HIGH ENERGY AND SMOOTHNESS ASYMPTOTIC EXPANSION OF THE SCATTERING AMPLITUDE

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Abstract

We find an explicit expression for the kernel of the scattering matrix for the Schrödinger operator containing at high energies all terms of power order. It turns out that the same expression gives a complete description of the diagonal singularities of the kernel in the angular variables. The formula obtained is in some sense universal since it applies both to short- and long-range electric as well as magnetic potentials.

1. INTRODUCTION

1. High energy asymptotics of the scattering matrix $S(\lambda) : L_2(\mathbb{S}^{d-1}) \to L_2(\mathbb{S}^{d-1})$ for the Schrödinger operator $H = -\Delta + V$ in the space $\mathcal{H} = L_2(\mathbb{R}^d)$, $d \geq 2$, with a real short-range potential (bounded and satisfying the condition $V(x) = O(|x|^{-\rho})$, $\rho > 1$, as $|x| \to \infty$) is given by the Born approximation. To describe it, let us introduce the operator $\Gamma_0(\lambda)$,

$$(\Gamma_0(\lambda) f)(\omega) = 2^{-1/2} k^{(d-2)/2} \hat{f}(k\omega), \quad k = \lambda^{1/2} \in \mathbb{R}_+ = (0, \infty),\ \omega \in \mathbb{S}^{d-1}, \tag{1.1}$$

of the restriction (up to the numerical factor) of the Fourier transform $\hat{f}$ of a function f to the sphere of radius k. Set $R_0(z) = (-\Delta - z)^{-1}$, $R(z) = (H - z)^{-1}$. By the Sobolev trace theorem and the limiting absorption principle the operators $\Gamma_0(\lambda)\langle x\rangle^{-r} : \mathcal{H} \to L_2(\mathbb{S}^{d-1})$ and $\langle x\rangle^{-r} R(\lambda + i0)\langle x\rangle^{-r} : \mathcal{H} \to \mathcal{H}$ are correctly defined as bounded operators for any $r > 1/2$ and their norms are estimated by $\lambda^{-1/4}$ and $\lambda^{-1/2}$, respectively. Therefore it is easy to deduce (see, e.g., [16, 27]) from the usual stationary representation

$$S(\lambda) = I - 2\pi i \Gamma_0(\lambda)(V - V R(\lambda + i0) V) \Gamma_0^*(\lambda) \tag{1.2}$$

for the scattering matrix (SM) and the resolvent identity that

$$S(\lambda) = I - 2\pi i \sum_{n=0}^{N} (-1)^n \Gamma_0(\lambda) V (R_0(\lambda + i0) V)^n \Gamma_0^*(\lambda) + \sigma_N(\lambda), \tag{1.3}$$

where $||\sigma_N(\lambda)|| = O(\lambda^{-(N+2)/2})$ as $\lambda \to \infty$. Moreover, the operators σ_N belong to suitable Schatten - von Neumann classes $\mathfrak{S}_{\alpha(N)}$ and $\alpha(N) \to 0$ as $N \to \infty$.

Nevertheless the Born expansion (1.3) has at least three drawbacks. First, the structure of the n^{th} term is extremely complicated already for relatively small n. Second, (1.3) definitely fails for long-range potentials, and, finally, it fails as $\lambda \to \infty$ for a perturbation of the operator $-\Delta$ by first order differential operators even with short-range coefficients (magnetic potentials).

2. In the particular case when $A = 0$ and V belongs to the Schwartz class a convenient form of the high-energy expansion of the kernel of SM (called often the scattering amplitude) was obtained in [4] (see also the earlier paper [9]). The method of [4] relies on a preliminary study of the scattering solutions of the Schrödinger equation defined, for example, by the formula

$$\psi_\pm(\xi) = u_0(\xi) - R(|\xi|^2 \mp i0)Vu_0(\xi), \quad u_0(x,\xi) = \exp(i\langle x, \xi\rangle), \quad \xi = \hat{\xi}|\xi| \in \mathbb{R}^d.$$

It is shown in [4] that (at least on all compact sets of x) the function $\psi_\pm(x,\xi)$ has the asymptotic expansion $\psi_\pm(x,\xi) = e^{i\langle x,\xi\rangle}\mathrm{b}_\pm(x,\xi)$ where

$$\mathrm{b}_\pm(x,\xi) = \mathrm{b}_\pm^{(N)}(x,\xi) = \sum_{n=0}^{N}(2i|\xi|)^{-n}b_n^{(\pm)}(x,\xi), \quad b_0(x,\xi) = 1, \quad N \to \infty. \tag{1.4}$$

The function $\mathrm{b}_\pm(x,\xi)$ is determined by the transport equation (see subs. 2.3 below), and the coefficients $b_n^{(\pm)}(x,\xi) = b_n^{(\pm)}(x,\hat{\xi})$ are quite explicit. Therefore it is easy to deduce from (1.2) that, for any N, the kernel of SM admits the asymptotic expansion

$$s(\omega,\omega';\lambda) = \delta(\omega,\omega') - \pi i(2\pi)^{-d}k^{d-2}$$
$$\times \sum_{n=0}^{N}(2ik)^{-n}\int_{\mathbb{R}^d} e^{ik\langle\omega'-\omega,x\rangle}V(x)b_n^{(-)}(x,\omega')dx + O(k^{d-3-N}), \tag{1.5}$$

where $\delta(\cdot)$ is of course the Dirac-function on the unit sphere. We emphasize that the functions $b_n^{(-)}(x,\omega')$ are growing as $|x| \to \infty$ in the direction of ω' and the rate of growth increases as n increases. Thus, expansion (1.5) loses the sense (for sufficiently large N) if $V(x)$ decreases only as some power of $|x|^{-1}$.

The generalization of the results of [4] to the case of short-range potentials V satisfying the condition $\partial^\alpha V(x) = O(|x|^{-\rho_v-|\alpha|})$ for some $\rho_v > 1$ was suggested in [24] where the asymptotics of the scattering amplitude was also deduced from that of the scattering solutions. We note finally the paper [3] where the leading term of the high-energy asymptotics of the scattering amplitude was found for short-range magnetic potentials.

3. In the present paper we suggest a new method which allows us to find an explicit function $s_0(\omega,\omega';\lambda)$ which describes with arbitrary accuracy the kernel $s(\omega,\omega';\lambda)$ of the SM $S(\lambda)$ at high energies (as $\lambda \to \infty$) both for short- and long-range electric and magnetic potentials. It turns out that the same function $s_0(\omega,\omega';\lambda)$ gives also all diagonal singularities of the kernel $s(\omega,\omega';\lambda)$ in the angular variables $\omega,\omega' \in \mathbb{S}^{d-1}$. We emphasize that our approach allows us to avoid a study of solutions of the Schrödinger equation.

We consider the Schrödinger operator

$$H = (i\nabla + A(x))^2 + V(x) \tag{1.6}$$

in the space $\mathcal{H}$ with electric $V(x)$ and magnetic $A(x) = (A_1(x), \dots, A_d(x))$ potentials satisfying the assumptions

$$\left.\begin{array}{rcll} |\partial^\alpha V(x)| & \leq & C_\alpha(1+|x|)^{-\rho_v-|\alpha|}, & \rho_v > 0, \\ |\partial^\alpha A(x)| & \leq & C_\alpha(1+|x|)^{-\rho_a-|\alpha|}, & \rho_a > 0, \end{array}\right\} \tag{1.7}$$

for all multi-indices α. We suppose that potentials are real, that is $V(x) = \overline{V(x)}$ and $A_j(x) = \overline{A_j(x)}$, $j = 1, \dots, d$. Set $\rho = \min\{\rho_v, \rho_a\}$, and

$$V_0(x) = V(x) + |A(x)|^2, \quad V_1(x) = V_0(x) + i\mathrm{div}\, A(x).$$

Then

$$H = -\Delta + 2i\langle A(x), \nabla\rangle + V_1(x). \tag{1.8}$$

We emphasize that the cases $\rho > 1$ (short-range potentials) and $\rho \in (0, 1]$ (long-range potentials) are treated in almost the same way.

Let us formulate our main result. The answer is given in terms of approximate solutions of the Schrödinger equation

$$-\Delta\psi(x,\xi) + 2i\langle A(x), \nabla\rangle\psi(x,\xi) + V_1(x)\psi(x,\xi) = |\xi|^2\psi(x,\xi). \tag{1.9}$$

To be more precise, we denote by $u_\pm(x,\xi) = u_\pm^{(N)}(x,\xi)$ explicit functions (see Section 2, for their construction)

$$u_\pm(x,\xi) = e^{i\Theta_\pm(x,\xi)}\mathrm{b}_\pm(x,\xi) \tag{1.10}$$

such that

$$(-\Delta + 2i\langle A(x), \nabla\rangle + V_1(x) - |\xi|^2)u_\pm(x,\xi) = e^{i\Theta_\pm(x,\xi)}r_\pm(x,\xi) =: q_\pm(x,\xi) \tag{1.11}$$

and $r_\pm(x,\xi) = r_\pm^{(N)}(x,\xi)$ tends to zero faster than $|x|^{-p}$ as $|x| \to \infty$ and $|\xi|^{-q}$ as $|\xi| \to \infty$ where $p = p(N) \to \infty$ and $q = q(N) \to \infty$ as $N \to \infty$ off any conical neighborhood of the direction $\hat{x} = \mp\hat{\xi}$. Note that the phase $\Theta_\pm(x,\xi) = \langle x, \xi\rangle$ if $A(x) = 0$ and $V(x)$ is a short-range function and $\Theta_\pm(x,\xi)$ satisfies approximately the eikonal equation in the general case. The function $\mathrm{b}_\pm(x,\xi)$ is obtained as an approximate solution of the corresponding transport equation.

As is well known (see [1]), off the diagonal $\omega = \omega'$, the kernel $s(\omega, \omega'; \lambda)$ is a C^∞-function of $\omega, \omega' \in \mathbb{S}^{d-1}$ where it tends to zero faster than any power of λ^{-1} as $\lambda \to \infty$. Thus, it suffices to describe the structure of $s(\omega, \omega'; \lambda)$ in a neighborhood of the diagonal $\omega = \omega'$. Let $\omega_0 \in \mathbb{S}^{d-1}$ be an arbitrary point, Π_{ω_0} be the plane orthogonal to ω_0 and $\Omega_\pm(\omega_0, \delta) \subset \mathbb{S}^{d-1}$ be determined by the condition $\pm\langle\omega, \omega_0\rangle > \delta > 0$. Set

$$x = \omega_0 z + y, \quad y \in \Pi_{\omega_0}, \tag{1.12}$$

and

$$\begin{aligned} s_0(\omega, \omega'; \lambda) = s_0^{(N)}(\omega, \omega'; \lambda) = \mp\pi i k^{d-2}(2\pi)^{-d} \\ \times\Big(\int_{\Pi_{\omega_0}} \Big(\overline{u_+(y, k\omega)}(\partial_z u_-)(y, k\omega') - u_-(y, k\omega')\overline{(\partial_z u_+)(y, k\omega)}\Big)dy \\ -2i\int_{\Pi_{\omega_0}} \langle A(y), \omega_0\rangle\overline{u_+(y, k\omega)}u_-(y, k\omega')dy\Big) \end{aligned} \tag{1.13}$$

for $\omega, \omega' \in \Omega_\pm = \Omega_\pm(\omega_0, \delta)$. Then, for any p, q and sufficiently large $N = N(p, q)$, the kernel

$$\tilde{s}^{(N)}(\omega, \omega'; \lambda) = s(\omega, \omega'; \lambda) - s_0^{(N)}(\omega, \omega'; \lambda) \tag{1.14}$$

belongs to the class $C^p(\Omega \times \Omega)$ where $\Omega = \Omega_+ \cup \Omega_-$, and its C^p-norm is $O(\lambda^{-q})$ as $\lambda \to \infty$. Thus, all singularities of $s(\omega, \omega'; \lambda)$ both for high energies and in smoothness are described by the explicit formula (1.13). Let $S_0(\lambda)$ be integral operator with kernel $s_0(\omega, \omega'; \lambda)$. In view of representation (1.10), formula (1.13) shows that we actually consider the singular part $S_0(\lambda)$ of the SM as a Fourier integral or, more precisely, a pseudo-differential operator (PDO) acting on the unit sphere and determined by its amplitude.

By our construction of functions (1.10), $u_+(x, \xi) = \overline{u_-(x, -\xi)}$ if $A(x) = 0$. Therefore in the case $A = 0$ the singular part $s_0(\omega, \omega'; \lambda)$ satisfies the same symmetry relation (the time reversal invariance)

$$s(\omega, \omega'; \lambda) = s(-\omega', -\omega; \lambda)$$

as kernel of the SM itself. Kernel (1.13) is also gauge invariant. This means that, for a smooth function $\varphi(x)$, the integrand in (1.13) is not changed if the functions $u_\pm$ are replaced by $e^{-i\varphi} u_\pm$ and the magnetic potential A is replaced by $A - \nabla\varphi$. We emphasize however that throughout the paper we do not use any particular gauge.

Formula (1.13) gives the singular part of the scattering amplitude off any neighborhood of the hyperplane Π_{ω_0}. Since $\omega_0 \in \mathbb{S}^{d-1}$ is arbitrary, this determines the singular part of $s(\omega, \omega'; \lambda)$ for all $\omega, \omega' \in \mathbb{S}^{d-1}$. We note that the leading diagonal singularity of $s(\omega, \omega', \lambda)$ was found in [26] for $\rho_v \in (1/2, 1]$ and $A = 0$.

4. Our approach to the proof of formula (1.13) relies (even in the short-range case, considered earlier in [29]) on the expression of the SM via modified wave operators

$$W_\pm(H, H_0; J_\pm) = s - \lim_{t \to \pm\infty} e^{iHt} J_\pm e^{-iH_0 t}, \tag{1.15}$$

where PDO $J_\pm$ are constructed in terms of the functions $u_\pm(x, \xi)$. Following [11], we kill neighborhoods of "bad" directions $\hat{x} = \mp\hat{\xi}$ by appropriate cut-off functions $\zeta_\pm(x, \xi)$. Let

$$T_\pm = H J_\pm - J_\pm H_0 \tag{1.16}$$

be the "effective" perturbation. The SM $S(\lambda)$ corresponding to wave operators (1.15) admits (see [12, 26, 27, 22]) the representation

$$S(\lambda) = S_1(\lambda) + S_2(\lambda), \tag{1.17}$$

where

$$S_1(\lambda) = -2\pi i \Gamma_0(\lambda) J_+^* T_- \Gamma_0^*(\lambda) \tag{1.18}$$

and

$$S_2(\lambda) = 2\pi i \Gamma_0(\lambda) T_+^* R(\lambda + i0) T_- \Gamma_0^*(\lambda). \tag{1.19}$$

Both these expressions are correctly defined which will be discussed in Sections 5 and 4, respectively.

With the help of the so called propagation estimates [19, 14, 13] we show in Section 4 that the operator $S_2(\lambda)$ has smooth kernel rapidly decaying as $\lambda \to \infty$. Therefore we call

$S_2(\lambda)$ the regular part of the SM. The singular part $S_1(\lambda)$ is given by explicit expression (1.18) not depending on the resolvent of the operator H. However it contains the cut-off functions $\zeta_\pm$. In Section 5 we get rid of these auxiliary functions and, neglecting C^∞-kernels decaying faster than any power of λ^{-1}, transform the kernel of $S_1(\lambda)$ to the invariant expression (1.13). Some consequences of (1.13) are discussed in Section 6.

2. THE EIKONAL AND TRANSPORT EQUATIONS

In this section we give a standard construction of approximate but explicit solutions of the Schrödinger equation. This construction relies on a solution of the corresponding eikonal and transport equations by iterations.

1. Let us plug expression (1.10) into the Schrödinger equation (1.9). Then

$$\begin{aligned}(-\Delta + 2i\langle A(x), \nabla\rangle + V_1(x) - |\xi|^2)(e^{i\Theta}\mathrm{b}) \\ = e^{i\Theta}(|\nabla\Theta|^2\mathrm{b} - i(\Delta\Theta)\mathrm{b} - 2i\langle\nabla\Theta, \nabla\mathrm{b}\rangle - \Delta\mathrm{b} \\ -2\langle A, \nabla\Theta\rangle\mathrm{b} + 2i\langle A, \nabla\mathrm{b}\rangle + V_1\mathrm{b} - |\xi|^2\mathrm{b}), \quad \nabla = \nabla_x. \end{aligned} \tag{2.1}$$

We require that the phase $\Theta(x,\xi)$ and the amplitude $\mathrm{b}(x,\xi)$ be approximate solutions of the eikonal and transport equations, that is

$$|\nabla\Theta|^2 - 2\langle A, \nabla\Theta\rangle + V_0 - |\xi|^2 = \mathrm{q}_0(x,\xi), \tag{2.2}$$

and

$$-2i\langle\nabla\Theta, \nabla\mathrm{b}\rangle + 2i\langle A, \nabla\mathrm{b}\rangle - \Delta\mathrm{b} + (-i\Delta\Theta + i\mathrm{div}\, A + \mathrm{q}_0)\mathrm{b} = r(x,\xi). \tag{2.3}$$

It follows from (2.1) that, for such functions Θ and b, equality (1.11) is satisfied with the same function $r(x,\xi)$ as in (2.3). When considering (2.2), (2.3), we always remove either a conical neighborhood of the direction $\hat{x} = -\hat{\xi}$ (for the sign " + ") or $\hat{x} = \hat{\xi}$ (for the sign " − "). We choose $\Theta(x,\xi) = \Theta_\pm(x,\xi)$ in such a way that $\mathrm{q}_0(x,\xi) = \mathrm{q}_0^{(\pm)}(x,\xi)$ defined by (2.2) is a short-range function of x, and it tends to 0 as $|\xi| \to \infty$. Then we construct $\mathrm{b}(x,\xi) = \mathrm{b}_\pm(x,\xi)$ so that $r(x,\xi) = r_\pm(x,\xi)$ decays as $|x| \to \infty$ as an arbitrary given power of $|x|^{-1}$. It turns out that $r(x,\xi)$ has a similar decay also in the variable $|\xi|^{-1}$.

