Abstract. — The Cremona group of birational transformations of \( \mathbb{P}^2 \mathbb{C} \) acts on the space \( \mathcal{F}(2) \) of holomorphic foliations on the complex projective plane. Since this action is not compatible with the natural graduation of \( \mathcal{F}(2) \) by the degree, its description is complicated. The fixed points of the action are essentially described by Cantat-Favre in [3]. In that paper we are interested in problems of "aberration of the degree" that is pairs \((\phi, \mathcal{F}) \in \text{Bir}(\mathbb{P}^2 \mathbb{C}) \times \mathcal{F}(2)\) for which \(\deg \phi^* \mathcal{F} < (\deg \mathcal{F} + 1)\deg \phi + \deg \phi - 2\), the generic degree of such pull-back. We introduce the notion of numerical invariance (\(\deg \phi^* \mathcal{F} = \deg \mathcal{F}\)) and relate it in small degrees to the existence of transversal structure for the considered foliations.

1. Introduction

Let us consider on the complex projective plane \( \mathbb{P}^2 \mathbb{C} \) a foliation \( \mathcal{F} \) of degree \( d \) and a birational map \( \phi \) of degree \( k \geq 2 \). If the pair \((\mathcal{F}, \phi)\) is generic then \(\deg \phi^* \mathcal{F} = (d + 1)k + k - 2\). For example if \( \mathcal{F} \) and \( \phi \) are both of degree 2, then \(\phi^* \mathcal{F} \) is of degree 6. Nevertheless one has the following statement which says that "aberration of the degree" is not exceptional:

**Theorem A.** — For any foliation \( \mathcal{F} \) of degree 2 on \( \mathbb{P}^2 \mathbb{C} \) there exists a quadratic birational map \( \psi \) of \( \mathbb{P}^2 \mathbb{C} \) such that \(\deg \psi^* \mathcal{F} \leq 3\).

Holomorphic singular foliations on compact complex projective surfaces have been classified up to birational equivalence by Brunella, McQuillan and Mendes ([1]). Let \( \mathcal{F} \) be a holomorphic singular foliation on a compact complex projective surface \( S \). Let \( \text{Bir}(\mathcal{F}) \) (resp. \( \text{Aut}(\mathcal{F}) \)) denote the group of birational (resp. biholomorphic) maps of \( S \) that send leaf to leaf. If \( \mathcal{F} \) is of general type, then \( \text{Bir}(\mathcal{F}) = \text{Aut}(\mathcal{F}) \) is a finite group. In [3] Cantat and Favre classify the pairs \((S, \mathcal{F})\) for which \( \text{Bir}(\mathcal{F}) \) (resp. \( \text{Aut}(\mathcal{F}) \)) is infinite; in the case of \( \mathbb{P}^2 \mathbb{C} \) such foliations are given by closed rational 1-forms.

In this article we introduce a weaker notion: the numerical invariance. We consider on \( \mathbb{P}^2 \mathbb{C} \) a pair \((\mathcal{F}, \phi)\) of a foliation \( \mathcal{F} \) of degree \( d \) and a birational map \( \phi \) of degree \( k \geq 2 \). The foliation \( \mathcal{F} \) is **numerically invariant** under the action of \( \phi \) if \(\deg \phi^* \mathcal{F} = \deg \mathcal{F}\). We characterize such pairs \((\mathcal{F}, \phi)\) with \(\deg \mathcal{F} = \deg \phi = 2\) which

Second author supported by the Swiss National Science Foundation grant no PP00P2_128422 /1 and by ANR Grant "BirPol" ANR-11-JS01-004-01.
is the first degree with deep (algebraic and dynamical) phenomena, both for foliations and birational maps. We prove that a numerically invariant foliation under the action of a generic quadratic map is special:

**Theorem B.** — Let \( \mathcal{F} \) be a foliation of degree 2 on \( \mathbb{P}^2 \)C numerically invariant under the action of a generic quadratic birational map of \( \mathbb{P}^2 \)C. Then \( \mathcal{F} \) is transversely projective.

In that statement generic means outside an hypersurface in the space \( \overset{\circ}{\text{Bir}}_2 \) of quadratic birational maps of \( \mathbb{P}^2 \)C.

For any quadratic birational map \( \phi \) of \( \mathbb{P}^2 \) there exists at least one foliation of degree 2 on \( \mathbb{P}^2 \)C numerically invariant under the action of \( \phi \) and we give "normal forms" for such foliations. We don’t know if the foliations numerically invariant under the action of a non-generic quadratic birational map have a special transversal structure. Problem: for any birational map \( \phi \) of degree \( d \geq 3 \), does there exist a foliation numerically invariant under the action of \( \phi \)?

A foliation \( \mathcal{F} \) on \( \mathbb{P}^2 \)C is **primitive** if \( \deg \mathcal{F} \leq \deg \phi^* \mathcal{F} \) for any birational map \( \phi \). Foliations of degree 0 and 1 are defined by a rational closed 1-form (it is a well-known fact, see for example [2]). Hence a non-primitive foliation of degree 2 is also defined by a closed 1-form that is a very special case of transversely projective foliations. Generically a foliation of degree 2 is primitive. The following problem seems relevant: classify in any degree the primitive foliations numerically invariant under the action of birational maps of degree \( \geq 2 \); are such foliations transversely projective or is this situation specific to the degree 2 ? In this vein we get the following statement.

**Theorem C.** — A foliation \( \mathcal{F} \) of degree 2 on \( \mathbb{P}^2 \)C numerically invariant under the action of a generic cubic birational map of \( \mathbb{P}^2 \)C satisfies the following properties:

- \( \mathcal{F} \) is given by a closed rational 1-form (Liouvillian integrability);
- \( \mathcal{F} \) is non-primitive.

Is it a general fact, i.e. if \( \mathcal{F} \) is numerically invariant under the action of \( \phi \) and \( \deg \phi \gg \deg \mathcal{F} \) is \( \mathcal{F} \) Liouvillian integrable ?

The text is organized as follows: we first give some definitions, notations and properties of birational maps of \( \mathbb{P}^2 \)C and foliations on \( \mathbb{P}^2 \)C. In §3 we give a proof of Theorem A; we focus on foliations of degree 2 on \( \mathbb{P}^2 \)C that have at least two singular points and then on foliations of degree 2 on \( \mathbb{P}^2 \)C with exactly one singular point. The section 4 is devoted to the description of foliations of degree 2 numerically invariant under the action of any quadratic birational map. This allows us to prove Theorem B. At the end of the paper, §5 we describe the foliations of degree 2 numerically invariant under some cubic birational maps of \( \mathbb{P}^2 \)C and establish Theorem C.

