SCATTERING BY MAGNETIC FIELDS

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Abstract. Consider the scattering amplitude $s(\omega, \omega'; \lambda)$, $\omega, \omega' \in \mathbb{S}^{d-1}$, $\lambda > 0$, corresponding to an arbitrary short-range magnetic field $B(x)$, $x \in \mathbb{R}^d$. This is a smooth function of ω and ω' away from the diagonal $\omega = \omega'$ but it may be singular on the diagonal. If $d = 2$, then the singular part of the scattering amplitude (for example, in the transversal gauge) is a linear combination of the Dirac δ-function and of a singular denominator. Such structure is typical for long-range magnetic scattering. We refer to this phenomenon as to the long-range Aharonov-Bohm effect. On the contrary, for $d = 3$ scattering is essentially of short-range nature although, for example, the magnetic potential $A^{(tr)}(x)$ such that $\operatorname{curl} A^{(tr)}(x) = B(x)$ and $\langle A^{(tr)}(x), x\rangle = 0$ decays at infinity as $|x|^{-1}$ only. To be more precise, we show that, up to the diagonal Dirac function (times an explicit function of ω), the scattering amplitude has only a weak singularity in the forward direction $\omega = \omega'$. Our approach relies on a construction in the dimension $d = 3$ of a short-range magnetic potential $A(x)$ corresponding to a given short-range magnetic field $B(x)$.

1. Introduction

1.1. In the original paper [2] by Aharonov and Bohm (see also [4, 11]) the following mental experiment was discussed. Consider a thin straight solenoid of infinite length so that the magnetic field $B(x)$ is confined inside this solenoid and is zero outside of it. Consider a beam of particles (electrons) coming from infinity following some direction. Suppose that its interaction with the magnetic field inside the solenoid is blocked out by some shield, for example, by a strong repulsive electric field. Nevertheless the scattering amplitude turns out to be different from zero (the corresponding scattering matrix is not the identity operator). Therefore it can be expected that one may observe in experiments a non-trivial interference behind the solenoid between parts of the initial beam going around the solenoid from the left and right. Moreover, this interference picture should depend on the magnetic flux Φ through a cross-section of the solenoid. This contradicts of course the classical picture but is perfectly conformal with the principles of quantum mechanics. Indeed, the Schrödinger equation is formulated in terms of a magnetic potential $A(x)$ defined by the equation

$$\operatorname{curl} A(x) = B(x). \tag{1.1}$$

In view of translation invariance in the direction of the solenoid, the problem considered is two-dimensional. For definiteness, we suppose that the axis of the solenoid coincides with the x_3-axis, so that $B(x) = (0, 0, B(x))$, $x = (x_1, x_2)$, and (1.1)

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reduces to the equation

$$\partial A_2(x)/\partial x_1 - \partial A_1(x)/\partial x_2 = B(x) \tag{1.2}$$

for components of the potential $A(x) = (A_1(x), A_2(x), 0)$. According to the Stokes theorem, the flux Φ is defined by each of the following two equalities

$$\Phi = \int_{\mathbb{R}^2} B(x)dx = \lim_{R\to\infty} \int_{|x|=R} \langle A(x), dx\rangle \tag{1.3}$$

where $\langle\cdot,\cdot\rangle$ is the scalar product in $\mathbb{R}^2$ or, more generally, in $\mathbb{R}^d$. Therefore $A(x)$ is not zero provided $\Phi \neq 0$. It follows that scattering is non-trivial even though the magnetic field is zero in the region where particles can penetrate.

Actually, to solve the problem explicitly, Aharonov and Bohm [2] have simplified it in the following way. First, they chose an infinitely thin solenoid. Second, instead of an inpenetrable shield described mathematically by the Dirichlet boundary condition they required a regularity of the wave function on the solenoid itself (at $x = 0$). Thus, strictly speaking, a direct interaction of particles with a magnetic field was not completely excluded in the Aharonov-Bohm (A-B) model (this "drawback" was remedied in [11], see also [1]). To be more precise, in the paper [2] the Schrödinger operator H with magnetic potential

$$A(x) = -\alpha(-x_2, x_1, 0)|x|^{-2}, \quad \alpha = -(2\pi)^{-1}\Phi \in \mathbb{R}, \tag{1.4}$$

was considered (according to (1.3), in this case the magnetic field equals $-2\pi\alpha\delta(x)$ where $\delta(x)$ is the Dirac function). For such potentials, variables in the Schrödinger equation can be separated in polar coordinates (r, θ), and for every angular momentum $m = 0, \pm1, \pm2, \dots$ the radial equation

