Study at high frequencies of a stratified waveguide

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ABSTRACT. A waveguide in integrated optics is defined by its refractive index. The guide is assumed to be invariant in the propagation direction while in the transverse direction it is supposed to be a compact perturbation of an unbounded stratified medium. We are interested in the high frequency modes guided by this device.

We consider the problem under the assumptions of weak guidance, so that it reduces to a two dimensional eigenvalue problem for a scalar field. While a general study has been done in a previous paper [1], our goal here is to present an asymptotic study at high frequencies, which illustrates the dispersive character of the stratified guide. We will give the limit as the frequency tends to ∞ of the guided modes and characterize this limit as the solution of an eigenproblem. The technical difficulty lies in the stratified character of the unbounded reference medium.

1 Introduction

A waveguide in integrated optics is defined by its refractive index. In our paper we shall consider guides invariant in the propagation direction x_3, which are composed of a stratified medium with a compact perturbation in the transverse section, called the core of the guide, see Figure 1.

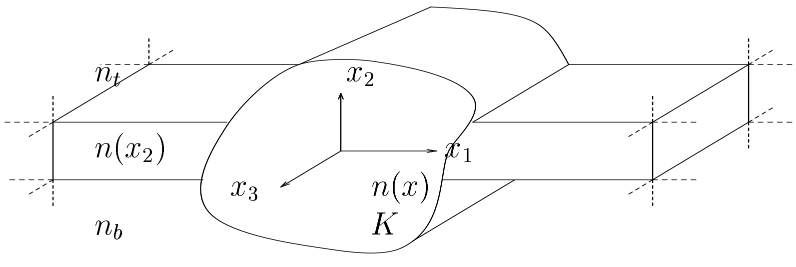

Figure 1: Stratified optical guide.

The stratified medium is the reference medium and is intended to guide electromagnetic waves in one layer. The unboundedness of the reference medium will put obstacles in the theoretical and numerical studies as well as carry particular phenomena at high frequency. The compact perturbation of the reference medium is designed to confine waves inside a layer in a neighborhood of the perturbation.

We will work under the assumption of weak guidance, so our problem reduces to a two dimensional eigenvalue problem for a scalar field. A careful study of the scalar model has to be carried out before starting with the vectorial model, since it allows to solve in a much simpler situation a lot of mathematical difficulties due to the stratification of the unbounded medium. The vectorial model will be studied in a forthcoming work. In [1], we have presented a general study of electromagnetic waves guided by such devices, which are waves of the form $\phi(x_1, x_2)e^{i(kc_0t-\beta x_3)}$ where x_1, x_2 denote the transverse coordinates, x_3 the longitudinal coordinate, c_0 the speed of light in the vacuum, k the wave number, and β the propagation constant of the mode. The guided modes correspond to waves of finite transverse energy which propagate without attenuation, i.e. with k and β real. We have determined existence conditions of guided modes and bounds for the number of guided modes. Here we will pursue by a high frequency analysis. The wave number k will be considered as a parameter and taken large, then we will look for dispersion relations $k \to \beta(k)$.

The paper is organized as follows. In the next section we introduce the notations and present our main results : when the frequency tends to infinity, (i.e. $k \to \infty$), the guided modes tend to the solutions of the Dirichlet eigenproblem on the set B_+ where the refractive index n achieves its maximum, or disappear in the lower bound of its essential spectrum. In particular we have at least as many guided modes at high frequency as solutions of the Dirichlet eigenproblem on B_+ .

In Section 3 we study the eigenvalue problem on the perturbed strip B_+ . We give necessary conditions to have eigenpairs of this problem and discuss thoroughly with numerical computations the range of validity of two different criteria. The convergence of the guided modes to the eigenpairs of the perturbed strip is then proved in Section 4. Finally Section 5 is devoted to get underestimates for the number of guided modes at high frequencies. In particular we prove that we can have guided modes which disappear in the lower bound of the essential spectrum of the Dirichlet eigenproblem on B_+ . This result is also illustrated by an example.

Let us introduce the standard notations we shall use all over the paper. $\mathbb{R}^+$ denotes the non negative real numbers and $\mathbb{R}^+_*$ the positive real numbers. For $m \in \mathbb{N}$ and $\Omega \subset \mathbb{R}^d$ ($d \in \mathbb{N}$), $H^m(\Omega)$ is the classical Sobolev space of functions with derivatives up to the order m in $L^2(\Omega)$ endowed with the scalar product $(\cdot,\cdot)_{m,\Omega}$, the norm $\|\cdot\|_{m,\Omega}$, and seminorm $|\cdot|_{m,\Omega}$. We denote by $\mathring{H}^m(\Omega)$ the closure of $\mathscr{D}(\Omega)$, the space of C^∞ functions with compact support in Ω , with respect to the norm $\|\cdot\|_{m,\Omega}$, and by $H^{-m}(\Omega)$ its dual space. We shall also use the standard differential operators div , ∇ , Δ .

2 Framework and results

We assume that the guide is invariant in one direction (say Ox_3) which will be the propagation direction, and that it is a perturbation of a stratified medium. If the function n denotes the refractive index, then n is a function of x_1, x_2 only, $n = n(x_1, x_2)$. Outside the perturbation, the function n is depending only on x_2 . A guide is represented in Figure 1. Since we shall study electromagnetic waves harmonic in x_3 , so later on we will only represent the transverse section of the guide.

The index function $\overline{n}$ defines a planar waveguide associated to the guide under consideration; it represents the stratified medium without perturbation, see Figure 2.

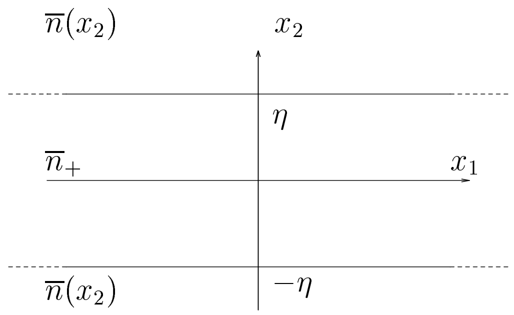

Figure 2: Planar waveguide associated.

Theoretical studies of the one dimensional problem (with respect to x_2) of the planar waveguide are considered in GUILLOT [3], SCHECHTER [9], or in WILCOX [10]. We assume that $\overline{n}$ is a piecewise continuous function defined in $\mathbb{R}$ and that, if $\overline{n}_+$ denotes the supremum of $\overline{n}$,

$$\begin{cases} \overline{n}(\xi) = \overline{n}_+ & \text{if} \quad |\xi| < \eta \\ \overline{n}(\xi) < \overline{n}_+ & \text{if} \quad |\xi| > \eta, \end{cases} \tag{2.1}$$

for some positive number η . This assumption corresponds to guides with one layer of maximal index to vertically confine waves. To that category belongs the canonical rib guide often used in the applications. In fact under this assumption (2.1) we have proved in [1] that the number of guided modes remains bounded as k tends to ∞ . Then it is natural to determine theirs limits.

The refractive index n is a piecewise continuous function defined in $\mathbb{R}^2$; moreover there exists a compact set $K = (-a, a) \times (-b, b) \subset \mathbb{R}^2$, $b \geq \eta$, such that

$$\text{for all } x \equiv (x_1, x_2) \notin K \quad n(x) = \overline{n}(x_2). \tag{2.2}$$

Let n_+ denote the supremum of n . We assume

$$n_+ = \overline{n}_+, \tag{2.3}$$

which is the case under consideration here. The case $n_+ > \overline{n}_+$ is known from optical fibers, see [2].

In the asymptotical study of guided modes we often refer to the Dirichlet eigenproblem on the set where n achieves its maximum. We set

$$B_+ = \text{Interior}\{x \in \mathbb{R}^2; n(x) = n_+\}. \tag{2.4}$$

To carry out our analysis we add an assumption on n. For real positive δ, let ω_δ be the set

$$\omega_\delta = \{x \in \mathbb{R}^2; \text{dist}(x, B_+) > \delta\}$$

and $\zeta(\delta) = n_+^2 - \sup_{x\in\omega_\delta} n^2(x)$. Then our hypothesis reads

$$B_+ \text{ is a Lipschitz domain and for all } \delta > 0,\ \zeta(\delta) > 0. \tag{2.5}$$

Remark 2.a The assumption (2.5) is technical. It expresses what is needed to describe the behavior of guided modes at high frequencies and can be loosened by considering it in a neighborhood of ∂B_+. The classical example of the three layers rib guide satisfies this assumption. In fact the two generic cases, either n has a jump at ∂B_+ or n is a regular function (C^1) fulfill the loosened assumption. ■

Looking for harmonic guided modes under the assumption of weak guidance consists in determining the real numbers β, k, the functions $u \in H^1(\mathbb{R}^2)$ such that

$$-\Delta u - k^2 n^2 u = -\beta^2 u \quad \text{in} \quad \mathbb{R}^2; \tag{2.6}$$

k is the wave number and β is the propagation constant of the mode. A classical way to study these modes is to fix k and to look for $(-\beta^2, u)$. We will keep the same point of view here, that is k will be considered as a parameter, while the unknowns will be the eigenpair $(-\beta^2 \equiv \lambda, u)$. By varying k we then get the dispersion relations $k \mapsto \lambda(k)$.

Thus we define the unbounded operator $A_k : \mathscr{D}(A_k) \subset L^2(\mathbb{R}^2) \to L^2(\mathbb{R}^2)$ by

$$\mathscr{D}(A_k) = \{v \in H^1(\mathbb{R}^2); \Delta v \in L^2(\mathbb{R}^2)\} \quad \text{and} \quad A_k v = -\Delta v - k^2 n^2 v \text{ for } v \in \mathscr{D}(A_k).$$

We consider now the problem: find $\lambda \in \mathbb{R}$ and $u \in \mathscr{D}(A_k)$, $u \not\equiv 0$, such that

$$A_k u = \lambda u. \tag{2.7}$$

The operator A_k has been extensively studied in [1]. It is a bounded from below selfadjoint operator and its spectrum satisfies $\sigma(A_k) \subset [-k^2 n_+^2, \infty)$. The spectrum consists of a continuum, the essential spectrum $\sigma_{\text{ess}}(A_k)$, and of a discrete set, the discrete spectrum $\sigma_d(A_k)$, which is the set of isolated eigenvalues of finite multiplicity. The essential spectrum is given by

$$\sigma_{\text{ess}}(A_k) = [\overline{\gamma}(k), \infty) \tag{2.8}$$

where

$$\overline{\gamma}(k) = \inf_{\substack{\varphi \in H^1(\mathbb{R}) \\ \varphi \neq 0}} \frac{\int_{\mathbb{R}} (\varphi'^2 - k^2 \overline{n}^2 \varphi^2)\, dy}{\int_{\mathbb{R}} \varphi^2\, dy}, \tag{2.9}$$

is the smallest eigenvalue of the Sturm-Liouville operator of the associated planar waveguide.

