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A Particle in a Magnetic Field of an Infinite Rectilinear Current

D. YAFAEV
Department of Mathematics, University of Rennes, Campus Beaulieu, 35042, Rennes, France.
e-mail: yafaev@univ-rennes1.fr

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Abstract. We consider the Schrödinger operator $\mathbf{H} = (i\nabla + A)^2$ in the space $L_2(\mathbb{R}^3)$ with a magnetic A potential created by an infinite rectilinear current. We show that the operator $\mathbf{H}$ is absolutely continuous, its spectrum has infinite multiplicity and coincides with the positive half-axis. Then we find the large-time behavior of solutions $\exp(-i\mathbf{H}t)f$ of the time dependent Schrödinger equation. Our main observation is that a quantum particle has always a preferable (depending on its charge) direction of propagation along the current. Similar result is true in classical mechanics.

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1. Introduction

They are very few examples of explicit solutions of the Schrödinger equation with a magnetic potential, the case of a constant magnetic field (see, e.g., [3]) probably being the single one. Here we consider the magnetic field $B(x, y, z)$ created by an infinite rectilinear current which we suppose to coincide with the axis z. We assume that the axes (x, y, z) are positively oriented. According to the Biot–Savart–Laplace law (see, e.g., [6])

$$B(x, y, z) = \alpha r^{-2}(-y, x, 0), \quad r = (x^2 + y^2)^{1/2}, \tag{1.1}$$

where $|\alpha|$ is proportional to the current strength and $\alpha > 0$ ($\alpha < 0$) if the current streams in the positive (negative) direction. The magnetic potential is defined by the equation

$$B(x, y, z) = \operatorname{curl} A(x, y, z)$$

and can be chosen as

$$A(x, y, z) = -\alpha(0, 0, \ln r). \tag{1.2}$$

Thus, the corresponding Schrödinger operator in the space $L_2(\mathbb{R}^3)$ has the form

$$\mathbf{H} = \mathbf{H}^{(\gamma)} = -\partial_x^2 - \partial_y^2 + (i\partial_z - \gamma \ln r)^2, \quad \gamma = ec^{-1}\alpha, \tag{1.3}$$

where e is the charge of a quantum particle of the mass $1/2$ and c is the speed of the light.

Since the magnetic potential (1.2) grows at infinity, the Hamiltonian $\mathbf{H}$ does not fit to the well elaborated framework of spectral and scattering theory. Actually, we are aware of only one paper [4] (see also the book [1]) on this subject where it was proven that the essential spectrum of the magnetic Schrödinger operator coincides with the positive half-line provided the field vanishes at infinity. Here we obtain much more advanced information on the operator $\mathbf{H}$ and perform in Section 2 its spectral analysis almost explicitly. We show that the operator $\mathbf{H}$ is absolutely continuous, its spectrum has infinite multiplicity and coincides with the positive half-axis. Then we find in Section 3 the large-time behavior of solutions $\exp(-i\mathbf{H}t)f$ of the time dependent Schrödinger equation. Our main observation is that a positively (negatively) charged quantum particle always moves in the direction of the current (in the opposite direction) and is localized in the orthogonal plane. Actually, somewhat similar results are true in classical mechanics. Since we were unable to find a solution of the classical problem in the literature, it is given in Section 4.

2. The Spectrum of the Operator H

Let us make the Fourier transform $\Phi = \Phi_z$ in the variable z. Then the operator $H = \Phi\mathbf{H}\Phi^*$ acts in the space $L_2(\mathbb{R}^2 \times \mathbb{R})$ as

$$H = -\Delta + (p + \gamma \ln r)^2,$$

where Δ is always the Laplacian in the variables (x, y) and $p \in \mathbb{R}$ is the variable dual to z. Thus,

$$(Hu)(x, y, p) = (h(p)u)(x, y, p),$$

where

$$h(p) = -\Delta + \ln^2(\mathrm{e}^p r^\gamma) \tag{2.1}$$

acts in the space $L_2(\mathbb{R}^2)$. Clearly, the spectrum of each operator $h(p)$ is positive and discrete. If we separate variables in the polar coordinates (r, φ) and denote by $\mathcal{H}_m \subset L_2(\mathbb{R}^2)$ the subspace of functions $f(r)\mathrm{e}^{im\varphi}$, where $f \in L_2(\mathbb{R}_+; r\,\mathrm{d}r)$ and $m = 0, \pm 1, \pm 2, \ldots$ is the orbital quantum number, then

