Mathematical and Numerical Study of the Guided Modes of Stratified Dielectric Waveguides

Recent progress in integrated optics have seen the emergency of a great variety of dielectric waveguides, which study is far more difficult than that of optical fibers. An important difference is that the cladding of waveguides used in integrated optics is generally stratified while that of an optical fiber is homogeneous. This provides original physical phenomena, especially at high frequency, which have to be handled by specific theoretical and numerical techniques.

Classically we suppose that the waveguides are invariant in the direction Ox_3 and infinitely extended in all directions. Although our analysis concerns very general index profiles $n(x_1,\, x_2)$ we consider here the simpler case of a ridge waveguide (cf. Figure 1). The slab waveguide represented on Figure 2 will play an important role in the following.

The aim of this work is the description and numerical approximation of the guided modes of the ridge waveguide. A mode is a non trivial solution of Maxwell's equations which has the particular form:

$$(\mathbb{E}, \mathbb{H})\; (x_1, x_2, x_3, t) = Re\left((E, H)(x_1, x_2)\; e^{i(kc_0 t - \beta x_3)} \right) \tag{1}$$

where c_0 is the light velocity and $\mathbb{E}$ (resp. $\mathbb{H}$) the electric (resp. magnetic) field. We say that k is the wave number and β the propagation constant of the mode.

This mode is guided if β and k are reals, so that the wave (1) propagates without attenuation, and if its transverse energy is finite ($E,\ H \in [L^2(\mathbb{R}^2)]^3$).

1 Theoretical study of the scalar approximation

Under the classical hypotheses of the weak guidance approximation, the problem becomes:

$$\begin{cases} \text{Find } u \in L^2(\mathbb{R}^2),\ u \neq 0,\ \beta \in \mathbb{R},\ k \in \mathbb{R} \text{ such that:} \\ \qquad \triangle u + (k^2 n^2 - \beta^2)\, u = 0 \quad \text{in } \mathbb{R}^2 \end{cases} \tag{2}$$

where u approximates any transverse component of the electromagnetic field and $n(x)$ is the index profile of the ridge waveguide.

This means that $(-\beta^2)$ is an eigenvalue of the operator A_k defined by $A_k v = -\triangle v + k^2 n^2 v$ and u is the associated eigenfunction. Problem (2) is therefore a two dimensional eigenvalue problem.

We prove the:

Theorem 1 *A_k is a selfadjoint operator bounded from below and if $\sigma_{ess}(k)$ denotes its essential spectrum we have:*

$$\sigma_{ess}(k) = [-\gamma(k)^2, +\infty[\tag{3}$$

where $\gamma(k)$ is the propagation constant of the fundamental mode of the slab waveguide, if it exists, otherwise $\gamma(k) = kn_1$.

A difficulty of the study results from the nonlinear dependence of $\gamma(k)$ with respect to k, which is a consequence of the dispersive nature of the stratified medium. Then we use the Min-Max principle which provides a characterization of the eigenvalues of A_k located below the essential spectrum.

Theorem 2

(i) If the slab waveguide is symetric ($n_1 = n_2$), the fundamental mode of the ridge waveguide exists for every $k > 0$.

(ii) If $n_1 \neq n_2$, the fundamental mode of the ridge waveguide exists for large k but not for small k.

(iii) Let $N(k)$ denote the number of eigenvalues of A_k located below the essential spectrum. Then the function $k \longmapsto N(k)$ is bounded.

Notice that, for an optical fiber, $N(k)$ is a non-decreasing function of k and tends to infinity with k. Therefore (iii) is a specific property of ridge waveguides.

Remarks .

• Theorem 2 concerns the ridge waveguide described above. More generally, there exist guides such that $N(k) = 0$ when k is big: these guides admit guided modes at some frequencies but not at high frequency.

• We have also exhibited examples of guides such that A_k has eigenvalues embedded in its essential spectrum. In this unusual situation, the guided modes propagate faster than the fundamental mode of the slab waveguide.

2 The numerical method

For numerical purposes, we suppose that the field u or its normal derivative vanishes on Γ_1 and Γ_2. This assumption is justified since the guided modes of ridge waveguides are well confined in the vertical direction. Moreover, the eigenvalues obtained by using either a Dirichlet or a Neumann homogeneous boundary condition on Γ_1 and Γ_2 provide upper and lower bounds for the exact values.

On the contrary, the field decreases slowly horizontally. So we write on the artificial boundaries Σ_1 and Σ_2 an exact boundary condition using a series expansion of the solution in the exterior domain. That allows to put Σ_1 and Σ_2 very close to the core region.

This leads to a nonlinear eigenvalue problem set in a bounded domain. This problem is discretized by finite elements and solved by an iterative process which is proved to converge.

3 The vector case

Following Bamberger-Bonnet (*Mathematical analysis of the guided modes of an optical fiber*, SIAM J. MATH. ANAL., Vol. 21, No. 6, pp. 1487-1510, November 1990), we consider a formulation of the problem for the magnetic field:

$$\begin{cases} \text{Find } k \in \mathbb{R}^+, \ H \in [H^1(\mathbb{R}^2)]^3, \ H \neq 0 \text{ such that:} \\ \operatorname{Rot}_\beta\left((1/n^2)\operatorname{Rot}_\beta H\right) - \operatorname{Grad}_\beta\left((1/\bar{n}^2)\operatorname{Div}_\beta H\right) = k^2 H, \quad \text{in } \mathbb{R}^2. \end{cases} \tag{4}$$

where n is the index profile of the ridge waveguide and $\bar{n}$ that of the slab waveguide, and where Rot_β, $\operatorname{Grad}_\beta$, Div_β are the usual differential operators where the derivation with respect to x_3 is replaced by the multiplication by $(-i\beta)$.

The difficulty of the study of (4) results from the fact that Maxwell's equations remain coupled even in a perfect stratified medium (cf. Weder, *Spectral and scattering theory for wave propagation in perturbed stratified media*, Applied Mathematical Sciences 87, Springer-Verlag, 1991).

However, if we consider the following representation of the magnetic field:

$$H = \operatorname{Grad}_\beta \varphi + \operatorname{Rot}_\beta(\psi e_2) + {\operatorname{Rot}_\beta}^2(\zeta e_2). \tag{5}$$

where e_2 is the unit vector perpendicular to the direction of stratification, then, equation (4) reduces, outside the core region (i.e. for $|x|$ great enough), into the three following scalar equations:

$$\begin{cases} \triangle\varphi + (k^2\bar{n}^2 - \beta^2)\varphi = 0 \\ \operatorname{div}\left((1/\bar{n}^2)\operatorname{grad}\psi\right) + (k^2 - \beta^2/\bar{n}^2)\,\psi = 0 \\ \triangle\zeta + (k^2\bar{n}^2 - \beta^2)\zeta = 0 \end{cases} \tag{6}$$

Thanks to this decomposition, theoretical techniques used in the scalar case can be extended to the vector formulation. The numerical approximation is then performed, using standard Lagrange finite elements. Exact conditions on the lateral boundaries Σ_1 and Σ_2 can be explicited using series expansions of the potentials φ, ψ and ζ in the exterior domain.

References

[1] A. Bamberger and A.S. Bonnet, *Mathematical analysis of the guided modes of an optical fiber*, siam j. math. anal., Vol. 21, No. 6, pp. 1487-1510, November 1990.

[2] R. Weder, *Spectral and scattering theory for wave propagation in perturbed stratified media*, Applied Mathematical Sciences 87, Springer-Verlag, 1991.

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