$$-u_m'' + ((m+\alpha)^2 - 1/4)r^{-2}u_m = \lambda u_m$$

($\lambda > 0$ is the energy) can be solved in terms of the Bessel functions $\mathcal{I}_\nu$, namely $u_m(r) = r^{1/2}\mathcal{I}_{|m+\alpha|}(\lambda^{1/2}r)$. Using the asymptotics of these functions as $r \to \infty$, we see that the scattering matrix (SM) S for the operator H does not depend on λ and has two eigenvalues of infinite multiplicity

$$s_m = e^{i\alpha\pi} \quad \text{for} \quad m \leq -\alpha \qquad \text{and} \qquad s_m = e^{-i\alpha\pi} \quad \text{for} \quad m \geq -\alpha \tag{1.5}$$

with corresponding eigenfunctions $e^{im\theta}$. This implies of course that the SM is non-trivial, that is, $S \neq I$ (I is the identity operator) if $\alpha \notin 2\mathbb{Z}$. This fact is known as the A-B effect. It appeared to be surprising, at least from the point of view of classical physics, since the magnetic field $B(x) = 0$ for $x \neq 0$. Anyway the A-B effect is a perfect test for the validity of quantum mechanics. Its experimental confirmation is discussed in [7]. Note that if we introduce the Planck constant $\hbar$, then $s_m(\hbar) = \exp(i\alpha\pi\hbar^{-2})$ for $m \leq -\alpha\hbar^{-2}$ and $s_m(\hbar) = \exp(-i\alpha\pi\hbar^{-2})$ for $m \geq -\alpha\hbar^{-2}$. Therefore the SM $S(\hbar)$ has no limit as $\hbar \to 0$ which is consistent with the absence of the A-B effect in classical physics.

Actually, for the A-B potential, the SM is not only non-trivial, but its properties are typical for long-range scattering. Indeed, the scattering amplitude (kernel of the SM regarded as an integral operator) equals

$$s(\theta,\theta') = (2\pi)^{-1} \sum_{m=-\infty}^{\infty} s_m e^{im(\theta-\theta')}.$$

A simple calculation (see [11]) shows that, for eigenvalues (1.5), this expression can be written as

$$s(\theta,\theta') = \delta(\theta-\theta')\cos\pi\alpha + i\pi^{-1}e^{-i[\alpha](\theta-\theta')}\sin\pi\alpha\, P.V.(e^{i(\theta-\theta')}-1)^{-1}, \qquad (1.6)$$

where $[\alpha]$ denotes the greatest integer less than or equal to α. Thus, the scattering amplitude contains a singular denominator (understood in the sense of the principal value) if the magnetic flux $\Phi \notin 2\pi\mathbb{Z}$. We use the term "long-range Aharonov-Bohm" effect for this phenomenon since such singularity is absent for short-range (both electric and magnetic) potentials.

1.2. Recall that, under natural assumptions, the scattering amplitude $s(\omega,\omega';\lambda)$, where $\omega,\omega' \in \mathbb{S}^{d-1}$ if $x \in \mathbb{R}^d$, is a smooth function away from the diagonal but can be very singular for $\omega = \omega'$. This singularity is determined by the decay of a potential at infinity. For short-range potentials satisfying the condition

$$|A(x)| \le C(1+|x|)^{-\rho}, \quad \rho > 1, \qquad (1.7)$$

the SM is the sum of the identity operator and of an integral operator with a weak diagonal singularity. Moreover, if $\rho \in (1,d)$, then the estimate

$$s(\omega,\omega';\lambda) = O(|\omega-\omega'|^{-d+\rho}), \quad \omega \ne \omega', \quad \omega-\omega' \to 0,$$

holds (see, e.g., [14]). This singularity is weaker than that of kernel of the singular integral operator (cf. (1.6)). Moreover, it becomes weaker as long as a potential decays faster at infinity.

For long-range potentials the diagonal singularity of the scattering amplitude is stronger than in the short-range case, but the diagonal Dirac function disappears. Roughly speaking, for potentials $A(x)$ asymptotically homogeneous of degree $-\rho$, $\rho \in (0,1)$, the singular part of the scattering amplitude is given by the formula

$$s_0(\omega,\omega';\lambda) = G(\omega,\omega-\omega';\lambda)\exp\Big(i\Xi(\omega,\omega-\omega';\lambda)\Big), \qquad (1.8)$$

where G and Ξ are asymptotically homogeneous functions of $\omega-\omega'$ of degrees $-(d-1)(1+\rho^{-1})/2$ and $1-\rho^{-1}$, respectively (see [15]). Thus, the diagonal singularity of function (1.8) is stronger than that of kernel of a singular integral operator. Nevertheless due to oscillations of the second factor in (1.8), the operator with such kernel is bounded in $L_2(\mathbb{S}^{d-1})$.