**Acknowledgment.** — We thank Alcides Lins Neto for helpful discussions.

### 2. Some definitions, notations and properties

#### 2.1. About birational maps of \( \mathbb{P}^2 \)C.

A **rational map** \( \phi \) of \( \mathbb{P}^2 \)C is a "map" of the type

\[
\phi: \mathbb{P}^2 \longrightarrow \mathbb{P}^2, \quad (x : y : z) \longrightarrow (\phi_0(x,y,z) : \phi_1(x,y,z) : \phi_2(x,y,z))
\]

where the \( \phi_i \)'s are homogeneous polynomials of the same degree and without common factor. The **degree** of \( \phi \) is by definition the degree of the \( \phi_i \)'s. A **birational map** \( \phi \) of \( \mathbb{P}^2 \)C is a rational map having a rational
birationally maps of $\text{Bir}_d \subseteq \text{Aut}(\mathbb{P}^2_C)$ and called Cremona group. If $\phi$ is an element of $\text{Bir}(\mathbb{P}^2_C)$ denoted $\text{Bir}(\mathbb{P}^2_C)$ then $\mathcal{O}(\phi)$ is the orbit of $\phi$ under the action of $\text{Aut}(\mathbb{P}^2_C) \times \text{Aut}(\mathbb{P}^2_C)$:

$$\mathcal{O}(\phi) = \{ \ell \phi \ell' \mid \ell, \ell' \in \text{Aut}(\mathbb{P}^2_C) \}.$$

A very old theorem, often called Nœther Theorem, says that any element of $\text{Bir}(\mathbb{P}^2_C)$ can be written, up to the action of an automorphism of $\mathbb{P}^2_C$, as a composition of quadratic birational maps (\cite{4}). In \cite{5} Chapters 1 & 6] the structure of the set $\text{Bir}_d$ (resp. $\text{Bir}_d$) of birational maps of $\mathbb{P}^2_C$ of degree $d$ (resp. $d$) has been studied when $d = 2$ and $d = 3$.

**Theorem 2.1 (Corollary 1.10, Theorem 1.31, \cite{5}).** — One has the following decomposition

$$\text{Bir}_2 = \mathcal{O}(\sigma) \cup \mathcal{O}(\rho) \cup \mathcal{O}(\tau).$$

Furthermore

$$\text{Bir}_2 = \overline{\mathcal{O}(\sigma)},$$

where $\overline{\mathcal{O}(\sigma)}$ denotes the ordinary closure of $\mathcal{O}(\sigma)$, and

$$\dim \mathcal{O}(\tau) = 12, \quad \dim \mathcal{O}(\rho) = 13, \quad \dim \mathcal{O}(\sigma) = 14.$$

Note that there is a more precise description of $\text{Bir}_3$ in \cite{5} Chapter 1].

We will further do some computations with birational maps of degree 3. Let us consider the following family of cubic birational maps:

$$\Phi_{a,b} : (x : y : z) \rightarrow (x(x^2 + y^2 + axy + bxz + yz) : y(x^2 + y^2 + axy + bxz + yz) : xyz)$$

with $a, b \in \mathbb{C}, a^2 \neq 4$ and $2b \notin \{a \pm \sqrt{a^2 - 4}\}$. The structure of the set of cubic birational maps is much more complicated (\cite{5} Chapter 6]), nevertheless one has the following result.

**Theorem 2.2 (Proposition 6.35, Theorem 6.38, \cite{5}).** — The closure of

$$\mathcal{X} = \{ \mathcal{O}(\Phi_{a,b}) \mid a, b \in \mathbb{C}, a^2 \neq 4, 2b \notin \{a \pm \sqrt{a^2 - 4}\} \}$$

in the set of rational maps of degree 3 is an irreducible algebraic variety of dimension 18. Furthermore the closure of $\mathcal{X}$ in $\text{Bir}_3$ is $\text{Bir}_3$.

The "most degenerate model"\cite{11} is up to automorphisms of $\mathbb{P}^2_C$

$$\Psi : (x : y : z) \rightarrow (xz^2 + y^3 : yz^2 : z^3).$$

1. In the following sense: for any $\phi$ in $\text{Bir}_3$ the following inequality holds: $\dim \mathcal{O}(\phi) \geq \dim \mathcal{O}(\Psi) = 13$. 

2.2. About foliations. —

Definition 2.3. — Let \( \mathcal{F} \) be a foliation (maybe singular) on a complex manifold \( M \); the foliation \( \mathcal{F} \) is a **singular transversely projective** one if there exists

a) \( \pi: P \to M \) a \( \mathbb{P}^1 \)-bundle over \( M \),

b) \( \mathcal{G} \) a codimension one singular holomorphic foliation on \( P \) transversal to the generic fibers of \( \pi \),

c) \( \zeta: M \to P \) a meromorphic section generically transversal to \( \mathcal{G} \),
such that \( \mathcal{F} = \zeta^* \mathcal{G} \).

Let \( \mathcal{F} \) be a foliation on \( \mathbb{P}^2_\mathbb{C} \); assume that there exist three rational 1-forms \( \theta_0, \theta_1 \) and \( \theta_2 \) on \( \mathbb{P}^2_\mathbb{C} \) such that

i) \( \mathcal{F} \) is described by \( \theta_0 \), i.e. \( \mathcal{F} = \mathcal{F}_{\theta_0} \),

ii) the \( \theta_i \)'s form a \( \text{sl}(2; \mathbb{C}) \)-triplet, that is

\[
\begin{align*}
d\theta_0 &= \theta_0 \wedge \theta_1, \\
d\theta_1 &= \theta_0 \wedge \theta_2, \\
d\theta_2 &= \theta_1 \wedge \theta_2.
\end{align*}
\]

Then \( \mathcal{F} \) is a singular transversely projective foliation. To see it one considers the manifolds \( M = \mathbb{P}^2_\mathbb{C}, P = \mathbb{P}^2_\mathbb{C} \times \mathbb{P}^1_\mathbb{C} \), the canonical projection \( \pi: P \to M \) and the foliation \( \mathcal{G} \) given by

\[
\theta = dz + \theta_0 + z\theta_1 + c^2 \theta_2,
\]

where \( z \) is an affine coordinate of \( \mathbb{P}^1_\mathbb{C} \); in that case the transverse section is \( z = 0 \). When one can choose the \( \theta_i \)'s such that \( \theta_1 = \theta_2 = 0 \) (resp. \( \theta_2 = 0 \)) the foliation \( \mathcal{F} \) is **defined by a closed 1-form** (resp. is **transversely affine**).

Classical examples of singular transversely projective foliations are given by Riccati foliations.