$$L_2(\mathbb{R}^2) = \bigoplus_{m=-\infty}^{\infty} \mathcal{H}_m. \tag{2.2}$$

Every subspace $\mathcal{H}_m$ is invariant with respect to the operator $h(p)$. The spectrum of its restriction

$$h_m(p) = -r^{-1}\partial_r(r\partial_r) + m^2 r^{-2} + \ln^2(\mathrm{e}^p r^\gamma) \tag{2.3}$$

on $\mathcal{H}_m$ consists of positive simple eigenvalues $\lambda_{m,1}(p), \lambda_{m,2}(p), \ldots$ which are analytic functions of p. We denote by $\psi_{m,1}(r, p), \psi_{m,2}(r, p), \ldots$ the corresponding eigenfunctions which are supposed to be normalized and real.

Quite similarly, if $\mathfrak{H}_m \subset L_2(\mathbb{R}^3)$ is the subspace of functions $u(r, z)\mathrm{e}^{im\varphi}$ where $u \in L_2(\mathbb{R}_+ \times \mathbb{R}; r\,\mathrm{d}r\,\mathrm{d}z)$, then

$$L_2(\mathbb{R}^3) = \bigoplus_{m=-\infty}^{\infty} \mathfrak{H}_m. \tag{2.4}$$

Every subspace $\mathfrak{H}_m$ is invariant with respect to $\mathbf{H}$. We denote by $\mathbf{H}_m$ the restriction of $\mathbf{H}$ on $\mathfrak{H}_m$. Actually, decompositions (2.2), (2.4) are needed only to avoid crossings of different eigenvalues of the operators $h(p)$. It allows us to use always formulas of perturbation theory (see [5]) for simple eigenvalues.

Fixing γ, we often use the parameter $a = \mathrm{e}^{p/\gamma} \in (0, \infty)$ instead of p. Let us set

$$K(a) = -a^2\Delta + \gamma^2 \ln^2 r, \tag{2.5}$$

and let $w(a)$,

$$(w(a)u)(x, y) = au(ax, ay),$$

be the unitary operator of dilations in the space $L_2(\mathbb{R}^2)$. Then the operator (2.1) equals

$$h(p) = w(a)K(a)w^*(a), \tag{2.6}$$

where, as always, $a = \mathrm{e}^{p/\gamma}$. We denote by $\mu_{m,n}(a)$ and $\phi_{m,n}(r, a)$ eigenvalues and eigenfunctions $K_m(a)$ of the restrictions of the operators $K(a)$ on the subspaces $\mathcal{H}_m$. It follows from (2.6) that $\mu_{m,n}(a) = \lambda_{m,n}(p)$ and $\phi_{m,n}(a) = w^*(a)\psi_{m,n}(p)$. Below we usually fix m and omit it from the notation.

The following assertions are quite elementary.

LEMMA 2.1. *For every n, we have that $\mu_n'(a) > 0$, for all $a > 0$.*

Proof. Applying analytic perturbation theory to the family (2.5), we see that

$$\mu_n'(a) = 2a \int_{\mathbb{R}^2} |\nabla \phi_n(x, y, a)|^2\,\mathrm{d}x\,\mathrm{d}y. \tag{2.7}$$

This expression is obviously positive since otherwise $\phi_n(x, y, a) = \text{const}$. □

The next lemma realizes an obvious idea that the spectrum of $K(a)$ converges as $a \to 0$ to that of the multiplication operator by $\gamma^2 \ln^2 r$, which is continuous and starts from zero.

