GUIDED MODES OF INTEGRATED OPTICAL GUIDES. A MATHEMATICAL STUDY

A.S. Bonnet–BenDhia[(1)], G. Caloz[(2)], and F. Mahé[(2)]

Abstract. A waveguide in integrated optics is defined by its refractive index. The guide is assumed to be invariant in the propagation direction while in the transverse direction it is supposed to be a compact perturbation of an unbounded stratified medium. We are interested in the modes guided by this device, which are waves with a transverse energy confined in a neighborhood of the perturbation.

Our goal is to analyze the existence of such guided modes. Under the assumptions of weak guidance the problem reduces to a two dimensional eigenvalue problem for a scalar field. The associated operator is unbounded, selfadjoint, and bounded from below. Its spectrum consists of the discrete spectrum corresponding to the guided modes and of the essential spectrum corresponding to the radiation modes. We present existence results of guided modes and an asymptotic study at high frequencies, which shows that contrarily to the case of optical fibers, the number of guided modes can remain bounded. The major tools are the min-max principle and comparison results between different eigenvalue problems. The originality of the present study lies in the stratified character of the unbounded reference medium.

0. Introduction

During the last decade high speed transmission of information has undergone important changes with the development of optical fibers and more generally of waveguides. These have been the object of numerous studies either from a theoretical or from a numerical point of view, see for instance Marcuse [1982], Wilcox [1984], Bonnet [1988], Bamberger and Bonnet [1990], Weder [1991], and the references therein.

In the field of integrated optics the same properties of light propagation are used to improve the performances which up to now were reserved to electronic devices.

A microguide in integrated optics is defined by its refractive index. In our paper we shall consider waveguides invariant in the propagation direction, which are composed of a stratified medium with a compact perturbation, in the transverse section called the core of the guide, see Figure 1.

The stratified medium is the reference medium (sometimes called the cladding by analogy with optical fibers) and is intended to guide electromagnetic waves in one

Key words and phrases. Integrated optics, stratified medium, guided mode, selfadjoint operator, min-max principle.

Typeset by $\mathcal{A}\mathcal{M}\mathcal{S}$-TEX

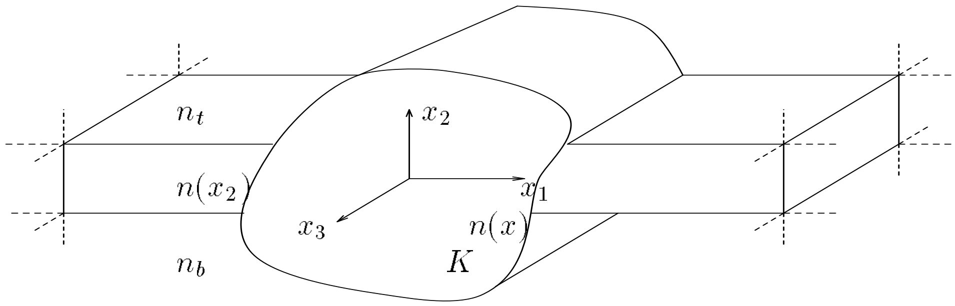

FIGURE 1: STRATIFIED OPTICAL GUIDE.

layer. The unboundedness of the reference medium will put obstacles in the theoretical and numerical studies as well as induce particular phenomena at high frequency. The compact perturbation of the reference medium is designed to confine waves inside a layer in a neighborhood of the perturbation.

Our goal is to study electromagnetic waves guided by such devices, which are waves of the form $\phi(x_1, x_2)e^{i(\omega t-\beta x_3)}$ where (x_1, x_2) denotes the transverse coordinates and x_3 the longitudinal coordinate. The guided modes correspond to waves of finite transverse energy which propagate without attenuation, i.e. ω and β are real. We will determine existence conditions for guided modes and bounds for the number of guided modes. In a subsequent paper, we will present numerical results based on the localized finite element method, see MAHÉ [1996].

To our knowledge there are no theoretical studies of guided modes for such perturbed stratified media. Contrarily to the case of optical fibers with homogeneous cladding, there are no simple geometries for which analytical computations can be done. Our study is carried out with spectral theory tools for selfadjoint operators.

Under the assumptions of weak guidance, the study of the Maxwell system reduces through an approximation of order 0 to the study of a scalar equation in the plane, see BONNET [1988], SNYDER and LOVE [1983], or VASSALO [1985]. For the dielectric microguides used in practice, this scalar approximation is generally not valid, due to the great variation between the refractive index of the air occupying the upper medium and the one of the dielectric materials. However the study of this scalar model has to be carried out before starting with the vectorial model, since it allows to solve in a much simpler situation a lot of mathematical difficulties due to the stratification of the unbounded medium. Moreover the results established in this paper will be used as technical tools for the study of the vectorial problem.

The paper is organized as follows. In Section 1 we present how the study of guided modes under the weak guidance assumption is reduced to a two dimensional scalar eigenvalue problem, for a one parameter dependent selfadjoint linear operator.

The properties of the operator are presented in Section 2. Its spectrum consists of an essential spectrum corresponding to a continuum of radiation modes and of a point spectrum corresponding to guided modes. We determine the lower bound of the essential spectrum, which is given by the fundamental mode of the reference medium.

In Section 3 we recall the Min–Max principle which will be extensively used to characterize the point spectrum below the essential spectrum. Thanks to the Min–Max principle we can compare the original problem to eigenproblems which are simpler to solve. Original comparison techniques are introduced.

Section 4 is devoted to the study of the first mode, also called the fundamental mode. We discuss thoroughly conditions of its existence with a particular attention at low frequencies.

An asymptotic study at high frequencies is developed in Section 5. We prove that the number of modes can be bounded as the wave number tends to infinity. In some cases all guided modes will disappear beyond a critical value, which is contrary to what happens for optical fibers.

In the appendix A we report the results for the one dimensional study of the reference stratified medium.

Problems of guided waves are also related to other research fields than optics. In elastodynamics we can refer to Bamberger, Dermenjian, and Joly [1990], Bamberger, Joly, and Kern [1991]. Hydrodynamics offers similarities with optics in stratified media. In Bonnet-BenDhia and Joly [1994], the authors study the waves guided by the seacost. The depth plays the role of the refractive index. Its variations confine the surface waves close to the coast. Like for electromagnetic waves in stratified media, the propagation of surface waves presents a dispersive character. Some results are similar in both fields and the technique used here is inspired from the one used in hydrodynamics. Nevertheless in optics the study is much more involved because the wave confinement depends on the refractive index which is a function of two variables while the guidance of the surface waves only depends on the depth. The difference becomes evident at high frequencies. For instance the example presented at the end of Section 5, where the fundamental mode disappears at high frequencies, has no equivalent in hydrodynamics.

Finally let us introduce the standard notations we shall use through the paper. $\mathbb{R}^+$ denotes the non negative real numbers and $\mathbb{R}^+_*$ the positive real numbers. For $m \in \mathbb{Z}$, $1 \le p \le \infty$, and $\Omega \subset \mathbb{R}^n$ $(n \in \mathbb{N})$, $W^{m,p}(\Omega)$ is the Sobolev space of functions with derivatives up to the order m in $L^p(\Omega)$ endowed with the norm $\|.\|_{m,p,\Omega}$ and seminorm $|.|_{m,p,\Omega}$. For $p = 2$, $W^{m,2}(\Omega)$ is a Hilbert space also denoted by $H^m(\Omega)$ endowed with the scalar product $(.,.)_{m,\Omega}$ and the norm $\|.\|_{m,\Omega}$. We shall also use the standard differential operators div, rot, ∇, Δ.

1. Preliminaries

The propagation of lightwave is described by the Maxwell equations

$$\begin{array}{ll} \mathrm{rot}\mathbb{E} = -\dfrac{\partial \mathbb{B}}{\partial t}, & \mathrm{rot}\mathbb{H} = \dfrac{\partial \mathbb{D}}{\partial t}, \\ \mathrm{div}\mathbb{D} = 0, & \mathrm{div}\mathbb{B} = 0, \end{array}$$

where $\mathbb{E}$ is the electric field, $\mathbb{H}$ the magnetic field, $\mathbb{D}$ the electric induction, and $\mathbb{B}$ the magnetic induction related by

$$\mathbb{B} = \mu\mathbb{H} \qquad \text{and} \qquad \mathbb{D} = \epsilon\mathbb{E}.$$

The magnetic permeability μ and the dielectric permittivity ϵ characterize the electromagnetic behavior of the material. It is classical in guided optics to assume that μ is a constant equal to μ_0, the permeability of the vacuum. The permittivity ϵ is related to the refractive index by the formula $\epsilon = \epsilon_0 n^2$, ϵ_0 being the permittivity of the vacuum.

Before presenting the mathematical model, we describe the guide. We assume that the guide is invariant in one direction (say Ox_3) which will be the propagation direction, and that the guide is a perturbation of a stratified medium. If the function n denotes the refractive index, then n is a function of x_1, x_2 only, $n = n(x_1, x_2)$. Outside the perturbation, the function n is depending only on x_2. A guide is represented in Figure 1.

We shall study electromagnetic waves harmonic in x_3, so later on we will only represent transverse section of the guide.

Let $\overline{n} \in L^\infty(\mathbb{R})$ be a function with positive values,

$$\overline{n} : \xi \in \mathbb{R} \to \overline{n}(\xi) \in \mathbb{R}_*^+$$

such that

$$\overline{n}(\xi) = \begin{cases} n_t & \text{if} \quad \xi > c \\ n_b & \text{if} \quad \xi < -c \end{cases}, \qquad \inf_{\xi \in \mathbb{R}} \overline{n}(\xi) > 0,$$

for some positive c. The values n_t, n_b play the same role and without loss of generality we choose

$$n_b > n_t$$

considering that the upper material is the air. The refractive index n satisfies the following assumptions

$$n \in L^\infty(\mathbb{R}^2), \qquad \inf_{x \in \mathbb{R}^2} n(x) > 0, \tag{1.1}$$

and there exists a compact set $K \subset \mathbb{R}^2$ such that

$$\text{for all } x \equiv (x_1, x_2) \notin K \quad n(x) = \overline{n}(x_2). \tag{1.2}$$

Let n_+ denote the essential supremum of n,

$$n_+ = \|n\|_{\infty, \mathbb{R}^2} (= \sup n). \tag{1.3}$$

The index function $\overline{n}$ defines a planar waveguide associated to the guide; it represents the stratified medium without perturbation.

Theoretical studies of the one dimensional problem of the planar waveguide are considered in Guillot [1985], Schechter [1981], or Wilcox [1984]. In the appendix A we report basic results of this problem, which are not given in the references above.

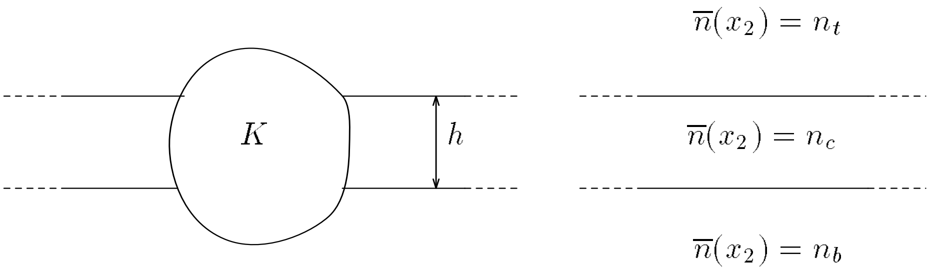

FIGURE 2: THREE LAYERS GUIDE AND PLANAR GUIDE ASSOCIATED.

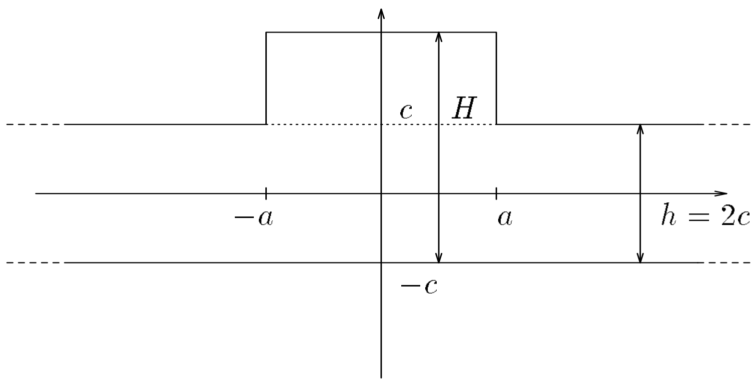

FIGURE 3: THREE LAYERS RIB GUIDE.

Remark 1.1. Actually the transverse section of the guide is bounded. We assume it unbounded because the dimensions of K are very small compared to the ones of the stratified medium and because we are looking for waves confined in a neighborhood of K. □

A model case, we shall refer to quite often, is the case of the three layers guide, see Figure 2. To that category belongs the rib guide often used in the applications, see Figure 3.

Remark 1.2. Notice that when n is constant outside K, the cladding surrounding the core is homogeneous and we have an optical fiber. For a mathematical study of fibers see BAMBERGER and BONNET [1990], POIRIER [1994].

When the cladding is composed of two different homogeneous materials, then we have a perturbed diopter see for instance BONNET and DJELLOULI [1988], GMATI [1992]. □

The electromagnetic waves which propagate inside the guide would be solution of the Maxwell equations. The ones we are looking for are the **modes** and have the following form

$$\begin{pmatrix} \mathbb{E} \\ \mathbb{H} \end{pmatrix}(x_1, x_2, x_3, t) = \Re\left(\begin{pmatrix} \mathbf{E} \\ \mathbf{H} \end{pmatrix}(x_1, x_2)\exp^{i(kc_0t-\beta x_3)}\right), \tag{1.4}$$

where c_0 is the speed of light in the vacuum, k is the wave number, and β is the propagation constant of the mode.

We say that a **mode is guided** if it propagates without attenuation and has a finite transverse energy, that is

$$k, \beta \in \mathbb{R} \quad \text{and} \quad \mathbf{E}, \mathbf{H} \in (L^2(\mathbb{R}^2))^3.$$

The guided mode with the largest propagation constant β is called the **fundamental mode**.

If the transverse energy is not finite or if β is complex, then we have a radiation mode. Roughly speaking a lightwave propagating inside the guide is decomposed into the guided modes which are guided inside the core and into the radiation modes which disperse into the cladding.

Remark 1.3. Given an incident lightwave (k is given), several guided modes (corresponding to different values of β) could propagate inside the guide at different speed.