If V is short-range and $A = 0$, then we can set $\Theta_\pm(x,\xi) = \langle x,\xi\rangle$ and consider the transport equation (2.3) only. However, the eikonal equation (2.2) is necessary for any non-trivial magnetic potential or (and) long-range electric potential V. The transport equation is always unavoidable because, as we shall see below, the function $\Delta\Theta_\pm$ decays at infinity as $|x|^{-1-\rho}$ only and hence, for example, the choice $\mathrm{b}_\pm = 1$ in (1.10) is not sufficient.

We seek $\Theta_\pm(x,\xi)$ in the form

$$\Theta_\pm(x,\xi) = \langle x,\xi\rangle + \Phi_\pm(x,\xi), \tag{2.4}$$

where $(\nabla\Phi_\pm)(x,\xi)$ tends to zero as $|x| \to \infty$ off any conical neighborhood of the direction $\hat{x} = \mp\hat{\xi}$. We construct a solution of equation (2.2) by iterations. Actually, we set

$$\Phi_\pm(x,\xi) = \Phi_\pm^{(N_0)}(x,\xi) = \sum_{n=0}^{N_0}(2|\xi|)^{-n}\phi_n^{(\pm)}(x,\hat{\xi}) \tag{2.5}$$

and plug expressions (2.4) and (2.5) into equation (2.2). Comparing coefficients at the same powers of $(2|\xi|)^{-n}$, $n = -1, 0, \dots, N_0 - 1$, we obtain the equations

$$\langle \hat{\xi}, \nabla \phi_0 \rangle = \langle \hat{\xi}, A \rangle, \quad \langle \hat{\xi}, \nabla \phi_1 \rangle + |\nabla \phi_0|^2 - 2\langle A, \nabla \phi_0 \rangle + V_0 = 0, \tag{2.6}$$

$$\langle \hat{\xi}, \nabla \phi_{n+1} \rangle + \sum_{m=0}^{n} \langle \nabla \phi_m, \nabla \phi_{n-m} \rangle - 2\langle A, \nabla \phi_n \rangle = 0, \quad n \geq 1, \tag{2.7}$$

so that the "error term" equals

$$\mathrm{q}_0(x, \xi) = \sum_{n+m \geq N_0} (2|\xi|)^{-n-m} \langle \nabla \phi_n, \nabla \phi_m \rangle - 2(2|\xi|)^{-N_0} \langle A, \nabla \phi_{N_0} \rangle.$$

Of course, if A is replaced by $\tilde{A} = A - \nabla \varphi$ for some function $\varphi \in C^\infty(\mathbb{R}^d)$, then the function $\tilde{\phi}_0 = \phi_0 - \varphi$ satisfies the first equation (2.6) with the magnetic potential $\tilde{A}$.

All equations (2.6), (2.7) have the form

$$\langle \hat{\xi}, \nabla \phi(x, \hat{\xi}) \rangle + f(x, \hat{\xi}) = 0 \tag{2.8}$$

and can be explicitly solved. Let the domain $\mathbf{\Gamma}_\pm(\epsilon, R) \subset \mathbb{R}^d \times \mathbb{R}^d$ be distinguished by the condition: $(x, \xi) \in \mathbf{\Gamma}_\pm(\epsilon, R)$ if either $|x| \leq R$ or $\pm\langle \hat{x}, \hat{\xi} \rangle \geq -1 + \epsilon$ for some $\epsilon > 0$. Of course, all constants below depend on ϵ and R. The following assertion is almost obvious (see [26], for details).

Lemma 2.1 *Suppose that*

$$|\partial_x^\alpha \partial_\xi^\beta f(x, \hat{\xi})| \leq C_{\alpha,\beta} |\xi|^{-|\beta|} (1 + |x|)^{-\rho - |\alpha|} \tag{2.9}$$

for $x \in \mathbf{\Gamma}_\pm(\epsilon, R)$ and some $\rho > 1$. Then the function

$$\phi^{(\pm)}(x, \hat{\xi}) = \pm \int_0^\infty f(x \pm t\hat{\xi}, \hat{\xi}) dt \tag{2.10}$$

satisfies equation (2.8) *and the estimates*

$$|\partial_x^\alpha \partial_\xi^\beta \phi^{(\pm)}(x, \hat{\xi})| \leq C_{\alpha,\beta} |\xi|^{-|\beta|} (1 + |x|)^{1 - \rho - |\alpha|}, \quad x \in \mathbf{\Gamma}_\pm(\epsilon, R). \tag{2.11}$$

If estimates (2.9) *are fulfilled for some $\rho \in (0, 1)$ only, then the function*

$$\phi^{(\pm)}(x, \hat{\xi}) = \pm \int_0^\infty \Big(f(x \pm t\hat{\xi}, \hat{\xi}) - f(\pm t\hat{\xi}, \hat{\xi}) \Big) dt \tag{2.12}$$

satisfies both equation (2.8) *and estimates* (2.11).

Proceeding by induction, we can solve by formulas (2.10) or (2.12) all equations (2.6) and (2.7). The case where V and A are both short-range is discussed specially in subs. 3. Here we focus on the long-range case. Let us formulate the corresponding result.

Proposition 2.2 *Let assumption* (1.7) *hold for some $\rho \in (0, 1)$ such that ρ^{-1} is not integer. Then estimates*

$$|\partial_x^\alpha \partial_\xi^\beta \phi_n^{(\pm)}(x, \hat{\xi})| \leq C_{\alpha,\beta} |\xi|^{-|\beta|} (1 + |x|)^{1 - n\rho - |\alpha|}, \quad n = 1, 2, \dots, \tag{2.13}$$

and

$$|\partial_x^\alpha \partial_\xi^\beta \mathrm{q}_0^{(\pm)}(x, \xi)| \leq C_{\alpha,\beta} |\xi|^{-N_0 - |\beta|} (1 + |x|)^{-N_0 \rho - |\alpha|}.$$

are fulfilled on the set $\mathbf{\Gamma}_\pm(\epsilon, R)$ for all multi-indices α and β. The function $\phi_0^{(\pm)}(x, \hat{\xi})$ satisfies the same estimate as $\phi_1^{(\pm)}(x, \hat{\xi})$.

Corollary 2.3 *The function* (2.5) *satisfies the estimates*

$$|\partial_x^\alpha \partial_\xi^\beta \Phi_\pm(x,\xi)| \leq C_{\alpha,\beta} |\xi|^{-|\beta|}(1+|x|)^{1-\rho-|\alpha|}, \qquad x \in \mathbf{\Gamma}_\pm(\epsilon, R). \tag{2.14}$$

Below the number N_0 in (2.4) is subject to the only restriction $N_0\rho \geq 2$.

Of course, in particular cases the procedure above can be simplified. For example, if $A = 0$ and V is long-range but $\rho_v > 1/2$, then

$$\Phi^{(\pm)}(x,\xi) = (2|\xi|)^{-1}\phi_1^{(\pm)}(x,\hat{\xi}) = \pm 2^{-1}\int_0^\infty \Big(V(x \pm t\xi) - V(\pm t\xi)\Big)dt.$$

2. An approximate solution of the transport equation (2.3) can be constructed by a procedure similar to the one given above. Using (2.4), we rewrite this equation as

$$-2i\langle \xi, \nabla \mathrm{b}\rangle + 2i\langle A - \nabla\Phi, \nabla \mathrm{b}\rangle - \Delta \mathrm{b} + (-i\Delta\Phi + i\mathrm{div}\, A + \mathrm{q}_0)\mathrm{b} = r. \tag{2.15}$$

We look for the function $\mathrm{b}_\pm(x,\xi)$ in the form (1.4) with bounded in ξ coefficients $b_n^{(\pm)}(x,\xi)$. Plugging this expression into (2.15), we obtain the following recurrent equations

$$\langle \hat{\xi}, \nabla b_{n+1}\rangle = 2i\langle A - \nabla\Phi, \nabla b_n\rangle - \Delta b_n + (-i\Delta\Phi + i\mathrm{div}\, A + \mathrm{q}_0)b_n, \quad n = 0, 1, \ldots, N. \tag{2.16}$$

Then

$$r(x,\xi) = r^{(N)}(x,\xi) = -(2i|\xi|)^{-N}\langle \hat{\xi}, \nabla b_{N+1}\rangle.$$

All these equations have the form (cf. (2.8))

$$\langle \hat{\xi}, \nabla b_{n+1}(x,\xi)\rangle + f_n(x,\xi) = 0,$$

where a short-range function f_n depends on $b_1, \ldots, b_n$ (and is a polynomial of $|\xi|^{-1}$). Therefore they can be solved by one of the formulas (2.10). Thus, using again Lemma 2.1, we obtain

Proposition 2.4 *Let assumption* (1.7) *hold, let* $\rho_1 = \min\{1, \rho\}$ *and let* $(x,\xi) \in \mathbf{\Gamma}_\pm(\epsilon, R)$. *Then functions* $b_n^{(\pm)}$, $n \geq 1$, *satisfy the estimates*

$$|\partial_x^\alpha \partial_\xi^\beta b_n^{(\pm)}(x,\xi)| \leq C_{\alpha,\beta}|\xi|^{-|\beta|}(1+|x|)^{-\rho_1 n - |\alpha|}.$$

The right-hand side of equation (2.3) *satisfies*

$$|\partial_x^\alpha \partial_\xi^\beta r_\pm^{(N)}(x,\xi)| \leq C_{\alpha,\beta}|\xi|^{-N-|\beta|}(1+|x|)^{-1-\rho_1(N+1)-|\alpha|}. \tag{2.17}$$

Corollary 2.5 *The function* (1.4) *satisfies the estimates*

$$|\partial_x^\alpha \partial_\xi^\beta \mathrm{b}_\pm(x,\xi)| \leq C_{\alpha,\beta}|\xi|^{-|\beta|}(1+|x|)^{-|\alpha|}. \tag{2.18}$$

Combining Propositions 2.2 and 2.4, we get the final result.

Theorem 2.6 *For the functions* $\Theta_\pm^{(N_0)}(x,\xi)$ *and* $\mathrm{b}_\pm^{(N)}(x,\xi)$ *constructed in Propositions* 2.2 *and* 2.4, *respectively, and for the functions* $u_\pm(x,\xi) = u_\pm^{(N)}(x,\xi)$ *defined by* (1.10), *equality* (1.11) *holds with the remainder* $r_\pm^{(N)}(x,\xi)$ *satisfying estimates* (2.17) *in the region* $\mathbf{\Gamma}_\pm(\epsilon, R)$.

We emphasize that in contrast to the parameter N_0 which is fixed, we need $N \to \infty$.

3. Of course, the functions $b_n^{(\pm)}(x,\xi)$ contain different powers of $|\xi|^{-1}$. However, in the short-range case $b_n^{(\pm)}$ depend on x and $\hat\xi$ only. Suppose first that $A=0$. Then $\Phi=0$ and equation (2.16) reduces to

$$\langle \hat\xi, \nabla b_{n+1}\rangle = -\Delta b_n + V b_n.$$

Thus, we obtain the following assertion.

Proposition 2.7 *Let* $A=0$ *and let* V *satisfy assumption* (1.7) *with* $\rho_v>1$. *Let* $u_\pm(x,\xi)=e^{i\langle x,\xi\rangle}\mathrm{b}_\pm(x,\xi)$ *where* $\mathrm{b}_\pm$ *is the sum* (1.4) *and the functions* $b_n^{(\pm)}(x,\hat\xi)$ *are defined by recurrent formulas* $b_0^{(\pm)}=1$ *and*

$$b_{n+1}^{(\pm)}(x,\hat\xi) = \mp\int_0^\infty \Big(-\Delta b_n^{(\pm)}(x\pm t\hat\xi,\hat\xi) + V(x\pm t\hat\xi) b_n^{(\pm)}(x\pm t\hat\xi,\hat\xi)\Big)dt.$$

Then for $(x,\xi)\in\mathbf{\Gamma}_\pm(\epsilon,R)$ *and* $\rho_2=\min\{2,\rho_v\}$

$$|\partial_x^\alpha\partial_\xi^\beta b_n^{(\pm)}(x,\hat\xi)| \le C_{\alpha,\beta}|\xi|^{-|\beta|}\,(1+|x|)^{-(\rho_2-1)n-|\alpha|} \tag{2.19}$$

and the remainder (1.11) *satisfies the estimates*

$$|\partial_x^\alpha\partial_\xi^\beta r_\pm^{(N)}(x,\xi)| \le C_{\alpha,\beta}|\xi|^{-N-|\beta|}\,(1+|x|)^{-(\rho_2-1)(N+1)-|\alpha|}. \tag{2.20}$$

Let us write down explicit expressions for the first two functions b_n:

$$b_1^{(\pm)}(x,\hat\xi) = \mp\int_0^\infty V(x\pm t\hat\xi)dt, \tag{2.21}$$

$$b_2^{(\pm)}(x,\hat\xi) = -\int_0^\infty t(\Delta V)(x\pm t\hat\xi)dt + \frac12\Big(\int_0^\infty V(x\pm t\hat\xi)dt\Big)^2. \tag{2.22}$$

If a magnetic potential is non-trivial, then

$$\Phi_\pm(x,\hat\xi) = \phi_0^{(\pm)}(x,\hat\xi) = \mp\int_0^\infty \langle\hat\xi, A(x\pm t\hat\xi)\rangle dt \tag{2.23}$$

and

$$\mathrm{q}_0^{(\pm)} = |\nabla\Phi_\pm|^2 - 2\langle A,\nabla\Phi_\pm\rangle + V_0.$$

Hence it follows from (2.16) that the coefficients $b_n^{(\pm)}(x,\hat\xi)$ are determined by formulas $b_0^{(\pm)}=1$ and

$$b_{n+1}^{(\pm)}(x,\hat\xi) = \mp\int_0^\infty f_n^{(\pm)}(x\pm t\hat\xi,\hat\xi)dt, \tag{2.24}$$

where

$$\begin{aligned} f_n^{(\pm)} &= 2i\langle A-\nabla\Phi_\pm, \nabla b_n^{(\pm)}\rangle - \Delta b_n^{(\pm)} \\ &+(|\nabla\Phi_\pm|^2 - 2\langle A,\nabla\Phi_\pm\rangle + V_1 - i\Delta\Phi_\pm) b_n^{(\pm)}. \end{aligned} \tag{2.25}$$

Let us formulate the result obtained.

Proposition 2.8 *Let* A *and* V *satisfy assumption* (1.7) *with* $\rho>1$, *and let* $\rho_2=\min\{2,\rho\}$. *Define* $\Theta(x,\xi)$ *by formulas* (2.4) *and* (2.23). *Let the functions* $b_n^{(\pm)}$ *be constructed by recurrent formulas* (2.24), (2.25) *and let* $\mathrm{b}_\pm$ *be the sum* (1.4). *Then estimates* (2.19) *on* $b_n^{(\pm)}$ *and* (2.20) *on the remainder* (1.11) *hold.*

3. WAVE OPERATORS AND THE SCATTERING MATRIX

1. Let us recall briefly some basic facts about PDO (see, e.g., [8] or [23]). Let

$$(Af)(x) = (2\pi)^{-d/2} \int_{\mathbb{R}^d} e^{i\langle x,\xi\rangle} a(x,\xi)\hat{f}(\xi)d\xi,$$

where $\hat{f} = \mathcal{F}f$ is the Fourier transform of f from, say, the Schwartz space $\mathcal{S}(\mathbb{R}^d)$ and the symbol $a \in C^\infty(\mathbb{R}^d \times \mathbb{R}^d)$. Sometimes it is more convenient to consider more general PDO determined by their amplitudes. We define such operators in terms of the corresponding sesquilinear forms

$$(\mathbf{A}f, g) = (2\pi)^{-d} \int_{\mathbb{R}^d}\int_{\mathbb{R}^d}\int_{\mathbb{R}^d} e^{i\langle x,\xi'-\xi\rangle} \mathbf{a}(x,\xi,\xi')\hat{f}(\xi')\overline{\hat{g}(\xi)}d\xi d\xi' dx, \tag{3.1}$$

where the amplitude $\mathbf{a}(x,\xi,\xi')$ is also a C^∞-function of all its variables.

It is standard to assume that a and $\mathbf{a}$ belong to Hörmander classes. Set $\langle x\rangle = (1+|x|^2)^{1/2}$, $\langle\xi\rangle = (1+|\xi|^2)^{1/2}$. By definition, the symbol a (or the corresponding operator A) belongs to the class $\mathcal{S}^{n,m}(\rho,\delta)$, $\rho > 0$, $\delta < 1$, if for all multi-indices α and β

$$|(\partial_x^\alpha \partial_\xi^\beta a)(x,\xi)| \le C_{\alpha,\beta}\langle x\rangle^{n-|\alpha|\rho+|\beta|\delta}\langle\xi\rangle^{m-|\beta|}.$$

The operators A from these classes send the Schwartz space $\mathcal{S}(\mathbb{R}^d)$ into itself. For the amplitudes $\mathbf{a}$ we do not have to keep track of the dependence on ξ and ξ'. Thus, $\mathbf{a} \in \mathcal{S}^n(\rho,\delta)$ if for all multi-indices α, β, β', any compact set $K \subset \mathbb{R}^d$ and $\xi,\xi' \in K$

$$|(\partial_x^\alpha \partial_\xi^\beta \partial_{\xi'}^{\beta'} \mathbf{a})(x,\xi,\xi')| \le C_{\alpha,\beta,\beta'}(K)\langle x\rangle^{n-|\alpha|\rho+(|\beta|+|\beta'|)\delta}.$$

Under this assumption the form (3.1) is well-defined as an oscillating integral for $\hat{f},\hat{g} \in C_0^\infty(\mathbb{R}^d)$. We omit in notation ρ and δ if $\rho = 1$ and $\delta = 0$. The definition (3.1) makes also sense if x belongs to some subset of $\mathbb{R}^d$.