The point spectrum $\sigma_p(A_k)$ consists of the discrete spectrum $\sigma_d(A_k)$ which is the set of eigenvalues below $\overline{\gamma}(k)$ and of eigenvalues embedded in the essential spectrum. We can characterize the discrete spectrum with the Min–Max principle, see [8]. Corresponding to the problem (2.7), we define the Min–Max quantities $\lambda_m(k), m \geq 1$, by

$$\lambda_m(k) = \inf_{H_m \in \mathscr{H}_m(H^1(\mathbb{R}^2))} \sup_{\substack{v \in H_m \\ v \neq 0}} \frac{a_k(v,v)}{(v,v)_{0,\mathbb{R}^2}}, \tag{2.10}$$

where $\mathscr{H}_m(H^1(\mathbb{R}^2))$ is the set of m-dimensional subspaces of $H^1(\mathbb{R}^2)$ and $a_k(\cdot,\cdot) : H^1(\mathbb{R}^2) \times H^1(\mathbb{R}^2) \to \mathbb{R}$ is given for $u, v \in H^1(\mathbb{R}^2)$ by

$$a_k(u,v) = \int_{\mathbb{R}^2} (\nabla u \nabla v - k^2 n^2 uv)\, dx.$$

Then

$$-k^2 n_+^2 \leq \lambda_1(k) \leq \lambda_2(k) \leq \ldots \leq \lambda_m(k) \leq \ldots \leq \overline{\gamma}(k)$$

and if $\lambda_j(k) = \overline{\gamma}(k)$ for some $j \geq 1$ then A_k has at most $(j-1)$ eigenvalues below $\overline{\gamma}(k)$. If $\lambda_j(k) < \overline{\gamma}(k)$, then $\lambda_1(k), \ldots, \lambda_j(k)$ are the first j eigenvalues of A_k, repeated with their multiplicity.

For a given k we have only a finite number of eigenvalues below $\overline{\gamma}(k)$, see [1] for instance. If $N(k)$ is the number of eigenvalues strictly below $\overline{\gamma}(k)$, in other words

$$N(k) = \sup\{m \in \mathbb{N}; \lambda_m(k) < \overline{\gamma}(k)\},$$

then $N(k)$ represents the number of guided modes. Using comparison principles we have got in [1] upper and lower bounds of $N(k)$ for some indices n satisfying to (2.1)–(2.3), from which existence results are derived. The relevant point here is to study the limit of $\lambda_m(k) + k^2 n_+^2$ as k tends to ∞.

Similarly to the set B_+ defined in (2.4), we associate the Min–Max quantities for $m \geq 1$

$$\mu_m = \inf_{H_m \subset \mathscr{H}_m(\mathring{H}^1(B_+))} \sup_{\substack{\varphi \in H_m \\ \varphi \neq 0}} \frac{\int_{B_+} |\nabla \varphi|^2\, dx}{\int_{B_+} \varphi^2\, dx}, \tag{2.11}$$

where $\mathscr{H}_m(\mathring{H}^1(B_+))$ is the set of all vector subspaces of $\mathring{H}^1(B_+)$ of dimension m. The quantities μ_m characterize the discrete spectrum of the operator $A : \mathscr{D}(A) \subset L^2(B_+) \to L^2(B_+)$ defined by

$$\mathscr{D}(A) = \{\varphi \in \mathring{H}^1(B_+); \Delta\varphi \in L^2(B_+)\} \text{ and } A\varphi = -\Delta\varphi \quad \forall\, \varphi \in \mathscr{D}(A). \tag{2.12}$$

In the next section we will present a study of the Dirichlet eigenproblem on the perturbed strip B_+ and give sufficient conditions to have eigenvalues.

In section 4 we will prove the major result of the paper, which is the following.

Theorem 2.1 *We assume that n and B_+ satisfy to the assumptions* (2.1)–(2.3) *and* (2.5). *Then for all m*

$$\lim_{k\to\infty} \lambda_m(k) + k^2 n_+^2 = \mu_m, \ \nearrow . \tag{2.13}$$

In particular if μ_m is an eigenvalue of A, then for k large enough $\lambda_m(k)$ is an eigenvalue of A_k.

3 The Dirichlet eigenproblem on a perturbed strip

Here we present results on the spectrum of A. Results on the perturbed strip can be found in the literature, see for instance the pioneering work [4] or [11] close to the present case. Our techniques are adapted from [1] in the case of B_+.

Proposition 3.1 *The operator A defined in* (2.12) *is selfadjoint positive and*

$$\sigma(A) \subset \left[\frac{\pi^2}{e_+^2}, \infty\right) \quad \textit{and} \quad \sigma_{\text{ess}}(A) = \left[\frac{\pi^2}{4\eta^2}, \infty\right).$$

where η is defined in (2.1) *and $e_+ = \sup_{x_1\in\mathbb{R}} e(x_1)$ with*

$$e(x_1) = \sup\{|x_2 - y_2|; (x_1, x_2) \in B_+, (x_1, y_2) \in B_+\}.$$

PROOF. Clearly the operator A is selfadjoint positive. Let us check that its spectrum is above $\frac{\pi^2}{e_+^2}$. We notice that for a given x_1 with $e(x_1) > 0$, the first eigenvalue of the operator $-\frac{d^2}{dy^2} : H^2(\mathscr{I}) \cap \mathring{H}^1(\mathscr{I}) \to L^2(\mathscr{I})$, $\mathscr{I}$ being the smallest interval containing $B_+ \cap \{(x_1, x_2); x_2 \in \mathbb{R}\}$, is $\frac{\pi^2}{e(x_1)^2}$. So for any $\varphi \in \mathscr{D}(A)$, we denote by $\tilde{\varphi}$ its extension by 0 over $\mathbb{R}^2$ and we have

$$\int_{\mathscr{I}} \left(\frac{\partial\tilde{\varphi}}{\partial x_2}(x_1, x_2)\right)^2 dx_2 \geq \frac{\pi^2}{e(x_1)^2} \int_{\mathscr{I}} \tilde{\varphi}^2(x_1, x_2)\, dx_2. \tag{3.1}$$

Then we integrate over x_1 to get

$$\int_{B_+} |\nabla\varphi|^2\, dx \geq \frac{\pi^2}{e_+^2} \int_{B_+} \varphi^2\, dx;$$

with the Min–Max principle, we conclude.

To determine the essential spectrum of A we proceed with standard arguments of singular sequences, see [1]. ∎

Proposition 3.2 *(i) The number* N_{B_+} *of eigenvalues of* A *below* $\frac{\pi^2}{4\eta^2}$ *is finite.*

(ii) If B_+ *contains the rectangle* $(-d, d) \times (-h, h)$ *with* $h > \eta$, *then the following lower bound holds*

$$M(d, h) \leq N_{B_+},$$

where

$$M(d,h) = \left| \left\{ (p,q) \in \mathbb{N}^* \times \mathbb{N}^*; \frac{p^2}{d^2} + \frac{q^2}{h^2} < \frac{1}{\eta^2} \right\} \right|. \tag{3.2}$$

(iii) If $e(x_1) \leq 2\eta$ *for all* x_1, *then*

$$\sigma(A) = \sigma_{\text{ess}}(A) \quad \textit{and} \quad N_{B_+} = 0.$$

PROOF.

(i) The key point here is to introduce a Neumann boundary condition to get a lower bound for the eigenvalues in a comparison principle. Let B_a be the set $\{x \in B_+; -a < x_1 < a\}$, see (2.2). The eigenproblem

$$\begin{array}{rcll} -\Delta\varphi & = & \alpha\varphi & \text{in } B_a, \\ \varphi & = & 0 & \text{on } \partial B_a \backslash \{(\pm a, x_2) \in B_+\}, \\ \dfrac{\partial\varphi}{\partial\nu} & = & 0 & \text{on } \{(\pm a, x_2) \in B_+\}, \end{array}$$

admits a sequence $\{\alpha_m\}_{m\geq 1}$ of eigenvalues tending to infinity. The eigenvalues α_m are characterized by the Rayleigh quotient

$$\alpha_m = \sup_{\varphi_1,\dots,\varphi_{m-1} \in L^2(B_a)} \inf_{\substack{\varphi \in H,\, \varphi \neq 0 \\ \varphi \in [\varphi_1,\dots,\varphi_{m-1}]^\perp}} \frac{\int_{B_a} |\nabla\varphi|^2\, dx}{\int_{B_a} \varphi^2\, dx} \tag{3.3}$$

where $H = \{\varphi \in H^1(B_a); \varphi = 0 \text{ on } \partial B_a \backslash \{(\pm a, x_2) \in B_+\}\}$.

