A TRACE FORMULA FOR THE DIRAC OPERATOR

D. R. YAFAEV

ABSTRACT. Our goal is to extend the theory of the spectral shift function to the case where only the difference of some powers of the resolvents of self-adjoint operators belongs to the trace class. As an example, a pair of Dirac operators is considered.

1. Introduction

The concept of the spectral shift function (SSF) first appeared in the work of I. M. Lifshits [13] in connection with the quantum theory of crystals. A mathematical theory of the SSF was soon thereafter constructed by M. G. Kreĭn in [11]. One of his results can be formulated in the following way. Let H_0 and H be self-adjoint operators with a trace class difference $V = H - H_0$. Then there exists a function $\xi(\lambda) = \xi(\lambda; H, H_0)$, $\xi \in L_1(\mathbb{R})$, known as the spectral shift function such that the trace formula

$$\mathrm{Tr}\Big(f(H) - f(H_0)\Big) = \int_{-\infty}^{\infty} \xi(\lambda) f'(\lambda) d\lambda, \quad \xi(\lambda) = \xi(\lambda; H, H_0), \tag{1.1}$$

holds at least for all functions $f \in C_0^\infty(\mathbb{R})$. Later in [12] M. G. Kreĭn returned to this problem. In particular, he showed that formulae similar to (1.1) remain true for a couple of unitary operators U_0 and U with a trace class difference. In terms of self-adjoint operators this means that formula (1.1) holds if the difference of the resolvents of the operators H_0 and H belongs to the trace class $\mathfrak{S}_1$ (such operators H_0 and H are called resolvent comparable). This allows one to state formula (1.1) for a sufficiently large class of differential operators. A relatively detailed presentation of the theory of the SSF can be found in [5] or [17].

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In 1957 [7, 8, 16] T. Kato and M. Rosenblum proved the existence of the wave operators (all relevant definitions can be found in [17]) for a pair of self-adjoint operators with a trace class difference. This result was extended in [2] by M. Sh. Birman and M. G. Kreĭn to unitary operators. More important, in the same paper [2] they have found a connection between scattering theory and the theory of the SSF. Actually, they showed that the corresponding scattering matrix $S(\lambda; H, H_0)$ minus the identity operator I belongs to the trace class and

$$\operatorname{Det} S(\lambda; H, H_0) = e^{-2\pi i \xi(\lambda; H, H_0)} \tag{1.2}$$

for almost all λ from the core of the spectrum of the operator H_0.

Later, in [9] T. Kato proved the existence of the wave operators for the pair H_0, H under the assumption that

$$R^m(z) - R_0^m(z) \in \mathfrak{S}_1 \tag{1.3}$$

for some positive integer m and all z with $\operatorname{Im} z \neq 0$. However the construction of the SSF under this assumption seemed to be an open problem. Our goal here is to fill in this gap. Apart from its conceptual naturalness, there are very simple applications to differential operators which require such an extension of the theory. As examples, we note the Schrödinger operator with a linear potential (see, e.g., [6, 14]) and the Dirac operator with a bounded potential considered in the last section of this paper.

We emphasize that the problem treated here is non-trivial only in the case where the spectra of the operators H_0 and H cover the whole real line (as in the examples above). If, for instance, $\lambda = 0$ is a common regular point of the operators H_0 and H and $H^{-m} - H_0^{-m} \in \mathfrak{S}_1$ for some odd m, then the trace formula (1.1) for the pair H_0, H can be deduced from that for the pair H_0^{-m}, H^{-m} (see [17], for details).

We note also that very general conditions of the existence of wave operators were obtained by M. Sh. Birman in the framework of the local trace class approach

[1]. On the other hand, the local theory of the SSF due to L.S. Koplienko [10] is somewhat less satisfactory.

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2. The spectral shift function

Let us first formulate the result of M. G. Kreĭn obtained in [12].