Actually, we need a more special class of PDO with oscillating symbols

$$a(x,\xi) = e^{i\Phi(x,\xi)}\alpha(x,\xi), \tag{3.2}$$

where

$$\Phi \in \mathcal{S}^{1-\rho,0}, \quad \rho \in (0,1), \quad \text{and} \quad \alpha \in \mathcal{S}^{n,m}.$$

We denote by $\mathcal{C}^{n,m}(\Phi)$ the class of symbols or operators obeying the conditions above. The definition of the class $\mathcal{C}^n(\Phi)$ in the case of oscillating amplitudes is quite similar. Since $\mathcal{C}^{n,m}(\Phi) \subset \mathcal{S}^{n,m}(\rho, 1-\rho)$, the standard PDO calculus works in the classes $\mathcal{C}^{n,m}(\Phi)$ if $\rho > 1/2$. In the general case the oscillating factors $\exp(i\Phi(x,\xi))$ or $\exp(i\Phi(x,\xi,\xi'))$ should be explicitly taken into account. We often use the notation $\langle x\rangle$ and $\langle\xi\rangle$ for the operators of multiplication by these functions in the coordinate and momentum representations, respectively.

Proposition 3.1 *Let $a \in \mathcal{C}^{n,0}(\Phi)$ for $n \le 0$. Then the operator $A\langle x\rangle^{-n}$ is bounded in the space $L_2(\mathbb{R}^d)$.*

We give only a hint of the proof since it is similar to that of [28] where $a(x,\xi)$ was supposed to be compactly supported in ξ. First, commuting A and $\langle x\rangle^{-n}$, we reduce the problem to the case $n=0$. Then, calculating A^*A, we see that, up to some remainder, it belongs to the class $\mathcal{S}^{0,0}$ and hence is bounded by a simple version of the Calderon-Vaillancourt theorem (see, e.g., [8]). Finally, the remainder can be estimated as in [28] by a direct integration by parts. Note that a result similar to Proposition 3.1 can be found in [15].

We need also a class $\Xi_\pm$ of symbols such that

$$a(x,\xi)=0 \quad \text{if} \quad \mp\langle\hat{x},\hat{\xi}\rangle\leq\varepsilon$$

for some $\varepsilon>0$. Moreover, we assume that $a(x,\xi)=0$ if $|x|\leq\varepsilon$ or $|\xi|\leq\varepsilon$ for symbols from this class. Then we set

$$\mathcal{S}_\pm^{n,m}(\rho,\delta)=\mathcal{S}^{n,m}(\rho,\delta)\cap\Xi_\pm, \quad \mathcal{C}_\pm^{n,m}(\Phi)=\mathcal{C}^{n,m}(\Phi)\cap\Xi_\pm.$$

2. Let $H_0=-\Delta$ and the operator H defined by (1.6) act in the space $\mathcal{H}=L_2(\mathbb{R}^d)$. Denote by E_0 and E their spectral projections. Note that, as shown in [10, 25] where the proof of [21] was extended to magnetic potentials, the operator H does not have positive eigenvalues. In the long-range case the wave operators (1.15) exist only for a special choice of identifications $J_\pm$. We construct $J_\pm$ as PDO.

Let $\sigma\in C^\infty(-\gamma,\gamma)$, $\gamma>1$, be such that $\sigma(\tau)=1$ if $\tau\in[-1,1-2\varepsilon]$ for some $\varepsilon\in(0,1/2)$ and $\sigma(\tau)=0$ if $\tau\in[1-\varepsilon,1]$. Let $\eta\in C^\infty(\mathbb{R}^d)$ be such that $\eta(x)=0$ in a neighborhood of zero and $\eta(x)=1$ for large $|x|$. We denote by ϑ a $C^\infty(\mathbb{R}_+)$-function which equals to zero in a neighborhood of 0 and $\vartheta(\lambda)=1$ for, say, $\lambda\geq\lambda_0$ (for some $\lambda_0>0$). Set

$$\zeta_\pm(x,\xi)=\sigma(\mp\eta(x)\langle\hat{\xi},\hat{x}\rangle)\vartheta(|\xi|^2).$$

Let $u_\pm(x,\xi)$ be the function (it depends on N_0 and N) defined in the previous section (see Theorem 2.6). Following [11], we construct $J_\pm$ by the formula

$$(J_\pm f)(x)=(2\pi)^{-d/2}\int_{\mathbb{R}^d}u_\pm(x,\xi)\zeta_\pm(x,\xi)\hat{f}(\xi)d\xi. \tag{3.3}$$

Thus, $J_\pm$ is a PDO with symbol (3.2) where $\Phi=\Phi_\pm$ and

$$\alpha_\pm(x,\xi)=\mathrm{b}_\pm(x,\xi)\zeta_\pm(x,\xi).$$

We emphasize however that in contrast to [11] the symbol $a_\pm(x,\xi)$ of the operator $J_\pm$ is quite an explicit function. This is essential for construction of the asymptotic expansion of the SM. Due to the cut-off functions $\zeta_\pm(x,\xi)$ and estimates (2.18) on $\mathrm{b}_\pm(x,\xi)$, we have that $\alpha_\pm\in\mathcal{S}^{0,0}$. The function $\Phi_\pm(x,\xi)$ is of course singular on the set $\hat{x}=\mp\hat{\xi}$. However, $\Phi_\pm\tilde{\zeta}_\pm\in\mathcal{S}^{1-\rho,0}$ and $e^{i\Phi_\pm\tilde{\zeta}_\pm}\zeta_\pm=e^{i\Phi_\pm}\zeta_\pm$ for a suitable cut-off function $\tilde{\zeta}_\pm(x,\xi)$ such that $\tilde{\zeta}_\pm(x,\xi)=1$ on the support of $\zeta_\pm$. Abusing somewhat terminology, we write $J_\pm\in\mathcal{C}^{0,0}(\Phi_\pm)$. By Proposition 3.1, the operator $J_\pm$ is bounded.

It is shown in [11, 26, 22] that the wave operators (1.15) exist which implies the intertwining property

$$W_\pm(H,H_0;J_\pm)H_0=HW_\pm(H,H_0;J_\pm).$$

Moreover, they are isometric on the subspace $E_0(\lambda_0,\infty)\mathcal{H}$ and are complete, that is

$$\mathrm{Ran}\,(W_\pm(H,H_0;J_\pm)E_0(\lambda_0,\infty)) = E(\lambda_0,\infty)\mathcal{H}.$$

In the short-range case

$$s-\lim_{t\to\pm\infty}(J_\pm-\vartheta(H_0))e^{-iH_0t}=0,$$

so that the wave operators $W_\pm(H,H_0;J_\pm)$ coincide with the usual wave operators $W_\pm(H,H_0)$ (times $\vartheta(H_0)$). The scattering operator is defined by the standard relation

$$\mathbf{S}=\mathbf{S}(H,H_0;J_+,J_-)=W_+^*(H,H_0;J_+)W_-(H,H_0;J_-).$$

It commutes with the operator H_0 and is unitary on the space $E_0(\lambda_0,\infty)\mathcal{H}$.

3. Let us calculate the perturbation (1.16). According to (1.11), we have that

$$\begin{aligned} g_\pm(x,\xi) &:= (-\Delta+2i\langle A(x),\nabla\rangle+V_1(x)-|\xi|^2)(u_\pm(x,\xi)\zeta_\pm(x,\xi)) \\ &= q_\pm(x,\xi)\zeta_\pm(x,\xi)-2\langle\nabla u_\pm(x,\xi),\nabla\zeta_\pm(x,\xi)\rangle \\ &\quad -u_\pm(x,\xi)(\Delta\zeta_\pm)(x,\xi)+2iu_\pm(x,\xi)\langle A(x),\nabla\zeta_\pm(x,\xi)\rangle. \end{aligned} \tag{3.4}$$

Now it follows from (3.3) that

$$\begin{aligned} (T_\pm f)(x) &= (2\pi)^{-d/2}\int_{\mathbb{R}^d} g_\pm(x,\xi)\hat f(\xi)d\xi \\ &= (2\pi)^{-d/2}\int_{\mathbb{R}^d} e^{i\langle x,\xi\rangle}\left(t_\pm^{(r)}(x,\xi)+t_\pm^{(s)}(x,\xi)\right)\hat f(\xi)d\xi \\ &=: (T_\pm^{(r)}f)(x)+(T_\pm^{(s)}f)(x), \end{aligned} \tag{3.5}$$

where $t_\pm^{(r)}=\exp(i\Phi_\pm)\tau_\pm^{(r)}$, $t_\pm^{(s)}=\exp(i\Phi_\pm)\tau_\pm^{(s)}$ and

$$\tau_\pm^{(r)}=r_\pm\zeta_\pm,\quad \tau_\pm^{(s)}=-2i\mathrm{b}_\pm\langle\xi+\nabla\Phi_\pm-A,\nabla\zeta_\pm\rangle-2\langle\nabla\mathrm{b}_\pm,\nabla\zeta_\pm\rangle-\mathrm{b}_\pm\Delta\zeta_\pm.$$

Due to the cut-off functions $\zeta_\pm$, $\nabla\zeta_\pm$ and $\Delta\zeta_\pm$, the next result follows directly from Propositions 2.2 and 2.4.

Proposition 3.2 *Let assumption* (1.7) *hold and let* $\rho_1=\min\{1,\rho\}$. *Then*

$$t_\pm^{(r)}\in\mathcal{C}^{-1-\rho_1(N+1),-N}(\Phi_\pm)\quad\text{and}\quad t_\pm^{(s)}\in\mathcal{C}_\pm^{-1,1}(\Phi_\pm).$$

4. Let $\mathfrak{N}=L_2(\mathbb{S}^{d-1})$, let the operator $\Gamma_0(\lambda):\mathcal{S}(\mathbb{R}^d)\to\mathfrak{N}$ be defined by formula (1.1) and let $(Uf)(\lambda)=\Gamma_0(\lambda)f$. Then $U:\mathcal{H}\to\tilde{\mathcal{H}}=L_2(\mathbb{R}_+;\mathfrak{N})$ extends by continuity to a unitary operator and UH_0U^* acts in the space $\tilde{\mathcal{H}}$ as multiplication by the independent variable λ. Since $\mathbf{S}H_0=H_0\mathbf{S}$, the operator $U\mathbf{S}U^*$ acts in the space $\tilde{\mathcal{H}}$ as multiplication by the operator-function $S(\lambda):\mathfrak{N}\to\mathfrak{N}$ known as the SM.

We need a stationary formula (see [12, 26, 27, 22]) for the SM $S(\lambda)$ in the case where identifications J_+ and J_- for $t\to+\infty$ and $t\to-\infty$ are different.

Proposition 3.3 *Let assumption* (1.7) *hold. Then the SM admits the representation* (1.17) *where* $S_1(\lambda)$ *and* $S_2(\lambda)$ *are given by formulas* (1.18) *and* (1.19), *respectively.*

Let us discuss here the precise meaning of the expression

$$A^\flat(\lambda) := \Gamma_0(\lambda) A \Gamma_0^*(\lambda),$$

where A is an operator acting on functions on $\mathbb{R}^d$ and $\Gamma_0(\lambda)$ is defined by (1.1). Put

$$\delta_\varepsilon(H_0 - \lambda) = (2\pi i)^{-1}(R_0(\lambda + i\varepsilon) - R_0(\lambda - i\varepsilon)), \tag{3.6}$$

and recall that

$$\lim_{\varepsilon\to 0}(\delta_\varepsilon(H_0 - \lambda) f_1, f_2) = (\Gamma_0(\lambda) f_1, \Gamma_0(\lambda) f_2)_{\mathfrak{N}}, \quad f_1, f_2 \in \mathcal{S}.$$

Therefore it is natural to define (see, e.g., [27]) the sesquilinear form $(A^\flat(\lambda) w_1, w_2)$ for $w_j \in C^\infty(\mathbb{S}^{d-1})$ by the relation

$$(A^\flat(\lambda) w_1, w_2)_{\mathfrak{N}} = 2k^{-d+2} \lim_{\varepsilon\to 0}(A \delta_\varepsilon(H_0 - \lambda)\psi_1, \delta_\varepsilon(H_0 - \lambda)\psi_2), \tag{3.7}$$

where $k = \lambda^{1/2}$,

$$\hat{\psi}_j(\xi) = w_j(\hat{\xi})\gamma_j(|\xi|), \quad j = 1, 2, \tag{3.8}$$

and $\gamma_j \in C_0^\infty(\mathbb{R}_+)$ is an arbitrary function such that $\gamma_j(k) = 1$. The form $(A^\flat(\lambda) w_1, w_2)$ is well defined if the limit (3.7) exists for all $w_j \in C^\infty(\mathbb{S}^{d-1})$. This is, of course, true if $G = \mathcal{F} A \mathcal{F}^*$ is an integral operator with kernel $G(\xi, \xi')$ which is continuous near the surface $|\xi| = |\xi'| = k$. In this case $A^\flat(\lambda)$ is also an integral operator in $L_2(\mathbb{S}^{d-1})$ with kernel

$$g(\omega, \omega'; \lambda) = 2^{-1} k^{d-2} G(k\omega, k\omega'). \tag{3.9}$$

Furthermore, by the Sobolev trace theorem, limit (3.7) exists and hence the operator $A^\flat(\lambda)$ is well-defined (and is bounded in the space $L_2(\mathbb{S}^{d-1})$) if

$$A = \langle x\rangle^{-r} B \langle x\rangle^{-r} \tag{3.10}$$

for a bounded operator B in $L_2(\mathbb{R}^d)$ and $r > 1/2$. This means that the operators $A^\flat(\lambda)$ are also well-defined for PDO of order $n < -1$.

We note that the stationary representation of the SM is determined exactly by the limits as the one in the right-hand side of (3.7).

To estimate in the next section the regular part $S_2(\lambda)$ of the SM, we need the following obvious remark.

Lemma 3.4 *Suppose that* (3.10) *is satisfied for* $r > d/2$ *and set* $u_0(x, \omega, \lambda) = \exp(ik\langle\omega, x\rangle)$. *Then the operator* $A^\flat(\lambda)$ *has continuous kernel*

$$g(\omega, \omega'; \lambda) = 2^{-1} k^{d-2} (2\pi)^{-d} (B\langle x\rangle^{-r} u_0(\omega', \lambda), \langle x\rangle^{-r} u_0(\omega, \lambda)). \tag{3.11}$$

Moreover, this function belongs to the class $C^p(\mathbb{S}^{d-1} \times \mathbb{S}^{d-1})$ *for* $p < r - d/2$ *and its* C^p*-norm is bounded by* Ck^{d-2+p}.

Proof – Representation (3.11) follows from (1.1), as well as from (3.7). The kernel $g(\omega,\omega';\lambda)$ is continuous because the function $\langle x\rangle^{-r}u_0(\omega,\lambda)\in L_2(\mathbb{R}^d)$ depends continuously on ω for any $r>d/2$. Finally, it remains to notice that $\partial u_0/\partial\omega_j = ikx_j u_0$ for any j. □

To treat the singular part $S_1(\lambda)$, we apply definition (3.7) to the PDO $\mathbf{A}=J_+^*T_-$ determined by its amplitude $\mathbf{a}(x,\xi,\xi')$. In this case, by Proposition 3.2, $\mathbf{a}$ is of order -1 and hence the operators $\mathbf{A}^\flat(\lambda)$ are defined only under special assumptions on $\mathbf{a}$. According to (3.4), (3.5), up to an integral operator with smooth kernel, $\mathbf{A}$ has the amplitude which, due to the functions $\nabla\zeta_-(x,\xi')$ and $\Delta\zeta_-(x,\xi')$, equals zero if $\langle\hat{x},\hat{\xi}'\rangle$ is close to 1 or -1 (in a neighborhood of the conormal bundle of each sphere $|\xi'|=k$). In this case, as shown in [28], the operators $\mathbf{A}^\flat(\lambda)$ are correctly defined by formula (3.7) in a space of functions on $\mathbb{S}^{d-1}$ (the case of PDO determined by their symbols was considered earlier in [17]). Moreover, they are also PDO, and an explicit expression for their amplitudes was given in [28]. However, our construction of the singular part of the scattering matrix in Section 5 is, at least formally, independent of the results of [28]. It is important that this construction allows us to get rid of the cut-off functions $\zeta_\pm$ and to obtain an arbitrary close approximation to the SM.

4. THE REGULAR PART

In this section we show that the regular part (1.19) of the SM is negligible.

1. Recall that the functions $u_\pm=u_\pm^{(N)}$ were constructed in Theorem 2.6 and that the corresponding operators $J_\pm=J_\pm^{(N)}$ and $T_\pm=T_\pm^{(N)}$ were defined by equations (3.3) and (3.5), respectively. Our main analytical result here is the following

Proposition 4.1 *For any p and q there exists N such that for $T_\pm=T_\pm^{(N)}$ the operators*

$$B_{p,q,N}(\lambda)=\langle x\rangle^p\langle\xi\rangle^q T_+^*R(\lambda+i0)T_-\langle\xi\rangle^q\langle x\rangle^p$$

are bounded uniformly in $\lambda\geq\lambda_0>0$.

This result will be checked in the following subsections. Let us first of all show that it implies regularity of the operator $S_2(\lambda)$.