LEMMA 2.2. *For every n, we have that*

$$\lim_{a\to 0} \mu_n(a) = 0.$$

Proof. Let $\varepsilon > 0$ be arbitrary and $\delta > 1$ be such that $\gamma^2 \ln^2 \delta = \varepsilon$. Suppose that functions $f_1, f_2, \dots, f_n \in C_0^\infty(\delta^{-1}, \delta)$ are obtained from, say, f_1, $\|f_1\| = 1$, by shifts and that they are disjointly supported. Set $u_p(x, y) = r^{-1/2} f_p(r) e^{im\varphi}$, then

$$(K_m(a)u_p, u_p) = a^2 \int_0^\infty (|f_p'(r)|^2 + (m^2 - 1/4)r^{-2}|f_p(r)|^2)\, dr + \\ + \gamma^2 \int_0^\infty \ln^2 r |f_p(r)|^2\, dr.$$

The first term here tends to zero as $a \to 0$ and the second is bounded by ε. Thus, for sufficiently small a, the operator $K_m(a)$ has at least n eigenvalues below 2ε. This implies that $\mu_n(a) < 2\varepsilon$. □

Let $\mathbb{B}_R = \{x^2 + y^2 \leqslant R^2\}$ and $\mathbb{S}_R = \{x^2 + y^2 = R^2\}$ be the disc and the circle of radius R. One of possible proofs of the next lemma relies on the Friedrichs inequality

$$R^2 \int_{\mathbb{B}_R} |\nabla u(x, y)|^2\, dx\, dy + R \int_{\mathbb{S}_R} |u(x, y)|^2\, dS_R \\ \geqslant c_1 \int_{\mathbb{B}_R} |u(x, y)|^2\, dx\, dy, \tag{2.8}$$

where R is arbitrary and $dS_R = R\, d\varphi$. This inequality is usually verified first for $R = 1$ and then one makes the dilation transformation $(x, y) \mapsto (Rx, Ry)$. We need also the standard Sobolev inequality where again the dilation transformation is taken into account:

$$R \int_{\mathbb{S}_R} |u(x, y)|^2\, dS_R \\ \leqslant R^2 \int_{\mathbb{R}^2 \setminus \mathbb{B}_R} |\nabla u(x, y)|^2\, dx\, dy + c_2 \int_{\mathbb{R}^2 \setminus \mathbb{B}_R} |u(x, y)|^2\, dx\, dy. \tag{2.9}$$

LEMMA 2.3. *For every n, we have that* $\lim_{a\to\infty} \mu_n(a) = \infty$.

Proof. It suffices to show, that the infimum of the operator $K(a)$ tends to ∞ as $a \to \infty$. It follows from inequality (2.8) that

$$a^2 \int_{\mathbb{B}_R} |\nabla u(x, y)|^2\, dx\, dy + a^2 R^{-1} \int_{\mathbb{S}_R} |u(x, y)|^2\, dS_R + \\ + \gamma^2 \int_{\mathbb{B}_R} \ln^2 r |u(x, y)|^2\, dx\, dy \geqslant c_1 a^2 R^{-2} \int_{\mathbb{B}_R} |u(x, y)|^2\, dx\, dy. \tag{2.10}$$

Inequality (2.9) implies that

$$a^2 \int_{\mathbb{R}^2 \setminus \mathbb{B}_R} |\nabla u(x, y)|^2\, dx\, dy - a^2 R^{-1} \int_{\mathbb{S}_R} |u(x, y)|^2\, dS_R +$$

$$+ \gamma^2 \int_{\mathbb{R}^2 \setminus \mathbb{B}_R} \ln^2 r |u(x, y)|^2 \, \mathrm{d}x \, \mathrm{d}y$$

$$\geqslant (\gamma^2 \ln^2 R - c_2 a^2 R^{-2}) \int_{\mathbb{R}^2 \setminus \mathbb{B}_R} |u(x, y)|^2 \, \mathrm{d}x \, \mathrm{d}y. \tag{2.11}$$

Let us now choose R such that $\gamma^2 R^2 \ln^2 R = a^2(c_1 + c_2)$. Combining estimates (2.10) and (2.11), we see that

$$(K(a)u, u) \geqslant \gamma^2 c_1 (c_1 + c_2)^{-1} \ln^2 R \|u\|^2$$

and the right-hand side tends to ∞ as $a \to \infty$. □

Of course, Lemma 2.3 and its proof remain valid for all dimensions and for arbitrary potentials tending to ∞ at infinity.