Theorem 2.1. *Let h_0 and h be self-adjoint operators such that*

$$(h-i)^{-1}-(h_0-i)^{-1}\in\mathfrak{S}_1. \tag{2.1}$$

Suppose that a function $g(\mu)$ has two bounded derivatives and

$$\partial^\alpha(g(\mu)-g_0\mu^{-1})=O(|\mu|^{-1-\epsilon-\alpha}),\quad \alpha=0,1,2,\quad \epsilon>0,$$

where the constant g_0 is the same for $\mu\to\infty$ and $\mu\to-\infty$. Then

$$g(H)-g(H_0)\in\mathfrak{S}_1$$

and there exists the SSF $\xi(\mu;h,h_0)$ such that

$$\int_{-\infty}^{\infty}|\xi(\mu;h,h_0)|(1+|\mu|)^{-2}d\mu<\infty \tag{2.2}$$

and

$$\operatorname{Tr}\Big(g(h)-g(h_0)\Big)=\int_{-\infty}^{\infty}\xi(\mu;h,h_0)g'(\mu)d\mu. \tag{2.3}$$

Our goal is to extend this result to the case where condition (1.3) is satisfied for some (not necessarily $m=1$) odd m.

Theorem 2.2. *Let, for a pair of self-adjoint operators H_0 and H, the assumption* (1.3) *hold for some odd m and all z with $\operatorname{Im} z\neq 0$. Suppose that a function $f(\lambda)$ has two bounded derivatives and*

$$\partial^\alpha(f(\lambda)-f_0\lambda^{-m})=O(|\lambda|^{-m-\epsilon-\alpha}),\quad \alpha=0,1,2,\quad \epsilon>0,$$

where the constant f_0 is the same for $\lambda \to \infty$ and $\lambda \to -\infty$. Then

$$f(H) - f(H_0) \in \mathfrak{S}_1 \tag{2.4}$$

and there exists a function (the SSF) $\xi(\lambda; H, H_0)$ satisfying the condition

$$\int_{-\infty}^{\infty} |\xi(\lambda; H, H_0)|(1 + |\lambda|)^{-m-1} d\lambda < \infty \tag{2.5}$$

such that the trace formula (1.1) *is true. Moreover, for the corresponding scattering matrix $S(\lambda; H, H_0)$, the operator $S(\lambda; H, H_0) - I \in \mathfrak{S}_1$ and relation* (1.2) *holds for almost all λ from the core of the spectrum of the operator H_0.*

Our proof of Theorem 2.2 relies on its reduction to Theorem 2.1 although, similarly to [12], we could have deduced it from the corresponding result for unitary operators. Actually, we construct a function φ such that the operators $h_0 = \varphi(H_0)$ and $h = \varphi(H)$ are resolvent comparable. To be more precise, we shall prove the following result.

Theorem 2.3. *Let, for a pair of self-adjoint operators H_0 and H, the assumption* (1.3) *hold for some odd m and all z with $\operatorname{Im} z \neq 0$. Suppose that a real function $\varphi \in C^2(\mathbb{R})$ and that $\varphi(\lambda) = \lambda^m$ for sufficiently large $|\lambda|$. Then the pair $h_0 = \varphi(H_0)$, $h = \varphi(H)$ satisfies condition* (2.1).

We postpone the proof of Theorem 2.3 until the next section. Here we use it for the construction of the SSF. Since, under the assumptions of Theorem 2.2, the operators $h_0 = \varphi(H_0)$ and $h = \varphi(H)$ are resolvent comparable, we can apply Theorem 2.1 to the pair h_0, h. Then we define the SSF for the pair H_0, H by the relation

$$\xi(\lambda; H, H_0) = \xi(\varphi(\lambda); \varphi(H), \varphi(H_0)). \tag{2.6}$$

Suppose that the function φ is invertible and that

$$\varphi'(\lambda) \geq c > 0.$$

Set $\mu = \varphi(\lambda)$, $\psi = \varphi^{-1}$, $g(\mu) = f(\psi(\mu))$. Then formula (2.3) implies that

$$\operatorname{Tr}\Big(f(H) - f(H_0)\Big) = \operatorname{Tr}\Big(g(h) - g(h_0)\Big) = \int_{-\infty}^{\infty} \xi(\mu; h, h_0) g'(\mu) d\mu.$$

This coincides with (1.1) if the SSF $\xi(\lambda; H, H_0)$ is defined by formula (2.6).