Theorem 4.2 *For any p and q there exists N such that for $T_\pm=T_\pm^{(N)}$, the operator* (1.19) *has kernel $s_2(\omega,\omega';\lambda)$ which belongs to the class $C^p(\mathbb{S}^{d-1}\times\mathbb{S}^{d-1})$ and the C^p-norm of this kernel is $O(\lambda^{-q})$ as $\lambda\to\infty$.*

Remark that

$$\Gamma_0(\lambda)\langle\xi\rangle^{-q_0}=(1+\lambda)^{-q_0/2}\Gamma_0(\lambda)$$

and hence

$$S_2(\lambda)=2\pi i(1+\lambda)^{-q_0}\Gamma_0(\lambda)\langle x\rangle^{-p_0}B_{p_0,q_0,N}(\lambda)\langle x\rangle^{-p_0}\Gamma_0^*(\lambda).$$

Let $p_0>d/2+p$ and $q_0\geq q-1+(d+p)/2$. We suppose here that $N=N(p_0,q_0)$ is the same as in Proposition 4.1, so that the operators $B_{p_0,q_0,N}(\lambda)$ are bounded uniformly in $\lambda\geq\lambda_0$. Then, as shown in Lemma 3.4, the kernel of the operator $S_2(\lambda)$ belongs to the class $C^p(\mathbb{S}^{d-1}\times\mathbb{S}^{d-1})$, and its C^p-norm is bounded by $Ck^{d-2+p-2q_0}$ which is estimated by Ck^{-2q}. This concludes the proof of Theorem 4.2.

In the following subsections we shall prove Proposition 4.1.

2. We need some results on the boundedness of combinations of PDO T with symbols $t \in \mathcal{C}_{\pm}^{n,m}(\Phi)$ (see subs. 1 of Section 3) where $\Phi \in \mathcal{S}^{1-\rho,0}$ with functions of the generator of dilations

$$\mathbb{A} = \frac{1}{2}\sum_{j=1}^{d}(x_j D_j + D_j x_j).$$

We denote by $\mathbb{P}_{\pm} = E_{\mathbb{A}}(\mathbb{R}_{\pm})$ the spectral projection of the operator $\mathbb{A}$.

First we formulate a strengthening of a result of [13].

Proposition 4.3 *Let $t \in \mathcal{C}_{\pm}^{n,m}(\Phi)$ for one of the signs and some n, m. Then there exists k such that the operator $\langle \mathbb{A}\rangle^{-k} T$ is bounded.*

Of course, this result is of interest only if at least one of the indices n or m is positive.

We start the proof with the following simple

Lemma 4.4 *Let $T \in \mathcal{C}_{\pm}^{n,m}(\Phi)$ for one of the signs and some n, m. Then there exist PDO $S_0 \in \mathcal{C}_{\pm}^{n-1,m-1}(\Phi)$ and $S_1 \in \mathcal{C}_{\pm}^{n-\rho,m-1}(\Phi)$ such that*

$$T = \mathbb{A} S_0 + S_1.$$

Proof – Suppose, for example, that $T \in \mathcal{C}_{-}^{n,m}(\Phi)$. Let us set

$$s_0(x,\xi) = (1 + \langle x,\xi\rangle)^{-1} t(x,\xi).$$

Since

$$1 + \langle x,\xi\rangle \geq c(1+|x|)\,(1+|\xi|)$$

on the support of t, we have that $s_0 \in \mathcal{C}_{-}^{n-1,m-1}(\Phi)$. An elementary calculation shows that the symbol of the operator $\mathbb{A} S_0$ equals $t - s_1$ where

$$s_1(x,\xi) = (1 + id/2) s_0(x,\xi) + i\langle x, \nabla_x\rangle s_0(x,\xi).$$

This function belongs to the class $\mathcal{C}_{-}^{n-\rho,m-1}(\Phi)$. $\square$

Applying this lemma to both operators S_0, S_1 and repeating this procedure several times, we obtain more general

Lemma 4.5 *Let the assumptions of Lemma 4.4 be satisfied. Then for any positive k there exist PDO $S_0 \in \mathcal{C}_{\pm}^{n-k,m-k}(\Phi)$, $S_1 \in \mathcal{C}_{\pm}^{n-k+1-\rho,m-k}(\Phi), \ldots, S_k \in \mathcal{C}_{\pm}^{n-k\rho,m-k}(\Phi)$ such that*

$$T = \mathbb{A}^k S_0 + \mathbb{A}^{k-1} S_1 + \cdots + S_k. \tag{4.1}$$

Let k be such that $k\rho \geq n$, $k \geq m$. Then, by Proposition 3.1, all operators $S_0, S_1, \cdots, S_k$ are bounded. Hence equation (4.1) implies that the operator $\langle \mathbb{A}\rangle^{-k} T$ is also bounded.

Proposition 4.3 can be formulated in a formally more general way.

Proposition 4.6 *Let $t \in \mathcal{C}_{\pm}^{n,m}(\Phi)$ for one of the signs and some n, m, and let p, q be arbitrary numbers. Then there exists k such that the operator*

$$\langle \mathbb{A}\rangle^{-k} T \langle \xi\rangle^{q} \langle x\rangle^{p}$$

is bounded.

The proof reduces to obvious commutations of the operator $\langle x\rangle^p$ with the operators $\langle\xi\rangle^q$ and T.

The following assertion is also motivated by the results of [13].

Proposition 4.7 *Let $t \in \mathcal{S}_{\pm}^{n,m}(\rho,\delta)$ for some n, m and $\rho > 0$, $\delta < 1$. Then the operator*

$$\langle\mathbb{A}\rangle^k \mathbb{P}_{\pm} T \langle\xi\rangle^q \langle x\rangle^p$$

is bounded for arbitrary p, q and k.

Commutating the operator $\langle x\rangle^p$ with the operators $\langle\xi\rangle^q$ and T and using that n, m are arbitrary, we reduce the problem to the case $p = q = 0$.

Let us check Proposition 4.7, for example, for the upper sign. We standardly diagonalize the operator $\mathbb{A}$ by the Mellin transform. Let the operator

$$\mathcal{M} : \mathcal{K} = L_2(\mathbb{R}) \otimes L_2(\mathbb{S}^{d-1}) \to L_2(\mathbb{R}^d)$$

be defined by the equality

$$(\mathcal{M}f)(r,\omega) = (2\pi)^{-1/2} r^{-d/2} \int_{-\infty}^{\infty} e^{is\ln r} f(s,\omega)ds.$$

Then $\mathcal{M}$ is unitary and

$$\mathbb{A} = \mathcal{M}\Sigma\mathcal{M}^*,$$

where Σ is the operator of multiplication by independent variable $s \in \mathbb{R}$ in the space $\mathcal{K}$.

It follows that

$$(\langle\mathbb{A}\rangle^k \mathbb{P}_+ f)(r,\omega) = (2\pi)^{-1/2} r^{-d/2} \int_0^{\infty} e^{is\ln r} (1+s^2)^{k/2} g(s,\omega)ds,$$

where $g = \mathcal{M}^* f$ and $||g|| = ||f||$. Hence

$$(\langle\mathbb{A}\rangle^k \mathbb{P}_+ f, Th) = (2\pi)^{-(d+1)/2} \lim_{\epsilon\to 0} \int_0^{\infty} dr \varphi(\epsilon r) r^{d/2-1}$$
$$\times \int_{\mathbb{S}^{d-1}} d\omega \int_0^{\infty} ds e^{is\ln r} (1+s^2)^{k/2} g(s,\omega) \int_{\mathbb{R}^d} e^{-ir\langle\omega,\xi\rangle} \overline{t(r\omega,\xi)} \overline{\hat{h}(\xi)} d\xi,$$

where $\varphi \in C_0^{\infty}(\mathbb{R}_+)$ and $\varphi(0) = 1$. Interchanging here the order of integrations, we find that

$$(\langle\mathbb{A}\rangle^k \mathbb{P}_+ f, Th) = (2\pi)^{-(d+1)/2} \lim_{\epsilon\to 0} \int_{\mathbb{S}^{d-1}} d\omega \int_0^{\infty} ds$$
$$\times (1+s^2)^{k/2} g(s,\omega) \int_{\mathbb{R}^d} K_\epsilon(\xi,s,\omega) \overline{\hat{h}(\xi)} d\xi, \tag{4.2}$$

where

$$K_\epsilon(\xi,s,\omega) = \int_0^{\infty} \varphi(\epsilon r) \tilde{t}(r\omega,\xi) e^{-ir\langle\omega,\xi\rangle + is\ln r} dr \tag{4.3}$$

and $\tilde{t}(r\omega,\xi) = r^{d/2-1}\overline{t(r\omega,\xi)}$.

Now the proof of Proposition 4.7 reduces to a direct integration by parts. To that end, we formulate

Lemma 4.8 *Let*

$$K_\epsilon(\xi,s,\omega)=\int_0^\infty \varphi(\epsilon r)t(r\omega,\xi,s)e^{-ir\langle\omega,\xi\rangle+is\ln r}dr, \tag{4.4}$$

where $t\in\Xi_+$ *uniformly in* $s\geq 0$ *and for some* n *and* γ

$$|(\partial_x^\alpha t)(x,\xi,s)|\leq C_\alpha\langle x\rangle^{n-|\alpha|\rho}(|\xi|+s)^{-\gamma},\quad \rho>0. \tag{4.5}$$

Then

$$K_\epsilon(\xi,s,\omega)=\int_0^\infty (\epsilon\varphi'(\epsilon r)t_0(r\omega,\xi,s)+\varphi(\epsilon r)t_1(r\omega,\xi,s))e^{-ir\langle\omega,\xi\rangle+is\ln r}dr, \tag{4.6}$$

where $t_j\in\Xi_+$, $j=0,1$, *and satisfy the estimate* (4.5) *with* $n_0=n$, $\gamma_0=\gamma$ *and* $n_1=n-\rho/2$, $\gamma_1=\gamma+\rho/2$.

Proof – Integrating by parts in (4.4), we find that (4.6) holds with

$$t_0(x,\xi,s)=it(x,\xi,s)r(-\langle x,\xi\rangle+s)^{-1}$$

and $t_1(x,\xi,s)=(\partial_r t_0)(x,\xi,s)$. Since $t\in\Xi_+$, we have that on the support of t

$$|-\langle x,\xi\rangle+s|\geq\varepsilon|x|\,|\xi|+s.$$

Taking into account (4.5), we obtain the necessary estimates on t_0 and t_1. □

Applying this lemma p times to the function (4.3), we find that

$$K_\epsilon(\xi,s,\omega)=\int_0^\infty (\epsilon^p\varphi^{(p)}(\epsilon r)t_0(r\omega,\xi,s)+\ldots+\varphi(\epsilon r)t_p(r\omega,\xi,s))e^{-ir\langle\omega,\xi\rangle+is\ln r}dr, \tag{4.7}$$

where

$$|t_k(x,\xi,s)|\leq C\langle x\rangle^{n+d/2-1-k\rho/2}(|\xi|+s)^{-k\rho/2}. \tag{4.8}$$

Using expression (4.7) for the kernel $K_\epsilon(\xi,s,\omega)$, we can now pass to the limit $\epsilon\to 0$ in (4.2). Due to the estimates (4.8), for sufficiently large p, all terms containing the derivatives of φ disappear in this limit, and we obtain

$$\begin{aligned}(\langle\mathbb{A}\rangle^k\mathbb{P}_+f,Th)&=(2\pi)^{-(d+1)/2}\int_{\mathbb{S}^{d-1}}d\omega\int_0^\infty ds\\ &\times(1+s^2)^{k/2}g(s,\omega)\int_{|\xi|\geq\varepsilon}K_0(\xi,s,\omega)\overline{\hat{h}(\xi)}d\xi,\end{aligned} \tag{4.9}$$

where

$$|K_0(\xi,s,\omega)|\leq C_N(|\xi|+s)^{-N},\quad N=p\rho/2.$$

Since N is arbitrary large, it follows from (4.9) that

$$\begin{aligned}|(\langle\mathbb{A}\rangle^k\mathbb{P}_+f,Th)|&\leq C\int_{\mathbb{S}^{d-1}}d\omega\int_0^\infty ds(1+s^2)^{k/2}|g(s,\omega)|\\ \times\int_{|\xi|\geq\varepsilon}(|\xi|+s)^{-N}|\hat{h}(\xi)|d\xi&\leq C_1||g||\,||\hat{h}||=C_1||f||\,||h||.\end{aligned}$$

This concludes the proof of Proposition 4.7.

3. The following resolvent estimates were deduced in [19, 14, 13] from the famous Mourre estimate [18].

Proposition 4.9 *Let assumption* (1.7) *hold. Then for* $\operatorname{Re} z > 0$, $\operatorname{Im} z \geq 0$ *the operator-functions*

$$\langle \mathbb{A} \rangle^{-p} R(z) \langle \mathbb{A} \rangle^{-p}, \quad p > 1/2, \tag{4.10}$$

$$\langle \mathbb{A} \rangle^{-1+p_2} \mathbb{P}_- R(z) \langle \mathbb{A} \rangle^{-p_1}, \quad \langle \mathbb{A} \rangle^{-p_1} R(z) \mathbb{P}_+ \langle \mathbb{A} \rangle^{-1+p_2} \tag{4.11}$$

for each $p_1 > 1/2$, $p_2 < p_1$ *and*

$$\langle \mathbb{A} \rangle^{p} \mathbb{P}_- R(z) \mathbb{P}_+ \langle \mathbb{A} \rangle^{p} \tag{4.12}$$

for arbitrary p *are continuous in norm with respect to* z.

The proof of Proposition 4.9 in [13] relies on the two following analytical facts. Since the operator H does not have positive eigenvalues, for any $\lambda > 0$, sufficiently small ε and $\Lambda = (\lambda - \varepsilon, \lambda + \varepsilon)$, the Mourre estimate

$$iE(\Lambda)[H, \mathbb{A}]E(\Lambda) \geq cE(\Lambda), \quad c > 0, \tag{4.13}$$

holds. Let

$$B_1 = [H - H_0, \mathbb{A}], \quad B_2 = [B_1, \mathbb{A}], \dots, B_{n+1} = [B_n, \mathbb{A}].$$

Then all operators $B_n(H_0 + I)^{-1}$, $n = 1, 2, \dots$, are bounded. As shown in [13], these two results imply Proposition 4.9.

Remark 4.10 *If a family of operators* $H^{(\kappa)}$ *satisfies* (4.13) *with a common constant* c *and the corresponding operators* $B_n^{(\kappa)}$ *are bounded uniformly in* κ, *then the norms of the operators* (4.10) − (4.12) *are bounded on any compact set of* z *uniformly in* κ.

To extend Proposition 4.9 to high energies, we make the dilation transformation

$$(\mathbf{G}^{(\kappa)} f)(x) = \kappa^{-d/2} f(\kappa^{-1} x).$$

The operators $\mathbb{A}$ and $\mathbb{P}_\pm$ commute with $\mathbf{G}^{(\kappa)}$ and

$$H\mathbf{G}^{(\kappa)} = \kappa^{-2} \mathbf{G}^{(\kappa)} H^{(\kappa)}, \tag{4.14}$$

where H is given by (1.8),

$$H^{(\kappa)} = -\Delta + 2i\langle A^{(\kappa)}(x), \nabla \rangle + V_1^{(\kappa)}(x)$$

and

$$A^{(\kappa)}(x) = \kappa A(\kappa x), \quad V_1^{(\kappa)}(x) = \kappa^2 V_1(\kappa x).$$

Let us now apply Proposition 4.9 to the family of operators $H^{(\kappa)}$ for $\kappa \leq \kappa_0$ and sufficiently small κ_0. Clearly,

$$i[H_0, \mathbb{A}] = 2H_0, \quad i[V_1^{(\kappa)}, \mathbb{A}] = -\kappa^3 \langle x, (\nabla V_1)(\kappa x) \rangle,$$

$$[\langle A^{(\kappa)}, \nabla \rangle, \mathbb{A}] = -i\kappa \langle A(\kappa x), \nabla \rangle - \kappa^2 \sum_{j=1}^{d} \langle x, (\nabla A_j)(\kappa x) \rangle D_j,$$

and hence under assumption (1.7)

$$||V_1^{(\kappa)}|| \leq C\kappa^2, \quad ||\langle A^{(\kappa)}, \nabla \rangle (H_0 + I)^{-1}|| \leq C\kappa, \quad ||[V_1^{(\kappa)}, \mathbb{A}]|| \leq C\kappa^2,$$
$$||[\langle A^{(\kappa)}, \nabla \rangle, \mathbb{A}](H_0 + I)^{-1}|| \leq C\kappa.$$

Therefore estimate (4.13) for the operators $H^{(\kappa)}$ is satisfied in a neighborhood of, say, the point $\lambda = 1$ with a constant c which does not depend on κ. Quite similarly, it can be checked that all operators $B_n^{(\kappa)}(H_0 + I)^{-1}$ are bounded by $C_n\kappa$. Thus, it follows from Proposition 4.9 and Remark 4.10 that estimates (4.10) – (4.12) for the resolvents of the operators $H^{(\kappa)}$ are satisfied in a neighborhood of the point $\lambda = 1$ uniformly in $\kappa \in (0, \kappa_0)$.

Let us now set $\kappa = \lambda^{-1/2}$. Then it follows from (4.14) that, for example,

$$||\langle \mathbb{A} \rangle^{-p} R(\lambda + i0) \langle \mathbb{A} \rangle^{-p}|| = \lambda^{-1} ||\langle \mathbb{A} \rangle^{-p} (H^{(\kappa)} - 1 - i0)^{-1} \langle \mathbb{A} \rangle^{-p}||$$

and similarly for the operators (4.11), (4.12). Thus, we obtain

Proposition 4.11 *Let assumption* (1.7) *hold. Then the norms of the operators* (4.10) – (4.12) *at* $z = \lambda + i0$ *are bounded by* $C\lambda^{-1}$ *as* $\lambda \to \infty$.

4. Now we are able to check Proposition 4.1. Let us first show that the operators

$$\langle x \rangle^p \langle \xi \rangle^q (T_+^{(r)})^* R(\lambda + i0) T_-^{(r)} \langle \xi \rangle^q \langle x \rangle^p$$

are uniformly bounded provided N is large enough. Note that the operators $\langle x \rangle^\sigma T_\pm^{(r)} \langle \xi \rangle^q \langle x \rangle^p$ are bounded by Propositions 3.1 and 3.2 if $(N+1)\rho_1 \geq \sigma + p - 1$ and $N \geq q$. Thus, it suffices to use that

$$||\langle x \rangle^{-\sigma} R(\lambda + i0) \langle x \rangle^{-\sigma}|| = O(\lambda^{-1/2}), \quad \sigma > 1/2,$$

which follows, for example, from the result of Proposition 4.11 about operator (4.10).