Definition 2.4. — A **Riccati equation** is a differential equation of the following type

\[
\mathcal{E}_R: y' = a(x)y^2 + b(x)y + c(x)
\]

where \( a, b \) and \( c \) are meromorphic functions on an open subset \( \mathcal{U} \) of \( \mathbb{C} \). To the equation \( \mathcal{E}_R \) one associates the meromorphic differential form

\[
\omega_{\mathcal{E}_R} = dy - (a(x)y^2 + b(x)y + c(x)) \, dx
\]

defined on \( \mathcal{U} \times \mathbb{C} \). In fact \( \omega_{\mathcal{E}_R} \) induces a foliation \( \mathcal{F}_{\omega_{\mathcal{E}_R}} \) on \( \mathcal{U} \times \mathbb{P}^1_\mathbb{C} \) that is transverse to the generic fiber of the projection \( \mathcal{U} \times \mathbb{P}^1_\mathbb{C} \to \mathcal{U} \). One can check that

\[
\begin{align*}
\theta_0 &= \omega_{\mathcal{E}_R}, \\
\theta_1 &= -2a(x) \, dx, \\
\theta_2 &= -2a(x) \, dx
\end{align*}
\]

is a \( \text{sl}(2; \mathbb{C}) \)-triplet associated to the foliation \( \mathcal{F}_{\omega_{\mathcal{E}_R}} \).

We say that \( \omega_{\mathcal{E}_R} \) is a **Riccati 1-form** and \( \mathcal{F}_{\omega_{\mathcal{E}_R}} \) is a **Riccati foliation**.

Let \( S \) be a ruled surface, that is a surface \( S \) endowed with \( f: S \to \mathcal{C} \), where \( \mathcal{C} \) denotes a curve and \( f^{-1}(c) \simeq \mathbb{P}^1_\mathbb{C} \). Let us consider a singular foliation \( \mathcal{F} \) on \( S \) transverse to the generic fibers of \( f \). The foliation \( \mathcal{F} \) is transversely projective.

Recall that a foliation \( \mathcal{F} \) is **radial** at a point \( m \) of the surface \( M \) if in local coordinates \((x, y)\) around \( m \) the foliation \( \mathcal{F} \) is given by a holomorphic 1-form of the following type

\[
\omega = xy \, dx - y \, dx + \text{h.o.t.}
\]

Let us denote by \( \mathbb{P}(n;d) \) the set of foliations of degree \( d \) on \( \mathbb{P}^n_\mathbb{C} \) (see [2]). The following statement gives a criterion which asserts that an element of \( \mathbb{P}(2; 2) \) is transversely projective.
Proposition 2.5. — Let $\mathcal{F} \in \mathcal{F}(2;2)$ be a foliation of degree 2 on $\mathbb{P}_C^2$. If a singular point of $\mathcal{F}$ is radial, then $\mathcal{F}$ is transversely projective.

Proof. — Assume that the singular point is the origin $0$ in the affine chart $z = 1$, the foliation $\mathcal{F}$ is thus defined by a 1-form of the following type

$$\omega = x \, dy - y \, dx + q_1 \, dx + q_2 \, dy + q_3 (x \, dy - y \, dx)$$

where the $q_i$'s denote quadratic forms. Let us consider the complex projective plane $\mathbb{P}_C^2$ blown up at the origin; this space is denoted by $\text{Bl}(\mathbb{P}_C^2, 0)$. Let $\pi: \text{Bl}(\mathbb{P}_C^2, 0) \to \mathbb{P}_C^2$ be the canonical projection. Then $\pi^* \mathcal{F}$ is transverse to the generic fibers of $\pi$, and in fact transverse to all the fibers excepted the strict transforms of the lines $xq_1 + yq_2 = 0$. Hence the foliation $\pi^* \mathcal{F}$ is transversely projective; since this notion is invariant under the action of a birational map, $\mathcal{F}$ is transversely projective.

Remark 2.6. — The same argument can be involved for foliations of degree 2 on $\mathbb{P}_C^2$ having a singular point with zero 1-jet.

Remark 2.7. — The closure of the set $\Delta_R$ of foliations in $\mathcal{F}(2;2)$ having a radial singular point is irreducible, of codimension 2 in $\mathcal{F}(2;2)$.

3. Proof of Theorem

We establish Theorem in two steps: we first look at foliations that have at least two singular points and then at foliations with exactly one singular point.

3.1. Foliations of degree 2 on $\mathbb{P}_C^2$ with at least two singularities. —

Proposition 3.1. — For any $\mathcal{F} \in \mathcal{F}(2;2)$ with at least two distinct singularities there exists a quadratic birational map $\psi \in \mathcal{O}(\rho)$ such that $\deg \psi^* \mathcal{F} \leq 3$.

Proof. — In homogeneous coordinates $\mathcal{F}$ is described by a 1-form

$$\omega = q_1yz \left( \frac{dy}{y} - \frac{dz}{z} \right) + q_2xz \left( \frac{dz}{z} - \frac{dx}{x} \right) + q_3xy \left( \frac{dx}{x} - \frac{dy}{y} \right)$$

where

$$q_1 = a_0 x^2 + a_1 y z + a_2 z^2 + a_3 x y + a_4 x z + a_5 y z, \quad q_2 = b_0 x^2 + b_1 y^2 + b_2 z^2 + b_3 x y + b_4 x z + b_5 y z,$$

$$q_3 = c_0 x^2 + c_1 y z + c_2 z^2 + c_3 x y + c_4 x z + c_5 y z.$$

Up to an automorphism of $\mathbb{P}_C^2$ one can suppose that $(1:0:0)$ and $(0:1:0)$ are singular points of $\mathcal{F}$, that is $a_1 = b_0 = c_0 = c_1 = 0$. If $c_3 \neq 0$, resp. $c_3 = 0$ and $b_4 \neq 0$, resp. $c_3 = b_4 = 0$, then let us consider the quadratic birational map $\psi$ of $\mathcal{O}(\rho)$ defined as follows

$$\psi: (x : y : z) \mapsto \left( xy : z^2 + \frac{b_3 - c_4 + \sqrt{(b_3 - c_4)^2 + 4b_4 c_3}}{2c_3} y z : y z \right),$$

resp.

$$\psi: (x : y : z) \mapsto \left( xy : z^2 + y z : \frac{b_3 - c_4}{b_4} y z \right),$$

resp. $\psi = \rho$. A direct computation shows that $\psi^* \omega = \omega' \omega'$ where $\omega'$ denotes a homogeneous 1-form of degree 4. The foliation $\mathcal{F}'$ defined by $\omega'$ has degree at most 3. □
3.2. Foliations of degree 2 on $\mathbb{P}^2_C$ with exactly one singularity. — Such foliations have been classified:

**Theorem 3.2 ([6]).** — Up to automorphisms of $\mathbb{P}^2_C$ there are four foliations of degree 2 on $\mathbb{P}^2_C$ having exactly one singularity. They are described in affine chart by the following 1-forms:

- $\Omega_1 = x^2 \, dx + y^2 (x \, dy - y \, dx)$,
- $\Omega_2 = x^2 \, dx + (x + y^2) (x \, dy - y \, dx)$,
- $\Omega_3 = xy \, dx + (x^2 + y^2) (x \, dy - y \, dx)$,
- $\Omega_4 = (x + y^2 - x^2 y) \, dy + x(x + y^2) \, dx$.