It follows from estimate (2.2) and the conditions on $\varphi(\lambda)$ that the function (2.6) satisfies estimate (2.5). The class of functions $f(\lambda)$ for which formula (1.1) is true is obtained from the class of functions $g(\mu)$ by the change of variables $\mu = \varphi(\lambda)$. Finally, formula (1.2) follows from the same formula for the pair h_0, h, definition (2.6) of the SSF and the invariance principle for scattering matrices. This concludes the proof of Theorem 2.2 given Theorem 2.3.

We emphasize that the trace formula (1.1) fixes the SSF $\xi(\lambda; H, H_0)$ up to an additive constant only. This constant remains undetermined also by condition (2.5). Nevertheless (see [17]) defining the SSF via the corresponding perturbation determinant fixes $\xi(\lambda; H, H_0)$ up to an *integer* constant. It is exactly for this "correct" choice of a constant that relation (1.2) holds.

3. Proof of Theorem 2.3

Our proof of Theorem 2.3 relies on the theory of Double Operator Integrals (DOI). Let us briefly recall its basic notions (see [3, 4], for details). Let H_0 and H be a pair of self-adjoint operators in a Hilbert space $\mathcal{H}$. Denote by E_0 and E their spectral families. Below C are different positive constants whose precise values are inessential. Let us use the following result which can be deduced from Theorem 5.2 of [4].

Proposition 3.1. *Let*

$$K = \Phi(T) = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} K(\lambda, \mu) dE(\mu) T dE_0(\lambda), \tag{3.1}$$

where the kernel $K(\lambda, \mu)$ is bounded, i.e.,

$$|K(\lambda, \mu)| \le C < \infty, \tag{3.2}$$

it is differentiable in λ *and*

$$\Big|\frac{\partial K(\lambda,\mu)}{\partial\lambda}\Big| \leq C(1+\lambda^2)^{-1}. \tag{3.3}$$

Assume, moreover, that

$$\lim_{\lambda\to+\infty} K(\lambda,\mu) = \lim_{\lambda\to-\infty} K(\lambda,\mu) \tag{3.4}$$

(*these limits exist by virtue of* (3.3)). *Then the transformer* $\Phi : \mathfrak{S}_1 \to \mathfrak{S}_1$ *defined by* (3.1) *is bounded.*

Below we always suppose that $z = ia$, $a \in \mathbb{R}$, in (1.3). Let us set

$$g(\lambda) = g_a(\lambda) = (\lambda - ia)^{-m}$$

and

$$T = T_a = g_a(H) - g_a(H_0).$$

We need the representation of the difference $f(H) - f(H_0)$ in terms of the DOI:

$$f(H) - f(H_0) = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \frac{f(\lambda)-f(\mu)}{g(\lambda)-g(\mu)} dE(\mu) T dE_0(\lambda). \tag{3.5}$$

A technical problem with a proof of Theorem 2.3 is that the denominator $g(\lambda)-g(\mu)$ in (3.5) has "extra" zeros at the points

$$\mu_k = -\lambda_k = a\tan(\pi k/m), \quad k = 1,\dots,m-1.$$

Therefore we shall split Theorem 2.3 into two separate assertions.

Proposition 3.2. *Let* $f \in C_0^2(\mathbb{R})$ *so that* $\operatorname{supp} f \subset [-r,r]$ *for some* $r > 0$. *Let the assumption* (1.3) *be satisfied for some odd* m, $z = ia$ *and a sufficiently large* (*compared to* r) *number* a. *Then inclusion* (2.4) *holds.*

Proposition 3.3. *Let a number* $r > 0$ *be given. Suppose that* $\theta \in C^2(\mathbb{R})$, $\theta(\lambda) = 0$ *for* $|\lambda| \leq r$, $\theta(\lambda) = 1$ *for* $|\lambda| \geq 2r$ *and set*

$$f(\lambda) = \theta(\lambda)(\lambda^m - i)^{-1}. \tag{3.6}$$

Let the assumption (1.3) *be satisfied for some odd* m, $z = ia$ *and a sufficiently small* (*compared to* r) *number* a. *Then inclusion* (2.4) *holds.*

Propositions 3.2 and 3.3 imply of course Theorem 2.3. Indeed, suppose that $\varphi(\lambda) = \lambda^m$ for $|\lambda| \geq r$. Then

$$(\varphi(H) - i)^{-1} - (\varphi(H_0) - i)^{-1} = (f_0(H) - f_0(H_0)) + (f(H) - f(H_0)), \qquad (3.7)$$

where $f_0(\lambda) = (1 - \theta(\lambda))(\varphi(\lambda) - i)^{-1}$ has finite support and f is given by formula (3.6). Both terms in the right-hand side of (3.7) belong to the trace class.