Let us further consider the singular part $T_\pm^{(s)}$ of $T_\pm$. Recall that, according to Proposition 3.2, $T_\pm^{(s)} \in \mathcal{C}_\pm^{-1,1}(\Phi_\pm)$. We need to prove the uniform boundedness of four operators

$$\langle x \rangle^p \langle \xi \rangle^q (T_+^{(s)})^* \mathbb{P}_- R(\lambda + i0) \mathbb{P}_+ T_-^{(s)} \langle \xi \rangle^q \langle x \rangle^p, \tag{4.15}$$

$$\langle x \rangle^p \langle \xi \rangle^q (T_+^{(s)})^* \mathbb{P}_+ R(\lambda + i0) \mathbb{P}_- T_-^{(s)} \langle \xi \rangle^q \langle x \rangle^p \tag{4.16}$$

and

$$\langle x \rangle^p \langle \xi \rangle^q (T_+^{(s)})^* \mathbb{P}_\pm R(\lambda + i0) \mathbb{P}_\pm T_-^{(s)} \langle \xi \rangle^q \langle x \rangle^p. \tag{4.17}$$

The operator (4.15) can be factorized into a product of three operators

$$\langle x \rangle^p \langle \xi \rangle^q (T_+^{(s)})^* \langle \mathbb{A} \rangle^{-k}, \quad \langle \mathbb{A} \rangle^k \mathbb{P}_- R(\lambda + i0) \mathbb{P}_+ \langle \mathbb{A} \rangle^k \quad \text{and} \quad \langle \mathbb{A} \rangle^{-k} T_-^{(s)} \langle \xi \rangle^q \langle x \rangle^p.$$

The first and the third factors are bounded for sufficiently large k by Proposition 4.6 while the second operator has the form (4.12), and hence it is bounded by $C\lambda^{-1}$ by Proposition 4.11.

The operator (4.16) can be factorized into a product of three operators

$$\langle x \rangle^p \langle \xi \rangle^q (T_+^{(s)})^* \mathbb{P}_+ \langle \mathbb{A} \rangle^\sigma, \quad \langle \mathbb{A} \rangle^{-\sigma} R(\lambda + i0) \langle \mathbb{A} \rangle^{-\sigma} \quad \text{and} \quad \langle \mathbb{A} \rangle^\sigma \mathbb{P}_- T_-^{(s)} \langle \xi \rangle^q \langle x \rangle^p.$$

The first and the third factors are bounded for each σ by Proposition 4.7 while the second operator has the form (4.10), and hence it is bounded for any $\sigma > 1/2$ by $C\lambda^{-1}$ by Proposition 4.11.

Finally, we factorize the operator (4.17) (for the sign " + ", for example) into a product of three operators $\langle x\rangle^p\langle\xi\rangle^q(T_+^{(s)})^*\mathbb{P}_+\langle\mathbb{A}\rangle^\sigma$, $\langle\mathbb{A}\rangle^{-\sigma}R(\lambda+i0)\mathbb{P}_+\langle\mathbb{A}\rangle^{-1+\sigma-\varepsilon}$, $\varepsilon>0$, and $\langle\mathbb{A}\rangle^{1-\sigma+\varepsilon}T_-^{(s)}\langle\xi\rangle^q\langle x\rangle^p$. The first factor is bounded for any σ by Proposition 4.7. The second operator has the form (4.11), and hence it is bounded for any $\sigma>1/2$ by $C\lambda^{-1}$ by Proposition 4.11. The last factor is bounded by Proposition 4.6 if σ is sufficiently large.

The cross-terms containing $T_+^{(r)}$ and $T_-^{(s)}$ can be considered quite similarly. We need to prove the uniform boundedness of two operators

$$\langle x\rangle^p\langle\xi\rangle^q(T_+^{(r)})^*R(\lambda+i0)\mathbb{P}_\tau T_-^{(s)}\langle\xi\rangle^q\langle x\rangle^p,$$

where $\tau=$ " + " or $\tau=$ " − ". First, using Proposition 3.2, for any l we can choose N such that the operator $\langle x\rangle^p\langle\xi\rangle^q(T_+^{(r)})^*\langle\mathbb{A}\rangle^l$ is bounded and hence it suffices to consider the operators

$$\langle\mathbb{A}\rangle^{-l}R(\lambda+i0)\mathbb{P}_\tau T_-^{(s)}\langle\xi\rangle^q\langle x\rangle^p. \tag{4.18}$$

If $\tau=$ "−", then the operators (4.18) are uniformly bounded for any $l>1/2$ according to Proposition 4.7 and the estimate of Proposition 4.11 on the operator (4.10). If $\tau=$ " + ", then according to Proposition 4.6 the operator $\langle\mathbb{A}\rangle^{-k}T_-^{(s)}\langle\xi\rangle^q\langle x\rangle^p$ is bounded for sufficiently large k. So it remains to use that the operator $\langle\mathbb{A}\rangle^{-l}R(\lambda+i0)\mathbb{P}_+\langle\mathbb{A}\rangle^k$ has the form (4.11), and hence it is bounded by $C\lambda^{-1}$ for $l>k+1$ by Proposition 4.11.

This concludes our proof of Proposition 4.1 and hence of Theorem 4.2.

5. THE SINGULAR PART

1. Let us discuss the precise meaning of the formula (1.13). Recall that $\omega_0\in\mathbb{S}^{d-1}$ is an arbitrary point, $\Pi=\Pi_{\omega_0}$ is the hyperplane orthogonal to ω_0 and $\Omega_\pm=\Omega_\pm(\omega_0,\delta)\subset\mathbb{S}^{d-1}$ is determined by the condition $\pm\langle\omega,\omega_0\rangle>\delta>0$. The coordinates (z,y) in $\mathbb{R}^d$ are defined by equation (1.12). Set

$$h_\pm(x,\xi)=e^{i\Phi_\pm(x,\xi)}\mathrm{b}_\pm(x,\xi), \tag{5.1}$$

so that

$$u_\pm(x,\xi)=e^{i\langle x,\xi\rangle}h_\pm(x,\xi).$$

Then (1.13) can be rewritten as

$$s_0(\omega,\omega';\lambda)=(2\pi)^{-d+1}\int_\Pi e^{ik\langle y,\omega'-\omega\rangle}\mathbf{a}_0(y,\omega,\omega';\lambda)dy, \tag{5.2}$$

where $\omega,\omega'\in\Omega_\pm$ and

$$\begin{aligned}\mathbf{a}_0(y,\omega,\omega';\lambda)=\pm 2^{-1}k^{d-2}\Big(&k\langle\omega+\omega',\omega_0\rangle\overline{h_+(y,k\omega)}h_-(y,k\omega')\\ &+ih_-(y,k\omega')\overline{(\partial_z h_+)(y,k\omega)}-i\overline{h_+(y,k\omega)}(\partial_z h_-)(y,k\omega')\\ &-2\langle A(y),\omega_0\rangle\overline{h_+(y,k\omega)}h_-(y,k\omega')\Big).\end{aligned} \tag{5.3}$$

Formula (5.2) shows that $S_0(\lambda)$ is, actually, regarded as a PDO with amplitude $\mathbf{a}_0(y,\omega,\omega';\lambda)$. Of course, we can replace the amplitude $\mathbf{a}_0(y,\omega,\omega';\lambda)$ by the corresponding symbol $a_0(y,\omega;\lambda)$ (as shown in [28], this procedure is possible for any $\rho>0$ due to the oscillating nature of $\mathbf{a}_0$). However, the expression for $a_0(y,\omega;\lambda)$ is much more complicated than (5.3), and the representation of $s_0(\omega,\omega';\lambda)$ in terms of $a_0(y,\omega;\lambda)$ is not symmetric.

It is convenient to define the operator $S_0(\lambda)$ via its sesquilinear form. Indeed, suppose, for example, that $\omega\in\Omega=\Omega_+$ and denote by Σ and ζ the orthogonal projections of Ω and of a point $\omega\in\Omega$ on the hyperplane Π which we identify with $\mathbb{R}^{d-1}$. We also identify below points $\omega\in\Omega$ and $\zeta\in\Sigma$ and functions

$$w(\omega)=\tilde{w}(\zeta) \tag{5.4}$$

on Ω and Σ. Set

$$\tilde{\mathbf{a}}_0(y,\zeta,\zeta';\lambda)=(1-|\zeta|^2)^{-1/2}(1-|\zeta'|^2)^{-1/2}\mathbf{a}_0(y,\omega,\omega';\lambda).$$

Then it follows from (5.2) that for arbitrary $w_j\in C_0^\infty(\Omega)$, $j=1,2$,

$$(S_0(\lambda)w_1,w_2)=(2\pi)^{-d+1}\int_\Pi\int_\Pi\int_\Pi e^{ik\langle y,\zeta'-\zeta\rangle}\tilde{\mathbf{a}}_0(y,\zeta,\zeta';\lambda)\tilde{w}_1(\zeta')\overline{\tilde{w}_2(\zeta)}d\zeta d\zeta' dy. \tag{5.5}$$

Since $\tilde{\mathbf{a}}_0\in\mathcal{S}^0(\rho,1-\rho)$, the right-hand side of the last equation is well-defined as an oscillating integral which gives the precise sense to its left-hand side. Indeed, integrating by parts in the variable ζ (or ζ'), we see that

$$\begin{aligned}(S_0(\lambda)w_1,w_2)=(2\pi)^{-d+1}\int_\Pi\int_\Pi\int_\Pi e^{ik\langle y,\zeta'-\zeta\rangle}\langle ky\rangle^{-n}\tilde{w}_1(\zeta')\\ \times\langle D_\zeta\rangle^n(\tilde{\mathbf{a}}_0(y,\zeta,\zeta';\lambda)\overline{\tilde{w}_2(\zeta)})d\zeta d\zeta' dy,\end{aligned}$$

and for sufficiently large n this integral is absolutely convergent. Of course, we can make the change of variables $y\mapsto k^{-1}y$ in (5.5) transforming PDO $S_0(\lambda)$ to the standard form, but this operation is not really necessary. It follows from (5.1) that amplitude (5.3) contains an oscillating factor $\exp(i\Xi)$ where

$$\Xi(y,\omega,\omega';k)=\Phi_-(y,k\omega')-\Phi_+(y,k\omega), \tag{5.6}$$

and hence the operator $S_0(\lambda)$ is bounded according to Proposition 3.1.

2. It follows from Theorem 4.2 that the operator (1.18) contains all power terms of the high-energy expansion of the SM as well as of its diagonal singularity. However, the obvious drawback of the expression (1.18) is that it depends on the cut-off functions $\zeta_\pm$. Our final goal is to show that, up to negligible terms, it can be transformed to the invariant expression (1.13). We recall that according to (3.7) the singular part of the SM is determined by its sesquilinear form

$$(S_1(\lambda)w_1,w_2)=-4i\pi k^{-d+2}\lim_{\varepsilon\to0}(T_-\mathcal{F}^*\delta_\varepsilon(|\xi|^2-\lambda)\hat{\psi}_1,J_+\mathcal{F}^*\delta_\varepsilon(|\xi|^2-\lambda)\hat{\psi}_2), \tag{5.7}$$

where the functions $\hat{\psi}_j$ and defined by (3.8) and (cf. (3.6))

$$\delta_\varepsilon(|\xi|^2-\lambda)=\varepsilon\pi^{-1}\Big((|\xi|^2-\lambda)^2+\varepsilon^2\Big)^{-1}.$$

Let us first consider the operator $J_+^* T_-$. Recall that J_+ and T_- are PDO defined by formulas (3.3) and (3.5), respectively. Therefore for all $f_1, f_2 \in \mathcal{S}$

$$(T_- f_1, J_+ f_2) = (2\pi)^{-d} \int_{\mathbb{R}^d} \Big(\int_{\mathbb{R}^d} \int_{\mathbb{R}^d} e^{i\langle x, \xi' - \xi \rangle} \mathbf{a}(x, \xi, \xi') \hat{f}_1(\xi') \overline{\hat{f}_2(\xi)} d\xi d\xi' \Big) dx, \tag{5.8}$$

where

$$\mathbf{a}(x, \xi, \xi') = \overline{j_+(x, \xi)} t_-(x, \xi') \tag{5.9}$$

and j_+, t_- are the symbols of the operators J_+, T_-, respectively. According to Propositions 2.2, 2.4 and 3.2, the amplitude $\mathbf{a}(x, \xi, \xi')$ belongs to the Hörmander class $\mathcal{S}^{-1}(\rho, 1 - \rho)$. To obtain a convenient representation for (5.8), we have to change the order of integrations over x and ξ, ξ' in (5.8) and then calculate the integral over x. To that end, let us introduce as usual a function $\varphi(\epsilon x)$ such that $\varphi \in C_0^\infty(\mathbb{R}^d)$ and $\varphi(0) = 1$. Then

$$(T_- f_1, J_+ f_2) = (2\pi)^{-d} \lim_{\epsilon \to 0} \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} G^{(\epsilon)}(\xi, \xi') \hat{f}_1(\xi') \overline{\hat{f}_2(\xi)} d\xi d\xi', \tag{5.10}$$

where

$$G^{(\epsilon)}(\xi, \xi') = \int_{\mathbb{R}^d} e^{i\langle x, \xi' - \xi \rangle} \mathbf{a}(x, \xi, \xi') \varphi(\epsilon x) dx. \tag{5.11}$$

We set $\zeta = \zeta_-$, then $\zeta_+(x, \xi) = \zeta(x, -\xi)$. It follows from (3.3), (3.5) and (5.9), (5.11) that

$$G^{(\epsilon)}(\xi, \xi') = \int_{\mathbb{R}^d} \overline{u_+(x, \xi)} \zeta(x, -\xi) g_-(x, \xi') \varphi(\epsilon x) dx. \tag{5.12}$$

Note that, at least formally, $G = (2\pi)^d \mathcal{F} J_+^* T_- \mathcal{F}^*$ is integral operator with kernel $G^{(0)}(\xi, \xi')$.

Let us first consider the function $G^{(\epsilon)}(\xi, \xi')$ for $\xi \neq \xi'$. Choosing j such that $d^{1/2} |\xi_j - \xi_j'| \geq |\xi - \xi'|$ and integrating by parts in (5.11) n times, we find that

$$G^{(\epsilon)}(\xi, \xi') = (\xi_j - \xi_j')^{-n} \int_{\mathbb{R}^d} e^{i\langle x, \xi' - \xi \rangle} D_{x_j}^n \Big(\mathbf{a}(x, \xi, \xi') \varphi(\epsilon x) \Big) dx.$$

If $n\rho > d - 1$, then the limit as $\epsilon \to 0$ of this expression exists and equals

$$G(\xi, \xi') = (\xi_j - \xi_j')^{-n} \int_{\mathbb{R}^d} e^{i\langle x, \xi' - \xi \rangle} D_{x_j}^n \mathbf{a}(x, \xi, \xi') dx. \tag{5.13}$$

This integral is absolutely convergent. Moreover, function (5.13) can be differentiated p times in ξ or ξ' for $p < n\rho - d + 1$. Since n is arbitrary, it also follows from (5.9) and (5.13) that

$$|(\partial_\xi^\beta \partial_{\xi'}^{\beta'} G)(k\omega, k\omega')| \leq C k^{-2q}, \quad |\beta| + |\beta'| = p,$$

for $\omega \neq \omega'$ and any q.

Applying (5.7) to functions w_1 and w_2 with disjoint supports, we now see that off the diagonal $\omega = \omega'$ the kernel $s_1(\omega, \omega', \lambda)$ of the operator $S_1(\lambda)$ satisfies the relation (cf. (3.9))

$$s_1(\omega, \omega'; \lambda) = -\pi i k^{d-2} G(k\omega, k\omega'), \quad \omega \neq \omega'. \tag{5.14}$$

Combining these results with Theorem 4.2, we obtain

Theorem 5.1 *Let assumption* (1.7) *hold, and let* $\omega \in \Omega$, $\omega' \in \Omega'$ *for some open sets* $\Omega, \Omega' \subset \mathbb{S}^{d-1}$ *such that* $\operatorname{dist}(\Omega, \Omega') > 0$. *Then for any* p *and* q *the kernel* $s(\omega, \omega', \lambda)$ *of the SM belongs to the space* $C^p(\Omega \times \Omega')$ *and its* C^p*-norm is bounded by* $C\lambda^{-q}$ *as* $\lambda \to \infty$.