Let us again fix the orbital quantum number m. In terms of eigenvalues $\lambda_n(p) = \lambda_{m,n}(p)$ of the operators $h(p) = h_m(p)$ Lemmas 2.1–2.3 mean that $\lambda_n'(p) > 0$ for all $p \in \mathbb{R}$ and

$$\lim_{p \to -\infty} \lambda_n(p) = 0, \qquad \lim_{p \to \infty} \lambda_n(p) = \infty \tag{2.12}$$

if $\gamma > 0$. Note that eigenvalues of the operators (2.1) satisfy the identity

$$\lambda_n(p, \gamma) = \lambda_n(-p, -\gamma),$$

so we may assume without loss of generality that $\gamma > 0$.

Let Λ_n be multiplication operator by the function $\lambda_n(p)$ in the space $L_2(\mathbb{R})$. It follows from the results on the eigenvalues $\lambda_n(p)$ that the spectrum of Λ_n is absolutely continuous, simple and coincides with the positive half axis. Let us introduce a unitary mapping

$$\Psi \colon L_2(\mathbb{R}_+ \times \mathbb{R}; r \, \mathrm{d}r \, \mathrm{d}z) \to \bigoplus_{n=1}^{\infty} L_2(\mathbb{R})$$

by the formula

$$(\Psi f)_n(p) = \int_0^\infty f(r, p) \psi_n(r, p) r^{1/2} \, \mathrm{d}r.$$

Then

$$\Psi \Phi \mathbf{H} \Phi^* \Psi^* = \bigoplus_{n=1}^{\infty} \Lambda_n \tag{2.13}$$

(of course, $\mathbf{H} = \mathbf{H}_m$ and $\Lambda_n = \Lambda_{m,n}$), and we obtain the following theorem:

THEOREM 2.4. *The spectra of all operators $\mathbf{H}_m$ and $\mathbf{H}$ are absolutely continuous, have infinite multiplicity and coincide with the positive half axis.*

As a by-product of our considerations, we have constructed a complete set of eigenfunctions of the operator $\mathbf{H}$. They are parametrized by the orbital quantum number m, the momentum p in the direction of the z-axis and the number n of an eigenvalue $\lambda_{m,n}(p)$ of the operator $h_m(p)$ defined by (2.3) on the subspace $L_2(\mathbb{R}_+; r\,\mathrm{d}r)$. Thus, if we set

$$\mathbf{u}_{m,n,p}(r, z, \varphi) = \mathrm{e}^{ipz}\mathrm{e}^{im\varphi}\psi_{m,n}(r, p),$$

then

$$\mathbf{H}\mathbf{u}_{m,n,p} = \lambda_{m,n}(p)\mathbf{u}_{m,n,p}.$$

3. Time Evolution

Explicit formulas obtained in the previous section allow us to find the asymptotics for large t of solutions $u(t) = \exp(-i\mathbf{H}t)u_0$ of the time-dependent Schrödinger equation. It follows from (1.3) that

$$\overline{\exp(-i\mathbf{H}^{(\gamma)}t)u_0} = \exp(i\mathbf{H}^{(-\gamma)}t)\overline{u_0}.$$

Therefore it suffices to consider the case $\gamma > 0$. Moreover, on every subspace $\mathfrak{H}_m$ with a fixed orbital quantum number m, the problem reduces to the asymptotics of the function $u(t) = \exp(-i\mathbf{H}_m t)u_0$.

Suppose that

$$(\Phi u_0)(r, p) = \psi_n(r, p)f(p), \tag{3.1}$$

where $f \in C_0^\infty(\mathbb{R})$. Then it follows from formula (2.13) that

$$u(r, z, t) = (2\pi)^{-1/2}\int_{-\infty}^{\infty} \mathrm{e}^{ipz-i\lambda_n(p)t}\psi_n(r, p)f(p)\,\mathrm{d}p. \tag{3.2}$$

The stationary points of this integral are determined by the equation

$$z = \lambda_n'(p)t. \tag{3.3}$$

Since, by Lemma 2.1, $\lambda_n'(p) > 0$ for $\gamma > 0$, Equation (3.3) has a solution only if $zt > 0$. We need the following information on the eigenvalues $\mu_n(a)$ of the operator (2.5).