The velocity of the mode is $v = kc_0/\beta$, which is the velocity of a planar wave in a material of constant index $\eta = \beta/k$. The ratio β/k is called the **effective index** of the mode. □

We want to look for solutions of the form (1.4) to the Maxwell system under the assumption of weak guidance, (i.e. large wave number and weak variations of the index). Then approximations of such solutions are got with the expressions

$$\mathbf{E} = \begin{pmatrix} u \\ 0 \\ 0 \end{pmatrix} \quad \text{and} \quad \mathbf{H} = \frac{\beta}{kc_0\mu_0}\begin{pmatrix} 0 \\ u \\ 0 \end{pmatrix}$$

or

$$\mathbf{E} = \begin{pmatrix} 0 \\ u \\ 0 \end{pmatrix} \quad \text{and} \quad \mathbf{H} = -\frac{\beta}{kc_0\mu_0}\begin{pmatrix} u \\ 0 \\ 0 \end{pmatrix}$$

where $u \in L^2(\mathbb{R}^2)$ is a distribution solution of the scalar equation

$$-\Delta u - k^2 n^2 u = -\beta^2 u \quad \text{in} \quad \mathbb{R}^2; \tag{1.5}$$

for zero order approximation of the Maxwell system, see for instance Bonnet [1988], Snyder and Love [1983], or Vassalo [1985].

Thus looking for guided modes under the assumption of weak guidance consists in determining the real numbers β, k, the functions $u \in L^2(\mathbb{R}^2)$ such that $(-\beta^2, u)$ is an eigenpair of the operator $(-\Delta - k^2n^2)$. A way to study these modes is to fix k and to look for $(-\beta^2, u)$. By varying k we then get the dispersion relation $k \mapsto \beta(k)$. This choice has physical motivation because k represents the wave number of a lightwave running through the guide.

Remark 1.4. Mathematicians generally prefer to consider β as a parameter and k as a function of β, see Wilcox [1984] or Weder [1991]. In the scalar case (1.5) both points of view are similar but for the Maxwell system the study has been done so far only when β is a parameter, see Bamberger and Bonnet [1990]. □

2. General properties

Let n be a refractive index, that is $n \in L^\infty(\mathbb{R}^2)$ satisfies (1.1) and (1.2). For a given non negative value of the parameter k, the problem we shall study reads: find $\beta \in \mathbb{R}^+$ and $u \in H^1(\mathbb{R}^2)$, $u \not\equiv 0$, such that

$$\text{for all } v \in H^1(\mathbb{R}^2) \quad a(k;u,v) = -\beta^2 (u,v)_{0,\mathbb{R}^2}, \tag{2.1}$$

where the bilinear form $a(k;.,.) : H^1(\mathbb{R}^2) \times H^1(\mathbb{R}^2) \to \mathbb{R}$ is given by: for $u, v \in H^1(\mathbb{R}^2)$

$$a(k;u,v) = \int_{\mathbb{R}^2} (\nabla u \nabla v - k^2 n^2 uv)\, dx. \tag{2.2}$$

In fact (2.1) is the **variational formulation** of our guided mode problem (1.5).

Let the unbounded operator $A_k : \mathscr{D}(A_k) \subset L^2(\mathbb{R}^2) \to L^2(\mathbb{R}^2)$ be given by

$$\mathscr{D}(A_k) = \{v \in H^1(\mathbb{R}^2); \Delta v \in L^2(\mathbb{R}^2)\} \quad \text{and} \quad \mathrm{A}_k v = -\Delta v - k^2 n^2 v \text{ for } v \in \mathscr{D}(A_k);$$

we consider the problem: find $\beta \in \mathbb{R}^+$ and $u \in \mathscr{D}(A_k)$, $u \not\equiv 0$, such that

$$A_k u = -\beta^2 u. \tag{2.3}$$

Problem (2.3) is the **spectral formulation** of our guided mode problem. We can check that (2.1) and (2.3) are equivalent, since for all $u \in \mathscr{D}(A_k)$ and $v \in H^1(\mathbb{R}^2)$

$$(A_k u, v)_{0,\mathbb{R}^2} = a(k;u,v).$$

To study the eigenproblem (2.3) or (2.1) we need a rather detailed study of the bilinear form $a(k;.,.)$ and of the operator A_k. The notations related to spectral theory are the ones used in Schechter [1971]. We will recall the major ones when needed.

Proposition 2.1. *The bilinear form $a(k;.,.)$ defined in (2.2) is continuous symmetric and bounded from below*

$$\text{for all } v \in H^1(\mathbb{R}^2) \quad a(k;v,v) \geq \|\nabla v\|^2_{0,\mathbb{R}^2} - k^2 n_+^2 \|v\|^2_{0,\mathbb{R}^2}. \quad \square \tag{2.4}$$

The proof of Proposition 2.1 is immediately deduced from the definition of the bilinear form $a(k;.,.)$ and the assumptions on n.

Proposition 2.2. *The following properties for the operator A_k hold:*

(1) $\mathscr{D}(A_k) = H^2(\mathbb{R}^2)$,

(2) *A_k is bounded from below, selfadjoint, and its spectrum satisfies*

$$\sigma(A_k) \subset [-k^2 n_+^2, \infty).$$

Proof. The property (1) is a direct consequence of the definition of $\mathscr{D}(A_k)$. From the estimate (2.4) we deduce that for $v \in \mathscr{D}(A_k)$

$$(A_k v, v)_{0,\mathbb{R}^2} \geq -k^2 n_+^2 \|v\|_{0,\mathbb{R}^2}^2,$$

that is A_k is bounded from below. Since $a(k;.,.)$ is symmetric and satisfies (2.4), we deduce that A_k is regularly accretive and that A_k is selfadjoint, see Schechter [1971], p.24 and p.33. The estimate for the spectrum is immediate since for $\lambda < -k^2 n_+^2$, the bilinear form $a(k;u,v) - \lambda(u,v)_{0,\mathbb{R}^2}$ is coercive. $\square$

One difficulty in the study of the spectrum of A_k relies in the fact that A_k is not with compact resolvent. Indeed let $\lambda < -k^2 n_+^2$, then λ is in the resolvent set of A_k and clearly $(A_k - \lambda I)^{-1}$ is not compact since the Sobolev embeddings $H^2(\mathbb{R}^2) \subset H^1(\mathbb{R}^2) \subset L^2(\mathbb{R}^2)$ are not compact. Consequently $(A_k - \lambda I)^{-1}$ is not compact for all λ in $\rho(A_k)$.

We know since A_k is selfadjoint that the spectrum of A_k consists of a continuum, the essential spectrum $\sigma_{ess}(A_k)$, and of a discrete set, the discrete spectrum $\sigma_d(A_k)$, which is the set of isolated eigenvalues of finite multiplicity. We determine now the essential spectrum of A_k while the discrete spectrum will be studied later.

The essential spectrum $\sigma_{ess}(A_k)$ is characterized as follows, see Schechter [1971], p.34: λ belongs to $\sigma_{ess}(A_k)$ if and only if there exists a sequence $\{u_p\}_{p\geq 1} \subset \mathscr{D}(A_k)$ such that

$$\|u_p\|_{0,\mathbb{R}^2} = 1, \quad (A_k - \lambda I)u_p \to 0, \quad u_p \rightharpoonup 0 \text{ weakly in } L^2(\mathbb{R}^2).$$

Such a sequence is called a **singular sequence** associated to $A_k - \lambda I$.

Proposition 2.3. *Let $\overline{A}_k : \mathscr{D}(\overline{A}_k) \subset L^2(\mathbb{R}^2) \to L^2(\mathbb{R}^2)$ be the operator associated to the refractive index of the non perturbed stratified medium. Then*

$$\sigma_{ess}(A_k) = \sigma_{ess}(\overline{A}_k).$$

Proof. The operator A_k can be decomposed into

$$A_k = \overline{A}_k + B$$

where for $v \in H^2(\mathbb{R}^2)$

$$\overline{A}_k v = -\Delta v - k^2 \overline{n}^2 v$$

and for $w \in L^2(\mathbb{R}^2)$

$$Bw = -k^2(n^2 - \overline{n}^2)w.$$

To prove the result it is sufficient now to check that B is $\overline{A}_k$-compact, that is whenever a sequence $\{u_p\}_{p\geq 1} \subset H^2(\mathbb{R}^2)$ satisfies

$$\|u_p\|_{0,\mathbb{R}^2} + \|\overline{A}_k u_p\|_{0,\mathbb{R}^2} \leq C, \quad p = 1,2,3,\dots$$

then $\{Bu_p\}_{p\geq 1}$ has a subsequence convergent in $L^2(\mathbb{R}^2)$, see Schechter [1971], p.16. Indeed let $\{u_p\}_{p\geq 1}$ be such a sequence. Then $\{u_p\}_{p\geq 1}$ is bounded in $H^1(\mathbb{R}^2)$. Since $n^2 - \overline{n}^2$ has a compact support there exists a subsequence $\{u_{p_j}\}_{j\geq 1}$ such that $\{(n^2 - \overline{n}^2)u_{p_j}\}_{j\geq 1}$ converges in $L^2(\mathbb{R}^2)$. $\square$

Proposition 2.4. *Let $\overline{A}_k$ be defined in Proposition 2.3. Then*

$$\sigma_{ess}(A_k) = \sigma_{ess}(\overline{A}_k) = [\overline{\gamma}(k), \infty) \tag{2.5}$$

where

$$\overline{\gamma}(k) = \inf_{\substack{\varphi \in H^1(\mathbb{R}) \\ \varphi \neq 0}} \frac{\int_{\mathbb{R}} (\varphi'^2 - k^2 \overline{n}^2 \varphi^2)\, dy}{\int_{\mathbb{R}} \varphi^2\, dy}, \tag{2.6}$$

is a characteristic value of the associated planar waveguide (see the appendix A).

Proof. The first equality in (2.5) has been proved in Proposition 2.3. Let us check now the second one.

We check first that $\sigma_{ess}(\overline{A}_k) \subset [\overline{\gamma}(k), \infty)$. Given $\varphi \in H^1(\mathbb{R}^2)$, we have

$$\overline{a}(k; \varphi, \varphi) = \int_{\mathbb{R}} \int_{\mathbb{R}} \left[\left(\frac{\partial \varphi}{\partial x_2} \right)^2 - k^2 \overline{n}^2 \varphi^2 \right] dx_2\, dx_1 + \int_{\mathbb{R}^2} \left(\frac{\partial \varphi}{\partial x_1} \right)^2 dx$$

and so by the definition of $\overline{\gamma}(k)$

$$\overline{a}(k; \varphi, \varphi) \geq \overline{\gamma}(k) \int_{\mathbb{R}^2} \varphi^2\, dx.$$

Consequently $\sigma(\overline{A}_k) \subset [\overline{\gamma}(k), \infty)$.

To prove the second inclusion, we will build a singular sequence $\{u_p\}_{p \geq 1}$ for all $\lambda > \overline{\gamma}(k)$. It is sufficient to take $\lambda > \overline{\gamma}(k)$ since $\sigma_{ess}(\overline{A}_k)$ is closed. We set $\nu = \sqrt{\lambda - \overline{\gamma}(k)} \in \mathbb{R}_*^+$. It is proved in the appendix A that $\overline{\gamma}(k) \leq -k^2 n_b^2$; if $\overline{\gamma}(k) < -k^2 n_b^2$, then $\overline{\gamma}(k)$ is an eigenvalue of $-\frac{\mathrm{d}^2}{\mathrm{d}y^2} - k^2 \overline{n}^2$ with an eigenfunction $\overline{\varphi} \in H^2(\mathbb{R})$; if $\overline{\gamma}(k) = -k^2 n_b^2$ then there exists a singular sequence $\{\varphi_p\}_{p \geq 1} \subset H^2(\mathbb{R})$ associated to $-\frac{\mathrm{d}^2}{\mathrm{d}y^2} - k^2 \overline{n}^2 - \overline{\gamma}(k)$. If $\overline{\gamma}(k) < -k^2 n_b^2$ we set

$$u_p(x) = \Re \left[\frac{1}{\sqrt{p}} \theta(\frac{x_1}{p}) e^{i \nu x_1} \overline{\varphi}(x_2) \right]$$

and if $\overline{\gamma}(k) = -k^2 n_b^2$ we set

$$u_p(x) = \Re \left[\frac{1}{\sqrt{p}} \theta(\frac{x_1}{p}) e^{i \nu x_1} \varphi_p(x_2) \right],$$

where $\theta \in C_0^\infty(\mathbb{R})$ is chosen such that

$$\|\theta\|_{0,\mathbb{R}} \|\overline{\varphi}\|_{0,\mathbb{R}} = 1 \quad \text{and} \quad \|\theta\|_{0,\mathbb{R}} = 1 \quad \text{respectively.}$$

With algebraic manipulations we can check that $\{u_p\}_{p \geq 1}$ is a singular sequence. $\square$

Remark 2.1. The values of λ in $\sigma_{ess}(\overline{A}_k)$ correspond to radiation modes, that is to solutions of (1.5) which are not in $L^2(\mathbb{R}^2)$. Indeed consider for positive ν

$$u(x) = \Re\left[\overline{\varphi}(x_2)e^{i\nu x_1}\right] \quad \text{and} \quad -\beta^2 = \overline{\gamma}(k) + \nu^2,$$

where $\overline{\varphi}(x_2)$ is the function introduced in the proof of Proposition 2.4. Then u satisfies the equation (1.5) with $\overline{n}$.

In fact we have proved in Proposition 2.4 that $\sigma_{ess}(\overline{A}_k) = \sigma(\overline{A}_k) = [\overline{\gamma}(k), \infty)$. We can already notice that the existence of a discrete spectrum can occur if the reference medium is perturbed. $\square$

We have represented in Figure 4, the already known results on the spectrum of A_k.

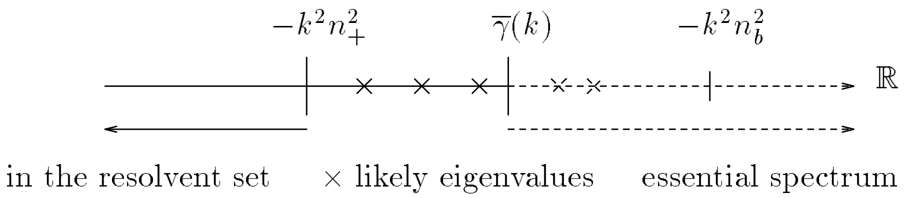

FIGURE 4: THE SPECTRUM OF A_k.

It is time now to turn our attention to the point spectrum of A_k, $\sigma_p(A_k)$, which gives the guided modes. It consists of the discrete spectrum $\sigma_d(A_k)$ which are eigenvalues below $\overline{\gamma}(k)$ (the lower bound of the essential spectrum of A_k) and of eigenvalues embedded in the essential spectrum.

Since $A_k \in \mathscr{L}(H^2(\mathbb{R}^2); L^2(\mathbb{R}^2))$ is selfadjoint and bounded from below by $-k^2n_+^2$, we deduce that

$$\sigma_p(A_k) \subset [-k^2n_+^2, \infty).$$

We have the following sharper result.

Proposition 2.5. *Let λ be an eigenvalue of A_k, then necessarily*

$$-k^2n_+^2 \leq \lambda \leq -k^2n_b^2. \tag{2.7}$$

Proof. We already have argued on the first inequality. To check the upper bound, we follow the method used in WEDER [1991]. $\square$

In the next two remarks we make some comments and give physical interpretation of the eigenvalues.