**Proposition 3.3.** — There exists a quadratic birational map $\psi_1 \in \mathcal{O}(\mathbb{P})$ such that $\deg \psi_1^* \mathcal{O}_{\Omega_1} = 2$; furthermore $\mathcal{O}_{\Omega_1}$ has a rational first integral and is non-primitive.

For $k = 2, 3$, there is no birational map $\phi_k$ such that $\deg \phi_k^* \mathcal{O}_{\Omega_k} = 0$ but there is a $\psi_k \in \mathcal{O}(\tau)$ such that $\deg \psi_k^* \mathcal{O}_{\Omega_k} = 1$. In particular $\mathcal{O}_{\Omega_2}$ and $\mathcal{O}_{\Omega_3}$ are non-primitive.

**Remark 3.4.** — If $\phi = (x^2 : xy : xz + y^2)$, then $\deg \phi^* \mathcal{O}_{\Omega_2} = \deg \phi^* \mathcal{O}_{\Omega_3} = 2$. A contrario we will see later there is no quadratic birational map $\phi$ such that $\deg \phi^* \mathcal{O}_{\Omega_4} = 2$ (see Corollary 4.15).

**Corollary 3.5.** — For any element $\mathcal{O}$ of $\mathcal{O}(2; 2)$ with exactly one singularity there exists a quadratic birational map $\psi$ such that $\deg \psi^* \mathcal{O} \leq 3$.

**Proof of Proposition 3.3** — The foliation $\mathcal{O}_{\Omega_1}$ is given in homogeneous coordinates by

$$\Omega_1' = (x^2 - y^3) \, dx + xy^2 \, dy - x^3 \, dz;$$

if $\psi_1: (x : y : z) \to (x^2 : xy : yz)$ then

$$\psi_1^* \Omega_1' \wedge (y(2xz - y^2) \, dx + x(y^2 - xz) \, dy - x^2 y \, dz) = 0.$$  

The foliation $\mathcal{O}_{\Omega_1}$ has a rational first integral and is non-primitive, it is the image of a foliation of degree 0 by a cubic birational map:

$$(x^3 : x^2 y : x^2 z + y^3 / 3) \wedge (z \, dx - x \, dz) = 0.$$  

The foliation $\mathcal{O}_{\Omega_2}$ is described in homogeneous coordinates by

$$\Omega_2' = (x^2 z - xy z - y^3) \, dx + x(xz + y^2) \, dy - x^3 \, dz;$$

let us consider the birational map $\psi_2: (x : y : z) \to (x^2 : xy : xz - 2x^2 - 2xy - y^2)$ then

$$\psi_2^* \Omega_2' \wedge ((xz - yz) \, dx + xz \, dy - x^2 \, dz) = 0.$$  

One can verify that

$$\left( 2 + \frac{1}{x} + 2 \frac{y}{x} + \frac{y^3}{x^2} \right) \exp \left( -\frac{y}{x} \right)$$

is a first integral of $\mathcal{O}_{\Omega_2}$; it is easy to see that $\mathcal{O}_{\Omega_2}$ has no rational first integral so there is no birational map $\phi_2$ such that $\deg \phi_2^* \mathcal{O}_{\Omega_2} = 0$.

The foliation $\mathcal{O}_{\Omega_3}$ is given in homogeneous coordinates by the 1-form

$$\Omega_3 = y(xz - x^2 - y^2) \, dx + x(x^2 + y^2) \, dy - x^2 y \, dz;$$

if $\psi_3: (x : y : z) \to (x^2 : xy : xz + y^2 / 2)$ then

$$\psi_3^* \Omega_3 \wedge (y(z - x) \, dx + x^2 \, dy - xy \, dz) = 0.$$
The function
\[ \left( \frac{y}{x} \right) \exp \left( \frac{1}{2} \frac{y^2}{x^2} - \frac{1}{x} \right) \]
is a first integral of \( F_\Omega \), and \( F_{\Omega_3} \) has no rational first integral so there is no birational map \( \phi_3 \) such that \( \deg \phi_3^* F_{\Omega_3} = 0 \).

Let us consider the birational map of \( \mathbb{P}^2 \) given by
\[ \psi_4 : (x : y : z) \mapsto (-x^2 : xy^2 - xz) \]
In homogeneous coordinates \( \Omega_4 = x(xz + y^2)dx + (xyz + y^2z - x^2y)dy + (xyz - y^3 - x^3)dz \); a direct computation shows that
\[ \psi_4^* \Omega_4 \wedge \left( (3y^3z - x^2y^2 + x^3z - 2xyz^2)dx + (x^3y - 4y^4 - x^2z^2 + 3xyz^2)dy + x(2y^3 - x^3 - yxz)dz \right) = 0. \]
The foliation \( F_{\Omega_4} \) has no invariant algebraic curve so \( F_{\Omega_4} \) is not transversely projective ([6, Proposition 1.3]). In fact a foliation of degree 2 without invariant algebraic curve is primitive; as a consequence \( F_{\Omega_4} \) is a primitive foliation.

\[ \square \]

4. Numerical invariance

In the sequel num. inv. means numerically invariant.

In this section we determine the foliations \( \mathcal{F} \) of \( \mathbb{F}(2; 2) \) num. inv. under the action of \( \sigma \) (resp. \( \rho \), resp. \( \tau \)). Note that if \( \phi \) is a birational map of \( \mathbb{P}^2 \) and \( \ell \) an element of \( \text{Aut}(\mathbb{P}^2) \) then \( \deg(\phi \ell)^* \mathcal{F} = \deg \phi^* \mathcal{F} \); hence following Theorem [2.1] we get the description of foliations num. inv. under the action of a quadratic birational map of \( \mathbb{P}^2 \).

**Lemma 4.1.** — An element \( \mathcal{F} \) of \( \mathbb{F}(2; 2) \) is num. inv. under the action of \( \sigma \) if and only if it is given up to permutations of coordinates and standard affine charts by 1-forms of the following type

- either \( \omega_1 = y(x + ey)dx + (\beta x + \delta y + \alpha x^2 + \gamma xy)dy \),
- or \( \omega_2 = (\delta + \beta y + \kappa y^2)dx + (\alpha + \varepsilon x + \gamma x^2)dy \),

where \( \alpha, \beta, \gamma, \delta, \varepsilon, \kappa \) (resp. \( \alpha, \beta, \gamma, \delta, \varepsilon, \kappa \)) are complex numbers such that \( \deg F_{\omega_1} = 2 \) (resp. \( \deg F_{\omega_2} = 2 \)).