3. Our study of the function (5.12) in a neighborhood of the diagonal $\xi = \xi'$ relies on integration by parts. To that end, we need the following simple

Lemma 5.2 *Suppose that*

$$G^{(\epsilon)}(\xi,\xi') = \int_{\mathbb{R}^d} e^{i\langle x,\xi'-\xi\rangle}\mathbf{a}(x,\xi,\xi')(\partial_j\zeta)(x,\xi')\varphi(\epsilon x)dx \tag{5.15}$$

where $\mathbf{a} \in \mathcal{S}^p(\rho,\delta)$ *for some* p *and* $\rho > 0$, $\delta < 1$. *Then*

$$G^{(\epsilon)}(\xi,\xi') = -\int_{\mathbb{R}^d} \partial_j\Big(e^{i\langle x,\xi'-\xi\rangle}\mathbf{a}(x,\xi,\xi')\Big)\zeta(x,\xi')\varphi(\epsilon x)dx + R^{(\epsilon)}(\xi,\xi'),$$

and for all $\hat{f}_1, \hat{f}_2 \in C_0^\infty(\mathbb{R}^d)$

$$\lim_{\epsilon\to 0}\int_{\mathbb{R}^d}\int_{\mathbb{R}^d} R^{(\epsilon)}(\xi,\xi')\hat{f}_1(\xi')\overline{\hat{f}_2(\xi)}d\xi d\xi' = 0. \tag{5.16}$$

Proof - Integrating by parts in (5.15), we see that

$$R^{(\epsilon)}(\xi,\xi') = -\epsilon\int_{\mathbb{R}^d} e^{i\langle x,\xi'-\xi\rangle}\mathbf{a}(x,\xi,\xi')\zeta(x,\xi')(\partial_j\varphi)(\epsilon x)dx. \tag{5.17}$$

Let us insert this expression into (5.16) and use the formula

$$e^{-i\langle x,\xi\rangle} = \langle x\rangle^{-n}\langle D_\xi\rangle^n e^{-i\langle x,\xi\rangle}.$$

Integrating in (5.16) by parts in ξ sufficiently large number of times, we see that its left-hand side is a product of ϵ and of an absolutely convergent integral which is uniformly bounded in ϵ. $\square$

Let us plug (3.4) into (5.12) and denote by $G_j^{(\epsilon)}(\xi,\xi')$, $j = 1,2,3,4$, the integrals corresponding to the four functions in the right-hand side of (3.4):

$$\begin{aligned}
G_1^{(\epsilon)}(\xi,\xi') &= \int_{\mathbb{R}^d} \overline{u_+(x,\xi)}\zeta(x,-\xi)q_-(x,\xi')\zeta(x,\xi')\varphi(\epsilon x)dx,\\
G_2^{(\epsilon)}(\xi,\xi') &= -2\int_{\mathbb{R}^d} \overline{u_+(x,\xi)}\zeta(x,-\xi)\langle\nabla u_-(x,\xi'),\nabla\zeta(x,\xi')\rangle\varphi(\epsilon x)dx,\\
G_3^{(\epsilon)}(\xi,\xi') &= -\int_{\mathbb{R}^d} \overline{u_+(x,\xi)}\zeta(x,-\xi)u_-(x,\xi')\Delta\zeta(x,\xi')\varphi(\epsilon x)dx,\\
G_4^{(\epsilon)}(\xi,\xi') &= 2i\int_{\mathbb{R}^d} \overline{u_+(x,\xi)}\zeta(x,-\xi)u_-(x,\xi')\langle A(x),\nabla\zeta(x,\xi')\rangle\varphi(\epsilon x)dx.
\end{aligned}$$

Let us consider first the function $G_1^{(\epsilon)}$ where $q_- = e^{i\Theta_-}r_-$. By virtue of Theorem 2.6, the function $\overline{\mathrm{b}_+(x,\xi)}\zeta(x,-\xi)$ satisfies estimates (2.18) for all $x,\xi \in \mathbb{R}^d$ and the function $r_-(x,\xi')\zeta(x,\xi')$ satisfies estimates (2.17) for all $x,\xi' \in \mathbb{R}^d$. Hence we can pass to the limit $\epsilon \to 0$ in the expression for $G_1^{(\epsilon)}$. Moreover, the integrand in $G_1^{(0)}(\xi,\xi')$ is estimated by

$$C|\xi|^{-N}(1+|x|)^{-1-\rho_1(N+1)},$$

where N can be chosen arbitrary large. Using also the estimates on derivatives of these functions and estimates (2.14) on the phase functions $\Phi_\pm$, we see that $G_1^{(0)}(\xi,\xi')$ is a smooth function of ξ,ξ' rapidly decreasing as $|\xi| = |\xi'| \to \infty$.

Let ω and ω' belong to some conical neighborhood of a point $\omega_1 \in \mathbb{S}^{d-1}$ where, for example, $\langle \omega_1, \omega_0 \rangle > 0$. Then

$$\zeta(x, -\xi)(\nabla\zeta)(x, \xi') = (\nabla\zeta)(x, \xi')$$

so that the function $\zeta(x, -\xi)$ in the integrals $G_j^{(\epsilon)}(\xi, \xi')$, $j = 2, 3, 4$, can be omitted. All these integrals will be transformed by integration by parts. Here we use Lemma 5.2 which shows that the terms containing derivatives of φ disappear in the limit $\epsilon \to 0$. Such terms will be denoted by $R^{(\epsilon)}(\xi, \xi')$.

Integrating in the integral $G_3^{(\epsilon)}(\xi, \xi')$ by parts, we find that

$$G_2^{(\epsilon)}(\xi, \xi') + G_3^{(\epsilon)}(\xi, \xi') = R_{23}^{(\epsilon)}(\xi, \xi') + \int_{\mathbb{R}^d} \langle u_-(x, \xi')\overline{(\nabla u_+)(x, \xi)} - \overline{u_+(x, \xi)}(\nabla u_-)(x, \xi'), \nabla\zeta(x, \xi')\rangle \varphi(\epsilon x) dx. \tag{5.18}$$

Due to the function $\nabla\zeta(x, \xi')$, the integrals (5.18) as well as $G_4^{(\epsilon)}(\xi, \xi')$ are actually taken over the half-space $z \geq 0$ only. Therefore integrating once more by parts and taking into account the equality $\zeta(y, \xi') = 1$, we obtain that

$$\begin{aligned} G_2^{(\epsilon)}(\xi, \xi') + G_3^{(\epsilon)}(\xi, \xi') &= \tilde{R}_{23}^{(\epsilon)}(\xi, \xi') \\ + \int_{z \geq 0} \Big(\overline{u_+(x, \xi)}(\Delta u_-)(x, \xi') &- u_-(x, \xi')\overline{(\Delta u_+)(x, \xi)}\Big)\zeta(x, \xi')\varphi(\epsilon x) dx \\ + \int_{\Pi} \Big(\overline{u_+(y, \xi)}(\partial_z u_-)(y, \xi') &- u_-(y, \xi')\overline{(\partial_z u_+)(y, \xi)}\Big)\varphi(\epsilon y) dy \end{aligned} \tag{5.19}$$

and

$$\begin{aligned} G_4^{(\epsilon)}(\xi, \xi') = -2i \int_{z \geq 0} &\operatorname{div}\Big(A(x)\overline{u_+(x, \xi)}u_-(x, \xi')\Big)\zeta(x, \xi')\varphi(\epsilon x) dx \\ &-2i \int_{\Pi} \langle A(y), \omega_0 \rangle \overline{u_+(y, \xi)} u_-(y, \xi')\varphi(\epsilon y) dy + R_4^{(\epsilon)}(\xi, \xi'). \end{aligned} \tag{5.20}$$

It is now convenient to formulate an intermediary result.

Proposition 5.3 *The function* (5.12) *is the sum*

$$G^{(\epsilon)} = G_1^{(\epsilon)} + G_2^{(\epsilon)} + G_3^{(\epsilon)} + G_4^{(\epsilon)}.$$

Here $G_1^{(\epsilon)}(\xi, \xi')$ has (for any fixed ξ, ξ') the limit $G_1^{(0)}(\xi, \xi')$ which, for arbitrary p, q and sufficiently large $N = N(p, q)$, belongs to the class $C^p(\mathbb{R}^d \times \mathbb{R}^d)$ and $G_1^{(0)}(\xi, \xi')$, together with its derivatives up to order p, is bounded by $|\xi|^{-q}$ as $|\xi| = |\xi'| \to \infty$. The functions $G_2^{(\epsilon)} + G_3^{(\epsilon)}$ and $G_4^{(\epsilon)}$ satisfy equalities (5.19) *and* (5.20)*, respectively.*

4. In the following we need to calculate the limits

$$\lim_{\varepsilon \to 0} (G\delta_\varepsilon(|\xi|^2 - \lambda)\hat{\psi}_1, \delta_\varepsilon(|\xi|^2 - \lambda)\hat{\psi}_2), \tag{5.21}$$

for two classes of operators G defined by their kernels (5.11) where however the integral is taken either over the hyperplane $z = 0$ or over the half-space $z \geq 0$. We always suppose that $\mathbf{a}(x,\xi,\xi')$ is supported as a function of ξ and ξ' in a small neighborhood of some point $\xi_0\in\mathbb{R}^d$, $\langle\xi_0,\omega_0\rangle>0$, $|\xi_0|=k$. In this neighborhood we choose coordinates $|\xi|$ and the orthogonal projection ζ of the point $\hat\xi$ on the hyperplane Π.

The operators from the first class are formally defined by their kernels

$$G(\xi,\xi')=\int_\Pi e^{i\langle y,\xi'-\xi\rangle}\mathbf{a}(y,\xi,\xi')dy,$$

where $\mathbf{a}\in\mathcal{S}^p(\rho,\delta)$ for some p and $\rho>0$, $\delta<1$. As always, the precise definition of the operator G is given in terms of its sesquilinear form

$$(G\hat f_1,\hat f_2)=\lim_{\epsilon\to0}\int_{\mathbb{R}^d}\int_{\mathbb{R}^d}d\xi d\xi'\hat f_1(\xi')\overline{\hat f_2(\xi)}\int_\Pi dye^{i\langle y,\xi'-\xi\rangle}\varphi(\epsilon y)\mathbf{a}(y,\xi,\xi'),\tag{5.22}$$

where $\hat f_1,\hat f_2\in\mathcal{S}$. Let us set

$$\tilde{\mathbf{a}}(y,|\xi|,\zeta,|\xi'|,\zeta')=(1-|\zeta|^2)^{-1/2}(1-|\zeta'|^2)^{-1/2}\mathbf{a}(y,\xi,\xi')$$

and integrate by parts n times in the variable ζ (or ζ'). If n is sufficiently large, after that we can pass to the limit $\epsilon\to0$ in (5.22), which yields

$$\begin{aligned}(G\hat f_1,\hat f_2)=\int_0^\infty\int_0^\infty|\xi|^{d-1}d|\xi|\,|\xi'|^{d-1}d|\xi'|\int_\Sigma\int_\Sigma d\zeta d\zeta'\int_\Pi dye^{i\langle y,|\xi'|\zeta'-|\xi|\zeta\rangle}\\ \times\hat f_1(\xi')\langle|\xi|y\rangle^{-n}\langle D_\zeta\rangle^n\Big(\tilde{\mathbf{a}}(y,|\xi|,\zeta,|\xi'|,\zeta')\overline{\hat f_2(\xi)}\Big).\end{aligned}\tag{5.23}$$

If $n(1-\delta)>p+d-1$, then this integral is absolutely convergent.

Representation (5.23) allows us to pass directly to the limit $\varepsilon\to0$ in the expression (5.21) where according to (3.8), (5.4)

$$\hat\psi_j(\xi)=\tilde w_j(\zeta)\gamma_j(|\xi|),\quad j=1,2,\tag{5.24}$$

and $\gamma_j\in C_0^\infty(\mathbb{R}_+)$, $\gamma_j(k)=1$. Making the change of variables $|\xi|^2=s$ and $|\xi'|^2=s'$, we get

$$\begin{aligned}&\lim_{\varepsilon\to0}(G\delta_\varepsilon(|\xi|^2-\lambda)\hat\psi_1,\delta_\varepsilon(|\xi|^2-\lambda)\hat\psi_2)\\ &=2^{-2}\lambda^{d-2}\int_\Pi\int_\Pi\int_\Pi d\zeta d\zeta'dye^{ik\langle y,\zeta'-\zeta\rangle}\tilde w_1(\zeta')\langle ky\rangle^{-n}\langle D_\zeta\rangle^n\Big(\tilde{\mathbf{a}}(y,k,\zeta,k,\zeta')\overline{\tilde w_2(\zeta)}\Big).\end{aligned}\tag{5.25}$$

We emphasize that the crucial point in the proof above was that, for regularization of integrals, we integrated by parts in the variable ζ only. If we integrated by parts in $|\xi|$, then a derivative would have fallen on $\delta_\varepsilon(|\xi|^2-\lambda)$ and hence destroyed the limit (5.25). Thus, we have proven the following result.

Proposition 5.4 *Let an operator G be defined by its form* (5.22) *where* $\mathbf{a}\in\mathcal{S}^p(\rho,\delta)$ *for some p and $\rho>0$, $\delta<1$. Then the relation* (5.25) *holds for all $\lambda>0$.*

Of course, we can integrate in (5.25) back by parts, understanding the expression obtained as an oscillating integral. Proposition 5.4 means that the operator $(\mathcal{F}^*G\mathcal{F})^\flat(\lambda)$ exists for all $\lambda>0$ and is the integral operator on the unit sphere with kernel (cf. (3.9))

$$g(\omega,\omega';\lambda)=2^{-1}k^{d-2}\int_\Pi e^{ik\langle y,\omega'-\omega\rangle}\mathbf{a}(y,k\omega,k\omega')dy,\quad\omega,\omega'\in\Omega_+.$$

The operators from the second class are formally defined by their kernels

$$G(\xi,\xi')=\int_{z\geq 0} e^{i\langle x,\xi'-\xi\rangle}\mathbf{a}(x,\xi,\xi')dx, \tag{5.26}$$

where again $\mathbf{a}\in\mathcal{S}^p(\rho,\delta)$ for some p and $\rho>0$, $\delta<1$. Our main assumption is that

$$\mathbf{a}(x,\xi,\xi')=0 \quad \text{if} \quad \langle\eta,x\rangle\geq c_0|\eta|\,|x| \tag{5.27}$$

for $\eta=\xi+\xi'$ and some $c_0\in(0,1)$. As before, the precise definition of the operator G is given in terms of the corresponding sesquilinear form and is quite similar to (5.22). We write it choosing in the half-space $z\geq 0$ the coordinates (y,t), $y\in\Pi$, $t\geq 0$, such that $x=\eta t+y$. Let us set

$$\tilde{\mathbf{a}}(y,t,|\xi|,\zeta,|\xi'|,\zeta')=\sigma(\xi,\xi')\mathbf{a}(\eta t+y,\xi,\xi')e^{i(|\xi'|^2-|\xi|^2)t}, \tag{5.28}$$

where

$$\sigma(\xi,\xi')=|\xi|\sigma_0(\zeta,\zeta')+|\xi'|\sigma_0^{-1}(\zeta,\zeta') \quad \text{and} \quad \sigma_0(\zeta,\zeta')=(1-|\zeta|^2)^{1/4}(1-|\zeta'|^2)^{-1/4}.$$

Then, for $\hat{f}_1,\hat{f}_2\in\mathcal{S}$,

$$\begin{aligned}(G\hat{f}_1,\hat{f}_2)=\lim_{\epsilon\to 0}\int_0^\infty\int_0^\infty |\xi|^{d-1}d|\xi|\,|\xi'|^{d-1}d|\xi'|\int_\Sigma\int_\Sigma d\zeta d\zeta'\hat{f}_1(\xi')\overline{\hat{f}_2(\xi)}\\ \times\int_\Pi dy e^{i\langle y,|\xi'|\zeta'-|\xi|\zeta\rangle}\int_0^\infty dt\varphi(\epsilon(\eta t+y))\tilde{\mathbf{a}}(y,t,|\xi|,\zeta,|\xi'|,\zeta').\end{aligned} \tag{5.29}$$

Note that, by (5.27), $\mathbf{a}(\eta t+y,\xi,\xi')=0$ if $t\geq c_1|y|$ for a suitable $c_1>0$, so that the last integral (5.29) is actually taken over a finite interval $t\in(0,c_1|y|)$.

Let us integrate by parts in the variable ζ (or ζ') n times and then pass to the limit $\epsilon\to 0$ in (5.29). We must show that if the function $\varphi(\epsilon(\eta t+y))$ is differentiated m times, $1\leq m\leq n$, then the limit of the corresponding expression is zero. Using the condition $\mathbf{a}\in\mathcal{S}^p(\rho,\delta)$, we see that the integral (5.29) over y and t is estimated by

$$\epsilon^m\int_\Pi dy\langle y\rangle^{-n}\int_0^{c_1|y|}dt t^m|\varphi^{(m)}(\epsilon(\eta t+y))|(1+t+|y|)^{p+\delta(n-m)}.$$

Since $\varphi\in C_0^\infty(\mathbb{R}^d)$, we have that $\varphi^{(m)}(\epsilon(\eta t+y))=0$ for sufficiently large $\epsilon|y|$. Hence this expression is estimated by

$$\epsilon^m\int_{|y|\leq c\epsilon^{-1}}\langle y\rangle^{-(1-\delta)(n-m)+1+p}dy.$$

It suffices to consider the case $(1-\delta)(n-m)\leq p+d$, when the integral over y is not bounded as $\epsilon\to 0$. In this case the last expression is estimated by ϵ^s where

$$s=(1-\delta)n+\delta m-p-d\geq n-(1-\delta)^{-1}(p+d)>0$$

if n is sufficiently large. Thus, the function φ disappears in the limit $\epsilon\to 0$ in (5.29), which yields

$$\begin{aligned}(G\hat{f}_1,\hat{f}_2)=\int_0^\infty\int_0^\infty |\xi|^{d-1}d|\xi|\,|\xi'|^{d-1}d|\xi'|\int_\Sigma\int_\Sigma d\zeta d\zeta'\int_\Pi dy e^{i\langle y,|\xi'|\zeta'-|\xi|\zeta\rangle}\\ \times\hat{f}_1(\xi')\langle|\xi|y\rangle^{-n}\langle D_\zeta\rangle^n\Big(\mathbf{a}^\natural(y,|\xi|,\zeta,|\xi'|,\zeta')\overline{\hat{f}_2(\xi)}\Big),\end{aligned} \tag{5.30}$$

where

$$\mathbf{a}^{\natural}(y,|\xi|,\zeta,|\xi'|,\zeta') = \int_0^\infty \tilde{\mathbf{a}}(y,t,|\xi|,\zeta,|\xi'|,\zeta')dt. \tag{5.31}$$

Due to assumption (5.27), this integral is taken over an interval $(0, c_1|y|)$, and hence the integral (5.30) is absolutely convergent for sufficiently large n.

The rest of the proof is essentially similar to that of Proposition 5.4. Representation (5.30) allows us to pass directly to the limit $\varepsilon \to 0$ in the expression (5.21), where the functions $\hat{\psi}_j(\xi)$ are defined by (5.24). Let us formulate the final result taking into account formulas (5.28) and (5.31).