Remark 2.2. If the eigenvalue $-\beta^2$ is below the essential spectrum, i.e.

$$-k^2n_+^2 \leq -\beta^2 \leq \overline{\gamma}(k),$$

then the velocity of the associated mode $v = kc_0/\beta$ has the bounds

$$\frac{c_0}{n_+} \leq v \leq \overline{v}(k),$$

where $\overline{v}(k)$ is given by the relation $-\overline{\gamma}(k)\overline{v}(k)^2 = c_0^2 k^2$. The lower bound is the velocity of a planar wave in a material of index n_+ and the upper bound is the velocity of the slowest wave propagating in the perfectly stratified medium.

When the cladding is a planar waveguide with guided modes, see the appendix A, the slowest velocity is the one of the fundamental mode $\overline{v}(k)$, which is depending on k nonlinearly. This is a difficulty of our problem, which shows the dispersive character of the stratified medium: two lightwaves with two different frequencies will propagate at different speed. This property will have a direct effect on the number of modes at high frequencies, see Section 5. □

Remark 2.3. In the case of optical fiber and diopter we have $\overline{\gamma}(k) = -k^2 n_b^2$ and so from Proposition 2.5 we cannot have eigenvalues in the essential spectrum. In the general case of waveguide we can find eigenvalues in the essential spectrum, which means we can find guided modes propagating faster than some mode in the stratified medium.

We will not present examples of guides with eigenvalues embedded in the essential spectrum in this paper; such eigenvalues lie in the interval $[\overline{\gamma}(k), -k^2 n_b^2]$. An example is given in Bonnet-BenDhia and Mahé [1995]. The guided modes embedded into the radiation modes are very sensitive to imperfections of the guide and are less interesting from a practical point of view than the modes below.

The question of existence of eigenvalues embedded in the essential spectrum is a problem encountered in various fields of application. Non existence results have been proved in acoustic in a stratified medium see Weder [1988] or for the Schrödinger operators see Reed and Simon [1978]. Examples of embedded eigenvalues are presented for perturbed guides by Witsch [1990], for electromagnetic gratings by Bonnet and Starling [1994], in elastodynamics by Joly and Weder [1994]. □

To go further in the study of the point spectrum of A_k below the essential spectrum $[\overline{\gamma}(k), \infty)$, we shall introduce in the next section the techniques adequate for selfadjoint operators bounded from below, namely the Min–Max principle and comparison principles.

3. The Min–Max principle and comparison principles

A major result to characterize eigenvalues of selfadjoint operators bounded from below is the Min–Max principle, see Reed and Simon [1978], Dunford and Schwartz [1963].

Corresponding to the problem (2.3), we define the Min–Max quantities $\lambda_m(k), m \geq 1$, by

$$\lambda_1(k) = \inf_{\substack{v \in H^1(\mathbb{R}^2) \\ v \neq 0}} \frac{a(k; v, v)}{(v, v)_{0,\mathbb{R}^2}} \tag{3.1}$$

and for $m > 1$

$$\lambda_m(k) = \inf_{H_m \in \mathscr{H}_m(H^1(\mathbb{R}^2))} \sup_{\substack{v \in H_m \\ v \neq 0}} \frac{a(k; v, v)}{(v, v)_{0,\mathbb{R}^2}}, \tag{3.2}$$

where $\mathscr{H}_m(H^1(\mathbb{R}^2))$ is the set of m-dimensional subspaces of $H^1(\mathbb{R}^2)$. Then

$$-k^2 n_+^2 \le \lambda_1(k) \le \lambda_2(k) \le \cdots \le \lambda_m(k) \le \cdots \le \overline{\gamma}(k)$$

and if $\lambda_j(k) = \overline{\gamma}(k)$ for some $j \ge 1$ then A_k has at most $(j-1)$ eigenvalues below $\overline{\gamma}(k)$. If $\lambda_j(k) < \overline{\gamma}(k)$, then $\lambda_1(k), \ldots, \lambda_j(k)$ are the first j eigenvalues of A_k. Moreover the numbers $\lambda_m(k), m > 1$, can be characterized by

$$\lambda_m(k) = \sup_{v_1, \ldots, v_{m-1} \in L^2(\mathbb{R}^2)} \inf_{\substack{w \in H^1(\mathbb{R}^2), w \neq 0 \\ w \in [v_1, \ldots, v_{m-1}]^\perp}} \frac{a(k; w, w)}{(w, w)_{0,\mathbb{R}^2}},$$

where $[v_1, \ldots, v_{m-1}]^\perp$ is the orthogonal complement in $L^2(\mathbb{R}^2)$ to $v_1, \ldots, v_{m-1}$. In particular when $\lambda_{m-1}(k) < \overline{\gamma}(k)$ then

$$\lambda_m(k) = \inf_{\substack{w \in H^1(\mathbb{R}^2), w \neq 0 \\ w \in [\varphi_1, \ldots, \varphi_{m-1}]^\perp}} \frac{a(k; w, w)}{(w, w)_{0,\mathbb{R}^2}},$$

where $\varphi_1, \ldots, \varphi_{m-1}$ are eigenvectors associated to the eigenvalues $\lambda_1(k), \ldots, \lambda_{m-1}(k)$. In other words, if $N(k)$ is the number of eigenvalues strictly below $\overline{\gamma}(k)$, then

$$N(k) = \sup\{m \in \mathbb{N}; \lambda_m(k) < \overline{\gamma}(k)\}, \tag{3.3}$$

and $N(k)$ is called the **number of guided modes**.

We present now a comparison method useful when we try to get bounds on the number of guided modes. We compare the solutions of problems stated in some subdomains Ω of $\mathbb{R}^2$ to the one stated in $\mathbb{R}^2$. The method has been already used to study wave phenomena in Bonnet-BenDhia and Joly [1994] when Ω is bounded. In our applications the domains Ω are not necessarily bounded.

Let $n \in L^\infty(\mathbb{R}^2)$ be an index function satisfying to (1.1) and (1.2), and $\Omega \subset \mathbb{R}^2$ be an open set containing K. We assume that the boundary $\partial\Omega$ of Ω is regular. The exterior normal unit vector on $\partial\Omega$ is denoted by ν and the complement of Ω in $\mathbb{R}^2$ by Ω^C.

On the set Ω we consider eigenproblems with homogeneous Dirichlet or Neumann boundary conditions.

We define the problem: find $\lambda \in \mathbb{R}$, $u \in H_0^1(\Omega)$, $u \not\equiv 0$, such that

$$\text{for all } v \in H_0^1(\Omega) \quad a_\Omega(k; u, v) = \lambda (u, v)_{0,\Omega}, \tag{3.4}$$

where the bilinear form $a_\Omega(k;u,v) : H^1(\Omega) \times H^1(\Omega) \to \mathbb{R}$ is given by

$$a_\Omega(k;u,v) = \int_\Omega \left\{ \nabla u \nabla v - k^2 n^2 uv \right\} \, dx. \tag{3.5}$$

We denote by A_k^D the operator given by the spectral formulation. Associated to the problem (3.4) we define the quantities

$$\lambda_m^D(k) = \inf_{H_m \in \mathscr{H}_m(H_0^1(\Omega))} \sup_{\substack{v \in H_m \\ v \neq 0}} \frac{a_\Omega(k;v,v)}{(v,v)_{0,\Omega}} \tag{3.6}$$

and

$$N^D(k) = \sup \left\{ m \in \mathbb{N}; \lambda_m^D(k) < \overline{\gamma}(k) \right\}, \tag{3.7}$$

where $\overline{\gamma}(k)$ is the lowest bound of the essential spectrum of A_k.

We define now the problem: find $\lambda \in \mathbb{R}$, $u \in H^1(\Omega)$, $u \not\equiv 0$, such that

$$\text{for all } v \in H^1(\Omega) \quad a_\Omega(k;u,v) = \lambda(u,v)_{0,\Omega}. \tag{3.8}$$

We denote by A_k^N the operator given by the spectral formulation. As before we associate to the eigenproblem (3.8) the quantities

$$\lambda_m^N(k) = \inf_{H_m \in \mathscr{H}_m(H^1(\Omega))} \sup_{\substack{v \in H_m \\ v \neq 0}} \frac{a_\Omega(k;v,v)}{(v,v)_{0,\Omega}} \tag{3.9}$$

and

$$N^N(k) = \sup \left\{ m \in \mathbb{N}; \lambda_m^N(k) < \overline{\gamma}(k) \right\}. \tag{3.10}$$

Proposition 3.1. *We assume that Ω is a piecewise C^1 open set. For all $m \in \mathbb{N}^*$, $k \in \mathbb{R}_+^*$*

$$\lambda_m(k) \leq \lambda_m^D(k) \quad \text{and} \quad N^D(k) \leq N(k).$$

Proof. The proof is immediate when we identify $H_0^1(\Omega)$ to a subspace of $H^1(\mathbb{R}^2)$. □

In the case of the Neumann boundary conditions our results are not so general. The extension by 0 of $\varphi \in H^1(\Omega)$ to $\tilde{\varphi} \in H^1(\mathbb{R}^2)$ does not work anymore.

Let $a, b \in \mathbb{R}^+$ be such that $K \subset (-a,a) \times (-b,b)$. We define the three different sets

$$\Omega_1 = \{x \in \mathbb{R}^2; |x_2| < d\}, \tag{3.11}$$

$$\Omega_2 = \{x \in \mathbb{R}^2; |x_1| < a\}, \tag{3.12}$$

$$\Omega_3 = (-a,a) \times (-d,d), \tag{3.13}$$

where $d = \max(b,c)$.

Proposition 3.2. *Let Ω be one of the three sets defined above in (3.11), (3.12), or (3.13). Then for all $m \in \mathbb{N}^*$, $k \in \mathbb{R}_+^*$, the following holds*

$$\min(\lambda_m^N(k), \overline{\gamma}(k)) \leq \lambda_m(k) \leq \lambda_m^D(k) \tag{3.14}$$

and

$$N^D(k) \leq N(k) \leq N^N(k). \tag{3.15}$$

Proof. Both estimates involving the Dirichlet problem in (3.14), (3.15) are proved in Proposition 3.1 for any piecewise C^1 open set. Since the estimate (3.15) is an immediate consequence of (3.14), it suffices to check

$$\min(\lambda_m^N(k), \overline{\gamma}(k)) \leq \lambda_m(k), \tag{3.16}$$

for Ω in (3.11), (3.12), and (3.13).
<u>Case 1: $\Omega = \Omega_1$.</u> Given $u \in H^1(\mathbb{R}^2)$ we have, since $d \geq c$,

$$\int_{\Omega^C} \left(|\nabla u|^2 - k^2 n^2 u^2\right)\, dx \geq -k^2 n_b^2 \int_{\Omega^C} u^2\, dx.$$

Then from the second characterization of $\lambda_m(k)$ we get

$$\lambda_m(k) \geq \sup_{V_{m-1} \in \mathscr{V}_{m-1}(L^2(\mathbb{R}^2))} \inf_{\substack{v \in V_{m-1}^\perp \cap H^1(\mathbb{R}^2) \\ v \neq 0}} \frac{\int_\Omega \left(|\nabla u|^2 - k^2 n^2 u^2\right)\, dx - k^2 n_b^2 \int_{\Omega^C} u^2\, dx}{\int_\Omega u^2\, dx + \int_{\Omega^C} u^2\, dx}, \tag{3.17}$$

where $\mathscr{V}_{m-1}(L^2(\mathbb{R}^2))$ is the set of m-dimensional subspaces of $L^2(\mathbb{R}^2)$. Notice that for $v \in L^2(\mathbb{R}^2)$ with $v = 0$ in Ω, the above quotient reduced to $-k^2 n_b^2$. Therefore

$$\lambda_m(k) \geq \sup_{V_{m-1} \in \mathscr{V}_{m-1}(L^2(\Omega))} \inf_{\substack{v \in V_{m-1}^\perp \cap H^1(\Omega) \\ v \neq 0}} \alpha(u)$$

with

$$\alpha(u) = \min\left(\frac{\int_\Omega \left(|\nabla u|^2 - k^2 n^2 u^2\right)\, dx}{\int_\Omega u^2\, dx}, -k^2 n_b^2 \right).$$

Notice that here we have used the estimate, for $a_1, a_2, a_3, a_4 \in \mathbb{R}$, $a_3, a_4 > 0$,

$$\frac{a_1 + a_2}{a_3 + a_4} \geq \min\left(\frac{a_1}{a_3}, \frac{a_2}{a_4}\right).$$

We deduce then

$$\lambda_m(k) \geq \min(\lambda_m^N(k), -k^2 n_b^2).$$

Since we already know that $\overline{\gamma}(k) \leq -k^2 n_b^2$, then

$$\lambda_m(k) \geq \min(\lambda_m^N(k), \overline{\gamma}(k)).$$

<u>Case 2: $\Omega = \Omega_2$.</u> Let $u \in H^1(\mathbb{R}^2)$ be given. Then for almost all x_1, $|x_1| > a$, we get from the definition (2.6) of $\overline{\gamma}(k)$

$$\int_{\mathbb{R}} \left(|\frac{\partial u}{\partial x_2}(x_1, x_2)|^2 - k^2 n^2 u^2(x_1, x_2) \right) dx_2 \geq \overline{\gamma}(k) \int_{\mathbb{R}} u^2(x_1, x_2)\, dx_2.$$

By integration in x_1 over $(-\infty, -c) \cup (c, \infty)$,

$$\int_{\Omega^C} \left(|\nabla u|^2 - k^2 n^2 u^2 \right) dx \geq \overline{\gamma}(k) \int_{\Omega^C} u^2\, dx.$$

We argue as in the case 1 to check (3.16).
<u>Case 3: $\Omega = \Omega_3$.</u> We proceed in two steps. We compare first $\lambda_m^N(k)$ with $\lambda_m^N(k)(\Omega_2)$ (the one in case 2) with the argument of case 1. So

$$\lambda_m^N(k)(\Omega_2) \geq \min\left(\lambda_m^N(k), -k^2 n_b^2\right).$$

Then from the analysis of case 2, we can check (3.16). $\square$

Remark 3.1. If we denote by $\lambda_m^N(k)$, $N^N(k)$ the quantities (3.9), (3.10) associated to Ω_2, and $\lambda_m^{NN}(k)$, $N^{NN}(k)$ the ones associated to Ω_3 then in the proof of Proposition 3.2 we have checked that

$$\lambda_m^{NN}(k) \leq \lambda_m^N(k) \tag{3.18}$$

and

$$N(k) \leq N^N(k) \leq N^{NN}(k). \quad \square \tag{3.19}$$

In the next proposition we present some additional results for both eigenvalue problems (3.4) and (3.8) when Ω is one of the sets (3.11), (3.12), or (3.13).

Proposition 3.3. *The following assertions hold.*
(i) When $\Omega = \Omega_1 = \{x \in \mathbb{R}^2; |x_2| < d\}$, then

$$\sigma_{ess}(A_k^D) = [\overline{\gamma}^D(k), \infty), \qquad \overline{\gamma}(k) \leq \overline{\gamma}^D(k), \tag{3.20}$$

$$\sigma_{ess}(A_k^N) = [\overline{\gamma}^N(k), \infty), \qquad \overline{\gamma}^N(k) \leq \overline{\gamma}(k), \tag{3.21}$$

where $\overline{\gamma}^D(k)$, $\overline{\gamma}^N(k)$ are defined in (A.26), (A.27) for $m = 1$.

