, *Chow groups are finite dimensional, in some sense*, Math. Ann. **331** (2005), no. 1, 173–201.

[53] Klaus Künnemann, *On the Chow motive of an abelian scheme*, Motives (Seattle, WA, 1991), Proc. Sympos. Pure Math., vol. 55, Amer. Math. Soc., Providence, RI, 1994, pp. 189–205.

[54] Michael Larsen and Valery A. Lunts, *Motivic measures and stable birational geometry*, Mosc. Math. J. **3** (2003), no. 1, 85–95, 259.

[55] ______, *Rationality criteria for motivic zeta functions*, Compos. Math. **140** (2004), no. 6, 1537–1560.

[56] Marc Levine, *Tate motives and the vanishing conjectures for algebraic K-theory*, Algebraic K-theory and algebraic topology (Lake Louise, AB, 1991), NATO Adv. Sci. Inst. Ser. C Math. Phys. Sci., vol. 407, Kluwer Acad. Publ., Dordrecht, 1993, pp. 167–188.

[57] ______, *Mixed motives*, Mathematical Surveys and Monographs, vol. 57, American Mathematical Society, Providence, RI, 1998.

[58] I. G. Macdonald, *Symmetric functions and Hall polynomials*, second ed., Oxford Mathematical Monographs, The Clarendon Press Oxford University Press, New York, 1995, With contributions by A. Zelevinsky, Oxford Science Publications.

[59] T. Matsusaka, *The criteria for algebraic equivalence and the torsion group*, Amer. J. Math. **79** (1957), 53–66.

[60] Carlo Mazza, *Schur functors and motives*, K-Theory **33** (2004), no. 2, 89–106.

[61] J. S. Milne, *Values of zeta functions of varieties over finite fields*, Amer. J. Math. **108** (1986), no. 2, 297–360.

[62] ______, *Motives over finite fields*, Motives (Seattle, WA, 1991), Proc. Sympos. Pure Math., vol. 55, Amer. Math. Soc., Providence, RI, 1994, pp. 401–459.

[63] J. P. Murre, *On the motive of an algebraic surface*, J. Reine Angew. Math. **409** (1990), 190–204.

[64] ———, *On a conjectural filtration on the Chow groups of an algebraic variety. I. The general conjectures and some examples*, Indag. Math. (N.S.) **4** (1993), no. 2, 177–188.

[65] ———, *Algebraic cycles on abelian varieties: application of abstract Fourier theory*, The arithmetic and geometry of algebraic cycles (Banff, AB, 1998), NATO Sci. Ser. C Math. Phys. Sci., vol. 548, Kluwer Acad. Publ., Dordrecht, 2000, pp. 307–320.

[66] Masayoshi Nagata, *On the nilpotency of nil-algebras*, J. Math. Soc. Japan **4** (1952), 296–301.

[67] Peter O'Sullivan, *The structure of certain rigid tensor categories*, C. R. Math. Acad. Sci. Paris **340** (2005), no. 8, 557–562.

[68] Bjorn Poonen, *The Grothendieck ring of varieties is not a domain*, Math. Res. Lett. **9** (2002), no. 4, 493–497.

[69] A. A. Roĭtman, *Rational equivalence of zero-dimensional cycles*, Mat. Sb. (N.S.) **89(131)** (1972), 569–585.

[70] Markus Rost, *Chow groups with coefficients*, Doc. Math. **1** (1996), No. 16, 319–393 (electronic).

[71] ———, *Some new results on the Chow groups of quadrics*, Preprint (1996), Ktheory/0165.

[72] ———, *The motive of a Pfister form*, Preprint (1998), http://www.math.uni-bielefeld.de/ rost/data/motive.pdf.

[73] Louis H. Rowen, *Ring theory. Vol. I*, Pure and Applied Mathematics, vol. 127, Academic Press Inc., Boston, MA, 1988.

[74] Neantro Saavedra Rivano, *Catégories Tannakiennes*, Springer-Verlag, Berlin, 1972, Lecture Notes in Mathematics, Vol. 265.

[75] A. J. Scholl, *Classical motives*, Motives (Seattle, WA, 1991), Proc. Sympos. Pure Math., vol. 55, Amer. Math. Soc., Providence, RI, 1994, pp. 163–187.

[76] Jean-Pierre Serre, *Groupes de Grothendieck des schémas en groupes réductifs déployés*, Inst. Hautes Études Sci. Publ. Math. (1968), no. 34, 37–52.

[77] A. M. Shermenev, *The motive of an abelian variety*, Funkt. Anal. **8** (1974), 47–53.

[78] C. Soulé, *Groupes de Chow et K-théorie de variétés sur un corps fini*, Math. Ann. **268** (1984), no. 3, 317–345.

[79] Michael Spieß, *Proof of the Tate conjecture for products of elliptic curves over finite fields*, Math. Ann. **314** (1999), no. 2, 285–290.

[80] Ross Street, *Ideals, radicals, and structure of additive categories*, Appl. Categ. Structures **3** (1995), no. 2, 139–149.

[81] John Tate, *Conjectures on algebraic cycles in l-adic cohomology*, Motives (Seattle, WA, 1991), Proc. Sympos. Pure Math., vol. 55, Amer. Math. Soc., Providence, RI, 1994, pp. 71–83.

[82] Vladimir Voevodsky, *Triangulated categories of motives over a field*, Cycles, transfers, and motivic homology theories, Ann. of Math. Stud., vol. 143, Princeton Univ. Press, Princeton, NJ, 2000, pp. 188–238.

[83] ———, *Motivic cohomology with $\mathbf{Z}/2$-coefficients*, Publ. Math. Inst. Hautes Études Sci. (2003), no. 98, 59–104.

[84] Claire Voisin, *Théorie de Hodge et géométrie algébrique complexe*, Cours Spécialisés [Specialized Courses], vol. 10, Société Mathématique de France, Paris, 2002.

[85] Jarosław Włodarczyk, *Toroidal varieties and the weak factorization theorem*, Invent. Math. **154** (2003), no. 2, 223–331.

Florian Ivorra, FB Mathematik, Universität Duisburg-Essen, 45117 Essen, Germany
E-mail : florian.ivorra@uni-due.de