For the proof of Propositions 3.2 and 3.3 we need the following elementary result.

Lemma 3.4. *Set*

$$p(\lambda, \mu; a) = (\lambda - ia)^{m-1} + (\lambda - ia)^{m-2}(\mu - ia) + \cdots + (\mu - ia)^{m-1}. \qquad (3.8)$$

Let m *be odd, and let* $r > 0$ *be some fixed number. Then for* $|\lambda| \leq r$, $|\mu| \leq r$ *and a sufficiently large* a

$$|p(\lambda, \mu; a)| \geq c > 0. \qquad (3.9)$$

Similarly, if either $|\lambda| \geq r$ *or* $|\mu| \geq r$ *and* a *is sufficiently small, then*

$$|p(\lambda, \mu; a)| \geq c(|\lambda| + |\mu|)^{m-1}, \quad c > 0. \qquad (3.10)$$

Proof. By virtue of the equality

$$p(\lambda, \mu; a) = (\lambda - ia)^{m-1}(1 + \sigma + \cdots + \sigma^{m-1}), \quad \sigma = (\mu - ia)(\lambda - ia)^{-1},$$

both estimates (3.9) and (3.10) (by the proof of (3.10) we assume that $|\mu| \leq |\lambda|$) reduce to the same estimate

$$|1 + \sigma + \cdots + \sigma^{m-1}| \geq c > 0. \qquad (3.11)$$

If $|\lambda| \leq r$, $|\mu| \leq r$ and a is sufficiently large, then

$$\sigma = (1 + i\mu a^{-1})(1 + i\lambda a^{-1})^{-1}$$

belongs to a neighbourhood of the point 1 which implies (3.11).

Note that the zeros of the function $1+\sigma+\cdots+\sigma^{m-1}$ are given by the formula

$$\sigma_k=\exp(2\pi ik/m),\quad k=1,\dots,m-1.$$

If $|\lambda|\geq r$, $|\lambda|\geq|\mu|$ and a is sufficiently small, then the values of

$$\sigma=(x-i\varepsilon)(1-i\varepsilon)^{-1},\quad x=\mu\lambda^{-1}\in[-1,1],\quad \varepsilon=a\lambda^{-1},$$

belong to a neighbourhood of the real axis and therefore are separated from all points σ_k. This again implies (3.11). □

For the proofs of Propositions 3.2 and 3.3, we consider DOI (3.5) with kernel

$$K(\lambda,\mu)=\frac{f(\lambda)-f(\mu)}{g(\lambda)-g(\mu)} \tag{3.12}$$

and verify the assumptions of Propositions 3.1. Since

$$g(\lambda)-g(\mu)=(\lambda-ia)^{-m}(\mu-ia)^{-m}(\mu-\lambda)p(\lambda,\mu;a), \tag{3.13}$$

we have that

$$K(\lambda,\mu)=-\Phi(\lambda,\mu)G(\lambda,\mu), \tag{3.14}$$

where

$$\Phi(\lambda,\mu)=\frac{f(\lambda)-f(\mu)}{\lambda-\mu} \tag{3.15}$$

and

$$G(\lambda,\mu)=\frac{(\lambda-ia)^m(\mu-ia)^m}{p(\lambda,\mu;a)}. \tag{3.16}$$

Obviously, the function $K(\lambda,\mu)$ tends to $f(\mu)g(\mu)^{-1}$ as $\lambda\to\pm\infty$ so that condition (3.4) is satisfied. Therefore for the proofs of Propositions 3.2 and 3.3 given below we have only to check the estimates (3.2) and (3.3).