Let us consider finally the intermediary case of potentials with Coulomb decay ($\rho = 1$) at infinity. Now the results for electric and magnetic potentials are qualitatively different. For electric potentials, the answer is again given (see, e.g., [5, 13]) by formula (1.8), where G is an asymptotically homogeneous function of degree $-d+1$ and Ξ has a logarithmic singularity at $\omega = \omega'$. As shown below, for arbitrary magnetic potentials satisfying the transversal condition

$$\langle A(x), x\rangle = 0, \qquad (1.9)$$

the singularity of the scattering amplitude is described by a formular similar to (1.6). However in this paper we pay a special attention to potentials corresponding to short-range magnetic fields, for which the answer depends crucially on dimension of the space.

1.3. Thus, we consider an arbitrary magnetic field satisfying the short-range condition

$$|B(x)| \le C(1+|x|)^{-r}, \quad r > 2, \qquad (1.10)$$

(C denotes different positive constants whose precise values are of no importance) and study properties of the corresponding SM. In particular, we find a diagonal singularity of the scattering amplitude. Our goal here is to reveal the difference between dimensions 2 and 3.

Of course by the study of the SM for a given magnetic field one has to take into account that, from a theoretical point of view, the SM is determined by a magnetic potential $A(x)$ satisfying equation (1.1). In its turn, a solution of this equation is not unique, that is the gradient of an arbitrary function can be added to $A(x)$ which leads to a gauge transformation of the Schrödinger operator. Although the SM are different in different gauges, they are connected by a simple formula, i.e., they are covariant with respect to gauge transformations. This allows us to speak about the SM corresponding to a given magnetic field.

As far as the A-B effect is concerned, the situation is similar in dimensions two and three. Consider, for example, a toroidal solenoid $\mathbf{T}$ in the space $\mathbb{R}^3$. The magnetic field is again concentrated inside the solenoid and is zero outside of it. Suppose again that this solenoid is surrounded by a slightly bigger toroidal solenoid which excludes a direct interaction of quantum particles with the magnetic field. By virtue of the Stokes theorem the corresponding magnetic potential is non-zero and hence the SM is non-trivial provided the magnetic flux Φ_s through a transverse cross-section of the solenoid is not zero.

On the contrary, it turns out that the long-range A-B effect always occurs in dimension two (if the total magnetic flux $\Phi \notin 2\pi\mathbb{Z}$), but under assumption (1.10) it cannot happen in dimension three. To be more precise, we show that in dimension $d = 2$, the diagonal singularity of the scattering amplitude is described by a formula similar to (1.6), i.e., it has a structure typical for long-range scattering (although the usual wave operators exist in this case). On the contrary, in dimension $d = 3$ the structure of the SM is almost the same as for scattering by short-range potentials. We show also that condition (1.10) is precise, that is the diagonal singularity of the scattering amplitude is described by a formula generalizing (1.6) to the case $d = 3$, if $B(x)$ decays at infinity as a homogeneous function of degree -2. Note that a condition similar to (1.10) distinguishes also short-range electric fields.

A priori the difference between dimensions $d = 3$ and $d = 2$ is not quite obvious. Indeed, in the case $d = 3$ a natural possibility is to choose a potential $A(x) = A^{(tr)}(x)$ satisfying transversal gauge condition (1.9). Note that this condition is fullfilled (for $x \neq 0$) for A-B potential (1.4). Potential $A^{(tr)}(x)$ decays always as $|x|^{-1}$ at infinity, and hence it can be expected that the SM has a structure typical for long-range scattering. However for $d = 3$ and a given magnetic field $B(x)$, we can also construct a short-range magnetic potential $A(x)$ satisfying equation (1.1) and condition (1.7). Moreover, if $B(x)$ has compact support, then $A(x)$ is also of compact support. This explains why scattering for $d = 3$ has a short-range nature.

The difference between dimensions $d = 3$ and $d = 2$ is of topological nature: if $d \geq 3$, then the set $\mathbb{R}^d \setminus \{0\}$ is simply connected whereas it is not true for $d = 2$. Gauge transformations cannot change the magnetic flux if $d = 2$. On the contrary, the absence of a similar invariant for $d = 3$ makes the three-dimensional problem essentially more flexible. If $d = 2$ and $\Phi \neq 0$, then a magnetic potential cannot even satisfy the condition $A(x) = o(|x|^{-1})$ as $|x| \to \infty$. Indeed, in this case the second integral in (1.3) (the circulation of $A(x)$ over the circle $|x| = R$) tends to 0 as $R \to \infty$, and equality (1.3) implies that necessarily $\Phi = 0$.