**Proof.** — The foliation \( \mathcal{F} \) is defined by a homogeneous 1-form \( \omega \) of degree 3. The map \( \sigma \) is an automorphism of \( \mathbb{P}^2 \setminus \{xyz = 0\} \) so if \( \sigma^* \omega = P \omega' \), with \( \omega' \) a 1-form of degree 3 and \( P \) a homogeneous polynomial then \( P = x^iy^jz^k \) for some integers \( i, j, k \) such that \( i + j + k = 4 \). Up to permutation of coordinates it is sufficient to look at the four following cases: \( P = x^4, P = x^3y, P = x^3y^2 \) and \( P = x^2yz \). Let us write \( \omega \) as follows
\[ \omega = q_1y^2z \left( \frac{dy}{y} - \frac{dz}{z} \right) + q_2xz \left( \frac{dz}{z} - \frac{dx}{x} \right) + q_3xy \left( \frac{dx}{x} - \frac{dy}{y} \right) \]
where
\[ q_1 = a_0x^2 + a_1y^2 + a_2z^2 + a_3xy + a_4xz + a_5yz, \quad q_2 = b_0x^2 + b_1y^2 + b_2z^2 + b_3xy + b_4xz + b_5yz, \quad q_3 = c_0x^2 + c_1y^2 + c_2z^2 + c_3xz + c_4xz + c_5yz. \]

Computation show that \( x^4 \) (resp. \( x^3y \)) cannot divide \( \sigma^* \omega \). If \( P = x^3yz \) then \( \sigma^* \omega = P \omega' \) if and only if
\[ c_0 = 0, \quad b_0 = 0, \quad a_2 = 0, \quad b_2 = 0, \quad a_1 = 0, \quad c_4 = 0, \quad b_4 = 0, \quad c_3 = 0, \quad b_3 = c_4, \quad c_5 = a_3; \]
\[ a_1 = 0, \quad c_0 = 0, \quad b_0 = 0, \quad a_2 = 0, \quad b_2 = 0, \quad a_1 = 0, \quad b_1 = 0, \quad c_4 = 0, \quad b_4 = 0, \quad c_3 = 0, \quad b_3 = c_4, \quad c_5 = a_3; \]
in that case we obtain $\omega_2$.

**Proposition 4.2.** — A foliation $\mathcal{F} \in \mathbb{F}(2; 2)$ num. inv. under the action of an element of $\mathcal{O}(\mathcal{F})$ is $\text{Aut}(\mathbb{P}^2_C)$-conjugate either to a foliation of type $\mathcal{F}_{0_1}$, or to a foliation of type $\mathcal{F}_{0_2}$; in particular it is transversely projective.

**Proof.** — Let $\phi$ be an element of $\mathcal{O}(\mathcal{F})$ such that $\text{deg} \phi^* \mathcal{F} = 2$; the map $\phi$ can be written $\ell_1 \sigma \ell_2$ where $\ell_1$ and $\ell_2$ denote automorphisms of $\mathbb{P}^2_C$. By assumption the degree of $(\ell_1 \sigma \ell_2)^* \mathcal{F} = \ell_1^* (\sigma^*(\ell_1^* \mathcal{F}))$ is 2. Hence $\text{deg} \sigma^*(\ell_1^* \mathcal{F}) = 2$ and the foliation $\ell_1^* \mathcal{F}$ is num. inv. under the action of $\sigma$. Since $\ell_1$ and $\mathcal{F}$ are conjugate and since the notion of transversal projectivity is invariant by conjugacy it is sufficient to establish the statement for $\phi = \sigma$. The proposition thus follows from the fact that 1-forms of Lemma 4.1 are Riccati ones (up to multiplication).

**Remark 4.3.** — For generic values of parameters $\alpha, \beta, \gamma, \epsilon, \kappa$ a foliation of type $\mathcal{F}_{0_1}$ given by the corresponding form $\omega_1$ is not given by a closed meromorphic 1-form. This can be seen by studying the holonomy group of $\mathcal{F}_{0_1}$ that can be identified with a subgroup of $\text{PGL}(2; \mathbb{C})$ generated by two elements $f$ and $g$. For generic values of the parameters $f$ and $g$ are also generic, in particular the group $\langle f, g \rangle$ is free. When $\mathcal{F}_{0_1}$ is given by a closed 1-form, then the holonomy group is an abelian one.

Remark that a contrario the foliations given by 1-forms of type $\omega$ either by $y \text{d}x + (\alpha + \epsilon x + \alpha^2) \text{d}y$, or by $y(1 - y) \text{d}x + (\beta x - \alpha y + \alpha^2 - \beta xy) \text{d}y$, or by $y(1 + y) \text{d}x + (\beta x + \alpha y + \alpha^2 + \beta xy) \text{d}y$, are conjugate either to a foliation of type $\mathcal{F}_{0_1}$, or to a foliation of type $\mathcal{F}_{0_2}$; in particular it is transversely projective.

**Remark 4.4.** — Let $\Delta_1$ denote the closure of the set of elements of $\mathbb{F}(2; 2)$ conjugate to a foliation of type $\mathcal{F}_{0_1}$. The following inclusion holds: $\Delta_2 \subset \Delta_1$.

Note also that $\Delta_1$ is contained in $\Delta_R$ (see Remark 2.7).

**Remark 4.5.** — The notion of num. inv. is not related to the dynamic of the map (see [3] for example): the foliations num. inv. by the involution $\sigma$ ("without dynamic") are conjugate to the foliations num. inv. by $A \sigma$, $A \in \text{Aut}(\mathbb{P}^2_C)$, which has a rich dynamic for a generic $A$.

The foliations of $\mathbb{F}(2; 2)$ invariant by $\sigma$ are particular cases of num. inv. foliations:

**Proposition 4.6.** — An element of $\mathbb{F}(2; 2)$ invariant by $\sigma$ is given up to permutations of coordinates and affine charts

- either by $y (1 + y) \text{d}x + (\beta x + \alpha y + \alpha^2 + \beta xy) \text{d}y$,
- or by $y (1 - y) \text{d}x + (\beta x - \alpha y + \alpha^2 - \beta xy) \text{d}y$,
- or by $y \text{d}x + (\alpha + \epsilon x + \alpha^2) \text{d}y$,

where the parameters are complex numbers such that the degree of the associated foliations is 2.

**Proof.** — With the notations of Lemma 4.1 one has

$$\sigma^* \omega_1 = -y(\epsilon + \kappa y) \text{d}x - (\gamma x + \alpha y + \delta x^2 + \beta xy) \text{d}y,$$

thus $\sigma^* \omega_1 \land \omega_1 = 0$ if and only if either $\gamma = \beta$, $\delta = \alpha$, $\epsilon = \kappa$, or $\gamma = -\beta$, $\delta = -\alpha$, $\epsilon = -\kappa$.