LEMMA 3.1. *For every n, we have that* $\lim_{a\to 0} a\mu_n'(a) = 0$.

Proof. It follows from Equation (2.7) that $a\mu_n'(a) \leqslant 2\mu_n(a)$. Therefore it remains to use Lemma 2.2. □

The following conjecture is physically quite plausible and is used mainly to formulate Theorem 3.3 below in a simpler form.

CONJECTURE 3.2. *For every n, we have that $(a\mu_n'(a))' > 0$ for all $a > 0$ and*

$$\lim_{a\to\infty} a\mu_n'(a) = \infty. \tag{3.4}$$

In terms of eigenvalues $\lambda_n(p)$ of the operators $h(p)$, Lemma 3.1 and Conjecture 3.2 mean that $\lambda_n''(p) > 0$ for all $p \in \mathbb{R}$ and

$$\lim_{p\to-\infty} \lambda_n'(p) = 0, \qquad \lim_{p\to\infty} \lambda_n'(p) = \infty. \tag{3.5}$$

Therefore equation $\lambda_n'(p) = \alpha$ has a unique solution $p_n = \nu_n(\alpha)$ for every $\alpha > 0$. Clearly,

$$\lambda_n''(\nu_n(\alpha))\nu_n'(\alpha) = 1. \tag{3.6}$$

Let

$$\Phi_n(\alpha) = \nu_n(\alpha)\alpha - \lambda_n(\nu_n(\alpha)), \tag{3.7}$$

$\theta(\alpha) = 1$ for $\alpha > 0$, $\theta(\alpha) = 0$ for $\alpha < 0$ and $\pm i = \mathrm{e}^{\pm\pi i/2}$. Applying to the integral (3.2) the stationary phase method and taking into account identity (3.6), we find that

$$\begin{aligned} u(r,z,t) \;=\; & \mathrm{e}^{i\Phi_n(z/t)t}\psi_n\left(r,\nu_n\left(\frac{z}{t}\right)\right)\nu_n'\left(\frac{z}{t}\right)^{1/2} f\left(\nu_n\left(\frac{z}{t}\right)\right)(it)^{-1/2}\theta\left(\frac{z}{t}\right) + \\ & + u_\infty(r,z,t), \end{aligned} \tag{3.8}$$

where

$$\lim_{t\to\pm\infty} \|u_\infty(\cdot,t)\| = 0. \tag{3.9}$$

Note that the norm in the space $L_2(\mathbb{R}_+ \times \mathbb{R})$ of the first term in the right-hand side of (3.8) equals $\|u_0\|$. The asymptotics (3.8) extends of course to all $f \in L_2(\mathbb{R})$ and to linear combinations of functions (3.1) over different n. Thus, we have proven

THEOREM 3.3. *Assume that Conjecture* 3.2 *is verified. Suppose $\gamma > 0$. Let $u(t) = \exp(-i\mathbf{H}_m t)u_0$ where u_0 satisfies* (3.1) *with $f \in L_2(\mathbb{R})$. Then the asymptotics as $t \to \pm\infty$ of this function is given by relations* (3.8), (3.9) *where Φ_n is the phase function* (3.7). *Moreover, if $f \in C_0^\infty(\mathbb{R})$ and $\mp z > 0$, then the function $u(r,z,t)$ tends to zero faster than any power of $|t|^{-1}$ as $t \to \pm\infty$.*

Conversely, for any $g \in L_2(\mathbb{R}_+)$ there exists the function u_0, namely

$$(\Phi u_0)(r,p) = \psi_n(r,\lambda_n'(p))\lambda_n''(p)^{1/2} g(\lambda_n'(p)),$$

such that $u(t) = \exp(-i\mathbf{H}_m t)u_0$ has the asymptotics as $t \to \pm\infty$

$$u(r,z,t) = \mathrm{e}^{i\Phi_n(z/t)t}\psi_n\left(r,\frac{z}{t}\right) g\left(\frac{z}{t}\right)(it)^{-1/2}\theta\left(\frac{z}{t}\right) + u_\infty(r,z,t),$$

where u_∞ satisfies (3.9).