As was already noted, condition (1.10) is precise, that is the long-range A-B effect occurs even in dimension three if $B(x)$ decays as $|x|^{-2}$ only. Thus, a long-range behaviour of a magnetic field in dimension 3 plays the same role as the topological obstruction in dimension 2 if the flux $\Phi \notin 2\pi\mathbb{Z}$. Of course, our results for $d = 3$ remain true for all dimensions $d \geq 3$ (in the general case one has to consider A as a 1-form and $B = dA$ as a 2-form). Electric potentials are supposed to be zero since they do not add anything new to the phenomena discussed here.

In the next section we discuss some elementary facts about pseudodifferential operators (PDO) acting on a manifold (the unit sphere). Then we recall in Section 3 some basic results of scattering theory and discuss the behaviour of the SM with respect to gauge transformations. The existence of the long-range A-B effect is established in Section 4. On the contrary, in Section 5 we prove its absence for $d = 3$ provided condition (1.10) is satisfied.

To a large extent, this paper can be considered as a survey article although the results of Section 5 are essentially new. Moreover, compared to [10] and [16], we change the point of view supposing that a magnetic field, rather than a magnetic potential, is given.

2. Pseudodifferential operators on the unit sphere

2.1. The definition of a PDO P on the unit sphere $\mathbb{S}^{d-1} \subset \mathbb{R}^d$ reduces to that on a domain $\Sigma \subset \mathbb{R}^{d-1}$ (see, e.g., [12]). Roughly speaking, for a neighbourhood Ω of an arbitrary point $\omega_0 \in \mathbb{S}^{d-1}$ and a diffeomorphism $\varkappa : \Omega \to \Sigma$, an operator $P : C_0^\infty(\Omega) \to C^\infty(\Omega)$ reduces by the change of variables $\zeta = \varkappa(\omega)$ to an operator $P_\varkappa : C_0^\infty(\Sigma) \to C^\infty(\Sigma)$. Suppose that, for all Ω and $\varkappa$, the operators $P_\varkappa$ are PDO on Σ, that is

$$(P_\varkappa u)(\zeta) = (2\pi)^{-(d-1)/2} \int_{\mathbb{R}^{d-1}} e^{i\langle y,\zeta\rangle} p_\varkappa(\zeta, y)\hat{u}(y)dy,$$

where $\hat{u}$ is the Fourier transform of u. Then P is called PDO on $\mathbb{S}^{d-1}$. We require that symbols $p_\varkappa \in C^\infty(\Sigma \times \mathbb{R}^{d-1})$ of the PDO $P_\varkappa$ belong to the (Hörmander) class $\mathcal{S}^m$ of symbols satisfying, for all multi-indices α and β, the estimates

$$|\partial_y^\alpha \partial_\zeta^\beta p_\varkappa(\zeta, y)| \leq C_{\alpha,\beta}(1 + |y|)^{m-|\alpha|}.$$

Then we say that the PDO P is also from the class $\mathcal{S}^m$. In terms of the standard PDO notation, ζ plays the role of the space variable and the variable y is the dual one.

Actually, it suffices to consider only special diffeomorphisms. For any $\omega_0 \in \mathbb{S}^{d-1}$, let Π_{ω_0} be the hyperplane orthogonal to ω_0, and let $\Omega = \Omega(\omega_0, \gamma) \subset \mathbb{S}^{d-1}$ be determined by the condition $\langle \omega, \omega_0 \rangle > \gamma > 0$. Let $\zeta = \varkappa(\omega)$ be the orthogonal projection of $\omega \in \Omega$ on Π_{ω_0}; in particular, we assume that $\varkappa(\omega_0) = 0$. We denote by Σ the orthogonal projection of Ω on the hyperplane Π_{ω_0} and identify points $\omega \in \Omega$ and $\zeta = \varkappa(\omega)$. The hyperplane Π_{ω_0} can be identified with $\mathbb{R}^{d-1}$. Let us also consider the unitary mapping $Z_\varkappa : L_2(\Omega) \to L_2(\Sigma)$ defined by the equality

$$(Z_\varkappa u)(\zeta) = (1 - |\zeta|^2)^{-1/4} u(\omega), \quad \zeta = \varkappa(\omega).$$

Note that compared to the standard definition it is convenient for us to add the factor $(1 - |\zeta|^2)^{-1/4}$ in order to make the operator $Z_\varkappa$ unitary. We suppose that the operator $P_\varkappa = Z_\varkappa P Z_\varkappa^*$ is a PDO on $\Sigma \subset \mathbb{R}^{d-1}$ with symbol $p_\varkappa(\zeta, y)$ from the class $\mathcal{S}^m = \mathcal{S}^m(\Sigma \times \mathbb{R}^{d-1})$. The symbol $p_\varkappa(\zeta, y)$ is invariant with respect to diffeomorphisms of Σ up to terms from the class $\mathcal{S}^{m-1}$. This invariant part, considered modulo functions from $\mathcal{S}^{m-1}$, is called the principal symbol of the PDO $P_\varkappa$ and will be denoted $p_\varkappa^{(pr)}$. The principal symbol of the PDO P is correctly defined (it means that it does not depend on a choice of $\varkappa$) on the cotangent bundle $T^*\mathbb{S}^{d-1}$ of $\mathbb{S}^{d-1}$ by the equality