Proof of Proposition 3.2. Note that $K(\lambda,\mu)=0$ if $|\lambda|\geq r$ and $|\mu|\geq r$. Let us consider first the region where $|\lambda|\leq R$, $|\mu|\leq R$ for some R. Here we proceed from representation (3.14). The function (3.15) is bounded because $f \in C^1$. Since

$$|f(\mu) - f(\lambda) - f'(\lambda)(\mu - \lambda)| \leq 2^{-1}(\mu - \lambda)^2 \operatorname{supp}_{|\nu| \leq R} |f''(\nu)|,$$

the function

$$\frac{\partial \Phi(\lambda, \mu)}{\partial \lambda} = \frac{f(\mu) - f(\lambda) - f'(\lambda)(\mu - \lambda)}{(\lambda - \mu)^2} \tag{3.17}$$

is also bounded for $|\lambda| \leq R$, $|\mu| \leq R$. According to bound (3.9) the function (3.16) as well as its derivatives in λ are bounded in this region provided a is sufficiently large.

It remains to consider the region $|\mu| \leq r$, $|\lambda| \geq R > r$ (and, similarly, $|\lambda| \leq r$, $|\mu| \geq R > r$). Here we use that the denominator in (3.12) is bounded from below because

$$\begin{aligned} |g(\lambda) - g(\mu)| &\geq (\mu^2 + a^2)^{-m/2} - (\lambda^2 + a^2)^{-m/2} \\ &\geq (r^2 + a^2)^{-m/2} - (R^2 + a^2)^{-m/2} \geq c > 0. \end{aligned} \tag{3.18}$$

Therefore the function (3.12) is bounded. Further, differentiating (3.12), we find that

$$\frac{\partial K(\lambda, \mu)}{\partial \lambda} = \frac{f'(\lambda)}{g(\lambda) - g(\mu)} - \frac{(f(\lambda) - f(\mu))g'(\lambda)}{(g(\lambda) - g(\mu))^2}. \tag{3.19}$$

Now it follows from (3.18) for $|\mu| \leq r$, $|\lambda| \geq R > r$ or for $|\lambda| \leq r$, $|\mu| \geq R > r$ that

$$\Big| \frac{\partial K(\lambda, \mu)}{\partial \lambda} \Big| \leq C(|f'(\lambda)| + (|f(\lambda)| + |f(\mu)|)|g'(\lambda)|) \leq C_1(1 + |\lambda|)^{-m-1}. \tag{3.20}$$

Thus, we have verified both estimates (3.2) and (3.3). □

Proof of Proposition 3.3. Note that $K(\lambda, \mu) = 0$ if $|\lambda| \leq r$ and $|\mu| \leq r$. Let us first consider the region where $|\lambda| \leq 3r$, $|\mu| \leq 3r$ (with the square $|\lambda| \leq r$, $|\mu| \leq r$ removed). Similarly to the proof of Proposition 3.2, we have that functions (3.15) and (3.17) are bounded. The boundedness in this region of the function (3.16), as well as of its derivatives in λ, follows, for sufficiently small a, from Lemma 3.4.

Next we consider the region $|\lambda| \geq 2r$, $|\mu| \geq 2r$ where

$$f(\lambda) - f(\mu) = (\lambda^m - i)^{-1} - (\mu^m - i)^{-1} = (\lambda^m - i)^{-1}(\mu^m - i)^{-1}(\mu - \lambda)p(\lambda, \mu; 0)$$

and the function $p(\lambda, \mu; 0)$ is defined by formula (3.8). Hence it follows from (3.12) and (3.13) that

$$K(\lambda, \mu) = \frac{(\lambda - ia)^m}{\lambda^m - i} \frac{(\mu - ia)^m}{\mu^m - i} \frac{p(\lambda, \mu; 0)}{p(\lambda, \mu; a)}.$$

By virtue of estimate (3.10), this function is bounded. For the proof of (3.3), we use two estimates

$$\Big| \frac{\partial}{\partial \lambda} \frac{(\lambda - ia)^m}{\lambda^m - i} \Big| \leq C\lambda^{-2}$$

and

$$\Big| \frac{\partial}{\partial \lambda} \frac{p(\lambda, \mu; 0)}{p(\lambda, \mu; a)} \Big| \leq C(|\lambda| + |\mu|)^{-2}. \tag{3.21}$$

The first of them is obvious. For the proof of the second, we remark that

$$\frac{\partial}{\partial \lambda} \frac{p(\lambda, \mu; 0)}{p(\lambda, \mu; a)} = p(\lambda, \mu; a)^{-2}$$

$$\Big((p(\lambda, \mu; a) - p(\lambda, \mu; 0)) p'_\lambda(\lambda, \mu; a) - (p'_\lambda(\lambda, \mu; a) - p'_\lambda(\lambda, \mu; 0)) p(\lambda, \mu; a) \Big). \tag{3.22}$$