(ii) When $\Omega = \Omega_2 = \{x \in \mathbb{R}^2; |x_1| < a\}$*, then*

$$\sigma_{ess}(A_k^D) = [-k^2 n_b^2 + \frac{\pi^2}{4a^2}, \infty), \tag{3.22}$$

$$\sigma_{ess}(A_k^N) = [-k^2 n_b^2, \infty). \tag{3.23}$$

(iii) When $\Omega = \Omega_3 = (-a, a) \times (-d, d)$*, then the spectrum* $\sigma(A_k^N)$ *or* $\sigma(A_k^D)$ *is an increasing sequence of eigenvalues tending to* ∞*.*

We will not present a proof of Proposition 3.3 here since all the arguments are classical or have been already used before.

As direct consequences of the comparison principles given above we get bounds on the number of guided modes.

Proposition 3.4. *For a fixed value* k *of the frequency a guide has a finite number of guided modes. Moreover there exists a* $k^* > 0$ *such that*

$$\text{for all } 0 < k < k^* \qquad N(k) \leq 1.$$

Proof. Let $k \in \mathbb{R}_+^*$ be given and $\Omega = (-a, a) \times (-d, d)$, see (3.13). From Proposition 3.2 we deduce $N(k) \leq N^N(k)$ and from Proposition 3.3, case 3), that $N^N(k) < \infty$.

We can prove the existence of k^* by contradiction. □

Let us apply the comparison principle in Proposition 3.2 to the three layers rib guide, see Figure 3, with the refractive index

$$n(x) = \begin{cases} n_b & \text{if } x < -c, \\ n_+ & \text{if } x \in (-a, a) \times (-c, H - c) \text{ or } -c < x_2 < c, \\ n_t & \text{else,} \end{cases}$$

where $H > 2c$. We assume that $n_+ > n_b \geq n_t$. We can operate in two different ways: either compare with respect to the domain $(-a, a) \times (-c, H - c)$ or with respect to $(-a, a) \times \mathbb{R}$. The second way gives more accurate bounds but needs more computations.

Let Ω be the set $(-a, a) \times (-c, H - c)$. It is not difficult to check that

$$\lambda_{p,q}^N(k) = -k^2 n_+^2 + \frac{p^2\pi^2}{4a^2} + \frac{q^2\pi^2}{H^2} \quad \text{for } p, q \in \mathbb{N},$$
$$\lambda_{p,q}^D(k) = -k^2 n_+^2 + \frac{p^2\pi^2}{4a^2} + \frac{q^2\pi^2}{H^2} \quad \text{for } p, q \in \mathbb{N}^*.$$

So we have the bounds

$$N^D(k) \leq N(k) \leq N^N(k) \tag{3.24}$$

with

(3.25)
$$N^D(k) = \text{Card}\left\{(p,q) \in \mathbb{N}^* \times \mathbb{N}^*; -k^2 n_+^2 + \frac{p^2\pi^2}{4a^2} + \frac{q^2\pi^2}{H^2} < \overline{\gamma}(k)\right\},$$

(3.26)
$$N^N(k) = \text{Card}\left\{(p,q) \in \mathbb{N} \times \mathbb{N}; -k^2 n_+^2 + \frac{p^2\pi^2}{4a^2} + \frac{q^2\pi^2}{H^2} < \overline{\gamma}(k)\right\}.$$

Notice that we can compute $\overline{\gamma}(k)$ with the formulae presented in the appendix A. Then with (3.25) and (3.26) it is simple to compute $N^D(k)$ and $N^N(k)$.

Let Ω be the set $(-a,a) \times \mathbb{R}$. We can compute numerically the eigenvalues when Ω is a strip. In the example A.1 of the appendix A we have presented a way of studying the three layers planar waveguide. If $\gamma_q(k,H)$ are the Min–Max associated to the three layers planar waveguide of thickness H, see (A.8) and (A.9), then

$$\lambda_{p,q}^N(k) = \gamma_q(k,H) + \frac{p^2\pi^2}{4a^2} \quad \text{for } p \in \mathbb{N},$$
$$\lambda_{p,q}^D(k) = \gamma_q(k,H) + \frac{p^2\pi^2}{4a^2} \quad \text{for } p \in \mathbb{N}^*.$$

So we have the bounds

$$N^D(k) \le N(k) \le N^N(k) \tag{3.27}$$

with

$$N^D(k) = \text{Card}\left\{(p,q) \in \mathbb{N}^* \times \mathbb{N}^*; \gamma_q(k,H) + \frac{p^2\pi^2}{4a^2} < \overline{\gamma}(k)\right\}, \tag{3.28}$$

$$N^N(k) = \text{Card}\left\{(p,q) \in \mathbb{N} \times \mathbb{N}; \gamma_q(k,H) + \frac{p^2\pi^2}{4a^2} < \overline{\gamma}(k)\right\}. \tag{3.29}$$

Remark 3.2. To illustrate both estimates (3.24) and (3.27), we present two computed examples. In both of them we have chosen

$$n_+ = 3, \quad n_b = 2, \quad n_t = 1.$$

Then we have modified the size of the *bump*.
Case 1. We choose: $a = 1$, $c = 1$, and $H = 3$. The estimates (3.24) and (3.27) give respectively

$$\begin{array}{ll} \text{if } k \in [0.1, 0.8] & 0 \le N(k) \le 1, \\ \text{if } k \in [0.9, 100] & 0 \le N(k) \le 2, \end{array} \quad \text{and} \quad \text{if } k \in [0.1, 100] \quad 0 \le N(k) \le 1.$$

Case 2. We choose: $a = 3$, $c = 1$, and $H = 6$. The estimates (3.24) give

$$\begin{array}{ll} \text{if } k \in [0.1, 0.2] & 0 \leq N(k) \leq 1, \\ \text{if } k \in [0.3, 0.4] & 0 \leq N(k) \leq 3, \\ \text{if } k \in [0.5, 0.8] & 1 \leq N(k) \leq 4, \\ \text{if } k \in [0.9, 1.2] & 1 \leq N(k) \leq 6, \\ \text{if } k \in [1.3, 6.6] & 3 \leq N(k) \leq 8, \\ \text{if } k \in [6.7, 100] & 4 \leq N(k) \leq 9, \end{array} \quad \text{and} \quad \begin{array}{ll} \text{if } k \in [0.1, 0.32] & 0 \leq N(k) \leq 1, \\ \text{if } k \in [0.33, 0.44] & 1 \leq N(k) \leq 2, \\ \text{if } k \in [0.45, 0.8] & 1 \leq N(k) \leq 3, \\ \text{if } k \in [0.9, 1.1] & 2 \leq N(k) \leq 4, \\ \text{if } k \in [1.2, 5.4] & 3 \leq N(k) \leq 5, \\ \text{if } k \in [5.5, 100] & 4 \leq N(k) \leq 6. \end{array}$$

The numbers presented above were obtained by solving the dispersion equation given in the example A.1. More details and further examples can be found in the work of MAHÉ [1993].

The results above illustrate Proposition 3.4. Moreover we notice that the number of modes is not increasing indefinitely with k. This result will be proved in a more general framework in Section 5. □

Remark 3.3. We will look in the next sections for solutions of (2.1) (or (2.3)) for given values of $k \in \mathbb{R}_*^+$. We could address also the problem of the dependence on k of the modes. Let $\lambda_m(k)$ be defined in (3.1) and (3.2). When $\lambda_m(k)$ is an eigenvalue of A_k, the function $k \mapsto \lambda_m(k)$ is called the m**th dispersion curve**. Extending the results of BONNET [1988], we can prove that the function $k \mapsto \lambda_m(k)$ is continuous and almost everywhere differentiable. □

4. The fundamental mode

From the Min–Max principle, we can immediately discuss some existence conditions for the fundamental mode.

Proposition 4.1. *If the refractive index n satisfies*

$$\text{for all } x \in \mathbb{R}^2 \quad n(x) \leq \overline{n}(x_2) \tag{4.1}$$

then there are no guided modes, $N(k) = 0$ for all $k > 0$.

Proof. By the definition of $\overline{\gamma}(k)$ in (2.6) and the hypothesis, we obtain for all $\varphi \in C_0^\infty(\mathbb{R}^2)$ and $x_1 \in \mathbb{R}$

$$\int_{\mathbb{R}} \left[\left(\frac{\partial \varphi}{\partial x_2} \right)^2 - k^2 n^2 \varphi^2 \right] dx_2 \geq \overline{\gamma}(k) \int_{\mathbb{R}} \varphi^2 \, dx_2.$$

We integrate the inequality over x_1 and use the density of $C_0^\infty(\mathbb{R}^2)$ in $H^1(\mathbb{R}^2)$ to get

$$\inf_{\substack{\varphi \in H^1(\mathbb{R}^2) \\ \varphi \neq 0}} \frac{a(k; \varphi, \varphi)}{(\varphi, \varphi)_{0,\mathbb{R}^2}} \geq \overline{\gamma}(k)$$

and we can conclude. □

Proposition 4.2. *We assume that the refractive index n satisfies*

$$\text{for all } x \in \mathbb{R}^2 \quad n(x) \geq \overline{n}(x_2) \tag{4.2}$$

and

$$\text{measure}(\tilde{K} = \{x \in \mathbb{R}^2; n(x) > \overline{n}(x_2)\}) \neq 0. \tag{4.3}$$

If the planar waveguide associated has a guided wave

$$\overline{\gamma}(k) < -k^2 n_b^2, \tag{4.4}$$

then the guide has at least one guided mode, that is

$$\lambda_1(k) < \overline{\gamma}(k) \tag{4.5}$$

and then $N(k) \geq 1$.

Proof. We will prove the following inequality

$$\inf_{\substack{v \in H^1(\mathbb{R}^2) \\ v \neq 0}} \frac{a(k; v, v)}{(v, v)_{0,\mathbb{R}^2}} < \overline{\gamma}(k).$$

It is sufficient thus to prove that there exists a function $v \in H^1(\mathbb{R}^2)$ such that

$$a(k; v, v) = \int_{\mathbb{R}^2} \left(|\nabla v|^2 - k^2 n^2 v^2\right) \, dx < \overline{\gamma}(k) \int_{\mathbb{R}^2} v^2 \, dx.$$

The assumption (4.4) implies that $\overline{\gamma}(k)$ is an eigenvalue of $-\frac{\mathrm{d}^2}{\mathrm{d}y^2} - k^2\overline{n}^2$ with an eigenfunction $\overline{\varphi}$, see the appendix A. We take

$$v(x) = \overline{\varphi}(x_2) e^{-\alpha|x_1|}$$

where α is a positive number which will be fixed later. Then

$$a(k; v, v) = \int_{\mathbb{R}^2} \left((\overline{\varphi}')^2 + (\alpha^2 - k^2\overline{n}^2)\overline{\varphi}^2\right) e^{-2\alpha|x_1|} \, dx - \int_{\mathbb{R}^2} k^2(n^2 - \overline{n}^2)\overline{\varphi}^2 e^{-2\alpha|x_1|} \, dx$$

and

$$a(k; v, v) = \frac{1}{\alpha} \int_{\mathbb{R}} (\overline{\gamma}(k) + \alpha^2)\overline{\varphi}^2 \, dx_2 - \int_K k^2(n^2 - \overline{n}^2)\overline{\varphi}^2 e^{-2\alpha|x_1|} \, dx.$$

We have $\alpha\|v\|^2_{0,\mathbb{R}^2} = \|\overline{\varphi}\|^2_{0,\mathbb{R}}$. So to check that $\lambda_1(k) < \overline{\gamma}(k)$ it suffices to notice that

$$\alpha\|\overline{\varphi}\|^2_{0,\mathbb{R}} < \int_{\tilde{K}} k^2(n^2 - \overline{n}^2)\overline{\varphi}^2 e^{-2\alpha|x_1|} \, dx$$

for $\alpha > 0$ small enough since (4.3) holds and $\overline{\varphi}$ is almost everywhere non zero in $\mathbb{R}$. $\square$

The next two corollaries are consequences of Proposition 4.2 for particular guides.

Corollary 4.3. *Let n be a refractive index such that (4.2) and (4.3) hold. We assume further that*

$$n_b = n_t \tag{4.6}$$

$$\|\overline{n}\|_{\infty,\mathbb{R}} > n_b, \tag{4.7}$$

$$\int_{-c}^{c} (\overline{n}^2 - n_b^2)\, dy \geq 0. \tag{4.8}$$

Then the guide has at least one guided mode, $N(k) \geq 1$, for all $k > 0$.

Proof. In the appendix A we prove that under the assumptions (4.6), (4.7), and (4.8), the fundamental mode of the planar waveguide associated exists. We can conclude with Proposition 4.2. □

Corollary 4.4. *Let n be a refractive index such that (4.2) and (4.3) hold. We assume further that*

$$n_b > n_t \tag{4.9}$$

$$\|\overline{n}\|_{\infty,\mathbb{R}} > n_b. \tag{4.10}$$

Then there exist two positive constants $\underline{k} \leq \overline{k}$ such that

$$N(k) = 0 \quad \textit{for all } k < \underline{k} \qquad\qquad N(k) \geq 1 \quad \textit{for all } k > \overline{k}.$$

Proof. In the appendix A we prove that there exists k_1 such that

$$\text{for } k < k_1 \quad \overline{\gamma}(k) = -k^2 n_b^2 \quad \text{and} \quad \text{for } k > k_1 \quad \overline{\gamma}(k) < -k^2 n_b^2.$$

For $k > k_1$ we get with Proposition 4.2 that $N(k) \geq 1$.

Let us prove now that $N(k) = 0$ for k sufficiently small. We define the refractive index $\bar{\bar{n}}$ by

$$\bar{\bar{n}}(x) = \begin{cases} n_t & \text{if } x_2 > d, \\ n_+ & \text{if } |x_2| \leq d, \\ n_b & \text{if } x_2 < -d, \end{cases}$$

where the real numbers a, b are such that $(-a, a) \times (-b, b) \supset K$ and $d = \max(b, c)$. Then clearly for all $x \in \mathbb{R}^2$ $\bar{\bar{n}}(x) \geq n(x)$. In fact $\bar{\bar{n}}$ is the index of a planar guide with $n_t \neq n_b$. So there exists $\underline{k} > 0$ such that

$$\text{for } k < \underline{k} \quad \bar{\bar{\gamma}}(k) = -k^2 n_b^2 \quad \text{and} \quad \text{for } k > \underline{k} \quad \bar{\bar{\gamma}}(k) < -k^2 n_b^2,$$

with obvious notation for $\bar{\bar{\gamma}}(k)$. So for $v \in H^1(\mathbb{R}^2)$ we have

$$\int_{\mathbb{R}^2} \left(|\nabla v|^2 - k^2 n^2 v^2\right) dx \geq \int_{\mathbb{R}^2} \left(|\nabla v|^2 - k^2 \bar{\bar{n}}^2 v^2\right) dx$$

and for $k < \underline{k}$ and $x_1 \in \mathbb{R}$

$$\int_{\mathbb{R}} \left[\left(\frac{\partial v}{\partial x_2} \right)^2 - k^2 \bar{\bar{n}}^2 v^2 \right] dx_2 \geq -k^2 n_b^2 \int_{\mathbb{R}} v^2 \, dx_2.$$

By integrating we deduce that

$$\int_{\mathbb{R}^2} \left(|\nabla v|^2 - k^2 n^2 v^2 \right) dx \geq -k^2 n_b^2 \int_{\mathbb{R}^2} v^2 \, dx$$

and then

$$\lambda_1(k) \geq -k^2 n_b^2 = \overline{\gamma}(k) \quad \text{for } k < \underline{k}.$$

This implies that $N(k) = 0$ for $k < \underline{k}$. $\square$

In the next two results we give some kind of characterization for guides having a fundamental mode at low frequencies.