One has $\sigma^* \omega_2 = -(\kappa + \beta y + \delta y^2) \text{d}x - (\gamma + \epsilon x + \alpha x^2) \text{d}y$ and $\omega_2 \land \sigma^* \omega_2 = 0$ if and only if $\gamma = \alpha$, $\delta = 0$ and $\kappa = 0$.

**Remark 4.7.** — The foliations associated to the two first 1-forms with parameters $\alpha, \beta$ of Proposition 4.6 are conjugate by the automorphism $(x, y) \mapsto (x, -y)$.

**Lemma 4.8.** — A foliation $\mathcal{F} \in \mathbb{F}(2; 2)$ is num. inv. under the action of $\rho$ if and only if $\mathcal{F}$ is given in affine chart
either by $\omega_3 = y(\kappa + \epsilon y + \lambda y^2)\, dx + (\beta + \kappa x + \gamma y + \lambda y^2)\, dy$,

- or by $\omega_4 = y(\mu + \delta y + \gamma y + \epsilon y^2)\, dx + (\alpha + \beta x + \lambda y + \delta x^2 + \kappa y - \epsilon y^2)\, dy$,

- or by $\omega_5 = (\lambda + \gamma y + \kappa y^2)\, dx + (\beta + \delta x + \alpha y^2)\, dy$,

where the parameters are such that the degree of the corresponding foliations is 2.

**Proof.** Let us take the notations of the proof of Lemma 4.1. The map $\rho$ is an automorphism of $\mathbb{P}^2_\mathbb{C}$ \{yz = 0\} so if $\rho^*\omega = P\omega'$ with $\omega'$ a 1-form of degree 3 and $P$ a homogeneous polynomial then $P = y^j z^k$ for some integer $j, k$ such that $j + k = 4$. We have to look at the four following cases: $P = z^4$, $P = y^2 z^2$, $P = y^4 z$ and $P = y^4$. Computations show that $y^4$ (resp. $y^3 z$) cannot divide $\rho^*\omega$. If $P = z^4$ then $\rho^*\omega = P\omega'$ if and only if

$$c_0 = 0, \quad b_0 = 0, \quad c_3 = 0, \quad b_4 = 0, \quad b_2 = 0, \quad a_0 = c_4, \quad b_3 = c_4, \quad a_4 = 2c_2 - b_5;$$

this gives the first case $\omega_3$. The equality $\rho^*\omega = y^2 z^2\omega'$ holds if and only if

$$b_0 = 0, \quad c_0 = 0, \quad b_4 = 0, \quad c_1 = 0, \quad a_1 = 0, \quad b_2 = 0, \quad a_0 = 2c_4 - b_3$$

and we obtain $\omega_4$. Finally one has $\rho^*\omega = y^2 z^2\omega'$ if and only if

$$c_1 = 0, \quad b_0 = 0, \quad c_3 = 0, \quad a_5 = 0, \quad a_4 = 0, \quad c_0 = 0, \quad b_4 = 0, \quad c_5 = a_3$$

which corresponds to $\omega_5$. \hfill $\square$

**Proposition 4.9.** The foliations of type $\mathcal{F}_{\omega_3}$ and $\mathcal{F}_{\omega_5}$ are transversely projective. In fact the $\mathcal{F}_{\omega_3}$ are transversely affine and the $\mathcal{F}_{\omega_5}$ are Riccati ones.

**Proof.** A foliation of type $\mathcal{F}_{\omega_3}$ is described by the 1-form

$$\theta_0 = dx - \frac{(\beta + \delta y + \alpha y^2) + (\kappa + \gamma y - \lambda y^2) x}{y(\kappa + \epsilon y + \lambda y^2)}\, dy$$

and it is transversely affine; to see it consider the $sl(2;\mathbb{C})$-triplet

$$\theta_0, \quad \theta_1 = \frac{\kappa + \gamma y - \lambda y^2}{y(\kappa + \epsilon y + \lambda y^2)}\, dy, \quad \theta_2 = 0.$$

A foliation of type $\mathcal{F}_{\omega_5}$ is given by

$$dy + \frac{\lambda + (\gamma + \kappa)y + \epsilon y^2}{\beta + \delta x + \alpha x^2}\, dx$$

and thus is a Riccati foliation. In fact the fibration $x/z = \text{constant}$ is transverse to $\mathcal{F}_{\omega_5}$ that generically has three invariant lines. \hfill $\square$

We don’t know if the $\mathcal{F}_{\omega_3}$ are transversely projective. For generic values of the parameters a foliation of type $\mathcal{F}_{\omega_3}$ hasn’t meromorphic uniform first integral in the affine chart $z = 1$. Thus if $\mathcal{F}_{\omega_3}$ is transversely projective then it must have an invariant algebraic curve different from $z = 0$ (see [7]). We don’t know if it is the case. A foliation of degree 2 is conjugate to a generic $\mathcal{F}_{\omega_3}$ (by an automorphism of $\mathbb{P}^2_\mathbb{C}$) if and only if it has an invariant line (say $y = 0$) with a singular point (say 0) and local model $2x\, dy - y\, dx$. The closure of the set of such foliations has codimension 2. Note that the three families $\mathcal{F}_{\omega_0}, \mathcal{F}_{\omega_3}$ and $\mathcal{F}_{\omega_5}$ have non trivial intersection. The set $\{\mathcal{F}_{\omega_0}\}$ contains many interesting elements such that the famous Euler foliation given by $y^2\, dx + (y - x)\, dy$; this foliation is transversely affine but is not given by a closed rational 1-form.

**Proposition 4.10.** A foliation $\mathcal{F} \in \mathbb{F}(2;2)$ num. inv. under the action of an element of $\Theta(\rho)$ is conjugate to a foliation either of type $\mathcal{F}_{\omega_3}$, or of type $\mathcal{F}_{\omega_5}$, or of type $\mathcal{F}_{\omega_0}$. \hfill $\square$
Let us look at special num. inv. foliations, those invariant by $\rho$.

**Proposition 4.11.** — An element of $\mathbb{F}(2; 2)$ invariant by $\rho$ is given by a 1-form of one of the following type

- $y(1 - y)\,dx + (\beta + x)\,dy$,
- $y^2\,dx + (-1 + y)\,dy$,
- $y(1 - y)(\gamma + \delta x)\,dx + (1 + y)(\alpha + \beta x + \delta x^2)\,dy$,
- $y(1 + y)(\gamma + \delta x)\,dx + (1 - y)(\alpha + \beta x + \delta x^2)\,dy$,
- $(1 - y^2)\,dx + (\beta + \delta x + \alpha x^2)\,dy$,

where the parameters are such that $\deg F = 4$. We are interested by the "intermediate" degrees of a numerically invariant foliation. If it is the case, this implies the existence of invariant algebraic curves, and that can be established with a direct and tedious computation.