Let us now take into account that $p(\lambda, \mu; a)$ is a polynomial in λ and μ of degree $m-1$, but the terms of order $m-1$ in $p(\lambda, \mu; a)$ and $p(\lambda, \mu; 0)$ cancel each other. Therefore their difference consists of terms $\lambda^p \mu^q$ where $p + q \leq m - 2$. This gives the estimate

$$|p(\lambda, \mu; a) - p(\lambda, \mu; 0)| \leq C(|\lambda| + |\mu|)^{m-2}. \tag{3.23}$$

Similarly, differentiating $p(\lambda, \mu; a) - p(\lambda, \mu; 0)$, we see that

$$|p'_\lambda(\lambda, \mu; a) - p'_\lambda(\lambda, \mu; 0)| \leq C(|\lambda| + |\mu|)^{m-3}. \tag{3.24}$$

Substituting (3.23) and (3.24) into (3.22) and using estimates (3.10) and

$$|p'_\lambda(\lambda, \mu; a)| \leq C(|\lambda| + |\mu|)^{m-2},$$

we obtain (3.21).

Let us finally consider the region where $|\lambda| \geq 3r$, $|\mu| \leq 2r$ or $|\mu| \geq 3r$, $|\lambda| \leq 2r$. Here we apply again representation (3.12) and estimate (3.18) which yields

$$|K(\lambda, \mu)| \leq C(|f(\lambda)| + |f(\mu)|) \leq C_1 < \infty.$$

The same bound (3.18) shows that the derivative (3.19) satisfies estimate (3.20), which yields (3.3). □

4. The Dirac operator

As an example to which Theorem 2.2 directly applies, we now consider the Dirac operator describing a relativistic particle of spin 1/2. Let $\mathcal{H} = L_2(\mathbb{R}^3; \mathbb{C}^4)$ and

$$H_{00} = -i \sum_{j=1}^{3} \boldsymbol{\alpha}_j \frac{\partial}{\partial x_j} + m\boldsymbol{\alpha}_0, \tag{4.1}$$

where $m > 0$ is the mass of a particle and 4×4 - Dirac matrices $\boldsymbol{\alpha}_0$, $\boldsymbol{\alpha}_1$, $\boldsymbol{\alpha}_2$, $\boldsymbol{\alpha}_3$ satisfy the anticommutation relations

$$\boldsymbol{\alpha}_j \boldsymbol{\alpha}_k + \boldsymbol{\alpha}_k \boldsymbol{\alpha}_j = 0, \quad j \neq k, \quad \boldsymbol{\alpha}_j^2 = I, \quad j, k = 0, 1, 2, 3.$$

These relations determine the matrices $\boldsymbol{\alpha}_j$ up to a unitary equivalence in the space $\mathbb{C}^4$. Their concrete choice is of no importance. If $\boldsymbol{\alpha}_j$ are replaced by $u\boldsymbol{\alpha}_j u^*$ where u is a unitary transformation of $\mathbb{C}^4$, then the operator H_{00} is replaced by a unitary equivalent operator of the same structure.

Making the Fourier transform Φ, we find that $H_{00} = \Phi^* A \Phi$ where A is multiplication by the matrix function (the symbol of H_{00})

$$A(\xi) = \sum_{j=1}^{3} \boldsymbol{\alpha}_j \xi_j + m\boldsymbol{\alpha}_0.$$

It is easy to see that $A(\xi)$ has the eigenvalues

$$a_1(\xi) = a_2(\xi) = -a_3(\xi) = -a_4(\xi) = (|\xi|^2 + m^2)^{1/2} \tag{4.2}$$

of multiplicity 2 so that

$$A(\xi) = T(\xi)\Lambda(\xi)T^*(\xi) \tag{4.3}$$

where the matrices $T(\xi)$ are unitary and $\Lambda(\xi) = \operatorname{diag}\{a_1(\xi), a_2(\xi), a_3(\xi), a_4(\xi)\}$. In particular, the operator H_{00} is self-adjoint on the Sobolev space $\mathsf{H}^1(\mathbb{R}^3; \mathbb{C}^4) =: \mathcal{D}(H_{00})$. The concrete form of the matrices $T(\xi)$ is inessential for us.