Proposition 4.5. *We assume the existence of a sequence $\{k_p\}_{p \geq 1}$ converging to 0 such that*

$$N(k_p) \geq 1 \quad \text{for all } p \in \mathbb{N}. \tag{4.11}$$

Then necessarily the following conditions hold

$$n_b = n_t, \tag{4.12}$$

$$\int_{\mathbb{R}^2} \left(n^2 - \overline{n}^2 \right) dx \geq 0. \tag{4.13}$$

Proof. The equality (4.12) is an immediate consequence of Proposition A.5. Let us check now that (4.13) holds. From the comparison principle we deduce that

$$N(k) \leq N^N(k),$$

where $N^N(k)$ is the number of eigenvalues for the Neumann problem in a given rectangle Ω containing K. From the assumption (4.11) we deduce that

$$N^N(k_p) \geq 1 \quad \text{for all } p \in \mathbb{N},$$

which gives the existence of a function $\varphi_p \in H^1(\Omega)$, $\|\varphi_p\|_{0,\Omega} = 1$, such that

$$\int_{\Omega} \left\{ |\nabla \varphi_p|^2 - k_p^2 n^2 \varphi_p^2 \right\} dx < \overline{\gamma}(k_p) \int_{\Omega} \varphi_p^2 \, dx. \tag{4.14}$$

With the definition (2.6) of $\overline{\gamma}(k_p)$ we deduce that

$$\int_\Omega |\nabla\varphi_p|^2\,dx \to 0 \quad \text{when } p\to\infty$$

and then

$$\varphi_p \to \frac{1}{|\Omega|^{1/2}} \quad \text{in } L^2(\Omega),$$

where $|\Omega|$ denotes the measure of Ω. Going back to (4.14) we write

$$\int_\Omega |\nabla\varphi_p|^2\,dx - k_p^2\int_\Omega \overline{n}^2\varphi_p^2\,dx - \overline{\gamma}(k_p)\int_\Omega \varphi_p^2\,dx < k_p^2\int_\Omega (n^2-\overline{n}^2)\varphi_p^2\,dx$$

and the definition of $\overline{\gamma}(k_p)$ implies

$$\int_\Omega |\frac{\partial\varphi_p}{\partial x_2}|^2\,dx - k_p^2\int_\Omega \overline{n}^2\varphi_p^2\,dx \geq \overline{\gamma}(k_p)\int_\Omega \varphi_p^2\,dx$$

so

$$\int_\Omega (n^2-\overline{n}^2)\varphi_p^2\,dx > 0 \quad \text{for all } p,$$

which implies (4.13), when passing to the limit. □

We can prove some sort of reciprocal of Proposition 4.5.

Proposition 4.6. *We assume that the associated planar waveguide satisfies to the following hypotheses*

$$n_b = n_t \quad \text{and} \quad \|\overline{n}\|_{\infty,\mathbb{R}} > n_b,$$

and that its fundamental mode exists at each frequency, that is

$$\int_{-c}^{c} (\overline{n}^2 - n_b^2)\,dy \geq 0. \tag{4.15}$$

Moreover we assume that

$$\int_{\mathbb{R}^2} (n^2-\overline{n}^2)\,dx > 0. \tag{4.16}$$

Then there exists a $\underline{k}>0$ *such that for* $0<k<\underline{k}$, $N(k)\geq 1$.

Proof. We argue like in the proof of Proposition 4.2, by proving the existence of a function $v\in H^1(\mathbb{R}^2)$ such that

$$a(k;v,v) = \int_{\mathbb{R}^2} \left(|\nabla v|^2 - k^2n^2v^2\right)\,dx < \overline{\gamma}(k)\int_{\mathbb{R}^2} v^2\,dx.$$

In Proposition A.4 we have proved that the assumption (4.15) is necessary and sufficient to have the existence of the fundamental mode of the associate planar waveguide, at each frequency. Therefore there exists a function $\overline{\varphi}_k \in H^1(\mathbb{R})$, with $\int_{-c}^{c} \overline{\varphi}_k \, dy = 2c$, such that

$$-\frac{d^2\overline{\varphi}_k}{dy^2} - k^2\overline{n}^2\overline{\varphi}_k = \overline{\gamma}(k)\overline{\varphi}_k \quad \text{in } \mathbb{R}.$$

We develop then like in the proof of Proposition 4.2. Then to prove the existence of a guided mode in the guide of index n for k small enough, it is sufficient to check that

$$\int_{\mathbb{R}^2} (n^2 - \overline{n}^2)\overline{\varphi}_k^2(x_2)\, dx > 0. \tag{4.17}$$

We have

$$\int_{\mathbb{R}^2} (n^2 - \overline{n}^2)\overline{\varphi}_k^2(x_2)\, dx = \int_{\mathbb{R}^2} (n^2 - \overline{n}^2)\, dx + \int_{\mathbb{R}^2} (n^2 - \overline{n}^2)(\overline{\varphi}_k^2(x_2) - 1)\, dx.$$

The proof is complete if we prove that

$$\int_{\mathbb{R}^2} (n^2 - \overline{n}^2)(\overline{\varphi}_k^2(x_2) - 1)\, dx \to 0 \quad \text{as } k \to 0.$$

Since $\overline{\gamma}(k) < -k^2 n_b^2$ we have

$$\int_{\mathbb{R}} \left| \frac{d\overline{\varphi}_k}{dy} \right|^2 dy \le k^2 \int_{\mathbb{R}} (\overline{n}^2 - n_b^2)\overline{\varphi}_k^2\, dy.$$

Then $\frac{d\overline{\varphi}_k}{dy} \to 0$ in $L^2(\mathbb{R})$. From the normalization of $\overline{\varphi}_k$ we deduce that $\overline{\varphi}_k$ tends to 1 in $L^2(-c, c)$ as $k \to 0$. $\square$

Remark 4.1. To illustrate the previous proposition we consider a guide whose shape is represented in Figure 5, with $n_b, n_+ \in \mathbb{R}$, $n_+ > n_b > 0$.

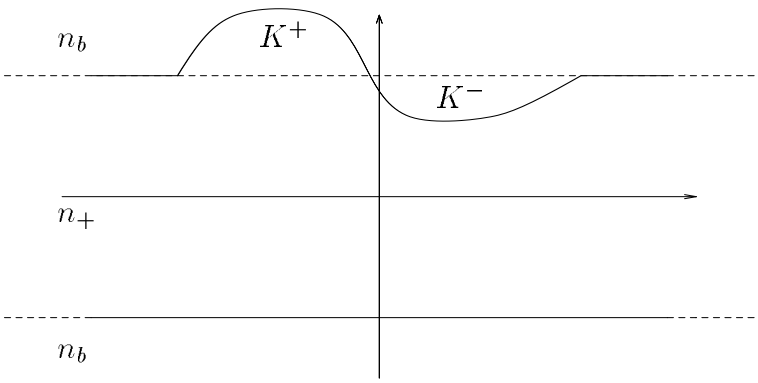

FIGURE 5: PARTICULAR WAVEGUIDE.

We define

$$K^+ = \{x \in \mathbb{R}^2; n^2(x) - \overline{n}^2(x_2) > 0\} \quad \text{and} \quad K^- = \{x \in \mathbb{R}^2; n^2(x) - \overline{n}^2(x_2) < 0\}.$$

If $|K^+|$, $|K^-|$ denote the measure of both sets K^+, K^-, then from Propositions 4.5 and 4.6, we get the following. If $|K^+| > |K^-|$ then $N(k) \geq 1$ for low frequencies. If $N(k) \geq 1$ for low frequencies then $|K^+| \geq |K^-|$.

Similar results were encountered for optical fibers see BONNET [1988] and in hydrodynamics see BONNET-BENDHIA and JOLY [1994]. □

Remark 4.2. The assumption (4.16) in Proposition 4.6 can be relaxed to the following

$$\int_{\mathbb{R}^2} (n^2 - \overline{n}^2)\, dx \geq 0 \quad \text{and} \quad \sup(n^2 - \overline{n}^2) > 0, \tag{4.18}$$

see Proposition A.4. □

5. Asymptotical study at high frequencies

Results at high frequencies, that is for large k, are developed now. When a guide has an index n whose maximum is reached only in a bounded domain, that is $n_+ > \|\overline{n}\|_{\infty,\mathbb{R}^2}$, then the perturbation K of the stratified medium traps the wave and the stratified medium does not play any role for large k. The results presented for optical fibers or for guides with a diopter cladding, see BAMBERGER and BONNET [1990], can be generalized to stratified media. In fact the number of guided modes is increasing to ∞ when k is increasing and the modes are more and more confined in the bounded set where the index is maximum.

Here our original results concern the case when the maximum of the index inside the core is equal to the maximum of the index in the stratified medium.

First upper bounds for the number of guided modes are derived. In particular we prove that the number of guided modes with the rib waveguide remains bounded for large k. Finally we present the example of a guide with no mode at high frequencies.

5.1 Upper bounds for the number of modes.

The main tool to get bounds for the number of modes is the comparison method. Here we will use it in the following way.

Proposition 5.1. *Let n be an index, that is $n \in L^\infty(\mathbb{R}^2)$ satisfies to (1.1) and (1.2). There exists C such that for $k \in \mathbb{R}^+_*$ the following bound for the number of modes $N(k)$ holds*

$$N(k) \leq C\left(\overline{\gamma}(k) + k^2 n_+^2\right); \tag{5.1}$$

we recall that here $\overline{\gamma}(k)$, defined in (2.6), is the lower bound of the essential spectrum of A_k.

Proof. We assume that the perturbation K is included in the rectangle $(-a, a) \times (-b, b)$. By Proposition 3.2, we know that

$$\min(\lambda_m^N(k), \overline{\gamma}(k)) \leq \lambda_m(k),$$

where $\lambda_m^N(k)$ is the mth eigenvalue for the Neumann problem (3.8) with $\Omega = (-a, a) \times (-d, d)$, see (3.13). Moreover by the Min–Max formulae, it is obvious that $\lambda_m'^N(k) \leq \lambda_m^N(k)$ where $\lambda_m'^N(k)$ is the mth eigenvalue for the Neumann problem (3.8) with n replaced by n_+ and $\Omega = (-a, a) \times (-d, d)$. Therefore

$$N(k) \leq \sup\{m \in \mathbb{N}; \lambda_m'^N(k) \leq \overline{\gamma}(k)\}.$$

The eigenvalues $\lambda_m'^N(k)$ are given in the numbering p, q by the expressions

$$\lambda_{p,q}'^N(k) = -k^2 n_+^2 + \frac{p^2\pi^2}{4a^2} + \frac{q^2\pi^2}{4b^2}, \quad \text{for } p,\ q \text{ in } \mathbb{N}. \tag{5.2}$$

Then the number of eigenvalues $\lambda_{p,q}'^N(k)$ inside the interval $[-k^2 n_+^2, \overline{\gamma}(k)]$ is bounded by its length $\overline{\gamma}(k) + k^2 n_+^2$ times a constant depending on a and b. And (5.1) is proved. □

Remark 5.1. Notice that in the proof above we have used part of the following result. If

$$\mathscr{N}(A) = \left| \{(p, q) \in \mathbb{N}^2; \left(\frac{p^2}{4a^2} + \frac{q^2}{4b^2}\right)\pi^2 < A\} \right|$$

then

$$\mathscr{N}(A) \sim A\frac{ab}{\pi} \quad \text{as } A \to \infty,$$

see Courant and Hilbert [1953], p.430. □

Remark 5.2. We can state Proposition 5.1 in other words. In fact the quantity $L = \overline{\gamma}(k) + k^2 n_+^2$ is the length of the interval $[-k^2 n_+^2, \overline{\gamma}(k)]$ where we can find guided modes, see Figure 4. The number of guided modes inside this interval is less than the number of eigenvalues for the Neumann problem in $(-a, a) \times (-b, b)$ with an index n_+. These eigenvalues are known and their number inside the interval is proportional to L, as L tends to ∞. □

Corollary 5.2. *Let n be an index with $K \subset (-a, a) \times (-b, b)$.*
(i) If the index satisfies to

$$n_+ > \|\overline{n}\|_{\infty,\mathbb{R}}, \tag{5.3}$$

then there exist $C, k_0 > 0$ such that for all $k \geq k_0$

$$N(k) \leq Ck^2. \tag{5.4}$$

(ii) If the index satisfies to

$$\begin{cases} \overline{n}(y) = n_+ \text{ for } y \in (-\eta, \eta), \\ \overline{n}(y) < n_+ \text{ else}, \end{cases} \tag{5.5}$$

then the number of guided modes remains bounded as k tends to ∞. Moreover we get the following estimate for the number of guided modes, for large k

$$N(k) \leq M_+ \equiv Card\{(m,p) \in \mathbb{N}^* \times \mathbb{N};\ \frac{m^2}{d^2} + \frac{p^2}{a^2} \leq \frac{1}{\eta^2}\}, \tag{5.6}$$

where $d = \max(b, \eta)$.

(iii) If the index satisfies to

$$\begin{cases} n_+^2 - \overline{n}(y)^2 \sim |y|^\alpha,\ \alpha > 0,\ \textit{in a neighborhood of } y = 0, \\ \overline{n}(y) \leq n_* < n_+ \ \textit{else}, \end{cases} \tag{5.7}$$

then there exist $C, k_0 > 0$ such that for all $k \geq k_0$

$$N(k) \leq C k^{4/(2+\alpha)}. \tag{5.8}$$

Proof.

(i) We notice that

$$\overline{\gamma}(k) + k^2 n_+^2 \leq \overline{\gamma}(k) + k^2 \overline{n}_+^2 + k^2 (n_+^2 - \overline{n}_+^2)$$

where $\overline{n}_+ = \|\overline{n}\|_{\infty,\mathbb{R}}$. Then from Proposition A.6 we deduce that $(\overline{\gamma}(k) + k^2 n_+^2)/k^2$ is bounded.