**Corollary 4.12.** — An element of $\mathbb{F}(2; 2)$ invariant by $\rho$ is defined by a closed 1-form.

**Remark 4.13.** — The third and fourth cases with parameters $\alpha, \beta, \gamma, \delta$ are conjugate by the automorphism $(x, y) \mapsto (x, -y)$.

From Lemmas 4.1 and 4.8 one gets the following statement.

**Proposition 4.14.** — A foliation num. inv. by an element of $\mathcal{O}(\phi)$, with $\phi = \sigma, \rho$, preserves an algebraic curve.

**Corollary 4.15.** — There is no quadratic birational map $\phi$ of $\mathbb{P}^2$, such that $\deg \phi^* J_{\Omega_1} = 2$.

**Proof.** — The foliation $J_{\Omega_1}$ has no invariant algebraic curve ([6 Proposition 1.3]); according to Proposition 4.14 it is thus sufficient to show that there is no birational map $\phi \in \mathcal{O}(\tau)$ such that $\deg \phi^* J_{\Omega_1} = 2$ that can be established with a direct and tedious computation.

**Remark 4.16.** — The map $\rho$ can be written $\ell_1 \sigma \ell_2 \sigma \ell_3$ with

$$\ell_1 = (z - y : y - x : y), \quad \ell_2 = (y + z : z : x), \quad \ell_3 = (x + z : y - z : z).$$

We are interested by the "intermediate" degrees of a numerically invariant foliation $F$, that is the sequence $\deg F$, $\deg(\ell_1 \sigma)^* F$, $\deg(\ell_1 \sigma \ell_2 \sigma \ell_3)^* F = \deg F$. A tedious computation shows that for generic values of the parameters the sequence is 2, 5, 2. We schematize this fact by the diagram

```
  5
 / \
2   2
```

A similar argument to Lemma 4.1 yields to the following result.

**Lemma 4.17.** — An element $F$ of $\mathbb{F}(2; 2)$ is num. inv. under the action of $\tau$ if and only if $F$ is given in affine chart by a 1-form of type

$$\omega_6 = (-\delta x + \alpha y - \varepsilon x^2 + \theta xy + \beta y^2 + \kappa x^2 y + \mu xy^2 + \lambda y^3)\,dx + (-3\alpha x + \xi x^2 + 2(\delta - \beta) xy + \alpha y^2 - \kappa x^3 - \mu y^2 y - \lambda xy^2)\,dy$$

where the parameters are such that $\deg F_{\omega_6} = 2$.

We don’t know the qualitative description of foliations of type $J_{\omega_6}$. For example we don’t know if the $J_{\omega_6}$ are transversely projective. If it is the case, this implies the existence of invariant algebraic curves, and that fact is unknown.
Proposition 4.18. — A foliation $\mathcal{F} \in \mathbb{P}(2; 2)$ num. inv. under the action of an element of $\Theta(\tau)$ is conjugate to $\mathcal{F}_{0\tau}$ for suitable values of the parameters.

Let us describe some special num. inv. foliations under the action of $\tau$, those invariant by $\tau$.

Proposition 4.19. — An element of $\mathbb{F}(2; 2)$ invariant by $\tau$ is given

- either by $(-\varepsilon x^2 + \theta xy + \beta y^2 + \varepsilon xy^2 - (\xi + \theta)y^3)\,dx + x(\xi x - 2\beta y - \varepsilon xy + (\xi + \theta)y^2)\,dy$,
- or by $(-\delta x + \alpha y + \delta y^2 + \kappa x^2 y + \mu xy^2 + \lambda y^3)\,dx - (3\alpha x + \delta xy - \alpha y^2 + \kappa x^3 + \mu x^2 y + \lambda xy^2)\,dy$,

where the parameters are complex numbers such that the degree of the associated foliations is 2.

The foliations associated to the first 1-form are transversely affine.

Proof. — The 1-jet at the origin of the 1-form

$$\omega = (-\varepsilon x^2 + \theta xy + \beta y^2 + \varepsilon xy^2 - (\xi + \theta)y^3)\,dx + x(\xi x - 2\beta y - \varepsilon xy + (\xi + \theta)y^2)\,dy$$

is zero so after one blow-up $\mathcal{F}_{0\omega}$ is transverse to the generic fiber of the Hopf fibration; furthermore as the exceptional divisor is invariant, $\mathcal{F}_{0\omega}$ is transversely affine.

Remark 4.20. — The map $\tau$ can be written $\ell_1\sigma\ell_2\sigma\ell_3\sigma\ell_2\sigma\ell_4$ with

$$\ell_1 = (x - y : x - 2y : -x + y - z), \quad \ell_2 = (x + z : x : y), \quad \ell_3 = (-y : x - 3y + z : x), \quad \ell_4 = (y - x : z - 2x : 2x - y).$$

Let us consider a foliation $\mathcal{F}$ num. inv. under the action of $\tau$; set $\mathcal{F}' = \ell_1^*\mathcal{F}$. We compute the intermediate degrees:

$$\deg(\ell_1^*\mathcal{F}) = 5, \quad \deg(\ell_2^*\mathcal{F}) = 4, \quad \deg(\ell_3^*\ell_2^*\mathcal{F}) = 5.$$

To summarize:

```
  5  5
 /   \
5   4
 /   \
 2  2
```

5. Higher degree

We will now focus on similar questions but with cubic birational maps of $\mathbb{P}^2$ and elements of $\mathbb{F}(2; 2)$. The generic model of such birational maps is:

$$\Phi_{a,b} : (x : y : z) \mapsto (x(x^2 + y^2 + axy + bxz + yz) : y(x^2 + y^2 + axy + bxz + yz) : xyz)$$

with $a, b \in \mathbb{C}$, $a^2 \neq 4$ and $2b \notin \{a \pm \sqrt{a^2 - 4}\}$.

Lemma 5.1. — An element $\mathcal{F}$ of $\mathbb{F}(2; 2)$ is num. inv. under the action of $\Phi_{a,b}$ if and only if $\mathcal{F}$ is given in affine chart

- either by $\omega_7 = y(\alpha + \gamma)\,dx - x(\alpha + \kappa)\,dy$,
- or by $\omega_8 = b(b^2 - ab + 1 + (a - 2b)y + y^2)\,dx + ((b^2 - ab + 1) + (ab - 2)x + x^2)\,dy$,

where the parameters are such that $\deg \mathcal{F}_{\omega_7} = \deg \mathcal{F}_{\omega_8} = 2$. 