Here we consider a pair of Dirac operators

$$H_0 = H_{00} + V_0, \quad H = H_0 + V \tag{4.4}$$

where V_0 is multiplication by a symmetric bounded 4×4 - matrix function $V_0(x)$ and a perturbation V which is also a symmetric 4×4 - matrix function satisfies the condition

$$|V(x)| \leq C(1 + |x|)^{-\rho}, \quad \rho > 3. \tag{4.5}$$

We do not make any special assumptions on the matrices $V_0(x)$ and $V(x)$. In particular, the spectrum of the operator H_0 might cover the whole real axis. A particular case

$$V_0(x) = \sum_{j=0}^{3} v_j^{(0)}(x)\boldsymbol{\alpha}_j + v^{(0)}(x), \quad V(x) = \sum_{j=0}^{3} v_j(x)\boldsymbol{\alpha}_j + v(x),$$

where $v_j^{(0)}, v^{(0)}$ and v_j, v are scalar functions, corresponds to an interaction of an electron (or of a positron) with magnetic and electric fields. Here $v^{(0)}$, v and $(v_1^{(0)}, v_2^{(0)}, v_3^{(0)})$, (v_1, v_2, v_3) are electric and magnetic potentials, respectively. We suppose that the "background" potential $V_0(x)$ is only a bounded function whereas the perturbation $V(x)$ satisfies the condition (4.5). We shall show that in this case condition (1.3) is satisfied for $m = 3$.

Differentiating the resolvent identity

$$R(z) - R_0(z) = -R(z)VR_0(z), \quad \operatorname{Im} z \neq 0,$$

we find that

$$R^m(z) - R_0^m(z) = -\sum_{l=1}^{m} R^l(z) V R_0^{m+1-l}(z). \tag{4.6}$$

If we replace the resolvents R and R_0 by R_{00} in the right-hand side of (4.6), then we obtain the sum of terms $R_{00}^l V R_{00}^{m+1-l}$. It is easy to see (this follows from Proposition 4.1 below) that, for $m \geq 3$, these operators belong to the trace class. Therefore it suffices to justify the replacement of R and R_0 by R_{00}. However the boundedness of the operator $(H_{00} - z)^n R^n(z)$ is equivalent to the inclusion

$$\mathcal{D}(H^n) \subset \mathcal{D}(H_{00}^n).$$

If $n > 1$, this inclusion requires boundedness of derivatives of the function V and is, in general, violated under the conditions above. To bypass this difficulty, we suggest a trick based on commutation of the operators $\langle x \rangle^{-r}$ and $R(z)$. We introduce also the whole scale of Schatten-von Neumann classes $\mathfrak{S}_p$.

We start however with operators $\langle x \rangle^{-r} R_{00}^n(z)$.

Proposition 4.1. *If*

$$p > 3/\min\{r, n\} =: p(r, n), \quad p \geq 1, \tag{4.7}$$

then

$$\langle x \rangle^{-r} R_{00}^n(z) \in \mathfrak{S}_p.$$

Proof. Since $R_{00}^n(z) = \Phi^*(A - z)^{-n}\Phi$, we have to check that the integral operator with kernel

$$\langle x \rangle^{-r} \exp(i < x, \xi >)(A(\xi) - z)^{-n}$$

belongs to the class $\mathfrak{S}_p$ with p determined by (4.7). According to (4.2), (4.3), the function

$$\langle \xi \rangle^n (A(\xi) - z)^{-n}$$

is bounded so that it suffices to consider the operator with kernel

$$\langle x \rangle^{-r} \exp(i < x, \xi >)\langle \xi \rangle^{-n}.$$

As is well-known (see, e.g., [15]), this operator belongs to the required class $\mathfrak{S}_p$. □

Next we extend this result to the operator H (or H_0).