Formulas (3.8), (3.9) show that a positively (negatively) charged quantum particle always moves in the magnetic field (1.1) in the direction of the current (in the opposite direction), and its motion is essentially free. Note however that for the free motion the phase in (3.8) would be $\Phi(\alpha) = \alpha^2/4$, whereas for the Hamiltonian $\mathbf{H}_m$ it is determined by the eigenvalues $\lambda_n(p)$ (see formula (3.7)). On the contrary, a particle remains localized in the plane orthogonal to the current.

As was already noted, Conjecture 3.2 is not really essential for Theorem 3.3. Remark first that $\lambda_n''(p)$ cannot vanish on an interval. Otherwise, $\lambda_n'(p)$ would be a constant on the same interval, and hence by analyticity $\lambda_n'(p) = \text{const}$ for all $p \in \mathbb{R}$. This contradicts conditions (2.12). If $\lambda_n''(p) < 0$ on some interval, this changes only the phase factor in (3.8). Finally, the condition (3.4), or equivalently the second condition (3.5), is required to guarantee that Equation (3.3) has solutions for arbitrary large z/t. We emphasize that the assertion that $u(r, z, t)$ 'lives' in the half-space $\pm z > 0$ for $\pm t > 0$ is true without Conjecture 3.2.

4. Classical Mechanics

Let us consider the motion of a classical particle of mass $m = 1/2$ and charge e in a magnetic field $B(x, y, z)$. It is natural to study somewhat more general case where

$$A(x, y, z) = (0, 0, \mathcal{A}(r)), \quad r = (x^2 + y^2)^{1/2},$$

and $\mathcal{A}(r)$ is an arbitrary $C^2(\mathbb{R}_+)$ function such that $\mathcal{A}(r) = \mathrm{o}(r^{-1})$ as $r \to 0$ and $|\mathcal{A}(r)| \to \infty$ as $r \to \infty$. For such magnetic potentials

$$B(x, y, z) = \mathcal{A}'(r) r^{-1} (y, -x, 0). \tag{4.1}$$

The force exercised by a magnetic field on a particle with a velocity $\mathbf{v}$ at a point $\mathbf{r} = (x, y, z)$ equals $ec^{-1}\mathbf{v} \times B(\mathbf{r})$ (see [6]). Therefore the Newton equation reads as

$$\mathbf{r}''(t) = e_0 \mathbf{r}'(t) \times B(\mathbf{r}(t)), \tag{4.2}$$

where $e_0 = 2ec^{-1}$. Clearly, the expression

$$\frac{\mathrm{d}|\mathbf{r}'(t)|^2}{\mathrm{d}t} = 2\langle \mathbf{r}'(t), \mathbf{r}''(t)\rangle = 2e_0 \langle \mathbf{r}'(t), \mathbf{r}'(t) \times B(\mathbf{r}(t))\rangle = 0$$

since the vectors $\mathbf{r}'(t)$ and $\mathbf{r}'(t) \times B(\mathbf{r}(t))$ are orthogonal. Therefore, as is well known, the kinetic energy

$$|\mathbf{r}'(t)|^2 = x'(t)^2 + y'(t)^2 + z'(t)^2 = K^2 \tag{4.3}$$

of a particle in a magnetic field does not depend on time.

For the magnetic field (4.1), Equation (4.2) is equivalent to the equations

$$x''(t) = e_0 z'(t) x(t) \mathcal{A}'(r(t)) r^{-1}(t), \tag{4.4}$$

$$y''(t) = e_0 z'(t) y(t) \mathcal{A}'(r(t)) r^{-1}(t), \tag{4.5}$$

$$z''(t) = -e_0(x'(t)x(t) + y'(t)y(t)) \mathcal{A}'(r(t)) r^{-1}(t). \tag{4.6}$$

It is convenient to rewrite these equations in cylindrical coordinates using the obvious identification $(x, y) \leftrightarrow x + iy = r\mathrm{e}^{i\varphi}$. Differentiating this identity, we find that

$$x''(t) + iy''(t) \\ = (r''(t) - \varphi'(t)^2 r(t) + i\varphi''(t) r(t) + 2i\varphi'(t) r'(t)) \mathrm{e}^{i\varphi(t)}. \tag{4.7}$$