Given $\varphi \in \overset{\circ}{H}{}^1(B_+)$, for almost all x_1, $|x_1| > a$, we have

$$\int_{-\eta}^{\eta} |\frac{\partial\varphi}{\partial x_2}(x_1,x_2)|^2\, dx_2 \geq \frac{\pi^2}{4\eta^2} \int_{-\eta}^{\eta} \varphi^2(x_1,x_2)\, dx_2 \tag{3.4}$$

and integrating over $(-\infty, -a) \cup (a, \infty)$ gives

$$\int_{C_a} |\nabla\varphi|^2\, dx \geq \frac{\pi^2}{4\eta^2} \int_{C_a} \varphi^2\, dx$$

with $C_a = \{(x_1, x_2) \in B_+; |x_1| > a\}$. The Max–Min characterization of μ_m is

$$\mu_m = \sup_{\varphi_1,\dots,\varphi_{m-1} \in L^2(B_+)} \inf_{\substack{\varphi \in \overset{\circ}{H}{}^1(B_+),\, \varphi \neq 0 \\ \varphi \in [\varphi_1,\dots,\varphi_{m-1}]^\perp}} \frac{\int_{B_+} |\nabla\varphi|^2\, dx}{\int_{B_+} \varphi^2\, dx}$$

and therefore with (3.4) we have

$$\mu_m \geq \sup_{\varphi_1,\ldots,\varphi_{m-1}\in L^2(B_+)} \inf_{\substack{\varphi\in \overset{\circ}{H}{}^1(B_+),\, \varphi\neq 0 \\ \varphi\in[\varphi_1,\ldots,\varphi_{m-1}]^\perp}} \frac{\int_{B_a} |\nabla\varphi|^2\,dx + \frac{\pi^2}{4\eta^2}\int_{C_a}\varphi^2\,dx}{\int_{B_a}\varphi^2\,dx + \int_{C_a}\varphi^2\,dx}.$$

We deduce then from the relation for $a_1, a_2, a_3, a_4 \in \mathbb{R}$, $a_3 > 0$, $a_4 > 0$,

$$\frac{a_1 + a_2}{a_3 + a_4} \geq \min\left(\frac{a_1}{a_3}, \frac{a_2}{a_4}\right),$$

that

$$\mu_m \geq \sup_{\varphi_1,\ldots,\varphi_{m-1}\in L^2(B_a)} \inf_{\substack{\varphi\in H,\, \varphi\neq 0 \\ \varphi\in[\varphi_1,\ldots,\varphi_{m-1}]^\perp}} \alpha(\varphi),$$

where

$$\alpha(\varphi) = \min\left(\frac{\int_{B_a}|\nabla\varphi|^2\,dx}{\int_{B_a}\varphi^2\,dx}, \frac{\pi^2}{4\eta^2}\right).$$

Finally we conclude that

$$\mu_m \geq \min\left(\alpha_m, \frac{\pi^2}{4\eta^2}\right).$$

(ii) We consider the Dirichlet eigenvalue problem in the set $R = (-d, d) \times (-h, h)$,

$$\begin{aligned} -\Delta\varphi &= \alpha\varphi && \text{in } R, \\ \varphi &= 0 && \text{on } \partial R. \end{aligned}$$

The eigenvalues are

$$\alpha_m = \frac{p^2\pi^2}{4d^2} + \frac{q^2\pi^2}{4h^2}, \qquad (p, q) \in \mathbb{N}^* \times \mathbb{N}^*.$$

With a comparison principle for Dirichlet problems, we conclude that the following bound holds $M(d, h) \leq N_{B_+}$.

(iii) It is an immediate consequence of Proposition 3.1. ■

When the perturbed strip B_+ does not contain a rectangle large enough, i.e. when $M(d, h) = 0$, then Proposition 3.2 is of no help to prove the existence of at least one eigenvalue below the essential spectrum. With a different method we can derive the following result.

Proposition 3.3 *We assume that the strip B_+ contains the set*

$$\widetilde{B}_+ = \{x \in \mathbb{R}^2; -g(x_1) < x_2 < g(x_1)\ \forall x_1 \in \mathbb{R}\},$$

where the function $g:\mathbb{R}\to\mathbb{R}$ *is positive continuous piecewise* C^1 *and satisfies*

$$\left\{\begin{array}{l} g(x_1)=\eta \quad \textit{for } |x_1|>a, \\ \displaystyle\int_{-a}^{a}\frac{(\eta^2-g^2)+\kappa\eta^2 g'^2}{g}\,dx_1<0, \end{array}\right. \tag{3.5}$$

with $\kappa=1/3+2/\pi^2$. *Then the operator* A *has at least one eigenvalue below its essential spectrum* $[\frac{\pi^2}{4\eta^2},\infty)$.

PROOF. By comparison principle it is sufficient to prove the existence of the first eigenvalue for the Laplacian operator $\tilde{A}$ defined in the strip $\widetilde{B}_+$. In fact we will construct a function $\varphi\in\overset{\circ}{H}{}^1(\widetilde{B}_+)$ such that

$$\int_{\widetilde{B}_+}|\nabla\varphi|^2\,dx<\frac{\pi^2}{4\eta^2}\int_{\widetilde{B}_+}\varphi^2\,dx; \tag{3.6}$$

then with the Min–Max principle we can conclude. We choose φ to be the function

$$\varphi(x)=\cos\left(\frac{\pi x_2}{2g(x_1)}\right)\zeta(x_1)$$

with

$$\zeta(x_1)=\left\{\begin{array}{ll} 1 & \text{if } -a<x_1<a, \\ e^{-\alpha(|x_1|-a)} & \text{elsewhere.}\end{array}\right.$$

It is not difficult to check that by choosing α small enough, the inequality (3.6) is satisfied if

$$\int_{-a}^{a}\int_{-g(x_1)}^{g(x_1)}\left[|\nabla\varphi|^2-\frac{\pi^2}{4\eta^2}\varphi^2\right]dx_2dx_1<0.$$

Then using the definition of φ, we can develop the above integral to get the inequality in (3.5). ■

Remark 3.a The criterion (3.5), although technical, can be interpreted in the following way. We introduce the relative perturbation function f by $g=\eta f$. Then the criterion reads

$$\int_{-a}^{a}\left[\frac{1-f^2}{f}+\kappa\eta^2\frac{f'^2}{f}\right]dx_1<0; \tag{3.7}$$

for η small the dominant term is the difference term $\int_{-a}^{a}[1/f-f]\,dx$.
Due to the term with a derivative of f, the criterion (3.7) can be bad when the boundary of B_+ has a step profile. ■

Remark 3.b We can have a variant of Proposition 3.3 if we consider the set $\widetilde{B}_+$ to be

$$\widetilde{B}_+=\{x\in\mathbb{R}^2;0<x_2+\eta<g(x_1)\;\forall x_1\in\mathbb{R}\},$$

where g is a continuous function piecewise C^1 larger than η. Then the conclusion of Proposition 3.3 still holds under the assumption

$$\left\{\begin{array}{l} g(x_1) = 2\eta \quad \text{for } |x_1| > a, \\ \displaystyle\int_{-a}^{a} \frac{(4\eta^2 - g^2) + \tilde{\kappa}\, 4\,\eta^2 g'^2}{g}\, dx_1 < 0, \end{array}\right. \tag{3.8}$$

with $\tilde{\kappa} = 1/3 + 1/2\pi^2$. We choose φ to be the function

$$\varphi(x) = \sin\left(\frac{\pi(x_2 + \eta)}{g(x_1)}\right)\zeta(x_1),$$

with $\zeta(\cdot)$ the function introduced in the proof of Proposition 3.3. ■

Corollary 3.4 *We assume that the strip* B_+ *contains the set*

$$\widetilde{B}_+ = \{x \in \mathbb{R}^2; -\eta < x_2 < \ell(x_1)\ \forall\, x_1 \in \mathbb{R}\},$$

where the piecewise continuous function ℓ *satisfies*

$$\left\{\begin{array}{l} \ell(x_1) \geq \eta \quad \forall\, x_1 \in \mathbb{R}, \\ \textit{measure of}\, \{x \in \widetilde{B}_+; \ell(x_1) > \eta\} > 0. \end{array}\right. \tag{3.9}$$

Then the operator A *has at least one eigenvalue below the essential spectrum.*

PROOF. Without loss of generality we can assume that there exists an $\varepsilon > 0$ such that

$$(-\varepsilon, \varepsilon) \times (\eta, \eta + \frac{\varepsilon^2}{\eta^2}) \subset \{x \in \widetilde{B}_+; \ell(x_1) > \eta\}.$$

We set

$$g(y) = \left\{\begin{array}{l} \eta \quad \text{if } |y| \geq \varepsilon, \\ \alpha(|y| - \varepsilon)^2 + \eta \quad \text{elsewhere.} \end{array}\right.$$

It is a simple matter to check that $\alpha > 0$ can be chosen such that (3.8) holds. ■

So far we have presented two different criteria, the rectangle criterion of Proposition 3.2 and the integral criterion (3.5) (or (3.8)). One does not imply the other.

With the rectangle criterion, it would be impossible to get Corollary 3.4. Indeed we can choose a strip B_+ such that the best choice of d and $h > \eta$ will lead to $M(d, h) = 0$, while the operator A has at least one eigenvalue below the essential spectrum.

The end of the section is devoted to numerical examples which illustrate precisely the range of both criteria.

Example 3.c We consider the two different strips B^i_+ , $i = 1, 2$, represented in Figure 3. More precisely here the functions g^i , f^i are given by

$$g^i = \eta f^i, \qquad f^i = 1 + \varepsilon \tilde{f}^i,$$

with

$$\tilde{f}^1(x_1) \;=\; \begin{cases} -\cos(x_1) & -\frac{3\pi}{2} < x_1 < \frac{3\pi}{2}, \\ 0 & \text{elsewhere}, \end{cases}$$
$$\tilde{f}^2(x_1) \;=\; \begin{cases} -1 & |x_1| \leq \frac{\pi}{2}, \\ 1 & \frac{\pi}{2} < |x_1| \leq \frac{3\pi}{2}, \\ 0 & \text{elsewhere}, \end{cases}$$

the real numbers η and ε are considered as parameters, $0 < \eta$ and $0 \leq \varepsilon < 1$.

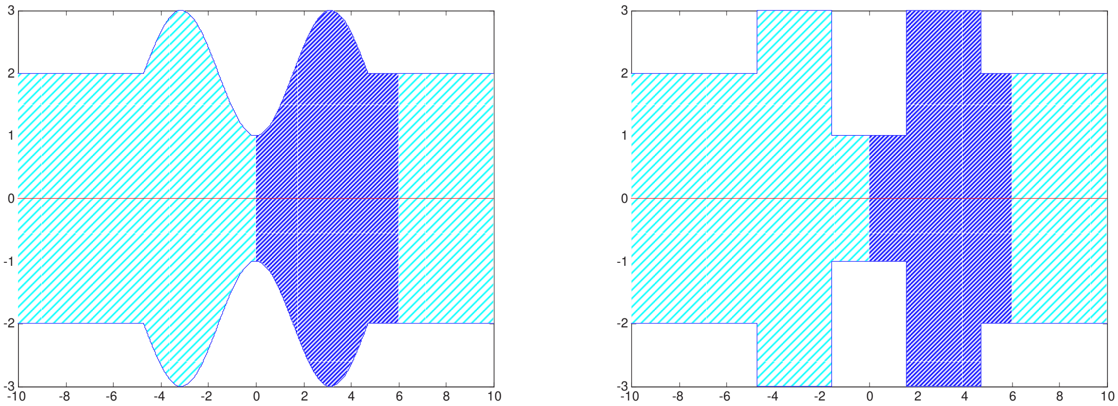

Figure 3: The two perturbed strips B^1_+ and B^2_+ with $\eta = 2$, $\varepsilon = 1$.