(ii) If we use the index comparison principle, then to prove the estimate (5.6) it is sufficient to check that the number of guided modes for the guide of index

$$\tilde{n}(x) = \begin{cases} n_+ & \text{if } x \in (-a,a) \times (-d,d), \\ \overline{n}(x_2) & \text{else.} \end{cases} \tag{5.9}$$

is bounded for large k by M_+. In the set $\Omega = (-a,a) \times \mathbb{R}$, we consider the Neumann eigenvalue problem

$$\begin{cases} -\Delta u - k^2 \tilde{n}^2 u = \mu u & \text{in } \Omega, \\ \frac{\partial u}{\partial \nu} = 0 & \text{on } \partial\Omega, \end{cases} \tag{5.10}$$

where ν is the unit outward normal vector on $\partial\Omega$. Since $\tilde{n}$ is in fact depending only on x_2 , it is not difficult with the separation of variables to check that the eigenvalues of (5.10) are

$$\mu_{m,p} = \tilde{\gamma}_m(k) + \frac{p^2 \pi^2}{4a^2} \quad p \in \mathbb{N},$$

where $\tilde{\gamma}_m(k)$ is the mth eigenvalue, when it exists, of the planar waveguide with index $\tilde{n}(0,.)$, see (A.9). Using the comparison principle with the Neumann problem, see Proposition 3.2, to check the bound given in (5.9), it is sufficient to prove that

$$\tilde{N}^N(k) \leq M_+, \tag{5.11}$$

where $\tilde{N}^N(k)$ is the number of eigenvalues of (5.10) below the essential spectrum of $\tilde{A}_k$ (and also of A_k), that is

$$\tilde{N}^N(k) = |\{(m,p) \in \mathbb{N}^* \times \mathbb{N}; \mu_{m,p} < \overline{\gamma}(k)\}|\,.$$

From Proposition A.8 we get

$$\overline{\gamma}(k) + k^2\overline{n}_+^2 \to \frac{\pi^2}{4\eta^2} \quad \text{as } k \text{ tends to } \infty$$

and

$$\tilde{\gamma}_m(k) + k^2\overline{n}_+^2 + \frac{p^2\pi^2}{4a^2} \to \frac{m^2\pi^2}{4d^2} + \frac{p^2\pi^2}{4a^2} \quad \text{as } k \text{ tends to } \infty.$$

Now if $1/\eta^2 < m^2/d^2 + p^2/a^2$, then for k large enough

$$\overline{\gamma}(k) < \tilde{\gamma}_m(k) + \frac{p^2\pi^2}{4a^2} = \mu_{m,p}$$

and so (5.11) holds.

(iii) We can adapt the proof of Proposition A.6 by setting $y_\epsilon = 0$ and $\rho_\epsilon = k^{-2/(\alpha+2)}$, to prove that under the assumptions (5.7) the term $\overline{\gamma}(k) + k^2n_+^2$ is estimated by

$$\overline{\gamma}(k) + k^2n_+^2 \le Ck^{4/(2+\alpha)},$$

see also Bonnet-BenDhia and Djellouli [1994]. Therefore (5.8) holds. □

Remark 5.3. From a physical point of view we can understand why the number of guided modes $N(k)$ remains bounded for some stratified guides while it tends to infinity for an optical fiber.

We have seen in Remark 2.2 that the velocity of a guided mode for a guide of index n satisfies

$$\frac{c_0}{n_+} \le v \le \overline{v}(k)$$

while the velocity of a guided mode for an optical fiber with an homogeneous cladding of index n_b satisfies

$$\frac{c_0}{n_+} \le v \le \frac{c_0}{n_b}.$$

In the case of the fiber the interval where v lies is independent of the frequency. When k tends to infinity then the size of the core becomes larger compared to the wavelength $2\pi/k$ and the number of guided modes tends to ∞.

Contrarily under the assumptions in (5.5) the length of the interval

$$\overline{v}(k) - \frac{c_0}{n_+} \to 0 \quad \text{for } k \to \infty,$$

see Proposition A.6. The difference between the velocity in the core and the one in the cladding tends to 0 and balance the fact that the size of the core is getting larger and larger compared to the wavelength. □

5.2 No mode at high frequencies.

Here we describe a guide with no mode at high frequencies. The idea is to choose a guide for which the set where n achieves its maximum is strictly included inside the strip $\mathbb{R} \times (-\eta, \eta)$.

Proposition 5.3. *We assume that*

$$n(x) < n_+ \quad \text{for almost all } x_2 \notin (-\eta, \eta) \tag{5.12}$$

and that

$$\begin{cases} \overline{n}(y) = n_+ & \text{if } |y| < \eta, \\ n(x) = \overline{n}(x_2) & \text{if } |x_1| > a, \\ n(x) = n_+ & \text{if } |x_1| < a \text{ and } |x_2| < \tau < \eta, \\ n(x) \le n_* < n_+ & \text{if } |x_1| < a \text{ and } |x_2| > \tau. \end{cases} \tag{5.13}$$

Then there exists $k_* > 0$ such that

$$\text{for } k > k_* \quad N(k) = 0. \tag{5.14}$$

Proof. We use the same arguments as the one in the proof of Corollary 5.2. We have

$$N(k) \le \tilde{N}^N(k) = \{(m, p) \in \mathbb{N}^* \times \mathbb{N}; \tilde{\gamma}_m(k) + \frac{p^2\pi^2}{4a^2} \le \overline{\gamma}(k)\},$$

where $\tilde{\gamma}_m(k) + p^2\pi^2/4a^2$ are the eigenvalues of a Neumann eigenproblem identical to (5.10).

From Proposition A.8, we get the following behavior of $\tilde{\gamma}_m(k)$ and $\overline{\gamma}(k)$, for $k \to \infty$,

$$\tilde{\gamma}_m(k) + k^2 n_+^2 \to \frac{\pi^2}{4\tau^2} \quad \text{and} \quad \overline{\gamma}(k) + k^2 n_+^2 \to \frac{\pi^2}{4\eta^2}.$$

We conclude that for k large enough $N(k) = 0$. $\square$

Example 5.1. Let n be the index defined in Figure 6.

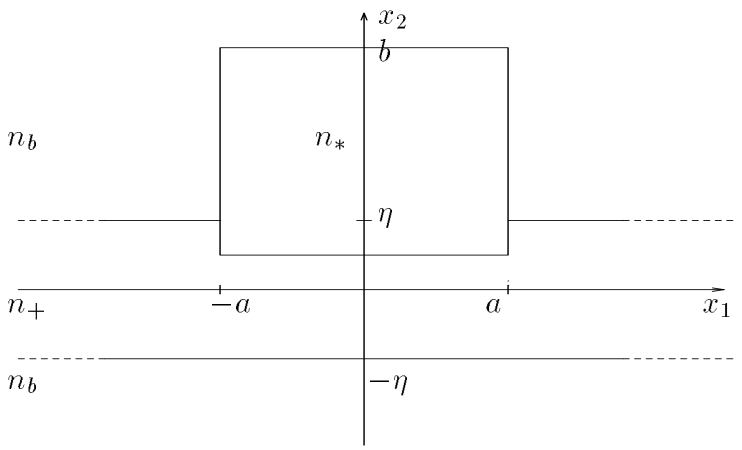

FIGURE 6: GUIDE WITH NO MODE AT HIGH FREQUENCIES.

We assume that $n_+ > n_* > n_b$ and for simplicity that $b-\eta = 2a$. The first eigenvalue μ for $-\Delta$ in the square $(-a,a)\times(\eta, 2a)$ with homogeneous Dirichlet boundary conditions is $\mu = \frac{\pi^2}{2a^2}$. If $\varphi \not\equiv 0$ is a corresponding eigenvector, we extend it by 0 and get $\tilde{\varphi} \in H^1(\mathbb{R}^2)$ satisfying

$$\frac{\int_{\mathbb{R}^2} \left(|\nabla\tilde{\varphi}|^2 - k^2n^2\tilde{\varphi}^2\right)\, dx}{\int_{\mathbb{R}^2} \tilde{\varphi}^2\, dx} = \frac{\pi^2}{2a^2} - k^2 n_*^2.$$

For all k, $-k^2n_+^2 < \overline{\gamma}(k) < -k^2n_b^2$. For a given k, we could choose n_* and a big enough to have

$$-k^2n_*^2 + \frac{\pi^2}{2a^2} < \overline{\gamma}(k),$$

with $n_+ > n_* > n_b$. From the comparison principle we deduce $\lambda_1(k) < \overline{\gamma}(k)$. So for that particular value of k we have a guided mode at least. For large k, we will have no guided modes. $\square$

Appendix A. The planar waveguide

Several times along the paper we needed results for the stratified medium associated to the guide. A planar waveguide associated to a guide with refractive index $n \in L^\infty(\mathbb{R}^2)$ satisfying (1.1) and (1.2), is characterized by the refractive index $\overline{n} \in L^\infty(\mathbb{R})$ such that

$$\overline{n}(y) = \begin{cases} n_t & \text{if } y > c \\ n_b & \text{if } y < -c \end{cases}, \quad \inf_{y\in\mathbb{R}} \overline{n}(y) > 0 \tag{A.1}$$

for some positive c. Let $\overline{n}_+$, $\overline{n}_-$ denote the supremum of n, the infimum of n respectively

$$\overline{n}_+ = \|\overline{n}\|_{\infty,\mathbb{R}}, \qquad \overline{n}_- = \|\overline{n}^{-1}\|_{\infty,\mathbb{R}}^{-1}. \tag{A.2}$$

In a planar waveguide the guided modes can be either transverse electric (TE, the electric field is in a plane transverse to the propagation direction) or transverse magnetic (TM). The problem to find TE guided modes of a planar guide is the following: determine the real numbers β, k, and the functions $u \in H^2(\mathbb{R})$, $u \not\equiv 0$, such that

$$-\frac{\mathrm{d}^2u}{\mathrm{d}y^2} - k^2\overline{n}^2u = -\beta^2 u \quad \text{in } \mathbb{R}. \tag{A.3}$$

The associated mode propagates with the velocity $v = kc_0/\beta$. We refer to Marcuse [1974] for a presentation of the model and do not emphasize on a similar study of TM modes.

Remark that the scalar equation (A.3) is derived directly from the Maxwell system without any assumption on the refractive index, contrarily to the twodimensional case where the scalar model is an approximation.

Like in the general case we consider k as a parameter. Then for a given non negative value of k, the variational problem consists in finding $\beta \in \mathbb{R}^+$ and $u \in H^1(\mathbb{R})$ such that

$$\text{for all } v \in H^1(\mathbb{R}) \quad b(k;u,v) = -\beta^2(u,v)_{0,\mathbb{R}}, \tag{A.4}$$

where the bilinear form $b(k;.,.) : H^1(\mathbb{R}) \times H^1(\mathbb{R}) \to \mathbb{R}$ is given for $u, v \in H^1(\mathbb{R})$ by

$$b(k;u,v) = \int_{\mathbb{R}} \left[\frac{\mathrm{d}u}{\mathrm{d}y}\frac{\mathrm{d}v}{\mathrm{d}y} - k^2\overline{n}^2 uv \right] dy.$$

Problem (A.4) can be written in the equivalent spectral form: find $\beta \in \mathbb{R}^+$ and $u \in H^2(\mathbb{R})$ such that

$$B_k u = -\beta^2 u, \tag{A.5}$$

where the unbounded operator $B_k : \mathscr{D}(B_k) = H^2(\mathbb{R}) \to L^2(\mathbb{R})$ is defined for $u \in H^2(\mathbb{R})$ by

$$B_k u = -\frac{\mathrm{d}^2 u}{\mathrm{d}y^2} - k^2\overline{n}^2 u.$$

Remark A.1. We insist on the fact that a mode of a planar waveguide is not a guided mode in the sense presented in Section 1 because here a mode u is only depending on one variable, $u = u(y) \in H^2(\mathbb{R})$.

Actually the study of the associated planar waveguide shows how the stratified medium confines the lightwave in the vertical direction.

Although the study follows the same steps as for the general problem, it is much easier here. □

The analogue of Propositions 2.1 and 2.2 holds for the bilinear form $b(k;.,.)$ and its associated unbounded operator B_k. The bilinear form $b(k;.,.)$ is continuous symmetric and bounded from below

$$\text{for all } u \in H^1(\mathbb{R}) \quad b(k;u,u) \geq -k^2\overline{n}_+^2 \|u\|_{0,\mathbb{R}}^2.$$

The corresponding operator B_k is bounded from below, selfadjoint, and its spectrum $\sigma(B_k)$ satisfies

$$\sigma(B_k) = [-k^2\overline{n}_+^2, \infty). \tag{A.6}$$

To study the spectrum $\sigma(B_k)$ we use a method similar to the one to study the spectrum of A_k in Section 2. It consists of the essential spectrum $\sigma_{ess}(B_k)$ and of the point spectrum $\sigma_p(B_k)$.

Proposition A.1. *The essential spectrum of B_k is the interval $[-k^2 n_b^2, \infty)$,*

$$\sigma_{ess}(B_k) = [-k^2 n_b^2, \infty).$$

Proof. We first check that $\sigma_{ess}(B_k) \supset [-k^2 n_b^2, \infty)$ by building a singular sequence associated to $B_k - \lambda I$. For $\lambda > -k^2 n_b^2$ we define the sequence $\{v_p\}_{p \geq 1}$ by

$$v_p(y) = \frac{1}{\sqrt{p}}\varphi(\frac{y}{p}) \cos \alpha y$$

with $\alpha = \sqrt{\lambda + k^2 n_b^2}$ and $\varphi \in C^\infty(\mathbb{R})$ with compact support in $(-\infty, -c)$. Let us check now that the sequence of $u_p = v_p/\|v_p\|_{0,\mathbb{R}}$ is a singular sequence associated to $B_k - \lambda I$. With some calculation we can show that

$$0 < \beta_1 \leq \|v_p\|_{0,\mathbb{R}} \leq \beta_2 \quad \text{for all } p,$$
$$\|B_k v_p - \lambda v_p\|_{0,\mathbb{R}} \to 0 \qquad \text{as } p \to \infty,$$

and that for all $v \in L^2(\mathbb{R})$

$$(v_p, v)_{0,\mathbb{R}} \to 0 \quad \text{as } p \to \infty.$$

This means that $\{u_p\}_{p\geq 1}$ is a singular sequence associated to $B_k - \lambda I$.

Let us check now that the inclusion $\sigma_{ess}(B_k) \subset [-k^2 n_b^2, \infty)$. To any $\lambda \in \sigma_{ess}(B_k)$ we can associate a singular sequence $\{u_p\}_{p\geq 1} \subset H^2(\mathbb{R})$. In particular

$$\lambda = \lim_{p\to\infty} \int_{\mathbb{R}} \left[\left(\frac{\mathrm{d}u_p}{\mathrm{d}y} \right)^2 - k^2 \overline{n}^2 u_p^2 \right] dy,$$

from which we deduce that

$$\lambda \geq \lim_{p\to\infty} \left[-k^2 n_b^2 - \int_{-c}^{c} k^2 (\overline{n}^2 - n_b^2) u_p^2 \, dy \right].$$

Since the sequence $\{u_p\}_{p\geq 1}$ is bounded in $H^1(\mathbb{R})$, u_p converges to 0 in $L^2(-c, c)$ and consequently

$$\lambda \geq -k^2 n_b^2. \qquad \square$$

We turn our attention now to the point spectrum of B_k, $\sigma_p(B_k)$.