$$p(\omega, z) = p_\varkappa^{(pr)}(\zeta, y), \quad |\omega| = 1, \quad \langle\omega, z\rangle = 0,$$

where $\zeta = \varkappa(\omega)$ and $z = {}^t\varkappa'(\omega)y$ is the orthogonal projection of y on the hyperplane Π_ω. Note also that kernels $g(\omega,\omega')$ and $g_\varkappa(\zeta,\zeta')$ of the operators P and $P_\varkappa$ regarded as integral operators in $L_2(\Omega)$ and $L_2(\Sigma)$, respectively, are related by the equation

$$g(\omega,\omega') = g_\varkappa(\zeta,\zeta')(1-|\zeta|^2)^{1/4}(1-|\zeta'|^2)^{1/4}, \quad \omega,\omega' \in \Omega.$$

It is required that $g(\omega,\omega')$ be a C^∞-function away from the diagonal $\omega = \omega'$.

2.2. We need information on the essential spectrum of a PDO with a homogeneous symbol of order zero. Below a function $f \in C^\infty$ is called asymptotically homogeneous of degree k if $f(z) = |z|^k f(\hat{z})$, $\hat{z} = z|z|^{-1}$, for $|z| \geq 1/2$. Of course, $1/2$ is chosen here for definiteness. Actually, only the behaviour of $f(z)$ for large $|z|$ is essential. Let us denote by $T_1^*\mathbb{S}^{d-1} \subset T^*\mathbb{S}^{d-1}$ the set of pairs (ω, z) such that $\omega, z \in \mathbb{S}^{d-1}$ and $\langle\omega, z\rangle = 0$.

Proposition 2.1. *Let P be a PDO on $\mathbb{S}^{d-1}$ from the class $\mathcal{S}^0$ with principal symbol $p(\omega, z)$ asymptotically homogeneous of degree* 0 (*in the variable z*). *Then the essential spectrum $\sigma_{ess}(P)$ of the operator P in the space $L_2(\mathbb{S}^{d-1})$ coincides with the image of the function $p(\omega, z)$ restricted to the set $T_1^*\mathbb{S}^{d-1}$.*

As is well known, kernel $g(\omega,\omega')$ of a PDO P regarded as an integral operator can be very singular on the diagonal $\omega = \omega'$. Let us find this singularity.

Proposition 2.2. *Under the assumptions of Proposition* 2.1, *the kernel $g(\omega,\omega')$ of a PDO P with principal symbol $p(\omega, z)$ admits the representation*

$$g(\omega,\omega') = p^{(av)}(\omega)\delta(\omega,\omega') + \mathrm{P.V.}q(\omega,\omega'-\omega), \tag{2.1}$$

up to terms of order $O(|\omega-\omega'|^{-d+1+\nu})$ for any $\nu < 1$ if $d = 2$ and for $\nu = 1$ if $d \geq 3$. Here $\delta(\omega,\omega')$ is the Dirac function on the unit sphere,

$$p^{(av)}(\omega) = |\mathbb{S}^{d-2}|^{-1}\int_{\mathbb{S}_\omega^{d-2}} p(\omega,\psi)d\psi, \quad \mathbb{S}_\omega^{d-2} = \mathbb{S}^{d-1}\cap\Pi_\omega,$$

$$q(\omega,\tau) = (2\pi i)^{-d+1}(d-2)!\int_{\mathbb{S}_\omega^{d-2}}(p(\omega,\psi) - p^{(av)}(\omega))(\langle\psi,\tau\rangle - i0)^{-d+1}d\psi,$$

so that, in particular, for all $\omega \in \mathbb{S}^{d-1}$

$$\int_{\mathbb{S}_\omega^{d-2}} q(\omega,\varphi)d\varphi = 0. \tag{2.2}$$

Note that the function $q(\omega,\omega'-\omega)$ in (2.1) is homogeneous of degree $-d+1$ in $\omega'-\omega$, so that due to condition (2.2) the integral operator with this kernel is correctly defined (as a bounded operator in $L_2(\mathbb{S}^{d-1})$) in the sense of principal value. Thus, up to an integral operator with a weak diagonal singularity, P is the sum P_0 of the operator of multiplication by $p^{(av)}(\omega)$ and of the singular integral operator. To be explicit,

$$(P_0 f)(\omega) = p^{(av)}(\omega) f(\omega) + \lim_{\varepsilon \to 0} \int_{|\omega' - \omega| > \varepsilon} q(\omega, \omega' - \omega) f(\omega') d\omega'.$$

We emphasize that a PDO P of order zero is determined by its principal symbol only up to terms from the class $\mathcal{S}^{-1}$. These operators are compact so that, by the Weyl theorem, the essential spectra of all such PDO P are the same. Similarly, singular parts of kernels of all such operators P are given by the same formula (2.1) and all remainders are $O(|\omega - \omega'|^{-d+1+\nu})$.