Proposition 5.5 *Let an operator G acting on functions of $\xi \in \mathbb{R}^d$ have kernel* (5.26) *where $\mathbf{a} \in \mathcal{S}^p(\rho,\delta)$ for some p and $\rho > 0$, $\delta < 1$, and let $\mathbf{a}$ satisfy assumption* (5.27). *Then the limit exists*

$$\lim_{\varepsilon\to 0}(G\delta_\varepsilon(|\xi|^2-\lambda)\hat{\psi}_1, \delta_\varepsilon(|\xi|^2-\lambda)\hat{\psi}_2)$$
$$= \int_\Sigma\int_\Sigma d\zeta d\zeta' \int_\Pi dy e^{ik\langle y,\zeta'-\zeta\rangle}\tilde{w}_1(\zeta')\overline{\tilde{w}_2(\zeta)}\mathbf{a}^\flat(y,\zeta,\zeta';k), \tag{5.32}$$

where the integral

$$\mathbf{a}^\flat(y,\zeta,\zeta';k) = 2^{-2}\lambda^{d-2}(\sigma_0(\zeta,\zeta')+\sigma_0^{-1}(\zeta,\zeta'))\int_0^\infty \mathbf{a}((\omega+\omega')t+y,k\omega,k\omega')dt$$

is taken over a finite interval and $\mathbf{a}^\flat \in \mathcal{S}^{p+1}(\rho,\delta)$. The right-hand side of (5.32) *is understood as an oscillating integral.*

Actually, we need only

Corollary 5.6 *Let G satisfy the assumptions of Proposition* 5.5 *and let*

$$\tilde{G}(\xi,\xi') = (|\xi|^2-|\xi'|^2)G(\xi,\xi').$$

Then the limit (5.21) *for the operator $\tilde{G}$ with this kernel equals zero.*

5. Now we are in a position to derive formula (1.13) for the singular part of the SM. To that end, we have to calculate the limit in the right-hand side of (5.7) and show that the expression obtained coincides, up to negligible terms, with the form $(S_0(\lambda)w_1,w_2)$. Let us proceed from representation (5.10) where the function $G^{(\epsilon)}(\xi,\xi')$ satisfies Proposition 5.3.

According to (3.9) the contribution of $G_1^{(\epsilon)}$ to $S_1(\lambda)$ is given by the expression $-\pi i k^{d-2} \times G_1^{(0)}(k\omega,k\omega')$ which is a smooth function of ω, ω' and rapidly decays as $k\to\infty$. Hence this term can be neglected.

Let us further consider the integrals (5.19) and (5.20) over Π. According to Proposition 5.4, the contribution of each integral to the kernel of $S_1(\lambda)$ equals its value at $\xi = k\omega$, $\xi' = k\omega'$ times (compare with (5.14)) the numerical factor $-\pi i k^{d-2}(2\pi)^{-d}$. The sum of this expressions coincides with (1.13).

It remains to show that the sum of the integrals over the half-space $z \geq 0$ in (5.19) and (5.20) is negligible. It follows from relation (1.11) that

$$\overline{u_+(x,\xi)}(\Delta u_-)(x,\xi') - u_-(x,\xi')\overline{(\Delta u_+)(x,\xi)} - 2i\,\mathrm{div}\left(A(x)\overline{u_+(x,\xi)}u_-(x,\xi')\right)$$
$$= \left(\overline{q_+(x,\xi)}u_-(x,\xi') - q_-(x,\xi')\overline{u_+(x,\xi)}\right) + (|\xi|^2-|\xi'|^2)\overline{u_+(x,\xi)}u_-(x,\xi').$$

To consider the integral

$$\int_{z\geq 0} e^{i\Theta_-(x,\xi')-i\Theta_+(x,\xi)}\Big(\overline{r_+(x,\xi)}\mathrm{b}_-(x,\xi') - r_-(x,\xi')\overline{\mathrm{b}_+(x,\xi)}\Big)\zeta(x,\xi')\varphi(\epsilon x)dx, \tag{5.33}$$

we use again that, by Proposition 2.4 and Corollary 2.5, the functions

$$r_-(x,\xi')\zeta(x,\xi') \quad \text{and} \quad \mathrm{b}_-(x,\xi')\zeta(x,\xi')$$

satisfy estimates (2.17) and (2.18), respectively, for all $x,\xi' \in \mathbb{R}^d$. The same result for the functions $\mathrm{b}_+(x,\xi)$ and $r_+(x,\xi)$ holds true in the half-space $z \geq 0$ which does not contain the "bad" direction $\hat{x} = -\hat{\xi}$. By Corollary 2.3, the function

$$\Phi_-(x,\xi') - \Phi_+(x,\xi)$$

satisfies estimates (2.14) for all $z \geq 0$ off a conical neighborhood of the direction $\hat{x} = \hat{\xi}'$ where $\zeta(x,\xi') = 0$. Therefore the integral (5.33) is a smooth function of ξ,ξ' rapidly decreasing as $|\xi| = |\xi'| \to \infty$. Hence, similarly to the function $G_1^{(\epsilon)}(\xi,\xi')$, this integral does not contribute to $S_0(\lambda)$.

Let us, further, consider the function

$$G_0^{(\epsilon)}(\xi,\xi') = (|\xi|^2 - |\xi'|^2)\tilde{G}_0^{(\epsilon)}(\xi,\xi'),$$

where

$$\tilde{G}_0^{(\epsilon)}(\xi,\xi') = \int_{z\geq 0} e^{i\langle x,\xi'-\xi\rangle}\overline{h_+(x,\xi)}h_-(x,\xi')\zeta(x,\xi')\varphi(\epsilon x)dx$$

and the functions $h_\pm(x,\xi)$ are defined by formula (5.1). The amplitude here belongs to the class $\mathcal{S}^0(\rho,1-\rho)$ and, due to the factor $\zeta(x,\xi')$, it satisfies condition (5.27). Therefore, by Corollary 5.6, the limit (5.21) for the operator G_0 equals zero. Finally, we note that the functions $\tilde{R}_{23}^{(\epsilon)}(\xi,\xi')$ and $R_4^{(\epsilon)}(\xi,\xi')$ in (5.19) and (5.20) disappear in the limit $\epsilon \to 0$.

Now we can formulate our main result on the asymptotics of the kernel $s(\omega,\omega';\lambda)$ of the SM.

Theorem 5.7 *Let assumption* (1.7) *hold, let p,q be arbitrary numbers and $N = N(p,q)$ be sufficiently large. Let the functions $\Theta_\pm^{(N_0)}(x,\xi)$ and $\mathrm{b}_\pm^{(N)}(x,\xi)$ be defined by formulas* (2.4), (2.5) *and* (1.4) *where the functions $\phi_n^{(\pm)}(x,\hat{\xi})$ and $b_n^{(\pm)}(x,\xi)$ are constructed in Propositions* 2.2 *and* 2.4, *respectively. Let $u_\pm^{(N)}(x,\xi)$ be defined by formula* (1.10). *Define, for $\omega,\omega' \in \Omega_\pm$, the kernel $s_0^{(N)}(\omega,\omega';\lambda)$ by formula* (1.13). *Then the remainder* (1.14) *belongs to the class $C^p(\Omega\times\Omega)$ and the C^p-norm of this kernel is $O(\lambda^{-q})$ as $\lambda\to\infty$.*

This result gives simultaneously the high-energy and smoothness expansion of the kernel of the SM. As was already mentioned, we actually formulate the result in terms of the corresponding amplitude $\mathbf{a}_0(y,\omega,\omega';\lambda)$ related to the kernel of the SM by formula (5.2). Indeed, it follows from (5.1), (5.3) and (5.6) that

$$\mathbf{a}_0(y,\omega,\omega';\lambda) = \pm 2^{-1}k^{d-1}\exp(i\Xi(y,\omega,\omega';k))\sum_{n=0}^{N}(2ik)^{-n}\sigma_n(y,\omega,\omega'), \tag{5.34}$$

where

$$\Xi(y,\omega,\omega';k)=\sum_{n=0}^{N_0}(2k)^{-n}\theta_n(y,\omega,\omega'), \tag{5.35}$$

$$\theta_n(y,\omega,\omega')=\phi_n^{(-)}(y,\omega')-\phi_n^{(+)}(y,\omega)$$

and the functions $\phi_n^{(\pm)}$ are constructed in Proposition 2.2. Note that $\theta_0\in\mathcal{S}^{1-\rho_a}$ and $\theta_n\in\mathcal{S}^{1-n\rho}$ for $n\geq 1$. The coefficients $\sigma_n(y,\omega,\omega')$ are expressed in terms of functions $\phi_n^{(\pm)}$ and $b_n^{(\pm)}$ constructed in Proposition 2.4. It is easy to see that $\sigma_n\in\mathcal{S}^{-n\rho_1}$ for $n\geq 0$. In particular, $S_0(\lambda)\in\mathcal{C}^0(\Xi)$.

6. APPLICATIONS

Theorem 5.7 allows us to replace the kernel $s(\omega,\omega';\lambda)$ of SM by the explicit function $s_0(\omega,\omega';\lambda)$ and thus find different limits of $s(\omega,\omega';\lambda)$ as $\lambda\to\infty$ or (and) $\omega-\omega'\to 0$. We emphasize that although approximation (1.13) is valid in all cases, these limits are very sensitive to the behavior of potentials at infinity.

1. Let us first distinguish the leading term $S_{00}(\lambda)$ of the operator $S_0(\lambda)$. Recall that $b_0^{(\pm)}(x,\xi)=1$ and hence, according to (5.1), (5.3),

$$\sigma_0(y,\omega,\omega')=\langle\omega+\omega',\omega_0\rangle \tag{6.1}$$

for any V and A. Therefore keeping only the term corresponding to $n=0$ in (5.34), we obtain

$$\mathbf{a}_{00}(y,\omega,\omega';\lambda)=\pm 2^{-1}k^{d-1}\langle\omega+\omega',\omega_0\rangle\exp(i\Xi(y,\omega,\omega';k)),\quad \omega,\omega'\in\Omega_\pm. \tag{6.2}$$

Now we define $S_{00}(\lambda)$ as the PDO on $\mathbb{S}^{d-1}$ with this amplitude or, to put it differently, $S_{00}(\lambda)$ is the integral operator with kernel

$$s_{00}(\omega,\omega';\lambda)=\pm 2^{-1}\langle\omega+\omega',\omega_0\rangle(2\pi)^{-d+1}$$
$$\times\int_{\Pi_{\omega_0}}\exp\Big(i\langle y,\omega'-\omega\rangle+i\Xi(y/k,\omega,\omega',k)\Big)dy. \tag{6.3}$$

To be quite precise, the operator $S_{00}(\lambda)$ is defined by (6.3) on Ω_+ and Ω_-. However taking into account Theorem 5.1 and that the point $\omega_0\in\mathbb{S}^{d-1}$ is arbitrary, we can naturally extend definition of $S_{00}(\lambda)$ to the whole sphere. As always, the operators with arbitrary smooth kernels which decay faster than any power of λ^{-1} as $\lambda\to\infty$ are neglected.

It follows from Proposition 3.1 that the operators $S_{00}(\lambda)$ are uniformly bounded in $L_2(\mathbb{S}^{d-1})$. The operators $S_{00}(\lambda)$ approximate $S(\lambda)$ both for high energies $\lambda\to\infty$ and in terms of smoothness of their kernels. Comparing representations (5.34) and (6.2), we obtain

Proposition 6.1 *Let assumption* (1.7) *be fulfilled and let* $\rho_1=\min\{1,\rho\}$. *Then the operator* $S(\lambda)-S_{00}(\lambda)$ *belongs to the class* $\mathcal{C}^{-\rho_1}(\Xi)$, *and hence it is compact. Moreover,*

$$||S(\lambda)-S_{00}(\lambda)||=O(\lambda^{-1/2}),\quad \lambda\to\infty.$$

This assertion can be made more precise if both A and V are short-range. If moreover $A = 0$, then $\Xi(y, \omega, \omega'; k) = 0$ and hence $S_{00}(\lambda) = I$. In the general case, it follows from (2.23) and (5.6) that

$$\Xi(y, \omega, \omega') = \theta_0(y, \omega, \omega') = \int_0^\infty \langle A(y + t\omega), \omega \rangle dt + \int_0^\infty \langle A(y - t\omega'), \omega' \rangle dt \tag{6.4}$$

and hence

$$\Xi(y, \omega, \omega') = O(|y|^{-\varepsilon}), \quad \varepsilon = \rho_a - 1 > 0,$$

as $|y| \to \infty$. Therefore we can expand $\exp(i\Xi)$ in the Taylor series which gives the representation

$$\mathbf{a}_0(y, \omega, \omega'; \lambda) = \pm 2^{-1} k^{d-1} \sum_{n=0}^{N} (2ik)^{-n} \tilde{\sigma}_n(y, \omega, \omega')$$

with $\tilde{\sigma}_0 = \sigma_0$ and $\tilde{\sigma}_n \in \mathcal{S}^{-n(\rho_2 - 1)}$. Of course, $\tilde{\sigma}_n = \sigma_n$ if $A = 0$. Note that, by virtue of (6.3), the first term in the expansion of $s(\omega, \omega'; \lambda)$ is always the Dirac-function. Thus, we arrive at

Proposition 6.2 *Let assumption* (1.7) *be fulfilled with* $\rho > 1$, *and let* $\rho_2 = \min\{2, \rho\}$. *Then the operator* $S(\lambda) - I$ *belongs to the class* $\mathcal{S}^{-\rho_2 + 1}$, *and hence it is compact.*

In the short-range case the coefficients $\sigma_n(y, \omega, \omega')$, $n \geq 1$, in (5.34) are determined by the coefficients $b_n^{(\pm)}(x, \hat{\xi})$ (see Proposition 2.8). Let us write down the first two coefficients supposing for simplicity that $A = 0$. According to (2.21)

$$\sigma_1(y, \omega, \omega') = \langle \omega + \omega', \omega_0 \rangle \Big(\int_0^\infty V(y + t\omega) dt + \int_0^\infty V(y - t\omega') dt \Big). \tag{6.5}$$

We shall write down the expression for the next coefficient keeping in mind that the functions b_1 and b_2 are given by formulas (2.21) and (2.22), respectively:

$$\sigma_2(\omega, \omega', y) = \langle \omega + \omega', \omega_0 \rangle (b_2^{(+)}(y, \omega) - b_1^{(+)}(y, \omega) b_1^{(-)}(y, \omega') + b_2^{(-)}(y, \omega')) \\ + 2\langle \omega_0, \nabla (b_1^{(+)}(y, \omega) + b_1^{(-)}(y, \omega')) \rangle.$$

The expressions for other coefficients σ_n can be obtained in a similar way.

Finally, we note that the leading contribution to the singularity of the SM of the term in (5.34) containing $\sigma_n(y, \omega, \omega')$ is determined by $\sigma_n(y, \omega, \omega)$. However this replacement leads to an error which depends on n. For example, it follows from (6.5) that

$$\sigma_1(y, \omega, \omega) = 2\langle \omega, \omega_0 \rangle \int_{-\infty}^\infty V(y + t\omega) dt,$$

which differs from (6.5) by a term of order $|\omega - \omega'| \langle y \rangle^{-\rho_v + 1}$.

2. Let us further consider the limit $\lambda \to \infty$. The assertion below follows from Proposition 2.2 and representations (5.35) and (6.3).

Lemma 6.3 *Under assumption* (1.7), *for any smooth functions* w_1, w_2, *the form* $(S_{00}(\lambda) w_1, w_2)$ *converges as* $\lambda \to \infty$ *to the form of the operator of multiplication by* $e^{i\theta_0(0, \omega, \omega)}$.

In the short-range case the expression for θ_0 is given by the formula (6.4). In the long-range case

$$\begin{aligned}\theta_0(y,\omega,\omega') &= \int_0^\infty (\langle A(y+t\omega),\omega\rangle - \langle A(t\omega),\omega\rangle)dt \\ &+ \int_0^\infty (\langle A(y-t\omega'),\omega'\rangle - \langle A(-t\omega'),\omega'\rangle)dt \end{aligned} \tag{6.6}$$

and, in particular, $\theta_0(0,\omega,\omega')=0$.

Let us now combine Proposition 6.1 and Lemma 6.3. Furthermore, using the unitarity of $S(\lambda)$, we can replace the weak convergence by the strong one.

Proposition 6.4 *Let assumption* (1.7) *be fulfilled. Then the SM $S(\lambda)$ has the strong limit as $\lambda\to\infty$ which is the identity, except the case when a short-range magnetic potential is present. In this case the strong limit of $S(\lambda)$ is the operator of multiplication by the function*

$$\exp\Big(i\int_{-\infty}^{\infty}\langle A(t\omega),\omega\rangle dt\Big).$$

According to Proposition 6.4, in the long-range case the behavior of $S(\lambda)$ as $\lambda\to\infty$ is simpler than in the short-range magnetic case. This is not surprising since the effect of a long-range interaction is to a large extent included via the operators $J_\pm$ in the definition of the SM.

We emphasize that, except in the case of a short-range V and $A=0$, the SM does not converge as $\lambda\to\infty$ in the sense of the norm.

3. Let us now study the diagonal singularity of $s(\omega,\omega';\lambda)$ as $\omega-\omega'\to 0$ and λ is fixed in the case, where at least one of potentials V or A is long-range. For example, if A is long-range, then, by (6.6), $\theta_0(y,\omega,\omega')$ tends to infinity as $|y|\to\infty$. Let us write the second coefficient in (5.35) as $\theta_1=\theta_1^{(0)}+\tilde\theta_1$ where

$$\theta_1^{(0)}(y,\omega,\omega') = \int_0^\infty (V(t\omega)-V(y+t\omega))dt + \int_0^\infty (V(-t\omega')-V(y-t\omega'))dt \tag{6.7}$$

and the part $\tilde\theta_1$ depends on the magnetic potential only. Clearly, for long-range V, the function $\theta_1^{(0)}$ also tends to infinity as $|y|\to\infty$.