Remark 5.2. — Remark that the foliations $\mathcal{F}_{\omega_7}$ do not depend on the parameters of $\Phi_{a,b}$, that is, the $\mathcal{F}_{\omega_7}$ are num. inv. by all $\Phi_{a,b}$, whereas the $\mathcal{F}_{\omega_8}$ only depend on $a$ and $b$.

Furthermore $\mathcal{F}_{\omega_7}$ is num. inv. by $\sigma$ and $\rho$.

Proposition 5.3. — Any $\mathcal{F} \in \mathbb{F}(2;2)$ num. inv. under the action of $\Phi_{a,b}$, and more generally any $\mathcal{F} \in \mathbb{F}(2;2)$ num. inv. under the action of a generic cubic birational map of $\mathbb{P}_C^2$, satisfies the following properties:

- $\mathcal{F}$ is given by a rational closed 1-form;
- $\mathcal{F}$ is non-primitive.

Proof. — Let us establish those properties for $\mathcal{F}_{\omega_7}$.

For generic values of $\alpha, \gamma$ and $\kappa$ one can assume up to linear conjugacy that $\mathcal{F}_{\omega_7}$ is given by

$$\eta' = y(1+y)dx - x(1+x)dy$$

that gives up to multiplication

$$\frac{dx}{x(1+x)} - \frac{dy}{y(1+y)}$$

which is closed. A foliation of type $\mathcal{F}_{\omega_7}$ is also described in homogeneous coordinates by the 1-form

$$\eta = yz(y+z)dx - xz(x+z)dy + xy(x-y)dz.$$

One has

$$\sigma^* \eta = xyz(-(y+z)dx + (x+z)dy + (x-y)dz)$$

so $\mathcal{F}_{\omega_7}$ is non-primitive.

The idea and result are the same for the foliations $\mathcal{F}_{\omega_8}$ (except that it gives a birational map $\phi$ such that $\deg \phi^* \mathcal{F}_{\omega_8} = 1$).

Let us consider an element $\mathcal{F}$ of $\mathbb{F}(2;2)$ num. inv. under the action of a birational map of degree $\geq 3$; is $\mathcal{F}$ defined by a closed 1-form ?

Remark 5.4. — The foliations $\mathcal{F}_{\omega_7}$ are contained in the orbit of the foliation $\mathcal{F}_{\eta'}$.

Remark 5.5. — Any map $\Phi_{a,b}$ can be written $\ell_1 \sigma \ell_2 \sigma \ell_3$ with $\ell_2 = (*y + *z : *y + *z : *x + *y + *z)$ (see [5, Proposition 6.36]). Let us consider the birational map $\xi = \sigma \ell_2 \sigma$ with

$$\ell_2 = (ay + bz : cy + ez : fx + gy + hz) \in \text{Aut}(\mathbb{P}_C^2).$$

As in Lemma[5.1] there are two families of foliations $\mathcal{F}_1, \mathcal{F}_2$ of degree 2, one that does not depend on the parameters of $\xi$ and the other one depending only on the parameters of $\xi$, such that $\xi^* \mathcal{F}_1$ and $\xi^* \mathcal{F}_2$ are of degree 2. One question is the following: what is the intermediate degree ? A computation shows that for generic parameters $\deg \sigma^* \mathcal{F}_1 = 4$ and that $\deg \sigma^* \mathcal{F}_2 = 2$. This implies in particular that $\mathcal{F}_{\omega_8}$ is num. inv. under the action of $\sigma$. For $\mathcal{F}_1$ and $\mathcal{F}_{\omega_7}$ one has

```
4
\downarrow
2 \quad 2
\downarrow
```

and for $\mathcal{F}_2$ and $\mathcal{F}_{\omega_8}$
```
2 ----> 2 ----> 2
```
Let us now consider the "most degenerate" cubic birational map
\[ \Psi: (x : y : z) \rightarrow (xz^2 + y^3 : yz^2 : z^3). \]

**Lemma 5.6.** An element \( F \) of \( \mathbb{F}(2; 2) \) is num. inv. under the action of \( \Psi \) if and only if \( F \) is given in affine chart by
\[ \omega_9 = \left( -\alpha + \beta y + \gamma z \right) dx + (\varepsilon - 3\beta x + \delta y - 3\gamma xy + \lambda y^2) dy \]
where the parameters are such that \( \deg F \omega_9 = 2 \). In particular \( F \) is transversely affine.

**Remark 5.7.** The map \( \psi \) can be written \( \ell_1 \sigma \ell_2 \sigma \ell_3 \sigma \ell_4 \sigma \ell_5 \sigma \ell_6 \sigma \ell_7 \sigma \ell_4 \sigma \ell_6 \sigma \ell_7 \) with
\[
\ell_1 = (z - y : y - x), \quad \ell_2 = (y + z : z : x), \quad \ell_3 = (-z : -y : x - y), \\
\ell_4 = (x + z : x : y), \quad \ell_5 = (-y : x - 3y + z : x), \quad \ell_6 = (-x : y - z : x + y), \\
\ell_7 = (x + y : z - y : y).
\]

As previously we consider the problem of the intermediate degrees; if \( F' = \ell_1^* F \), a computation shows that for generic parameters
\[
\deg \sigma^* F' = 4, \quad \deg (\sigma \ell_2 \sigma)^* F' = 3, \quad \deg (\sigma \ell_2 \sigma \ell_3 \sigma)^* F' = 5, \\
\deg (\sigma \ell_2 \sigma \ell_3 \sigma \ell_4 \sigma)^* F' = 3, \quad \deg (\sigma \ell_2 \sigma \ell_3 \sigma \ell_4 \sigma \ell_5 \sigma)^* F' = 5, \\
\deg (\sigma \ell_2 \sigma \ell_3 \sigma \ell_4 \sigma \ell_5 \sigma \ell_6 \sigma)^* F' = 3, \quad \deg (\sigma \ell_2 \sigma \ell_3 \sigma \ell_4 \sigma \ell_5 \sigma \ell_6 \sigma \ell_4 \sigma)^* F' = 4,
\]
that is

![Diagram](attachment:image.png)

We have not studied the quadratic foliations numerically invariant by (any) cubic birational transformation. It is reasonable to think that such foliations are transversely projective.

**References**


---

DOMINIQUE CERVEAU, Membre de l’Institut Universitaire de France. IRMAR, UMR 6625 du CNRS, Université de Rennes 1, 35042 Rennes, France. ● E-mail: dominique.cerveau@univ-rennes1.fr

JULIE DÉSERTI, Universität Basel, Mathematisches Institut, Rheinsprung 21, CH-4051 Basel, Switzerland. ● On leave from Institut de Mathématiques de Jussieu - Paris Rive Gauche, UMR 7586, Université Paris 7, Bâtiment Sophie Germain, Case 7012, 75205 Paris Cedex 13, France. ● E-mail: deserti@math.jussieu.fr