Proposition A.2. *The following assertions hold:*

(1) *if $\overline{n}_+ = n_b$ then B_k has no eigenvalue;*
(2) *if $\overline{n}_+ > n_b$ then $\sigma_p(B_k) \subset (-k^2 \overline{n}_+^2, -k^2 n_b^2)$.*

Proof. From the relation (A.6), we know that $\sigma_p(B_k)$ is in $[-k^2 \overline{n}_+^2, \infty)$. Let us check that $-k^2 \overline{n}_+^2$ cannot be an eigenvalue. If $-k^2 \overline{n}_+^2$ is an eigenvalue, this would imply the existence of a function $\varphi \in H^2(\mathbb{R})$ not identically 0 such that

$$\int_{\mathbb{R}} \left(\frac{\mathrm{d}\varphi}{\mathrm{d}y} \right)^2 dy = k^2 \int_{\mathbb{R}} (\overline{n}^2 - \overline{n}_+^2) \varphi^2 \, dy.$$

We deduce then that φ is identically 0. So

$$\sigma_p(B_k) \subset (-k^2 \overline{n}_+^2, \infty).$$

Let λ be an eigenvalue of B_k, $\lambda > -k^2n_b^2$, and $\varphi \in H^2(\mathbb{R})$ be a corresponding eigenvector. For $y < -c$, φ satisfies

$$-\frac{\mathrm{d}^2\varphi}{\mathrm{d}y^2} = (\lambda + k^2n_b^2)\varphi,$$

so φ has the form in $(-\infty, -c)$

$$\varphi(y) = \delta_1 \cos(\alpha y) + \delta_2 \sin(\alpha y)$$

with $\alpha = \sqrt{\lambda + k^2n_b^2}$. Since φ is in $L^2(\mathbb{R})$, then $\varphi \equiv 0$ in $(-\infty, -c)$. We deduce then from the uniqueness of initial value problems that $\varphi \equiv 0$ in $\mathbb{R}$. We argue in a same way when $\alpha = 0$. This means that

$$\sigma_p(B_k) \subset (-k^2\overline{n}_+^2, -k^2n_b^2). \qquad \square$$

Remark A.2. If $\beta \in \mathbb{R}^+$ and $u \in H^2(\mathbb{R})$ are a solution of (A.3), then

$$-k^2\overline{n}_+^2 < -\beta^2 < -k^2n_b^2.$$

So the velocity of the guided wave (for the planar waveguide) satisfies

$$\frac{c_0}{\overline{n}_+} < v < \frac{c_0}{n_b},$$

that is the guided wave propagates faster than in a medium of index $\overline{n}_+$ and slower than in a medium of index n_b. $\square$

Since our goal is to prove the existence of eigenvalues for B_k, from now on we shall assume that

$$\overline{n}_+ > n_b. \tag{A.7}$$

In the case of planar waveguide all the eigenvalues are below the essential spectrum and are characterized by the Min–Max principle. This was not the case in Section 2, see Remark 2.3.

We define the numbers $\gamma_m(k)$, $m \geq 1$, by

$$\gamma_1(k) = \inf_{\substack{v \in H^2(\mathbb{R}) \\ v \neq 0}} \frac{(B_k v, v)_{0,\mathbb{R}}}{(v, v)_{0,\mathbb{R}}} \tag{A.8}$$

and

$$\gamma_m(k) = \inf_{H_m \in \mathscr{H}_m(H^2(\mathbb{R}))} \sup_{\substack{v \in H_m \\ v \neq 0}} \frac{(B_k v, v)_{0,\mathbb{R}}}{(v, v)_{0,\mathbb{R}}}; \tag{A.9}$$

here $\mathscr{H}_m(H^2(\mathbb{R}))$ is the set of all m-dimensional subspaces of $H^2(\mathbb{R})$. From Proposition A.1 and the Min–Max principle given in Propostion 3.1, we know that $\gamma_m(k)$ is the mth eigenvalue of B_k if $\gamma_m(k) < -k^2n_b^2$.

Proposition A.3. *The number of eigenvalues of B_k is non decreasing in k.*

Proof. It is sufficient to check that the function $k \in \mathbb{R}^+ \mapsto \gamma_m(k) + k^2 n_b^2$ is decreasing. Given a $v \in H^2(\mathbb{R})$, the function

$$k \in \mathbb{R}^+ \mapsto \min\left(0, \frac{(B_k v, v)_{0,\mathbb{R}} + k^2 n_b^2 (v,v)_{0,\mathbb{R}}}{(v,v)_{0,\mathbb{R}}}\right)$$

is decreasing. With the Min–Max characterization of $\gamma_m(k)$ we can conclude. □

When the planar waveguide satisfies $n_b = n_t$, then we have a simple necessary and sufficient condition for the existence of at least one guided mode for any value of k.

Proposition A.4. *We assume that (A.7) holds and that*

$$n_b = n_t. \tag{A.10}$$

Then $\gamma_1(k) < -k^2 n_b^2$, i.e. the fundamental mode for the planar guide exists, for any $k > 0$ if and only if

$$\int_{-c}^{c} (\overline{n}^2 - n_b^2)\, dy \geq 0. \tag{A.11}$$

Proof. We first assume that

$$\int_{-c}^{c} (\overline{n}^2 - n_b^2)\, dy > 0.$$

For a given $k > 0$,

$$\gamma_1(k) = -k^2 n_b^2 + \inf_{\substack{v \in H^2(\mathbb{R}) \\ v \neq 0}} \frac{(B_k v, v)_{0,\mathbb{R}} + k^2 n_b^2 (v,v)_{0,\mathbb{R}}}{(v,v)_{0,\mathbb{R}}}.$$

We define the function v_R by

$$v_R(y) = \begin{cases} 1 & \text{if } |y| \leq c, \\ \frac{|y|-R}{c-R} & \text{if } c < |y| < R, \\ 0 & \text{if } |y| \geq R. \end{cases}$$

Then

$$b(k; v_R, v_R) + k^2 n_b^2 (v_R, v_R)_{0,\mathbb{R}} = \frac{2}{R-c} - k^2 \int_{-c}^{c} (\overline{n}^2 - n_b^2)\, dy < 0$$

for R sufficiently large. Since $H^2(\mathbb{R})$ is dense in $H^1(\mathbb{R})$, we conclude that $\gamma_1(k) < -k^2 n_b^2$. When (A.11) holds with the equality we choose a test function $w_R = v_R + \alpha w$ where $w \in H^1(\mathbb{R})$ satisfies

$$\int_{-c}^{c} (\overline{n}^2 - n_b^2) w\, dy = 0 \quad \text{and} \quad \int_{-c}^{c} (\overline{n}^2 - n_b^2) w^2\, dy > 0.$$

Then R and α can be chosen such that

$$b(k;w_R,w_R)+k^2{n_b}^2(w_R,w_R)_{0,\mathbb{R}}<0.$$

So the condition (A.11) is sufficient.

Let us check that (A.11) is necessary. If the fundamental mode exists for any k, then we can consider a sequence $\{k_p\}_{p\geq 1}$ converging to 0 and another sequence $\{v_p\}_{p\geq 1}\subset H^1(\mathbb{R})$ such that

$$\int_{\mathbb{R}}\left(\frac{\mathrm{d}v_p}{\mathrm{d}y}\right)^2 dy-k_p^2\int_{-c}^{c}(\overline{n}^2-n_b^2)v_p^2\,dy<0 \tag{A.12}$$

and

$$\int_{-c}^{c}v_p^2\,dy=1.$$

Thus the sequence $\{\frac{\mathrm{d}v_p}{\mathrm{d}y}\}_{p\geq 1}$ converges to 0 in $L^2(\mathbb{R})$. By compactness we deduce that the sequence $\{v_p\}_{p\geq 1}$ converges to a non zero constant in $L^2(-c,c)$. Then from (A.12) we conclude that (A.11) holds. □

Proposition A.5. *We assume that (A.7) holds and that*

$$n_b>n_t. \tag{A.13}$$

Then there exists $k_1>0$ such that

$$\text{for } 0<k<k_1\quad \gamma_1(k)=-k^2n_b^2, \tag{A.14}$$

which means that for $0<k<k_1$ B_k has no eigenvalues.

Proof. Ab absurdo suppose that there exists a positive sequence $\{k_p\}_{p\geq 1}$ tending to 0 such that $\gamma_1(k_p)<-k_p^2n_b^2$. Let $\{v_p\}_{p\geq 1}$ be a sequence of eigenvectors, $v_p\in H^2(\mathbb{R})$ eigenvector associated to $\gamma_1(k_p)$ and normalized by

$$\int_{-c}^{c}v_p^2\,dy=1.$$

Then

$$\int_{\mathbb{R}}\left(\frac{\mathrm{d}v_p}{\mathrm{d}y}\right)^2 dy+k_p^2\int_{\mathbb{R}}(n_b^2-\overline{n}^2)v_p^2\,dy<0$$

and with (A.13)

$$\int_{\mathbb{R}}\left(\frac{\mathrm{d}v_p}{\mathrm{d}y}\right)^2 dy+k_p^2\int_{c}^{\infty}(n_b^2-n_t^2)v_p^2\,dy<k_p^2\int_{-c}^{c}(\overline{n}_+^2-n_b^2)v_p^2\,dy.$$

Consequently $\{\frac{\mathrm{d}v_p}{\mathrm{d}y}\}_{p\geq 1}$ tends to 0 in $L^2(\mathbb{R})$ and $\{v_p\}_{p\geq 1}$ is bounded in $H^1(-c,\infty)$. This implies that $\{v_p\}_{p\geq 1}$ converges to 0 in $L^2(-c,\infty)$, which is a contradiction with the normalization equation. □

We turn our attention now on the number of guided modes as a function of the frequency k.

Proposition A.6. *The following assertions hold for any $m \geq 1$:*

$$\text{(A.15)} \qquad \text{the function } k \in \mathbb{R}^+ \to \gamma_m(k) + k^2\overline{n}_+^2 \text{ is increasing}$$

$$\text{(A.16)} \qquad \lim_{k\to\infty} \frac{\gamma_m(k)}{k^2} = -\overline{n}_+^2.$$

Proof. By the definition of $\overline{n}_+$, for any non zero $v \in H^2(\mathbb{R})$, the function

$$k \in \mathbb{R}^+ \mapsto \frac{(B_k v, v)_{0,\mathbb{R}} + k^2\overline{n}_+^2 (v,v)_{0,\mathbb{R}}}{(v,v)_{0,\mathbb{R}}}$$

is non negative increasing in k. Going back to the definition of $\gamma_m(k)$ in (A.8), (A.9), we conclude that (A.15) holds.

Let us check now the second assertion (A.16). For all $\epsilon > 0$, there exists an interval $B(y_\epsilon, \rho_\epsilon)$ such that

$$\frac{1}{\rho_\epsilon} \int_{B(y_\epsilon,\rho_\epsilon)} (\overline{n}_+^2 - \overline{n}(y)^2)\, dy \leq \epsilon.$$

Then for a given ϵ, we have

$$\gamma_m(k) + k^2\overline{n}_+^2 \leq \sup_{\substack{v\in H_m \\ v\neq 0}} \frac{\int_{B(y_\epsilon,\rho_\epsilon)} \left(\frac{\mathrm{d}v}{\mathrm{d}y}\right)^2 dy + k^2 \|v\|^2_{\infty, B(y_\epsilon,\rho_\epsilon)} \int_{B(y_\epsilon,\rho_\epsilon)} \left(\overline{n}_+^2 - \overline{n}(y)^2\right) dy}{\int_{B(y_\epsilon,\rho_\epsilon)} v^2\, dy}$$

where H_m is the space generated by the m functions

$$u_p(y) = \sin\left((y - y_\epsilon)\frac{p\pi}{2\rho_\epsilon}\right) \quad \text{when } p \text{ is even,}$$
$$u_p(y) = \cos\left((y - y_\epsilon)\frac{p\pi}{2\rho_\epsilon}\right) \quad \text{when } p \text{ is odd,}$$

for y in $B(y_\epsilon, \rho_\epsilon)$ and extended by 0 outside. Then

$$\text{(A.17)} \qquad \frac{\gamma_m(k)}{k^2} + \overline{n}_+^2 \leq \frac{m^2\pi^2}{4k^2\rho_\epsilon^2} + \frac{1}{\rho_\epsilon}\int_{B(y_\epsilon,\rho_\epsilon)} (\overline{n}_+^2 - \overline{n}^2(y))\, dy.$$

We can conclude with (A.17) by choosing k big enough. $\square$

From Proposition A.6 we immediately deduce the following.

Proposition A.7. *We assume that (A.7) holds. For any $m \geq 1$ there exists k_m such that*

$$\text{for all } k \leq k_m \ \gamma_m(k) = -k^2 n_b^2 \text{ and for all } k > k_m \ \gamma_m(k) < -k^2 n_b^2. \qquad \square$$

For $m \geq 1$, the number k_m in Proposition A.7 is called the m**th cutoff value**.

Under some further assumptions on $\overline{n}$ the results in Proposition A.6 can be precised in the following way.

Proposition A.8. *We assume that (A.7) holds and that there exists* $0 < \eta < c$ *with*

$$\begin{cases} \overline{n}(y) = \overline{n}_+ & \text{if } y \in [-\eta, \eta], \\ \overline{n}(y) < \overline{n}_+ & \text{else.} \end{cases} \tag{A.18}$$

Then

$$\gamma_m(k) + k^2\overline{n}_+^2 \leq \frac{m^2\pi^2}{4\eta^2} \tag{A.19}$$

$$\gamma_m(k) + k^2\overline{n}_+^2 \to \frac{m^2\pi^2}{4\eta^2} \quad \text{as } k \to \infty. \tag{A.20}$$

Proof. Working on the ball $B(0,\eta)$ we obtain the analogue of (A.17)

$$\gamma_m(k) + k^2\overline{n}_+^2 \leq \frac{m^2\pi^2}{4\eta^2} + \frac{k^2}{\eta}\int_{-\eta}^{\eta}(\overline{n}_+^2 - \overline{n}^2(y))\, dy,$$

from which we deduce (A.19) since $\overline{n}_+ = n(y)$ for $y \in [-\eta, \eta]$.