Remark 2.3. Let $d = 2$. Then $T_1^*\mathbb{S}$ consists of points $(\omega, \omega^{(+)})$ and $(\omega, \omega^{(-)})$ where $\omega \in \mathbb{S}$ is arbitrary and $\omega^{(+)}$ and $\omega^{(-)} = -\omega^{(+)}$ are obtained from ω by rotation at the angle $\pi/2$ and $-\pi/2$ in the positive (counterclockwise) direction. Integral over $\mathbb{S}^0_\omega$ reduces to a sum over two points $\omega^{(+)}$ and $\omega^{(-)}$ and

$$\langle \omega^{(\pm)}, \omega' - \omega \rangle = \pm |\omega' - \omega| \mathrm{sgn}\{\omega, \omega'\} + O(|\omega' - \omega|^2), \quad \omega' \to \omega,$$

where $\{\omega, \omega'\}$ is the oriented angle between an initial vector ω and a final vector ω'. Let us set

$$\left.\begin{array}{lcl} p^{(av)}(\omega) & = & 2^{-1}(p(\omega, \omega^{(+)}) + p(\omega, \omega^{(-)})), \\ p^{(s)}(\omega) & = & (2\pi i)^{-1}(p(\omega, \omega^{(+)}) - p(\omega, \omega^{(-)})). \end{array}\right\} \tag{2.3}$$

Then formula (2.1) for the singular part of kernel of the operator P can be written in the form

$$g(\omega, \omega') = p^{(av)}(\omega)\delta(\omega, \omega') + p^{(s)}(\omega)\mathrm{P.V.}|\omega' - \omega|^{-1}\mathrm{sgn}\{\omega, \omega'\}.$$

Proofs of Propositions 2.1 and 2.2 can be found in [16].

3. Scattering matrix

3.1. Let us discuss briefly some basic facts of scattering theory. Consider the Schrödinger operator

$$H = (i\nabla + A(x))^2, \quad x \in \mathbb{R}^d, \quad d \geq 2,$$

with a real magnetic potential $A(x) = (A_1(x), \ldots, A_d(x))$ satisfying condition (1.7). The dimension d is arbitrary in this section. We do not assume that the function $A(x)$ is differentiable so that, strictly speaking, H is correctly defined as a self-adjoint operator in the space $L_2(\mathbb{R}^d)$ in terms of the corresponding quadratic form. In general, equality (1.1) should be understood in the sense of distributions.

Let $H_0 = -\Delta$ be the "free" operator. Under assumption (1.7) the wave operators

$$W_\pm = W_\pm(H, H_0) = \operatorname*{s-lim}_{t \to \pm\infty} e^{iHt} e^{-iH_0 t} \tag{3.1}$$

exist, are unitary and possess the intertwining property

$$HW_\pm = W_\pm H_0.$$

The scattering operator $\mathbf{S} = W_+^* W_-$ is unitary and commutes with H_0. Let the unitary operator $F : L_2(\mathbb{R}^d) \to L_2(\mathbb{R}_+; L_2(\mathbb{S}^{d-1}))$ be defined by the formula

$$(Ff)(\lambda; \omega) = 2^{-1/2} \lambda^{(d-2)/4} \hat{f}(\lambda^{1/2}\omega), \quad \lambda > 0, \quad \omega \in \mathbb{S}^{d-1},$$

where $\hat{f} = \mathcal{F}f$ is the Fourier transform of $f \in L_2(\mathbb{R}^d)$. Clearly, $(FH_0f)(\lambda) = \lambda(Ff)(\lambda)$. Since $\mathbf{S}H_0 = H_0\mathbf{S}$, we have that

$$(F\mathbf{S}f)(\lambda) = S(\lambda)(Ff)(\lambda)$$

where the unitary operator $S(\lambda) : L_2(\mathbb{S}^{d-1}) \to L_2(\mathbb{S}^{d-1})$ is known as the scattering matrix (SM). The scattering amplitude $s(\omega, \omega'; \lambda)$, $\omega, \omega' \in \mathbb{S}^{d-1}$, is kernel of $S(\lambda)$ regarded as integral operator. The following assertion is well known (see, e.g., [14, 16]).