For different values of the parameters (η, ε) , that is for different sets B^1_+ and B^2_+ , we want to know whether the problem has at least one eigenvalue or no eigenvalue, whether the criteria presented above are valid or not.

In Figure 4 we have collected all our results. We distinguish subregions with their filling. In the squared subdomain both criteria are valid while in the vertically marked subdomain only the integral criterion is valid and in the horizontally marked subdomain the rectangle criterion only is valid. In the two obliquely marked subdomains the eigenproblem has at least one eigenvalue (sparse filling) or two eigenvalues (dense filling) . Finally the white subdomain represents the case when there is no eigenvalue below the essential spectrum.

In fact we have computed the boundaries of the subdomains by dichotomy procedure. The integral criterion reduces to numerical integrations, with affine regularization in the case 2, and the rectangle criterion needs an optimization process in the case 1 and is explicit in the case 2. The computation of the existence limit curves (for 1 or 2 eigenvalues) is much more involved. For a given pair (ε, η) we mesh the truncated strip $B^i = B^i_+ \cap (]0, 6[\times\mathbb{R})$

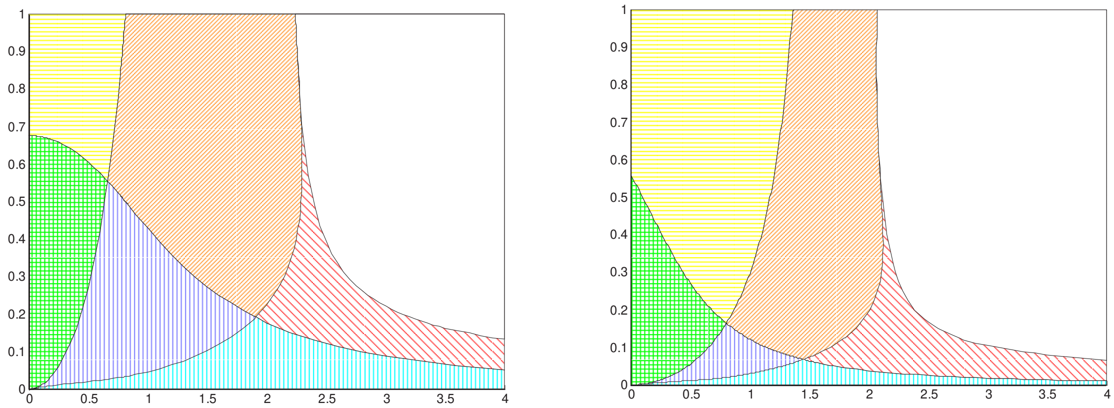

Figure 4: Results in $O\eta\varepsilon$ -plane for B^1_+ and B^2_+ .

(half of the grey region in Figure 3), we solve an equivalent eigenvalue problem on B^i by introducing transparent boundary condition on the boundary $\{(6, x_2); -\eta < x_2 < \eta\}$ and Neumann condition on the boundary $\{(0, x_2); -g_i(0) < x_2 < g_i(0)\}$. We refer to [5] for the method and to [6] for its implementation in our waveguide case, based on the finite element code MÉLINA, see [7].

Since we are using a finite element approximation it is not difficult to check with comparison principle that we overestimate the non existence region. Anyhow we are working with several thousands of triangles in B^i and piecewise polynomial of degree 1 approximation to have less than 1 percent of error. Due to the symmetry when we impose a zero boundary condition on $\{(0, x_2); -g_i(0) < x_2 < g_i(0)\}$, we can get the second eigenvalue.

Notice also that the rectangle criterion gives the second eigenvalue for symmetry reason. Both results in Figure 4 are essentially similar. Nevertheless we can figure out the effects of the regularity of B_+ on the existence region and on the validity of the integral criterion.

■

Remark 3.d To the light of the above example, we can look at the rectangle criterion of Proposition 3.2 and the integral criterion (3.5), when the strip B_+ is determined by the the functions

$$g = \eta f, \qquad f = 1 + \varepsilon \tilde{f}$$

and ε , η are getting smaller and smaller. If $\int_{-a}^{a} \tilde{f}\, dx > 0$, then for ε small enough the integral criterion (3.5) is satisfied. If ε is fixed and η small enough, then the rectangle criterion implies the existence of at least one eigenvalue, as soon as $\tilde{f}$ is strictly positive somewhere. ■

4 Convergence proofs

The goal of the section is to prove Theorem 2.1. In the case of the planar waveguide characterized by $\overline{n}$, we have studied in [1], Appendix A, the high frequency limit of the eigenvalues $\gamma_m(k)$ characterized by

$$\gamma_m(k) = \inf_{H_m \in \mathscr{H}_m(H^1(\mathbb{R}))} \sup_{\substack{\varphi \in H_m \\ \varphi \neq 0}} \frac{\int_{\mathbb{R}} (\varphi'^2 - k^2 \overline{n}^2 \varphi^2)\, dy}{\int_{\mathbb{R}} \varphi^2\, dy}, \tag{4.1}$$

where $\mathscr{H}_m(H^1(\mathbb{R}))$ is the set of all m-dimensional subspaces of $H^1(\mathbb{R})$. Recall that the quantity $\overline{\gamma}(k)$ in (2.9) is $\gamma_1(k)$. The following result holds.

Proposition 4.1 *We assume that* $\overline{n}$ *satisfies to* (2.1)*. Then for each* $m \geq 1$, $\gamma_m(k) + k^2 n_+^2$*, as a function of* $k > 0$*, is increasing and*

$$\lim_{k \to \infty} \gamma_m(k) + k^2 n_+^2 = \frac{m^2 \pi^2}{4\eta^2}.$$

Proposition 4.2 *Let* $\lambda_m(k)$ *and* μ_m, $m \geq 1$*, be the Min–Max quantities defined in* (2.10) *and* (2.11)*. Then* $\lambda_m(k) + k^2 n_+^2$*, as a function of* $k > 0$*, is increasing and bounded by* μ_m.

PROOF. For all $v \in H^1(\mathbb{R}^2)$, $v \neq 0$, and for $0 < k_1 < k_2$, we clearly have

$$\int_{\mathbb{R}^2} \left(|\nabla v|^2 + k_1^2 (n_+^2 - n^2) v^2 \right) dx \leq \int_{\mathbb{R}^2} \left(|\nabla v|^2 + k_2^2 (n_+^2 - n^2) v^2 \right) dx.$$

We divide both terms by $\int_{\mathbb{R}^2} v^2\, dx$ and take the $\inf \sup$ to get

$$\lambda_m(k_1) + k_1^2 n_+^2 \leq \lambda_m(k_2) + k_2^2 n_+^2.$$

Let us get now the bound μ_m. If $\mu_m = \pi^2/4\eta^2$, the result is immediate since we have the estimates

$$\lambda_m(k) + k^2 n_+^2 \leq \overline{\gamma}(k) + k^2 n_+^2 \leq \frac{\pi^2}{4\eta^2},$$

where the second estimate is given in Proposition 4.1. We assume now that $\mu_m < \pi^2/4\eta^2$, that is μ_m is an eigenvalue of the corresponding eigenproblem on the perturbed strip. Let $w^{(i)}$ be a normalized eigenvector associated to μ_i, $i = 1, \ldots, m$. To each $w^{(i)} \in \overset{\circ}{H}{}^1(B_+)$, we associate the function $\tilde{w}^{(i)} \in H^1(\mathbb{R}^2)$ defined in $\mathbb{R}^2$ by zero extension and we set $H_m = \operatorname{span}\{\tilde{w}^{(1)}, \ldots, \tilde{w}^{(m)}\}$. Then we have

$$\lambda_m(k) + k^2 n_+^2 \leq \sup_{\substack{v \in H_m \\ v \neq 0}} \frac{\int_{\mathbb{R}^2} |\nabla v|^2\, dx}{\int_{\mathbb{R}^2} v^2\, dx},$$

since $n_+^2 - n^2 = 0$ inside the support of $\varphi \in H_m$. Then clearly for any $v \in H_m$, $v \neq 0$,

$$\frac{\int_{\mathbb{R}^2} |\nabla v|^2 \, dx}{\int_{\mathbb{R}^2} v^2 \, dx} \leq \mu_m,$$

and the proof is complete. ∎

We deduce from the above result that the limit, $\lim_{k\to\infty} \lambda_m(k) + k^2 n_+^2$, exists and is bounded by μ_m. We will prove now that this limit is actually μ_m; to do it we first prove an estimate on corresponding eigenfunctions outside B_+.

Lemma 4.3 *We assume that n and B_+ satisfy to the assumptions* (2.1)–(2.3) *and* (2.5). *If $\{\lambda_m(k), \varphi_m(k)\}_{k\geq k_0}$ is a sequence of eigenpairs of A_k, normalized to $\|\varphi_m(k)\|_{0,\mathbb{R}^2} = 1$, then for all $\delta > 0$, the following holds*

$$\lim_{k\to\infty} \|\varphi_m(k)\|_{1,\omega_\delta} = 0.$$

PROOF. We start from the expression

$$-\Delta\varphi_m(k) + k^2(n_+^2 - n^2)\varphi_m(k) = (\lambda_m(k) + k^2 n_+^2)\varphi_m(k).$$

Let $\delta > 0$ be given and χ be a regular function such that $0 \leq \chi \leq 1$, $\chi(x) = 1$ for $x \in \omega_\delta$, $\chi(x) = 0$ for $x \in \omega_{\delta/2}^C$. Then we multiply the above equation by $\chi\varphi_m(k)$ and get

$$\int_{\mathbb{R}^2} \nabla\varphi_m(k)\nabla\left(\chi\varphi_m(k)\right) \, dx + k^2 \int_{\mathbb{R}^2} \chi(n_+^2 - n^2)\varphi_m^2(k) \, dx$$
$$= \left(\lambda_m(k) + k^2 n_+^2\right) \int_{\mathbb{R}^2} \chi\varphi_m^2(k) \, dx.$$

So we deduce

$$\int_{\mathbb{R}^2} \chi|\nabla\varphi_m(k)|^2 \, dx + \int_{\mathbb{R}^2} \nabla\varphi_m(k)\varphi_m(k)\nabla\chi \, dx + k^2 \int_{\mathbb{R}^2} \chi(n_+^2 - n^2)\varphi_m^2(k) \, dx$$
$$= \left(\lambda_m(k) + k^2 n_+^2\right) \int_{\mathbb{R}^2} \chi\varphi_m^2(k) \, dx. \quad (4.2)$$