Multiplying Equation (4.5) by i and taking its sum with Equation (4.4), we see that

$$x''(t) + iy''(t) = e_0 z'(t) \mathcal{A}'(r(t)) \mathrm{e}^{i\varphi(t)}. \tag{4.8}$$

Comparing the right-hand sides of (4.7) and (4.8), we obtain that

$$r''(t) - \varphi'(t)^2 r(t) = e_0 z'(t) \mathcal{A}'(r(t)), \tag{4.9}$$

$$\varphi''(t) r(t) + 2\varphi'(t) r'(t) = 0. \tag{4.10}$$

Since, moreover,

$$x'(t)x(t) + y'(t)y(t) = r'(t)r(t),$$

Equation (4.6) is equivalent to

$$z''(t) = -e_0 r'(t) \mathcal{A}'(r(t)). \tag{4.11}$$

Similarly, the conservation law (4.3) means that

$$r'(t)^2 + r(t)^2 \varphi'(t)^2 + z'(t)^2 = K^2. \tag{4.12}$$

Integrating Equations (4.10), (4.11), we find that

$$\varphi'(t) r(t)^2 = \sigma, \qquad \sigma = \varphi'(0) r(0)^2 \neq 0, \tag{4.13}$$

$$z'(t) = -e_0(\mathcal{A}(r(t)) + C), \qquad C = -e_0^{-1} z'(0) - \mathcal{A}(r(0)). \tag{4.14}$$

Plugging these expressions into (4.12), we see that

$$r'(t)^2 + V(r(t)) = K^2, \tag{4.15}$$

where

$$V(r) = \sigma^2 r^{-2} + e_0^2 (\mathcal{A}(r) + C)^2. \tag{4.16}$$

Clearly, (4.15) is the equation of one-dimensional motion (see [2]) with the effective potential energy $V(r)$ and the total energy K^2. It admits the separation of variables and can be integrated by the formula

$$t = \pm \int (K^2 - V(r))^{-1/2}\, \mathrm{d}r. \tag{4.17}$$

Note that $V(r) \to \infty$ as $r \to 0$ or $r \to \infty$. Let $r_{\min}$ and $r_{\max}$ be the roots of the equation $V(r) = K^2$ (it has exactly two roots for given initial data). It follows from (4.17) that the function $r(t)$ is periodic with period

$$T = 2 \int_{r_{\min}}^{r_{\max}} (K^2 - V(r))^{-1/2}\, \mathrm{d}r. \tag{4.18}$$

One can imagine, for example, that on the period $[0, T]$ the function $r(t)$ increases monotonically from $r_{\min}$ to $r_{\max}$ and then decreases from $r_{\max}$ to $r_{\min}$. Having found $r(t)$, we determine $\varphi(t)$ from Equation (4.13):

$$\varphi(t) = \varphi(0) + \sigma \int_0^t r(s)^{-2}\, \mathrm{d}s. \tag{4.19}$$

To find a motion in the variable z, we use Equation (4.14) which gives

$$z(t) - z(0) = -e_0 \int_0^t (\mathcal{A}(r(s)) + C)\, \mathrm{d}s. \tag{4.20}$$

Thus, we have integrated the system (4.4)–(4.6).

THEOREM 4.1. *In the variable r a classical particle moves periodically according to Equation* (4.17) *with period* (4.18)*. The angular variable is determined by Equation* (4.19) *so that $\varphi(t) = \varphi_0 t + \mathrm{O}(1)$, where*

$$\varphi_0 = \sigma T^{-1} \int_0^T r(s)^{-2}\, \mathrm{d}s,$$

as $|t| \to \infty$. The variable z is determined by Equation (4.20) *where C is the same constant as in* (4.14).