Proposition 3.1. *Let condition* (1.7) *hold. Then the operator* $T(\lambda) = S(\lambda) - I$ *is compact and it belongs to the trace class if* $\rho > d$*. If* $\rho > d + n$*,* $n = 0, 1, 2, \ldots$*, then* $T(\lambda)$ *is integral operator with kernel from the class* $C^n(\mathbb{S}^{d-1} \times \mathbb{S}^{d-1})$*.*

Let the condition

$$|\partial^\alpha A(x)| \le C_\alpha (1 + |x|)^{-\rho - |\alpha|} \tag{3.2}$$

hold for some $\rho \in (1, d)$ *and all multi-indices* α*. Then the operator* $T(\lambda)$ *has integral kernel which is a* C^∞*-function away from the diagonal* $\omega = \omega'$ *and is bounded by* $C(\lambda)|\omega - \omega'|^{-d+\rho}$ *as* $\omega' \to \omega$*.*

3.2. We consider also a class of long-range magnetic potentials satisfying the following

Assumption 3.2. A magnetic potential $A \in C^\infty$ and

$$A(x) = A^{(\infty)}(x) + A^{(reg)}(x), \tag{3.3}$$

where $A^{(\infty)} \in C^\infty(\mathbb{R}^d \setminus \{0\})$ is a homogeneous function of degree -1 satisfying the transversal condition

$$\langle A^{(\infty)}(x), x \rangle = 0, \quad x \neq 0, \tag{3.4}$$

and, for all α,

$$|\partial^\alpha A^{(reg)}(x)| = O(|x|^{-\rho - |\alpha|}), \quad \rho > 1. \tag{3.5}$$

It turns out that for such long-range potentials the usual wave operators exist. This fact was first observed in [6]; see also [10], for a different approach. Nevertheless the structures of the SM are completely different in the short- and long-range cases. In the long-range case it is natural to regard the SM as a PDO on the unit sphere. Its principal symbol can be expressed in terms of the circulation

$$I(x, \xi) = \int_{-\infty}^{\infty} \langle A^{(\infty)}(x + t\xi), \xi \rangle dt, \quad x \neq 0, \quad \xi \neq 0, \quad \langle x, \xi \rangle = 0, \tag{3.6}$$

of the homogeneous part $A^{(\infty)}$ of A over the straight line $x + t\xi$ where t runs over $\mathbb{R}$. It follows from condition (3.4) that

$$\langle A^{(\infty)}(x + t\xi), \xi \rangle = -t^{-1} \langle A^{(\infty)}(x + t\xi), x \rangle = O(|t|^{-2})$$

as $|t| \to \infty$, and hence integral (3.6) converges. Making in (3.6) the change of variables $t = |x||\xi|^{-1}s$, we arrive at the identity

$$I(x, \xi) = I(\hat{x}, \hat{\xi}), \quad \hat{x} = x/|x|, \quad \hat{\xi} = \xi/|\xi|, \tag{3.7}$$

i.e. the function $I(x, \xi)$ is homogeneous of degree 0 in both variables. Note also that

$$I(x, -\xi) = -I(x, \xi). \tag{3.8}$$

Theorem 3.3. *Let Assumption* 3.2 *be satisfied, and let* S_0 *be the PDO from the class* $\mathcal{S}^0$ *with principal symbol*

$$p(\omega, z) = \eta(z) \exp\Big(iI(-z, \omega)\Big), \quad \omega \in \mathbb{S}^{d-1},\ z \in \mathbb{R}^d,\ \langle \omega, z\rangle = 0, \tag{3.9}$$

where $I(x, \xi)$ *is integral* (3.6) *and* $\eta \in C^\infty$, $\eta(z) = 0$ *near zero,* $\eta(z) = 1$ *for* $|z| \geq 1/2$. *Then wave operators* (3.1) *exist and the corresponding SM admits, for every* p, *the representation*

$$S(\lambda) = S_0 + S_p(\lambda) + \tilde{S}_p(\lambda), \tag{3.10}$$

where $S_p(\lambda)$ *is a PDO from the class* $\mathcal{S}^{-\nu}$, $\nu = \min\{\rho - 1, 1\}$, *and kernel of the operator* $\tilde{S}_p(\lambda)$ *is a* C^p*-function of* $\omega, \omega' \in \mathbb{S}^{d-1}$.

This result almost coincides with Theorem 5.2 of [16]. On the other hand, it is a very particular case of the general result of [15] where a complete description of the amplitude of the PDO $S(\lambda)$ was obtained for all potentials satisfying condition (3.2) for some $\rho > 0$. Theorem 3.3 is specially adapted to magnetic potentials $A(x)$ arising naturally from magnetic fields. The operator S_0 can be considered as the first Born approximation to the SM. Of course the PDO S_0 is not determined uniquely by its principal symbol (3.9), but the difference of two PDO with the same principal symbol can be included in the operator $S_p(\lambda)$.