On the other hand since

$$\int_{\mathbb{R}^2} |\nabla\varphi_m(k)|^2 \, dx + k^2 \int_{\mathbb{R}^2} (n_+^2 - n^2)\varphi_m^2(k) \, dx = \left(\lambda_m(k) + k^2 n_+^2\right) \int_{\mathbb{R}^2} \varphi_m^2(x) \, dx,$$

we deduce

$$\int_{\mathbb{R}^2} |\nabla\varphi_m(k)|^2 \, dx \leq \mu_m,$$
$$k^2 \int_{\mathbb{R}^2} (n_+^2 - n^2)\varphi_m^2(k) \, dx \leq \mu_m.$$

Furthermore from the assumption (2.5) we know that in $\omega_{\delta/2}$, $n_+^2 - n^2$ is strictly positive, bounded from below by $\zeta(\delta/2) > 0$. Therefore

$$k^2\zeta(\delta/2)\int_{\omega_{\delta/2}} \varphi_m^2(k)\,dx \le k^2 \int_{\omega_{\delta/2}} (n_+^2 - n^2)\varphi_m^2(k)\,dx \le \mu_m$$

and then

$$\int_{\omega_{\delta/2}} \varphi_m^2(k)\,dx \le \frac{\mu_m}{\zeta(\delta/2)}k^{-2}.$$

Going back to (4.2), we deduce

$$\int_{\mathbb{R}^2} \chi|\nabla\varphi_m(k)|^2\,dx \le \mu_m\|\varphi_m(k)\|^2_{0,\omega_{\delta/2}} + |\chi|_{1,\infty,\mathbb{R}^2}\|\nabla\varphi_m(k)\|_{0,\omega_{\delta/2}}\|\varphi_m(k)\|_{0,\omega_{\delta/2}}$$

and then

$$\left(\int_\omega |\nabla\varphi_m(k)|^2\,dx\right)^{1/2} \le C(\delta)k^{-1/2}.$$

■

PROOF OF THEOREM 2.1. Let $L = N_{B_+}$ be the number of eigenvalues of A below its essential spectrum. We distinguish the cases $m \le L$ and $m > L$.

Step 1 $m \le L$. From Proposition 4.1, we deduce that for k large enough

$$\overline{\gamma}(k) + k^2 n_+^2 > \mu_L.$$

In Proposition 4.2, we have proved that $\lambda_m(k) + k^2 n_+^2 \le \mu_m$, for $m = 1, 2, \ldots$; so for k large enough and $m = 1, \ldots, L$, $\lambda_m(k)$ is an eigenvalue of A_k below its essential spectrum.

Our goal now is to get for $m = 1, \ldots, L$ and k large enough, an inequality of the form

$$\mu_m \le \lambda_m(k) + k^2 n_+^2 + \varepsilon(k),$$

with $\lim_{k\to\infty} \varepsilon(k) = 0$. Under the assumptions (2.1)–(2.3) and (2.5), we can construct a sequence of domains $\{B_j\}_{j\ge 1}$ such that

$$\begin{aligned} \overline{B_+} \subset B_j \subset \overline{B_j} \subset B_{j-1} \quad & \text{for all } j, \\ B_j \quad & \text{is regular}, \\ \cap_{j=1}^\infty B_j = B_+; & \end{aligned}$$

B_j can be a regularization of the set $\omega^C_{1/j}$. Let now μ^j_m be the Min–Max relations associated to the domain B_j . By continuity the sequence $\{\mu^j_m\}_{m\ge 1}$ converges to μ_m . Let $1 \le m \le L$ and $\varphi_m(k) \in H^1(\mathbb{R}^2)$ (for k large enough) be eigenvectors associated to $\lambda_m(k)$, $\|\varphi_m(k)\|_{0,\mathbb{R}^2} = 1$.

We define the function

$$\psi_m(k) = \varphi_m(k) - R_j \gamma_j \varphi_m(k) \in \overset{\circ}{H}{}^1(B_j)$$

with $\gamma_j \in \mathscr{L}(H^1(\mathbb{R}^2); H^{1/2}(\partial B_j))$ the trace operator on ∂B_j, $R_j : H^{1/2}(\partial B_j) \to H^1(B_j)$ a lifting operator which can be chosen such that

$$\|R_j v\|_{1,B_j} \leq C\|v\|_{1/2,\partial B_j}$$

for all $v \in H^{1/2}(\partial B_j)$ and with C independent of j. The function $\psi_m(k)$ is extended by 0 all over $\mathbb{R}^2$. Then from Lemma 4.3 we immediately deduce that

$$\|\psi_m(k) - \varphi_m(k)\|_{1,\mathbb{R}^2} \quad \to 0 \text{ as } k \to 0. \tag{4.3}$$

So the subspace

$$H_m = \text{span}\{\psi_1(k), \dots, \psi_m(k)\}$$

is of dimension m. We start with the estimate of the Rayleigh quotient

$$\frac{\int_{B_j} |\nabla\psi_i(k)|^2 \, dx}{\int_{B_j} |\psi_i(k)|^2 \, dx} \leq \frac{\int_{\mathbb{R}^2} |\nabla\varphi_i(k)|^2 \, dx}{\int_{\mathbb{R}^2} |\varphi_i(k)|^2 \, dx} + \rho^j(k),$$

where the term $\rho^j(k)$ tends to 0 as k tends to ∞ by (4.3); then we take the Min–Max to deduce

$$\mu_m^j \leq \lambda_m(k) + k^2 n_+^2 + \rho^j(k).$$

Letting k tend to ∞ we conclude.

<u>Step 2</u> $m > L$. If $\lambda_m(k) = \overline{\gamma}(k)$ for all k, then we can conclude since Proposition 4.1 yields that $\overline{\gamma}(k) + k^2 n_+^2 \to \pi^2/(4\eta^2)$ as k tends to ∞. Let now $\{\lambda_m(k_j), \varphi_m(k_j)\}_{j\geq 1}$ be a sequence of eigenpairs of A_k with $\lambda_m(k_j) < \overline{\gamma}(k_j)$. Then if $\lim_{j\to\infty} \lambda_m(k_j) + k_j^2 n_+^2 < \pi^2/(4\eta^2)$, it is not difficult to argue like in Step 1 to get a contradiction with the fact that $\mu_m = \pi^2/(4\eta^2)$. ∎

Proposition 4.4 *In the framework of Theorem* 2.1 *let* $m \leq N_{B_+}$ *and* $\{\varphi_m(k)\}_{k\geq 1}$ *be a sequence of normalized eigenfunctions associated to* $\{\lambda_m(k)\}_{k\geq 1}$, $\|\varphi_m(k)\|_{0,\mathbb{R}^2} = 1$. *Then there exists a function* $\varphi_m \in \overset{\circ}{H}{}^1(B_+)$, $\varphi_m \neq 0$, *such that, only to consider a subsequence,* $\varphi_m(k) \rightharpoonup \tilde{\varphi}_m$ *in* $H^1(\mathbb{R}^2)$, *where* $\tilde{\varphi}_m$ *is the extension by zero of* φ_m *over* $\mathbb{R}^2$, *and* (μ_m, φ_m) *is an eigenpair of* A.

PROOF. We prove first that the sequence $\{\varphi_m(k)\}_{k\geq 1}$ is bounded in $H^1(\mathbb{R}^2)$. From the variational formulation we get

$$\int_{\mathbb{R}^2} |\nabla\varphi_m(k)|^2 \, dx + k^2 \int_{\mathbb{R}^2} (n_+^2 - n^2)\varphi_m^2(k) \, dx \leq \frac{\pi^2}{4\eta^2} \int_{\mathbb{R}^2} \varphi_m^2(k) \, dx = \frac{\pi^2}{4\eta^2}.$$

Therefore only to consider a subsequence it converges weakly to some φ_m in $H^1(\mathbb{R}^2)$ and strongly in $L^2(K)$.

From the variational formulation for $(\lambda_m(k), \varphi_m(k))$ with the test function $\tilde{\psi} \in H^1(\mathbb{R}^2)$ the extension by zero of $\psi \in \overset{\circ}{H}{}^1(B_+)$ we deduce

$$\int_{B_+} \nabla\varphi_m(k)\nabla\psi\, dx = (\lambda_m(k) + k^2 n_+^2) \int_{B_+} \varphi_m(k)\psi\, dx;$$

taking the limit as $k \to \infty$ we get

$$\int_{B_+} \nabla\varphi_m \nabla\psi\, dx = \mu_m \int_{B_+} \varphi_m \psi\, dx.$$

With the assumptions on n, we easily obtain that $\varphi_m(k) \to 0$ in $\mathscr{D}'(B_+^C)$. Now since $\varphi_m(k) \rightharpoonup \varphi_m$ in $L^2(\mathbb{R}^2)$, with the uniqueness of the limit we get $\varphi_m = 0$ in B_+^C. Therefore the restriction of φ_m to B_+ is clearly in $\overset{\circ}{H}{}^1(B_+)$.
Finally we need to prove that $\varphi_m \not\equiv 0$ in B_+. We decompose $\mathbb{R}^2$ in the following way: $\mathbb{R}^2 = \Omega_a^- \cup \Omega_a \cup \Omega_a^+$ where $\Omega_a^- = \{(x_1, x_2) \in \mathbb{R}^2; x_1 < -a\}$, $\Omega_a = \{(x_1, x_2) \in \mathbb{R}^2; -a < x_1 < a\}$, $\Omega_a^+ = \{(x_1, x_2) \in \mathbb{R}^2; x_1 > a\}$. Then for $\psi \in H^1(\mathbb{R}^2)$ we have

$$\begin{aligned}\int_{\Omega_a} \nabla\varphi_m(k)\nabla\psi\, dx + k^2 \int_{\Omega_a} (n_+^2 - n^2)\varphi_m(k)\psi\, dx + R_a^+(\varphi_m(k), \psi)\\ +R_a^-(\varphi_m(k), \psi) = (\lambda_m(k) + k^2 n_+^2) \int_{\Omega_a} \varphi_m(k)\psi\, dx\end{aligned}$$

with

$$R_a^\pm(\varphi_m(k), \psi) = \int_{\Omega_a^\pm} \nabla\varphi_m(k)\nabla\psi\, dx - \int_{\Omega_a^\pm} (\lambda_m(k) + k^2\overline{n}^2)\varphi_m(k)\psi\, dx.$$