We shall find the leading term of the diagonal asymptotics of $s(\omega,\omega';\lambda)$. Therefore we keep only the term corresponding to $n=0$ in (5.34) which yields again expression (6.3). The asymptotics of $s(\omega,\omega';\lambda)$ as $\omega-\omega'\to 0$ is determined by the fall-off of potentials $V(x)$ and $A(x)$ at infinity and can be found by the stationary phase method (see, e.g., [7]). Let us formulate the result supposing first that $V(x)$ decays slower than $A(x)$, that is $\rho_v<\rho_a$. We assume that $\rho_v<1$, but ρ_a can be both smaller or larger than 1. We omit details of calculation since they are practically the same as in [26] for the case $\rho_v\in(1/2,1)$.

We assume for simplicity that $V(x)$ is a homogeneous function for sufficiently large $|x|$:

$$V(x)=V_\infty(x):=v(\hat x)|x|^{-\rho},\quad \rho=\rho_v,\quad v\in C^\infty(\mathbb{S}^{d-1}),\quad |x|\ge r_0. \tag{6.8}$$

It follows from (6.7) and (6.8) that for sufficiently large $|y|$

$$\theta_1^{(0)}(y,\omega,\omega) = \int_{-\infty}^{\infty} \Big(V(t\omega) - V(y+t\omega) \Big) dt = \mathbf{V}(y,\omega) + \nu(\omega), \tag{6.9}$$

where $\omega \in \Omega,\ y \in \Pi$,

$$\mathbf{V}(y,\omega) = \int_{-\infty}^{\infty} \Big(V_\infty(t\omega) - V_\infty(y+t\omega) \Big) dt = \mathbf{V}(\hat{y},\omega)|y|^{1-\rho} \tag{6.10}$$

is a homogeneous function of order $1-\rho$ and

$$\nu(\omega) = \int_{-r}^{r} \Big(V(t\omega) - V_\infty(t\omega) \Big) dt, \quad r \geq r_0,$$

does not depend on y (and r). Let us introduce the Hessian $\mathcal{H}(y,\omega)$ of the function (6.10), i.e., $\mathcal{H}(y,\omega)$ is the $(d-1)\times(d-1)$ - matrix with elements

$$\mathcal{H}_{jk}(y,\omega) = -\int_{-\infty}^{\infty} \partial^2 V_\infty(y+t\omega)/\partial y_j \partial y_k \, dt, \quad y \in \Pi.$$

Set also

$$\mathfrak{h}(y,\omega) = |\det \mathcal{H}(y,\omega)|^{-1/2} \exp\Big(i\pi \operatorname{sgn} \mathcal{H}(y,\omega)/4 \Big). \tag{6.11}$$

Let η be the orthogonal projection of $\omega' - \omega$ on the plane Π. Then $\langle y, \omega'-\omega\rangle = \langle y,\eta\rangle$. According to (5.35), (6.7) and (6.9)

$$\begin{aligned} F(y,\omega,\omega',\lambda) : &= k\langle y,\omega'-\omega\rangle + \Xi(y,\omega,\omega';k) \\ &= k\langle y,\eta\rangle + (2k)^{-1}\mathbf{V}(y,\omega) + f(y,\omega,\omega',\lambda), \end{aligned} \tag{6.12}$$

where

$$\begin{aligned} f(y,\omega,\omega',\lambda) = \theta_0(y,\omega,\omega') + (2k)^{-1}\Big(\nu(\omega) + \theta_1^{(0)}(y,\omega,\omega') - \theta_1^{(0)}(y,\omega,\omega) \\ + \tilde{\theta}_1(y,\omega,\omega') \Big) + \sum_{n=2}^{N_0} (2k)^{-n} \theta_n(y,\omega,\omega'). \end{aligned} \tag{6.13}$$

Thus, it follows from (6.3) and (6.12) that

$$s_{00}(\omega,\omega';\lambda) = \pm 2^{-1} k^{d-1} \langle \omega+\omega', \omega_0\rangle (2\pi)^{-d+1} \int_\Pi \exp\Big(iF(y,\omega,\omega',\lambda) \Big) dy. \tag{6.14}$$

Let us make here the change of variables $y = k^{-2\gamma}|\eta|^{-\gamma} u$ with $\gamma = \rho^{-1}$. Then

$$\begin{aligned} s_{00}(\omega,\omega';\lambda) = \pm 2^{-1} k^{-\sigma(d-1)} \langle \omega+\omega', \omega_0\rangle (2\pi)^{-d+1} \\ \times |\eta|^{-\gamma(d-1)} \int_\Pi \exp\Big(ik^{-\sigma}|\eta|^{1-\gamma} G(u,\omega,\omega',\lambda) \Big) du, \end{aligned} \tag{6.15}$$

where $\sigma = 2\gamma - 1$,

$$G(y,\omega,\omega',\lambda) = \langle y,\hat{\eta}\rangle + 2^{-1}\mathbf{V}(y,\omega) + k^{\sigma}|\eta|^{-1+\gamma} f(\lambda^{-\gamma}|\eta|^{-\gamma} y,\omega,\omega',\lambda) \tag{6.16}$$

and $\hat\eta = \eta|\eta|^{-1}$. Since $|\eta|^{1-\gamma} \to \infty$ as $\eta \to 0$, the asymptotics of the integral (6.15) is determined by the stationary points $y_1(\hat\eta), \dots, y_m(\hat\eta)$ satisfying the equation

$$\nabla_y G(y, \omega, \omega', \lambda) = 0. \tag{6.17}$$

It follows from definition (6.13) that the last term in (6.16) tends to zero as $\eta \to 0$. Therefore equation (6.17) can be solved by successive approximations starting from the solutions $y_1^{(0)}(\hat\eta), \dots, y_m^{(0)}(\hat\eta)$ of the simplified equation

$$2\hat\eta = \int_{-\infty}^{\infty} (\nabla_y V_\infty)(y^{(0)}(\hat\eta) + t\omega) dt, \quad y \in \Pi, \tag{6.18}$$

obtained from (6.17) by setting $f = 0$. Applying the stationary phase method to integral (6.15), we get

Theorem 6.5 *Let assumptions* (1.7) *where* $\rho_v < 1$, $\rho_v < \rho_a$ *and* (6.8) *be satisfied. Fix* $k > 0$, $\omega \in \mathbb{S}^{d-1}$, $\omega \notin \Pi$, *and let* η *be the orthogonal projection of* $\omega' - \omega$ *on the plane* Π. *Suppose that for a given* $\hat\eta$ *there is a finite number of points* $y_1^{(0)}(\hat\eta), \dots, y_m^{(0)}(\hat\eta)$ *satisfying equation* (6.18). *Assume that* $\det \mathcal{H}(u_j^{(0)}(\hat\eta), \omega) \neq 0$ *for all* $j = 1, \dots, m$, *and let* $\mathfrak{h}(y, \omega)$ *be defined by formula* (6.11). *Then the kernel of the SM admits as* $\omega' \to \omega$ *or, equivalently,* $\eta \to 0$ *the representation*

$$s(\omega, \omega'; \lambda) = |\langle \omega, \omega_0 \rangle| \, \tau(k) \, |\eta|^{-(d-1)(1+\gamma)/2}$$
$$\times \sum_{j=1}^{m} \mathfrak{h}(y_j^{(0)}(\hat\eta), \omega) \exp\Big(ik^{-\sigma} |\eta|^{1-\gamma} G(y_j(\hat\eta), \omega, \omega', \lambda) \Big) (1 + O(|\eta|^{\varepsilon})), \tag{6.19}$$

where $\gamma = \rho^{-1}$, $\sigma = 2\gamma - 1$, $\tau(k) = (\pi k^\sigma)^{-(d-1)/2}$ *and* $\varepsilon = \varepsilon(\rho) > 0$.

Remark It is possible, of course, that for some $\hat\eta$ there are no points u satisfying (6.18). In this case $s_{00}(\omega, \omega'; \lambda) \to 0$ as $\eta \to 0$ faster than any power of $|\eta|$ so that the kernel $s(\omega, \omega'; \lambda)$ of the SM remains bounded. However, typically (see [26], for concrete examples) equation (6.18) has one or several solutions.

Remark In the case $\rho_v \in (1/2, 1)$, $A = 0$ formula (6.19) reduces of course to the corresponding formula from the paper [26]. In this case the points $y_j(\hat\eta) = y_j^{(0)}(\hat\eta)$ are defined as solutions of the equation (6.18) and $f = (2k)^{-1}\nu(\omega)$.

Suppose now that $\rho_a < 1$ and $\rho_a < \rho_v$, that is the magnetic potential is dominating. We assume that, for some $\omega \in \mathbb{S}^{d-1}$, the magnetic potential $A(x)$ satisfies the condition

$$\langle A(x), \omega \rangle = A_\infty(x, \omega) := a(\hat x, \omega)|x|^{-\rho}, \quad \rho = \rho_a, \quad a(\cdot, \omega) \in C^\infty(\mathbb{S}^{d-1}), \tag{6.20}$$

for $|x| \geq r_0$. By virtue of (6.6), the role of (6.10) is played by the function

$$\mathbf{A}(y, \omega) = \int_{-\infty}^{\infty} \Big(A_\infty(y + t\omega, \omega) - A_\infty(t\omega, \omega) \Big) dt.$$

The functions $\nu(\omega)$, $\mathcal{H}(y, \omega)$ and $\mathfrak{h}(y, \omega)$ are defined quite similarly to the electric case if V, V_∞ are replaced by A, A_∞, respectively. Instead of (6.12), (6.13), we have now that

$$F(y, \omega, \omega', \lambda) = k\langle y, \eta \rangle + \mathbf{A}(y, \omega) + f(y, \omega, \omega', \lambda),$$

where

$$f(y,\omega,\omega',\lambda)=\theta_0(y,\omega,\omega')-\theta_0(y,\omega,\omega)+\nu(\omega)+\sum_{n=1}^{N_0}(2k)^{-n}\theta_n(y,\omega,\omega').$$

Choose any ω_0 such that $\omega\not\in\Pi_{\omega_0}=:\Pi$. Now we make in (6.14) the change of variables $y=k^{-\gamma}|\eta|^{-\gamma}u$ with $\gamma=\rho^{-1}$. Then the role of (6.15), (6.16) is played by the equalities

$$s_{00}(\omega,\omega';\lambda)=\pm 2^{-1}k^{(1-\gamma)(d-1)}\langle\omega+\omega',\omega_0\rangle(2\pi)^{-d+1}$$
$$\times|\eta|^{-\gamma(d-1)}\int_\Pi \exp\Big(ik^{1-\gamma}|\eta|^{1-\gamma}G(u,\omega,\omega',\lambda)\Big)du,$$

where

$$G(y,\omega,\omega',\lambda)=\langle y,\hat\eta\rangle+\mathbf{A}(y,\omega)+k^{-1+\gamma}|\eta|^{-1+\gamma}f(k^{-\gamma}|\eta|^{-\gamma}y,\omega,\omega',\lambda).$$

With such choice of G the equation (6.17) for stationary points is preserved and the role of (6.18) is played by the equation

$$\hat\eta+\int_{-\infty}^{\infty}(\nabla_y A_\infty)(y^{(0)}(\hat\eta)+t\omega,\omega)dt=0,\quad y\in\Pi, \tag{6.21}$$

Thus, instead of Theorem 6.5, we have

Theorem 6.6 *Let assumptions* (1.7) *where* $\rho_a<1$, $\rho_a<\rho_v$ *and* (6.20) *for some* $\omega\in\mathbb{S}^{d-1}$ *be satisfied. Suppose that for a given* $\hat\eta$ *there is a finite number of points* $y_1^{(0)}(\hat\eta),\dots,y_m^{(0)}(\hat\eta)$ *satisfying equation* (6.21) *and that* $\det\mathcal{H}(y_j^{(0)}(\hat\eta),\omega)\neq 0$ *for all* $j=1,\dots,m$. *Then the kernel of the SM admits as* $\omega'\to\omega$ *the representation* (6.19) *where* $\sigma=\gamma-1$ *and* $\tau(k)=(2\pi k^\sigma)^{-(d-1)/2}$.

Thus, compared to Theorem 6.5, in the magnetic case only the dependence on the spectral parameter k is different.

Remark Formula (6.19) does not exclude that $\lambda\to\infty$ as long as $k^{-\sigma}|\eta|^{1-\gamma}\to\infty$.

We emphasize that the diagonal singularity $|\eta|^{-(d-1)(1+\gamma)/2}$ in (6.19) is stronger than that of the singular integral operator. Nevertheless the SM is a bounded operator in the space $L_2(\mathbb{S}^{d-1})$ due to oscillations of the factors

$$\exp\Big(ik^{-\sigma}|\eta|^{1-\gamma}G(y_j(\hat\eta),\omega,\omega',\lambda)\Big).$$

4. Let us now consider a more special limit as $\omega'\to\omega$, $k\to\infty$ but $k(\omega-\omega')=\xi\neq 0$ remains fixed. This method allows us to reconstruct potentials V and A.

Let us first reconstruct the electric potential. By the proof, we suppose for definiteness that V is long-range although our construction remains of course true in the short-range case.

Proposition 6.7 *Let assumption* (1.7) *hold,* $A=0$ *and let* $\xi\neq 0$ *be an arbitrary fixed vector of* $\mathbb{R}^d$. *Suppose that* $k\to\infty$, $\omega'\to\omega$ *in such a way that* $k(\omega-\omega')=\xi$. *Then*

$$\lim k^{-d+2}s(\omega,\omega';\lambda)=-i\pi(2\pi)^{-d/2}\hat V(\xi). \tag{6.22}$$

Proof – According to Theorem 5.7 it suffices to check (6.22) for function (5.2). Let us choose $\Pi = \Pi_{\omega_0}$ in such a way that $\xi \in \Pi$. Then $\omega, \omega' \to \omega_0$ (or $-\omega_0$). Let the first coordinate axis in Π be directed along ξ. We integrate in (5.2) p times by parts in the first variable which yields

$$s_0(\omega, \omega'; \lambda) = (2\pi)^{-d+1} i^{-p} |\xi|^{-p} \int_\Pi e^{-i\langle y, \xi\rangle} (\partial_1^p \mathbf{a}_0)(y, \omega, \omega'; \lambda) dy.$$

Since $A = 0$, we have that $\theta_0 = 0$ in (5.35) and, according to (2.16), $b_1^{(\pm)}(x, \xi)$ contains at least the first power of $|\xi|^{-1}$. Therefore $\sigma_1 = 0$ in (5.34). It follows from (5.34) that, for sufficiently large p,

$$(\partial_1^p \mathbf{a}_0)(y, \omega, \omega'; \lambda) = i 4^{-1} k^{d-2} \Big(\langle \omega + \omega', \omega_0 \rangle (\partial_1^p \theta_1)(y, \omega, \omega') e^{i\Xi(y, \omega, \omega', k)} + \tilde\sigma(y, \omega, \omega'; k) \Big),$$

where $\tilde\sigma$ is a smooth function of all variables and

$$|\tilde\sigma(y, \omega, \omega'; k)| \le C k^{-1} (1 + |y|)^{-d+1-\varepsilon}, \quad \varepsilon > 0.$$

Therefore passing, for fixed ξ, to the limit $k \to \infty$, $\omega, \omega' \to \omega_0$ and taking into account formula (6.7) for $\theta_1 = \theta_1^{(0)}$, we find that

$$\begin{aligned} \lim k^{-d+2} s_0(\omega, \omega'; \lambda) &= -i\pi (2\pi)^{-d} i^{-p} |\xi|^{-p} \int_\Pi dy e^{-i\langle y, \xi\rangle} \int_{-\infty}^{\infty} dt (\partial_1^p V)(y + t\omega_0) \\ &= -i\pi (2\pi)^{-d} i^{-p} |\xi|^{-p} \int_{\mathbb{R}^d} e^{-i\langle x, \xi\rangle} (\partial_1^p V)(x) dx, \end{aligned}$$

which is equivalent to (6.22). □

Formula (6.22) goes back to [6] where very short-range potentials were considered. For long-range potentials V such that $\rho_v > 1/2$ this formula was obtained in [12].

The second assertion also relies on Theorem 5.7. Its proof is similar to that of Proposition 6.7 and hence we give only its sketch.

Proposition 6.8 *Let assumption* (1.7) *hold and let* $\xi \neq 0$ *be an arbitrary fixed vector of* $\mathbb{R}^d$. *Suppose again that* $k \to \infty$, $\omega' \to \omega \notin \Pi = \Pi_{\omega_0}$ *in such a way that* $k(\omega - \omega') = \xi$. *Then*

$$\lim k^{-d+1} s(\omega, \omega'; \lambda) = |\langle \omega, \omega_0 \rangle| (2\pi)^{-d+1} \int_\Pi e^{-i\langle y, \xi\rangle} e^{i\theta_0(y, \omega, \omega)} dy, \tag{6.23}$$

where

$$\theta_0(y, \omega, \omega) = \int_{-\infty}^{\infty} \langle A(y + t\omega), \omega \rangle dt, \quad \rho_a > 1,$$

and

$$\theta_0(y, \omega, \omega) = \int_{-\infty}^{\infty} (\langle A(y + t\omega), \omega \rangle - \langle A(t\omega), \omega \rangle) dt, \quad \rho_a \le 1.$$

Proof – Indeed, in this case we can neglect in (5.34) all terms $(2ik)^{-n} \sigma_n(y, \omega, \omega')$ for $n \ge 1$ which yields the approximation (6.3) for $s(\omega, \omega'; \lambda)$. Moreover, we can replace $\Xi(y, \omega, \omega'; k)$ by its leading term $\theta_0(y, \omega, \omega)$. □

Of course, the right-hand side of (6.23) is the $(d-1)$-dimensional Fourier transform of a simple function of the magnetic potential.

Finally, we note that there are other ways of understanding the high-energy limit discussed in the papers [5, 2, 20].

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