The function $k \in \mathbb{R} \mapsto \gamma_m(k) + k^2\overline{n}_+^2$ is increasing and bounded. Let μ_m be its limit. For any $k > k_m$ we can associate to $\gamma_m(k)$ an eigenfunction $v_m(k)$ such that

$$\text{for all } v \in H^1(\mathbb{R}) \quad b(k; v_m(k), v) = \gamma_m(k)(v_m(k), v)_{0,\mathbb{R}} \tag{A.21}$$

and

$$\|v_m(k)\|_{0,\mathbb{R}} = 1. \tag{A.22}$$

From the relation (A.21), we deduce that

$$\|\frac{\mathrm{d}v_m(k)}{\mathrm{d}y}\|_{0,\mathbb{R}}^2 \leq \mu_m$$

and

$$k^2\int_{\mathbb{R}}(\overline{n}_+^2 - \overline{n}(y)^2)v_m(k)(y)\, dy < \mu_m. \tag{A.23}$$

By compactness the family $\{v_m(k)\}_{k>k_m}$ has a subsequence $\{v_m(k_p)\}_{p\geq 1}$ converging to v_m in $L^2(-c,c)$. Now from (A.23) we also get that v_m has a support in $[-\eta,\eta]$ and

$$\int_{-\eta}^{\eta} v_m(y)^2\, dy = 1.$$

Then from the relation (A.21), we obtain that v_m is a non zero solution of

$$\begin{cases} -\dfrac{\mathrm{d}^2 v_m}{\mathrm{d}y^2} = \mu_m v_m \quad \text{in } (-\eta, \eta), \\ v_m(-\eta) = v_m(\eta) = 0, \end{cases}$$

which implies

$$\mu_m \leq \frac{m^2\pi^2}{4\eta^2} \quad \text{and there exists a } \ell \text{ with } \mu_m = \frac{\ell^2\pi^2}{4\eta^2}.$$

So $\mu_1 = \dfrac{\pi^2}{4\eta^2}$. We have no difficulty to check that $\ell = m$ with the Min–Max principle. $\square$

Remark A.3. It is not difficult to generalize Proposition A.8 when we replace the assumption (A.18) by the following

$$\begin{cases} \overline{n}(y) = \overline{n}_+ \quad \text{if } y \in [a_1, a_2] \cup [a_3, a_4], \\ \overline{n}(y) < \overline{n}_+ \quad \text{else}, \end{cases}$$

with $-c < a_1 \leq a_2 < a_3 \leq a_4 < c$. Then in (A.19) and (A.20) η is replaced by

$$\eta_* = \max(a_2 - a_1, a_4 - a_3). \quad \square$$

We end the appendix by getting upper and lower bounds for the eigenvalues of B_k in terms of eigenvalues of operators with compact resolvent. Let $\overline{n} \in L^\infty(\mathbb{R})$ be the refractive index of a planar waveguide satisfying to (A.1) and let d be bigger than c. To $\overline{n}$ and d, we associate the two operators B_k^D and B_k^N defined by

$$\text{(A.24)} \qquad \mathscr{D}(B_k^D) = H^2(-d,d) \cap H_0^1(-d,d), \qquad B_k^D u = -\frac{d^2u}{dy^2} - k^2\overline{n}^2 u$$

and

$$\text{(A.25)} \qquad \mathscr{D}(B_k^N) = \{u \in H^2(-d,d); u'(-d) = u'(d) = 0\}, \quad B_k^N u = -\frac{d^2u}{dy^2} - k^2\overline{n}^2 u.$$

Then we define the numbers for $m \geq 1$

$$\text{(A.26)} \qquad \gamma_m^D(k) = \inf_{H_m \in \mathscr{H}_m(H^2(-d,d)\cap H_0^1(-d,d))} \sup_{\substack{v \in H_m \\ v \neq 0}} \frac{(B_k^D v, v)_{0,(-d,d)}}{(v,v)_{0,(-d,d)}}$$

and

$$\text{(A.27)} \qquad \gamma_m^N(k) = \inf_{H_m \in \mathscr{H}_m(H^2(-d,d))} \sup_{\substack{v \in H_m \\ v \neq 0}} \frac{(B_k^N v, v)_{0,(-d,d)}}{(v,v)_{0,(-d,d)}}.$$

Proposition A.9. *Both operators B_k^D and B_k^N have an increasing sequence tending to ∞ of eigenvalues characterized by the relations (A.26) and (A.27) respectively.*

For $m \geq 1$, the following bounds hold

$$\min(\gamma_m^N(k), -k^2 n_b^2) \leq \gamma_m(k) \leq \gamma_m^D(k), \tag{A.28}$$

where $\gamma_m(k)$ are defined in (A.8) and (A.9). Moreover when the fundamental mode exists, that is $\gamma_1(k) < -k^2 n_b^2$, then we have

$$\gamma_1^N(k) < \gamma_1(k). \tag{A.29}$$

Proof. The first part of the result is deduced from the compactness of the Sobolev embedding $H^2(-d, d) \subset H^1(-d, d)$ and the Min–Max principle.

To check the bound (A.28), we argue like in Proposition 3.2.

Let us check the bound (A.29). The assumption $\gamma_1(k) < -k^2 n_b^2$ means that the fundamental mode exists. So let φ_k be an eigenfunction corresponding to the eigenvalue $\gamma_1(k)$ of B_k and ψ_k be an eigenfunction corresponding to the eigenvalue $\gamma_1^N(k)$ of B_k^N.

For $y \in (-\infty, -d] \cup [d, \infty)$, we have a formula for $\varphi_k(y)$, indeed

$$\varphi_k(y) = \varphi_k(-d) \exp^{\kappa_b (y+d)}, \quad \kappa_b = \sqrt{-\gamma_1(k) - k^2 n_b^2}, \quad \text{for } y \in (-\infty, -d],$$
$$\varphi_k(y) = \varphi_k(d) \exp^{-\kappa_t (y-d)}, \quad \kappa_t = \sqrt{-\gamma_1(k) - k^2 n_t^2}, \quad \text{for } y \in [d, \infty).$$

Then we deduce that

$$\frac{d\varphi_k}{dy}(-d) = \kappa_b \varphi_k(-d) \quad \text{and} \quad \frac{d\varphi_k}{dy}(d) = -\kappa_t \varphi_k(d)$$

and φ_k satisfies

$$\begin{cases} -\dfrac{d^2\varphi_k}{dy^2} - k^2 \overline{n}^2 \varphi_k = \gamma_1(k) \varphi_k \quad \text{on } [-d, d], \\ \qquad \dfrac{d\varphi_k}{dy}(-d) = \kappa_b \varphi_k(-d), \quad \dfrac{d\varphi_k}{dy}(d) = \kappa_t \varphi_k(d). \end{cases} \tag{A.30}$$

The following expression holds for $\gamma_1(k)$

$$\gamma_1(k) = \frac{\int_{-d}^{d} ({\varphi'_k}^2 - k^2 \overline{n}^2 \varphi_k^2)\, dy + \kappa_b \varphi_k^2(-d) + \kappa_t \varphi_k^2(d)}{\int_{-d}^{d} \varphi_k^2 \, dy}.$$

Using the Min–Max principle to characterize $\gamma_1^N(k)$ we get

$$\gamma_1(k) \geq \gamma_1^N(k) + \frac{\kappa_b \varphi_k^2(-d) + \kappa_t \varphi_k^2(d)}{\int_{-d}^{d} \varphi_k^2 \, dy}.$$

Consequently to have $\gamma_1(k) = \gamma_1^N(k)$, it is necessary that $\varphi_k(-d) = \varphi_k(d) = 0$. Since the function φ_k satisfies (A.30), that implies $\varphi_k \equiv 0$. Therefore we can conclude that (A.29) holds. $\square$

Example A.1. The above results can be illustrated in the case of the three layers planar waveguide, see Figure 2, with the refractive index

$$\overline{n}(y) = \begin{cases} n_t & \text{if } y > c, \\ n_c & \text{if } -c < y < c, \\ n_b & \text{if } y < -c, \end{cases}$$

where $n_c > n_b \geq n_t > 0$. It is a simple matter to check that

$$\gamma_m^N(k) = -k^2 n_c^2 + \frac{(m-1)^2\pi^2}{4c^2}$$

and

$$\gamma_m^D(k) = -k^2 n_c^2 + \frac{m^2\pi^2}{4c^2}.$$

To compute $\gamma_m(k)$, we solve the equation

$$-\frac{d^2 v}{dy^2} - k^2\overline{n}^2 v = \lambda v$$

successively in the intervals $(-\infty, -c), (-c, c)$, and (c, ∞), and impose the continuity of v and its derivative at $y = -c$, $y = c$. Then the eigenvalues $\gamma_m(k)$ are the solutions λ of the dispersion equation

$$\tan(2c\sqrt{\lambda + k^2 n_c^2}) = \frac{\sqrt{\lambda + k^2 n_c^2}\left(\sqrt{-\lambda - k^2 n_b^2} + \sqrt{-\lambda - k^2 n_t^2}\right)}{(\lambda + k^2 n_c^2) - \sqrt{-\lambda - k^2 n_b^2}\sqrt{-\lambda - k^2 n_t^2}}, \tag{A.31}$$

with $-k^2 n_c^2 \leq \lambda \leq -k^2 n_b^2$. The dispersion equation (A.31) can be written in the form

$$\tan x = F(x) \tag{A.32}$$

with $x = 2c\sqrt{\lambda + k^2 n_c^2}$ and

$$F(x) = \frac{x\left(\sqrt{\eta_1^2 - x^2} + \sqrt{\eta_2^2 - x^2}\right)}{x^2 - \sqrt{\eta_1^2 - x^2}\sqrt{\eta_2^2 - x^2}};$$

here $\eta_1 = 2kc\sqrt{n_c^2 - n_b^2}$, $\eta_2 = 2kc\sqrt{n_c^2 - n_t^2}$. The function F has a pole at the point

$$x_p = \frac{2kc\sqrt{(n_c^2 - n_t^2)(n_c^2 - n_b^2)}}{\sqrt{(n_c^2 - n_t^2) + (n_c^2 - n_b^2)}}$$

and is decreasing on both intervals $[0, x_p)$ and $(x_p, \eta_1]$. So there is a solution to (A.32) as soon as

$$\text{(A.33)} \qquad \big(\tan(\eta_1) \geq F(\eta_1) \quad \text{and} \quad \eta_1 \leq \pi/2\big) \quad \text{or} \quad \eta_1 > \pi/2.$$

In each interval $(m\pi, (m+1)\pi)$ there is at most one solution. To the solution $x \in (m\pi, (m+1)\pi)$ corresponds an eigenvalue $\gamma_\ell(k) = (x/2c)^2 - k^2 n_c^2$ with

$$-k^2 n_c^2 + \frac{m^2\pi^2}{4c^2} < \gamma_\ell(k) < -k^2 n_c^2 + \frac{(m+1)^2\pi^2}{4c^2}.$$

For small values of k the relation (A.33) does not hold. The lowest value for which (A.33) holds is the first cutoff value and is given by

$$k_1 = \frac{\arctan\left(\sqrt{\frac{n_b^2 - n_t^2}{n_c^2 - n_b^2}}\right)}{2c\sqrt{n_c^2 - n_b^2}}. \quad \square$$

References

A. Bamberger and A.S. Bonnet, *Mathematical analysis of the guided modes of an optical fiber*, Siam J. Math. Anal. **21** (1990), 1487–1510.

A. Bamberger, Y. Dermenjian, and P. Joly, *Mathematical analysis of the propagation of elastic guided waves in heterogeneous media*, J. of Differential Equations **88** (1990).

A. Bamberger, P. Joly, and M. Kern, *Propagation of elastic surface waves along a cylindrical cavity of arbitrary cross section*, Mathematical Modelling and Numerical Analysis **25** (1991).

A.S. Bonnet, *Analyse mathématique de la propagation de modes guidés dans les fibres optiques*, Thèse de doctorat (1988), Université de Paris 6.

A.S. Bonnet and R. Djellouli, *Étude mathématique des modes guidés d'une fibre optique, résultats complémentaires et extension au cas de couplages*, Rapport n° 182, C.M.A.P., École Polytechnique (1988).

A.S. Bonnet and F. Starling, *Diffraction et ondes guidées pour les modes TE ou TM dans les réseaux électromagnétiques*, Math. Meth. Appl. Sci. **17** (1994), 305–338.

A.S. Bonnet–BenDhia and R. Djellouli, *High-frequency asymptotics of guided modes in optical fibres*, IMA Journal of Applied Mathematics **52** (1994), 271–287.

A.S. Bonnet–BenDhia and P. Joly, *Mathematical analysis of guided water waves*, to appear in Siam J. Appl. Math. (1994).

A.S. Bonnet-BenDhia and F. Mahé, *Guided modes embedded into the radiation modes in integrated optical guides*, in preparation (1995).

R. Courant and D. Hilbert, *Methods of Mathematical Physics*, Interscience, 1953.

D. Dunford and J.T. Schwartz, *Linear Operators*, Interscience (second edition), 1963.

N. Gmati, *Guidage et diffraction d'ondes en milieu non borné*, Thèse de doctorat (1992), Université de Paris 6.

J.C. Guillot, *Complétude des modes TE et TM pour un guide d'ondes optiques planaires*, Rapport INRIA, n° 385 (1985).

P. Joly and R. Weder, *New results for guided waves in elastic heterogeneous media*, to appear in Math. Meth. in the Appl. Sci. (1994).

F. Mahé, *Étude mathématique et numérique de la propagation d'ondes électromagnétiques dans les microguides de l'optique intégrée*, Thèse de doctorat (1993), Université de Rennes I.

F. Mahé, *Propagation of electromagnetic waves in integrated optical guides. A numerical study*, in preparation (1996).

D. Marcuse, *Light transmission optics*, Electrical/Computer Science and Engineering Series, Van Nostrand Reinhold (second edition), 1982.

C. Poirier, *Guides d'ondes électromagnétiques: analyse mathématique et numérique*, Thèse de doctorat (1994), Université de Nantes.

M. Reed and B. Simon, *Analysis of operators*, vol. IV, Academic Press, 1978.

M. Schechter, *Spectra of partial differential operators*, North-Holland, 1971.

M. Schechter, *Operator Methods in Quantum Mechanics*, North-Holland, 1981.

A.W. Snyder and J.D. Love, *Optical waveguide theory*, Chapman and Hall, 1983.

C. Vassalo, *Théorie des guides d'ondes électromagnétiques*, Tomes 1 et 2, Editions Eyrolles, 1985.

R. Weder, *Absence of eigenvalues of the acoustic propagator in deformed wave guides*, Rocky Mount. J. Math. **18** (1988), 495–503.

R. Weder, *Spectral and scattering theory for wave propagation in perturbed stratified media*, Applied Mathematical Sciences, vol. 50, Springer Verlag, 1991.

C.H. Wilcox, *Sound propagation in stratified fluids*, Applied Mathematical Sciences, vol. 50, Springer Verlag, 1984.

K.J. Witsch, *Examples of embedded eigenvalues for the Dirichlet Laplacian in perturbed waveguides*, Math. Meth. in the Appl. Sci. **12** (1990), 91–93.

(1) Laboratoire de Simulation et de Modélisation des phénomènes de Propagation, ENSTA, Centre de l'Yvette, Chemin de la Hunière, 91120 Palaiseau, France

(2) IRMAR, Campus de Beaulieu, Université de Rennes I, 35042 Rennes Cedex

E-mail: caloz@univ-rennes1.fr fmahe@univ-rennes1.fr