Since $\lambda_m(k) + k^2 n_+^2 \to \mu_m < \pi^2/(4\eta^2)$, we can check by a simple calculation that there exists a real $\alpha > 0$ such that for $k \geq k_0$

$$R_a^\pm(\varphi_m(k), \varphi_m(k)) \geq (\overline{\gamma}(k) - \lambda_m(k)) \int_{\Omega_a^+} |\varphi_m(k)|^2\, dx \geq \alpha \|\varphi_m(k)\|^2_{1,\Omega_a^\pm}.$$

Then with this last estimate, we check that there exists $\beta > 0$ such that for $k \geq k_0$

$$R_a^\pm(\varphi_m(k), \varphi_m(k)) \geq \beta \|\varphi_m(k)\|^2_{1,\Omega_a^\pm}.$$

So we have $C > 0$ independent of k such that

$$\|\varphi_m(k)\|^2_{1,\mathbb{R}^2} \leq C \int_{\Omega_a} |\varphi_m(k)|^2\, dx.$$

Now we can use the normalization equation $\|\varphi_m(k)\|_{0,\mathbb{R}^2} = 1$, the inclusion $\Omega_a \cap B_+ \subset K$, and Lemma 4.3 to deduce that necessarily

$$\int_{\Omega_a \cap B_+} |\varphi_m|^2 \, dx > 0.$$

■

5 Further estimates at high frequencies

From the analysis in Sections 3 and 4, we can deduce bounds on the number of guided modes at high frequencies. Under the assumptions of Theorem 2.1, we have

$$\lim_{k\to\infty} \inf N(k) \geq N_{B_+}, \tag{5.1}$$

where $N(k)$ and N_{B_+} are the numbers of eigenvalues of A_k and A below their essential spectrum. In Propositions 3.2 and 3.3 we have underestimates of N_{B_+} . To get an upper bound on the number of guided modes under the assumptions (2.1)–(2.3), we can use a comparison principle see [1] for instance to check that the number of guided modes for the guide of index

$$\tilde{n}(x) = \begin{cases} n_+ & \text{if } x \in K, \\ \overline{n}(x_2) & \text{elsewhere,} \end{cases}$$

is bounded for large k by the number

$$\left| \left\{ (p,q) \in \mathbb{N}^* \times \mathbb{N};\ \frac{p^2}{b^2} + \frac{q^2}{a^2} < \frac{1}{\eta^2} \right\} \right|.$$

The inequality (5.1) is not an equality in general. This means that we can have guided modes with $\lim_{k\to\infty} \lambda_m(k) + k^2 n_+^2 \to \pi^2/(4\eta^2)$. It is this phenomenon we want to tackle here. In fact we will study a case with strict inequality and an other one with equality. In the case where the set B_+ is the non perturbed strip, that is

$$B_+ = \mathbb{R} \times (-\eta, \eta), \tag{5.2}$$

we prove that $N_{B_+} = 0$ and $\lim_{k\to\infty} N(k) = 1$. Indeed at high frequencies only one mode may exist which is less and less laterally confined, as shown in the example 5.a. Finally at the end of the section we present an example with no mode at high frequencies; in that case the inequality (5.1) is in fact an equality.

To the open set $\Omega = (-a, a) \times \mathbb{R}$, we associate the unbounded operator $A_k^N : \mathscr{D}(A_k^N) \subset L^2(\Omega) \to L^2(\Omega)$ where

$$\mathscr{D}(A_k^N) = \{ u \in H^2(\Omega);\ \frac{\partial u}{\partial \nu} = 0 \text{ on } \partial\Omega \} \quad \text{and} \quad A_k^N u = -\Delta u - k^2 n^2 u.$$

We first derive a technical result.

Lemma 5.1 *We assume that* $B_+ = \mathbb{R} \times (-\eta, \eta)$ *and there exists a sequence* $\{k_p\}_{p\geq 1}$ *tending to* ∞ *such that the operator* $A^N_{k_p}$ *has at least one eigenvalue below* $\overline{\gamma}(k_p)$. *Let* λ_p *denote such an eigenvalue and* φ_p *a corresponding eigenfunction normalized to* $\|\varphi_p\|_{0,\Omega} = 1$. *Then we can extract from* $\{\varphi_p\}_{p\geq 1}$ *a subsequence still denoted* $\{\varphi_p\}_{p\geq 1}$ *satisfying to*

$$\varphi_p \rightharpoonup \varphi \quad \textit{weakly in } H^1(\Omega) \textit{ for } p \to \infty, \tag{5.3}$$

$$\varphi_p \to \varphi \quad \textit{in } L^2(\Omega) \textit{ for } p \to \infty, \tag{5.4}$$

with

$$\varphi(x) = \begin{cases} \pm\frac{1}{\sqrt{2a\eta}} \cos\left(\pi \frac{x_2}{2\eta}\right) & \textit{if } -\eta < x_2 < \eta, \\ 0 & \textit{else}. \end{cases} \tag{5.5}$$

In particular

$$\lambda_p + k_p^2 n_+^2 \to \frac{\pi^2}{4\eta^2} \quad \textit{as } p \to \infty\,. \tag{5.6}$$

PROOF. We present a proof in 4 steps. First we bound the sequence $\{\varphi_p\}_{p\geq 1}$ in the norm $\|.\|_{1,\Omega}$. Then we extract a subsequence converging in $L^2(\Omega)$. To prove that the limit φ is given by (5.5) we check that φ is a function of x_2 only and finally we prove (5.6).

Step 1. By definition the function φ_p satisfies

$$\lambda_p = \int_\Omega \left(|\nabla \varphi_p|^2 - k_p^2 n^2 \varphi_p^2 \right)\, dx \tag{5.7}$$

and then

$$\int_\Omega |\nabla \varphi_p|^2 \, dx \leq \lambda_p + k_p^2 n_+^2. \tag{5.8}$$

So with Proposition 4.1 we deduce the bound

$$\int_\Omega |\nabla \varphi_p|^2 \, dx \leq \frac{\pi^2}{4\eta^2}.$$

Step 2. The sequence $\{\varphi_p\}_{p\geq 1}$ is bounded in $H^1(\Omega)$, so it is bounded in the space $H^1(K)$. By compact embedding $H^1(K) \subset L^2(K)$, we can extract a subsequence still denoted $\{\varphi_p\}_{p\geq 1}$ such that

$$\varphi_p \rightharpoonup \varphi \quad \text{in } H^1(\Omega) \qquad \text{and} \qquad \varphi_p \to \varphi \quad \text{in } L^2(K) \text{ for } p \to \infty\,.$$

If we prove that $\varphi_p \to 0$ in $L^2(\Omega \backslash K)$, we will have $\varphi = 0$ in $\Omega \backslash K$ and

$$\varphi_p \to \varphi \quad \text{in } L^2(\Omega)\,. \tag{5.9}$$

From the equality (5.7) we deduce

$$\int_\Omega \left(|\nabla \varphi_p|^2 + k_p^2 (n_+^2 - n^2) \varphi_p^2 \right)\, dx < \overline{\gamma}(k_p) + k_p^2 n_+^2$$

and with Proposition 4.1

$$\int_\Omega (n_+^2 - n^2)\varphi_p^2\,dx < \frac{\pi^2}{k_p^2 4\eta^2}. \tag{5.10}$$

Therefore in $\Omega\backslash K$ necessarily $\varphi_p \to 0$ as $p \to \infty$.

Step 3 . We prove now that φ is a function of x_2 only. In fact we will check that $\frac{\partial\varphi}{\partial x_1} \equiv 0$. Let $\overline{\gamma}(k_p, x_1)$ be the first eigenvalue of the operator associated to the planar waveguide of index $n(x_1, .)$, $x_1 \in \mathbb{R}$ given. From the Min–Max principle we get for all $\psi \in H^1(\Omega)$

$$\int_\Omega \left(\left| \frac{\partial\psi}{\partial x_2} \right|^2 - k_p^2 n^2 \psi^2 \right) dx \geq \int_\Omega \overline{\gamma}(k_p, x_1)\psi^2\,dx \tag{5.11}$$

and with (5.7) and $\psi = \varphi_p$

$$\begin{aligned} \int_\Omega \left| \frac{\partial\varphi_p}{\partial x_1} \right|^2 dx &\leq \int_\Omega \left[\overline{\gamma}(k_p) - \overline{\gamma}(k_p, x_1)\right] \varphi_p^2\,dx \\ = \int_{\Omega\backslash K} \left[\overline{\gamma}(k_p) - \overline{\gamma}(k_p, x_1)\right] \varphi_p^2\,dx &+ \int_K \left[\overline{\gamma}(k_p) - \overline{\gamma}(k_p, x_1)\right] \varphi_p^2\,dx. \end{aligned} \tag{5.12}$$

Let us check that both quantities in the right-hand side of (5.12) tend to 0. Since $\overline{\gamma}(k_p) \geq -k_p^2 n_+^2$ and $-\overline{\gamma}(k_p, x_1) \leq k_p^2 n_+^2$ we deduce from Proposition 4.1 that $|\overline{\gamma}(k_p) - \overline{\gamma}(k_p, x_1)|$ is bounded in k ; therefore since $\varphi_p \to 0$ in $L^2(\Omega\backslash K)$

$$\int_{\Omega\backslash K} |\overline{\gamma}(k_p) - \overline{\gamma}(k_p, x_1)| \varphi_p^2\,dx \to 0 \quad \text{for } p \to \infty\,.$$

We consider now the second term in the right-hand side of (5.12). The Cauchy-Schwartz inequality gives

$$\int_K |\overline{\gamma}(k_p) - \overline{\gamma}(k_p, x_1)|\, \varphi_p^2\,dx \leq \left(2b \int_{-a}^{a} |\overline{\gamma}(k_p) - \overline{\gamma}(k_p, x_1)|^2\,dx_1 \right)^{1/2} \|\varphi_p\|_{0,4,K}^2.$$

Since the sequence $\{\varphi_p\}_{p\geq 1}$ is bounded in $H^1(K)$, it is also bounded in $L^4(K)$. From Proposition 4.1 we know that for almost all $x_1 \in (-a, a)$

$$\overline{\gamma}(k_p) - \overline{\gamma}(k_p, x_1) \to 0 \quad \text{as } p \to \infty\,.$$

Consequently with the Lebesgue theorem of dominated convergence we get

$$\int_K |\overline{\gamma}(k_p) - \overline{\gamma}(k_p, x_1)| \varphi_p^2\,dx \to 0 \quad \text{as } p \to \infty\,.$$