Proposition 4.2. *If p satisfies* (4.7), *then*

$$\langle x\rangle^{-r}R^n(z)\in\mathfrak{S}_p.$$

Proof. If $n=1$, then

$$\langle x\rangle^{-r}R(z)=(\langle x\rangle^{-r}R_{00}(z))\cdot((H_{00}-z)R(z))$$

and the second factor in the right-hand side is a bounded operator. Let us justify the passage from n to $n+1$. Since

$$(H-z)\langle x\rangle^{-r}-\langle x\rangle^{-r}(H-z)=[H_{00},\langle x\rangle^{-r}],$$

we have that

$$\langle x\rangle^{-r}R^{n+1}=R\langle x\rangle^{-r}R^n+R[H_{00},\langle x\rangle^{-r}]R^{n+1}. \tag{4.8}$$

Let us write the first term in the right-hand side of (4.8) as

$$R\langle x\rangle^{-r}R^n=(R\langle x\rangle^{-r_0})(\langle x\rangle^{-nr_0}R^n) \tag{4.9}$$

where $r_0=r(n+1)^{-1}$. Here $R\langle x\rangle^{-r_0}\in\mathfrak{S}_p$ for $p>p(r_0,1)$, and $\langle x\rangle^{-nr_0}R^n\in\mathfrak{S}_p$ for $p>p(nr_0,n)$, by the inductive assumption. Therefore the product (4.9) belongs to the class $\mathfrak{S}_p$ where

$$p^{-1}<p(r_0,1)^{-1}+p(nr_0,n)^{-1}=p(r,n+1)^{-1}.$$

The second term in the right-hand side of (4.8) is even better since the matrix $[H_{00},\langle x\rangle^{-r}]$ is bounded by $\langle x\rangle^{-r-1}$. □

Now it easy to verify inclusion (1.3) for the Dirac operators.

Theorem 4.3. *Suppose that the symmetric matrix functions* $V_0(x)$ *and* $V(x)$ *are bounded and* $V(x)$ *satisfies condition* (4.5) *with* $\rho > 3$. *Then condition* (1.3) *is satisfied for all* $m \geq 3$.

Proof. Indeed, let us write all terms in the right-hand side of (4.6) as

$$R^l V R_0^{m+1-l} = (R^l \langle x\rangle^{-r_1})(\langle x\rangle^{\rho} V)(\langle x\rangle^{-r_2} R_0^{m+1-l}) \tag{4.10}$$

where $r_1 = l\rho(m+1)^{-1}$, $r_2 = (m+1-l)\rho(m+1)^{-1}$. According to Proposition 4.2 the first term in the right-hand side of (4.10) belongs to the class $\mathfrak{S}_p$ for $p > p_1 = p(r_1, l)$ and the last term belongs to the class $\mathfrak{S}_p$ for $p > p_2 = p(r_2, m+1-l)$. Since $p_1^{-1} + p_2^{-1} > 1$, the product (4.10) is trace class. $\square$

Corollary 4.4. *The WO* $W_{\pm}(H, H_0)$ *exist and are complete.*

Finally, combining Theorem 4.3 with Theorem 2.2, we obtain

Theorem 4.5. *Let the operators* H_0 *and* H *be given by formula* (4.4) *where* H_{00} *is the "free" Dirac operator* (4.1). *Suppose that the symmetric matrix functions* $V_0(x)$ *and* $V(x)$ *are bounded and* $V(x)$ *satisfies condition* (4.5) *with* $\rho > 3$. *Let a function* $f(\lambda)$ *have two bounded derivatives and*

$$\partial^\alpha (f(\lambda) - f_0 \lambda^{-3}) = O(|\lambda|^{-3-\epsilon-\alpha}), \quad \alpha = 0, 1, 2, \quad \epsilon > 0,$$

where the constant f_0 *is the same for* $\lambda \to \infty$ *and* $\lambda \to -\infty$. *Then inclusion* (2.4) *holds and there exists a function* (*the SSF*) $\xi(\lambda; H, H_0)$ *satisfying condition* (2.5) *where* $m = 3$ *such that the trace formula* (1.1) *is true. Moreover, for the corresponding scattering matrix* $S(\lambda; H, H_0)$, *the operator* $S(\lambda; H, H_0) - I \in \mathfrak{S}_1$ *and relation* (1.2) *holds for almost all* λ *from the core of the spectrum of the operator* H_0.

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IRMAR, Université de Rennes I, Campus de Beaulieu, 35042 Rennes Cedex, FRANCE

E-mail address: yafaev@univ-rennes1.fr