It follows from equation (4.14) that $z'(t) \geqslant 0$ or $z'(t) \leqslant 0$ for all t if and only if

$$z'(0) \geqslant \max_{r_{\min} \leqslant r \leqslant r_{\max}} (e_0 \mathcal{A}(r)) - e_0 \mathcal{A}(r(0))$$

or

$$z'(0) \leqslant \min_{r_{\min} \leqslant r \leqslant r_{\max}} (e_0 \mathcal{A}(r)) - e_0 \mathcal{A}(r(0)),$$

respectively. Otherwise, $\pm z'(t) \geqslant 0$ if and only if

$$\pm e_0(\mathcal{A}(r(t)) - \mathcal{A}(r(0))) \leqslant \pm z'(0),$$

so that a particle can move both in positive and negative directions in the variable z.

Nevertheless one gives simple sufficient conditions for the inequality

$$\pm(z(T) - z(0)) > 0. \tag{4.21}$$

Indeed, it follows from Equations (4.9) and (4.13) that

$$e_0 z'(t) = r''(t)\mathcal{A}'(r(t))^{-1} - \sigma^2 r(t)^{-3}\mathcal{A}'(r(t))^{-1}.$$

Integrating this equation and taking into account the periodicity of the function $r(t)$, we see that

$$\begin{aligned} e_0(z(T) - z(0)) &= \int_0^T r''(t)\mathcal{A}'(r(t))^{-1}\,\mathrm{d}t - \sigma^2\int_0^T r(t)^{-3}\mathcal{A}'(r(t))^{-1}\,\mathrm{d}t \\ &= \int_0^T r'(t)^2\mathcal{A}'(r(t))^{-2}\mathcal{A}''(r(t))\,\mathrm{d}t - \\ &\quad - \sigma^2\int_0^T r(t)^{-3}\mathcal{A}'(r(t))^{-1}\,\mathrm{d}t. \end{aligned} \tag{4.22}$$

In particular,

$$z(nT) - z(0) = n(z(T) - z(0)).$$

Let us formulate the results obtained.

THEOREM 4.2. *The increment of the variable z at every period is determined by Equation* (4.22). *In particular, if $\pm e_0\mathcal{A}'(r) < 0$ and $\pm e_0\mathcal{A}''(r) \geqslant 0$ for all r, then inequality* (4.21) *holds. In this case $z(t) = z_0 t + \mathrm{O}(1)$ with $z_0 = T^{-1}(z(T) - z(0))$, $\pm z_0 > 0$, as $|t| \to \infty$.*

Let us finally discuss the magnetic potential $\mathcal{A}(r) = -\alpha \ln r$ of an infinite rectilinear current. Such potentials satisfy all the assumptions of this section. Now the Equation (4.14) reduces to

$$z'(t) = 2\gamma \ln br(t), \qquad b = r(0)^{-1}\mathrm{e}^{(2\gamma)^{-1}z'(0)} > 0,$$

and effective potential (4.16) is given by

$$V(r) = \sigma^2 r^{-2} + 4\gamma^2 \ln^2 br,$$

where $\gamma = \alpha e c^{-1}$. Then $\pm z'(t) \geqslant 0$ for all t if and only if

$$\pm z'(0) \geqslant \pm 2\gamma \ln(r(0)/r_{\max})$$

for $\mp\gamma > 0$ and if and only if

$$\pm z'(0) \geqslant \pm 2\gamma \ln(r(0)/r_{\min})$$

for $\pm\gamma > 0$. Otherwise, $\pm z'(t) \geqslant 0$ for $\pm\gamma > 0$ if $r(t) \geqslant b^{-1}$ and $\pm z'(t) \geqslant 0$ for $\mp\gamma > 0$ if $r(t) \leqslant b^{-1}$. Equation (4.22) now takes the form

$$z(T) - z(0) = (2\gamma)^{-1} \int_0^T (r'(t)^2 + \sigma^2 r(t)^{-2})\, \mathrm{d}t.$$

This expression is strictly positive (negative) if $\alpha e > 0$ (if $\alpha e < 0$).

Thus, positively charged classical and quantum particles always move asymptotically in the direction of the current and never in the opposite direction. Similarly, negatively charged particles always move asymptotically against direction of the current and never in the same direction. In the plane orthogonal to the direction of the current classical and quantum particles are essentially localized.

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