Finally the estimate (5.12) leads to

$$\frac{\partial\varphi}{\partial x_1} = 0 \quad \text{in } \Omega\,. \tag{5.13}$$

Step 4 . We still need to check (5.5), (5.6). If $\psi \in H^1(\Omega)$ is chosen with a support inside $B_+ \cap \Omega$ then the variational formulation leads to

$$\int_\Omega \nabla\varphi_p \nabla\psi \, dx = (k_p^2 n_+^2 + \lambda_p) \int_\Omega \varphi_p \psi \, dx. \tag{5.14}$$

The sequence $\{k_p^2 n_+^2 + \lambda_p\}_{p\geq 1}$ admits a limit, say μ ; then taking the limit in (5.14) and using (5.13) we get

$$\int_\Omega \frac{\partial\varphi}{\partial x_2} \frac{\partial\psi}{\partial x_2} \, dx = \mu \int_\Omega \varphi\psi \, dx$$

for all ψ with support inside $B_+ \cap \Omega$. This means that (μ, φ) is an eigenpair of the eigenproblem

$$\begin{cases} -\dfrac{d^2\varphi}{dy^2} = \mu\varphi & \text{in } (-\eta, \eta), \\ \varphi(-\eta) = \varphi(\eta) = 0. \end{cases}$$

Since $\lambda_p + k_p^2 n_+^2 \leq \frac{\pi^2}{4\eta^2}$, necessarily $\mu = \frac{\pi^2}{4\eta^2}$ and φ is given by (5.5). ■

Proposition 5.2 *A guide with an index satisfying to* (5.2) *has at most one guided mode at high frequencies.*

PROOF. Let $N^N(k)$ be the number of eigenvalues of the operator A_k^N introduced before. From classical comparison principles, see Proposition 3.2 or [1] for instance, we know that $N(k) \leq N^N(k)$. So it is sufficient to verify that

$$\lim_{k\to\infty} \sup N^N(k) \leq 1.$$

Ab absurdo we assume that there is a sequence $\{k_p\}_{p\geq 1}$ tending to infinity for which the operator $A_{k_p}^N$ has at least two eigenvalues, $\lambda_1^N(p)$ and $\lambda_2^N(p)$, below $\overline{\gamma}(k_p)$. Corresponding normalized eigenvectors are denoted $\varphi_1(p)$ and $\varphi_2(p)$ and satisfy

$$\int_\Omega \varphi_1(p)\varphi_2(p) \, dx = 0 \quad \text{for all } p\,. \tag{5.15}$$

Applying Lemma 5.1 to both sequences $\{\varphi_1(p)\}_{p\geq 1}$, $\{\varphi_2(p)\}_{p\geq 1}$, we deduce for $i = 1, 2$, that $\varphi_i(p) \to \pm\varphi$ in $L^2(\Omega)$ as $p \to \infty$. Finally we have a contradiction with (5.15). ■

Corollary 5.3 *We assume that* n *satisfies to*

$$\overline{n}(\xi) = n_b \quad \textit{if} \quad \xi < -c, \quad \overline{n}(\xi) = n_t \quad \textit{if} \quad \xi > c, \tag{5.16}$$

where the numbers n_b *,* n_t *are such that* $n_+ > n_b > n_t$ *, to* (2.2)–(2.4) *and* (5.2). *If furthermore*

$$\forall\, x \in \mathbb{R}^2 \quad n(x) \geq \overline{n}(x_2), \qquad \text{measure}\Big(\{x \in \mathbb{R}^2; n(x) > \overline{n}(x_2)\}\Big) > 0, \tag{5.17}$$

and if the associated planar waveguide has a guided wave, then there exists $k_0 > 0$ *such that for* $k \geq k_0$

$$N(k) = 1.$$

PROOF. Under the assumptions (5.16), (2.2)–(2.4) and (5.2), we know that $N(k) \leq 1$. Now from Proposition 3.2 in [1] we deduce under our assumptions that $N(k) \geq 1$ for k large enough. ■

Example 5.a We illustrate our results in the case where $B_+ = \mathbb{R} \times (-0.5, 0.5)$ and n is given by, for $x = (x_1, x_2) \in \mathbb{R}^2$,

$$n(x) = \begin{cases} 3.44 & \text{for } x \in B_+, \\ 3.38 & \text{for } x \in (-1,1) \times (0.5,1), \\ 3.17 & \text{for } x_2 < -0.5, \\ 1 & \text{elsewhere.} \end{cases}$$

The assumptions of Corollary 5.3 are satisfied. Therefore for k large this guide has only one guided mode.

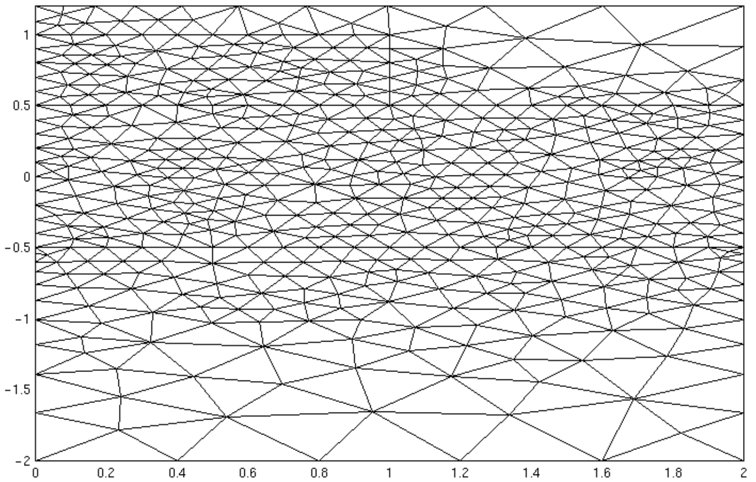

Figure 5: Mesh of the guide used in the computations.

We have computed for different values of k approximations of the fundamental mode $(\lambda_{1h}(k), \varphi_h(k))$ and given in Figure 6 its dispersion curve. In the next figures 7, 8, and 9, the corresponding normalized eigenvector $\varphi_h(k)$; here h represents the discretization parameter. The asymptotic behavior of the first mode can be described. Its energy is more and more confined in the set B_+ as k increases. Since B_+ is invariant by translation in x_1, the mode is not well horizontally confined for large k.

We have done the computations on the mesh presented in Figure 5, made up of 917 triangles and 517 degrees of freedom with a piecewise $\mathcal{P}_1$ approximation on the mesh.

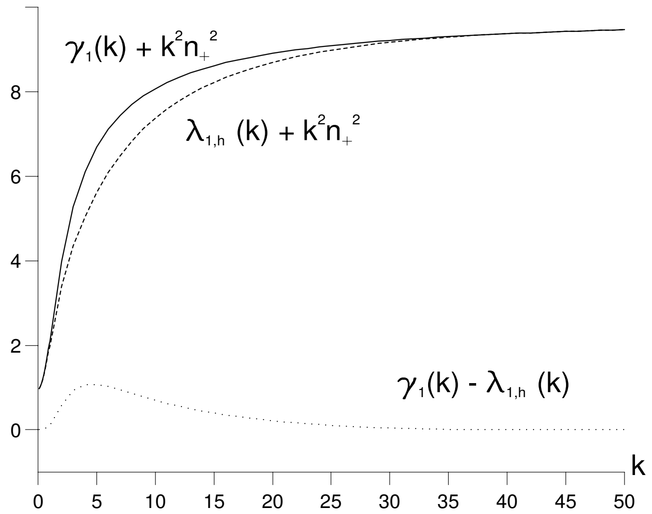

Figure 6: The dispersion curve for the fundamental mode.

For symmetry reason we have restricted the computations in the half plane $x_1 > 0$. As k increases the mode is better vertically confined in the strip B_+. That is why we have computed in an horizontal strip with boundary conditions $u = 0$. To compute the mode less and less laterally confined, that is for k large, the method is much more involved. Here we have used a localized finite element method which consists in using an *exact* representation of the solution on the vertical boundaries limiting the computation domain. We refer to [5] for a presentation of localized finite element methods and to [6] for its application to optical guides.

■

Remark 5.b In fact we can describe precisely what is happening for the eigenvector in the example above. With the same arguments developed in the proof of Lemma 5.1 we can prove the following result.

We assume that the index satisfies to (5.2) and a mode exists for large k. We denote by $\varphi(k)$ an eigenfunction associated to the eigenvalue $\lambda(k)$, normalized to $||\varphi(k)||_{0,(-a,a)\times\mathbb{R}} = 1$. Then the following properties hold

$$\lambda(k) + k^2 n_+^2 \to \frac{\pi^2}{4\eta^2} \quad \text{for } k \text{ tending to } \infty, \tag{5.18}$$

$$\int_{(-a,a)\times\mathbb{R}} (n_+^2 - n^2)\varphi(k)^2 \, dx < \frac{\pi^2}{k^2 4\eta^2}, \tag{5.19}$$

$$\int_{\mathbb{R}^2} \left|\frac{\partial\varphi(k)}{\partial x_1}\right|^2 dx \to 0 \quad \text{for } k \text{ tending to } \infty, \tag{5.20}$$

$$\varphi(k) \to \varphi \quad \text{in } L^2((-a,a)\times\mathbb{R}) \qquad \text{for } k \text{ tending to } \infty, \tag{5.21}$$

where φ has been defined in (5.5). ■

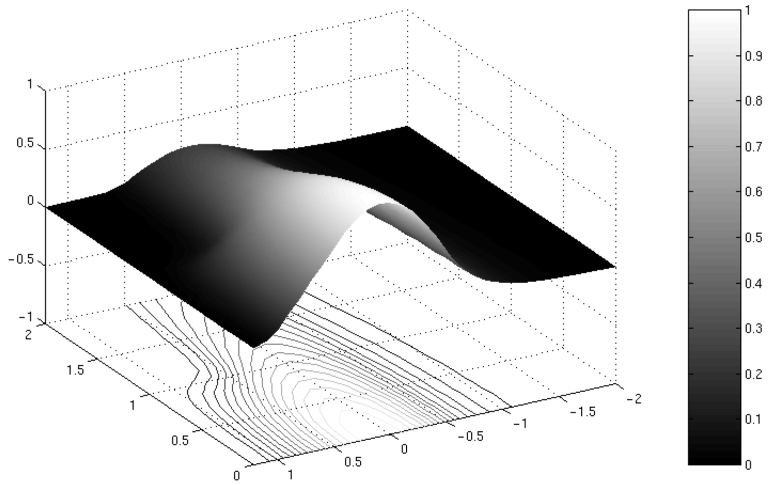

Figure 7: Fundamental mode with $k = 3$; $\lambda_{1,h}(3) + 9n_+^2 = 4.343$.

Finally we are interested in describing a guide with no mode at high frequencies. The idea is to choose a guide for which the set B_+ is strictly included inside the strip $\mathbb{R} \times (-\eta, \eta)$. With our previous developments we can present a less restrictive situation than the one in [1].

Proposition 5.4 *We assume that there exist two non empty open intervals I , J such that*

$$\begin{cases} I \times J \subset \mathbb{R} \times (-\eta, \eta), & \\ n(x) \leq n_* < n_+ & \textit{if } x \in I \times J, \\ B_+ \subset \mathbb{R} \times (-\eta, \eta) \backslash (I \times J), & \\ \zeta(\delta) \geq \zeta_0 > 0 & \forall\, \delta > 0. \end{cases} \tag{5.22}$$

Then there exists $k_* > 0$ such that

$$\textit{for } k > k_* \quad N(k) = 0. \tag{5.23}$$

PROOF. With the comparison principle relative to the index n , it suffices to check (5.23) for the guide with index $\hat{n}$ defined by

$$\hat{n}(x) = \begin{cases} \sup_{z \in I} n(z, x_2) & \text{if } x \in I \times (\mathbb{R} \backslash (-\eta, \eta)), \\ n_+ & \text{if } x \in (\mathbb{R} \times (-\eta, \eta)) \backslash I \times J, \\ n(x) & \text{elsewhere.} \end{cases} \tag{5.24}$$

Ab absurdo we assume there exist two sequences $\{k_p\}_{p \geq 1}$, $k_p > 0$, and $\{\varphi_p\}_{p \geq 1}$, $\varphi_p \in$

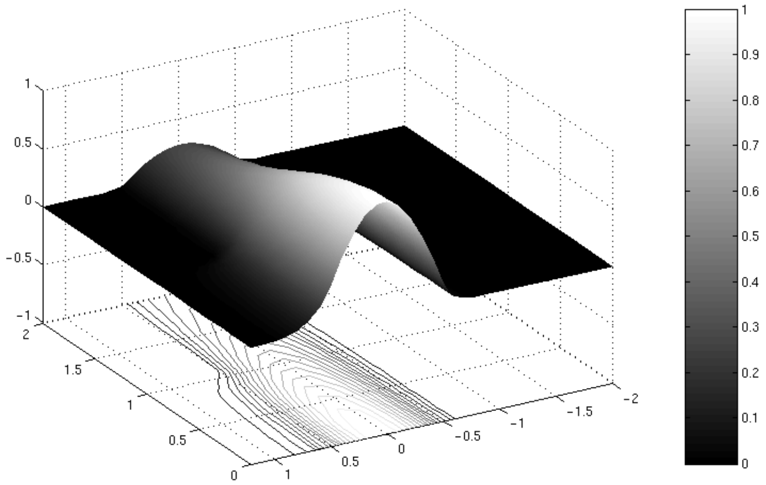

Figure 8: Fundamental mode with $k = 10$; $\lambda_{1,h}(10) + 100n_+^2 = 7.372$.

$H^1(\mathbb{R}^2)$, such that

$$\int_{(-a,a)\times\mathbb{R}} \varphi_p^2\, dx \;\;=\;\; 1, \tag{5.25}$$

$$\int_{\mathbb{R}^2} \left(|\nabla\varphi_p|^2 - k^2\hat{n}^2\varphi_p^2\right)\, dx \;\;<\;\; \overline{\gamma}(k_p) \int_{\mathbb{R}} \varphi_p^2\, dx. \tag{5.26}$$

Following the proof of Lemma 5.1, we can prove that

$$\|\varphi_p\|_{1,(-a,a)\times\mathbb{R}} \leq \left(1 + \frac{\pi^2}{4\eta^2}\right)^{1/2}, \tag{5.27}$$

$$\int_{\mathbb{R}^2} \left(\frac{\partial\varphi_p}{\partial x_1}\right)^2 dx \leq \int_{(-a,a)\times\mathbb{R}} [\overline{\gamma}(k_p) - \overline{\gamma}(k_p, x_1)]\varphi_p^2\, dx, \tag{5.28}$$

where $\overline{\gamma}(k_p, x_1)$ is the first eigenvalue of the operator associated to the planar waveguide of index $\hat{n}(x_1, .)$.

As in the proof of Lemma 5.1, we can prove that

$$\int_{((-a,a)\setminus I)\times\mathbb{R}} [\overline{\gamma}(k_p) - \overline{\gamma}(k_p, x_1)]\varphi_p^2\, dx \to 0 \quad \text{for } p \text{ tending to } \infty\,. \tag{5.29}$$

Using the estimate (5.27) we can extract a subsequence still denoted $\{\varphi_p\}_{p\geq 1}$ such that

$$\begin{array}{lcl} \varphi_p & \rightharpoonup & \varphi \quad \text{weakly in } H^1((-a,a)\times\mathbb{R}) \text{ for } p \to \infty, \\ \varphi_p & \to & \varphi \quad \text{in } L^2(K) \text{ for } p \to \infty. \end{array}$$

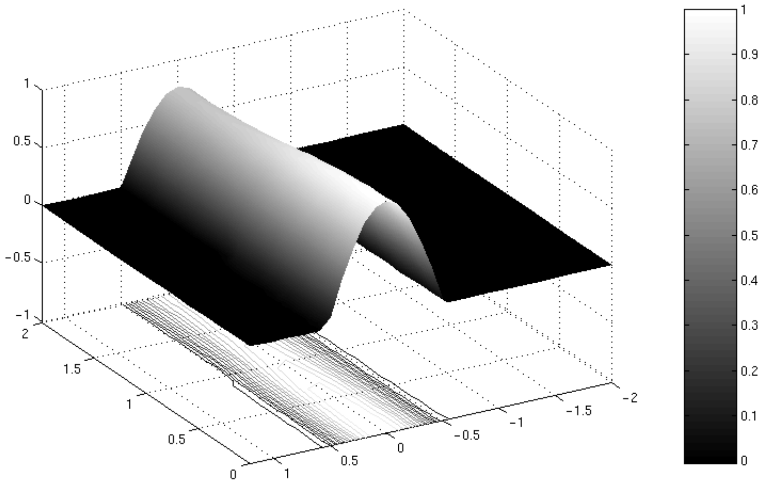

Figure 9: Fundamental mode with $k = 40$; $\lambda_{1,h}(40) + 1600n_+^2 = 9.369$.

Since $\varphi_p \to 0$ in $((-a,a) \times \mathbb{R})\backslash K$, (similarly to (5.10)), we deduce that

$$\varphi_p \to \varphi \quad \text{in } L^2((-a,a) \times \mathbb{R}) \text{ for } p \to \infty\,. \tag{5.30}$$

For $x_1 \in I$, $\overline{\gamma}(k_p, x_1) = \tilde{\gamma}(k_p)$, with

$$\tilde{\gamma}(k_p) + k^2 n_+^2 \to \frac{\pi^2}{4\eta_*^2}, \quad \eta_* < \eta. \tag{5.31}$$

Then with (5.29), (5.30), (5.31), we deduce

$$\int_{(-a,a)\times\mathbb{R}} [\overline{\gamma}(k_p) - \overline{\gamma}(k_p, x_1)]\varphi_p^2\,dx \to \left(\frac{\pi^2}{4\eta^2} - \frac{\pi^2}{4\eta_*^2}\right)\int_{I\times\mathbb{R}} \varphi^2\,dx$$

and with (5.28)

$$\varphi = 0 \quad \text{in } I \times \mathbb{R} \qquad \text{and} \qquad \frac{\partial\varphi}{\partial x_1} = 0 \quad \text{in } \mathbb{R}^2\,.$$

So with (5.30) we get $\varphi = 0$ in $L^2((-a,a) \times \mathbb{R})$ and we get a contradiction with (5.25).

■

Remark 5.c Let n be the index defined in Figure 10.

We assume that $n_+ > n_* > n_b$ and for simplicity that $b - \eta = 2a$. The first eigenvalue μ for $-\Delta$ in the square $(-a,a) \times (\eta, \eta + 2a)$ with homogeneous Dirichlet boundary conditions is $\mu = \frac{\pi^2}{2a^2}$. If $\varphi \not\equiv 0$ is a corresponding eigenvector, we extend it by 0 and

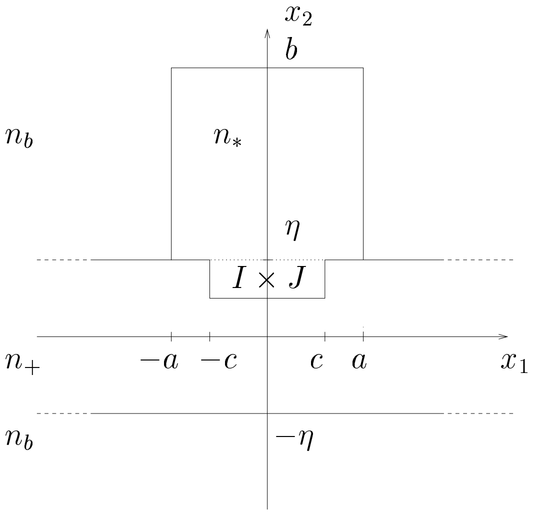

Figure 10: Guide with no mode at high frequencies

get $\tilde{\varphi} \in H^1(\mathbb{R}^2)$ satisfying

$$\frac{\int_{\mathbb{R}^2} \left(|\nabla \tilde{\varphi}|^2 - k^2 n^2 \tilde{\varphi}^2\right)\, dx}{\int_{\mathbb{R}^2} \tilde{\varphi}^2 \, dx} = \frac{\pi^2}{2a^2} - k^2 n_*^2.$$

For all k, $-k^2 n_+^2 < \overline{\gamma}(k) < -k^2 n_b^2$. For a given k, we could choose n_* and a big enough to have

$$-k^2 n_*^2 + \frac{\pi^2}{2a^2} < \overline{\gamma}(k),$$

with $n_+ > n_* > n_b$. From the comparison principle we deduce $\lambda_1(k) < \overline{\gamma}(k)$. So for that particular value of k we have a guided mode at least. For large k, we will have no guided modes. ■

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