3.3. Let us now discuss the behaviour of the SM with respect to gauge transformations defined by the formula

$$\tilde{H} = e^{i\phi} H e^{-i\phi} = (i\nabla + \tilde{A}(x))^2 \tag{3.11}$$

where

$$\tilde{A}(x) = A(x) + \operatorname{grad} \phi(x). \tag{3.12}$$

Of course $\operatorname{curl} \tilde{A}(x) = \operatorname{curl} A(x)$. We are particularly interested in functions $\phi(x)$ which are asymptotically homogeneous of degree zero.

Let us find a relation between the wave operators $W(H, H_0)$ and $W(\tilde{H}, H_0)$.

Proposition 3.4. *Let the wave operators* $W_\pm(H, H_0)$ *exist, and let a differentiable function* ϕ *be such that* $\phi(x) = \phi_0(x) + \phi_1(x)$ *where* $\phi_0(x) = \phi_0(\hat{x})$ *and* $\phi_1(x) = o(1)$ *as* $|x| \to \infty$. *Then the wave operators* $W_\pm(\tilde{H}, H_0)$ *also exist and*

$$W_\pm(\tilde{H}, H_0) = e^{i\phi(x)} W_\pm(H, H_0) \mathcal{F}^* e^{-i\phi_0(\pm\xi)} \mathcal{F}. \tag{3.13}$$

Proof. – Since

$$(\exp(-iH_0 t) f)(x) = e^{i|x|^2/(4t)} (2it)^{-d/2} \hat{f}(x/(2t)) + o(1),$$

we have that

$$\begin{aligned}(e^{-i\phi} \exp(-iH_0 t) f)(x) &= e^{i|x|^2/(4t)} (2it)^{-d/2} e^{-i\phi_0(\pm x/(2t))} \hat{f}(x/(2t)) + o(1) \\ &= e^{i|x|^2/(4t)} (2it)^{-d/2} \hat{f}^{(\pm)}(x/(2t)) + o(1), \quad t \to \pm\infty,\end{aligned}$$

where $\hat{f}^{(\pm)}(\xi) = e^{-i\phi_0(\pm\xi)} \hat{f}(\xi)$ and the remainder $o(1)$ tends to 0 in $L_2(\mathbb{R}^d)$ as $t \to \pm\infty$. This is equivalent to the relation

$$e^{-i\phi} \exp(-iH_0 t) f = \exp(-iH_0 t) f^{(\pm)} + o(1),$$

which, in view of definition (3.11), implies that

$$W_\pm(\tilde H, H_0)f = \lim_{t\to\pm\infty} e^{i\tilde H t}e^{-iH_0t}f = \lim_{t\to\pm\infty} e^{i\phi}e^{iHt}e^{-i\phi}e^{-iH_0t}f$$
$$= \lim_{t\to\pm\infty} e^{i\phi}e^{iHt}e^{-iH_0t}f^{(\pm)} = e^{i\phi}W_\pm(H,H_0)f^{(\pm)}.$$

This proves (3.13). □

As an immediate consequence of Proposition 3.4, we obtain a relation between the scattering operators and matrices.

Proposition 3.5. *Under the assumptions of Proposition* 3.4*, the scattering operators are related by the equation*

$$\mathcal{F}\mathbf{S}(\tilde H, H_0)\mathcal{F}^* = e^{i\phi_0(\xi)}\mathcal{F}\mathbf{S}(H,H_0)\mathcal{F}^*e^{-i\phi_0(-\xi)}. \tag{3.14}$$

The corresponding SM $S(\lambda) = S(H, H_0; \lambda)$ *and* $\tilde S(\lambda) = S(\tilde H, H_0; \lambda)$ *satisfy for all* $\lambda > 0$ *the relations*

$$\tilde S(\lambda) = e^{i\phi_0(\omega)}S(\lambda)e^{-i\phi_0(-\omega)} \tag{3.15}$$

or, in terms of the scattering amplitudes,

$$\tilde s(\omega,\omega';\lambda) = e^{i\phi_0(\omega)-i\phi_0(-\omega')}s(\omega,\omega';\lambda). \tag{3.16}$$

We emphasize that relations (3.14) – (3.16) for the scattering operators and matrices (but not (3.13) for the wave operators) depend only on the asymptotics ϕ_0 of the function ϕ. Probably, formulas (3.13), (3.14) appeared first in the paper [11] in the case $d = 2$ under some assumptions on A. Actually, these formulas do not require any assumptions at all.

Proposition 3.5 shows that the SM is not, in general, the identity operator for the zero magnetic field.

Example 3.6. Let

$$A(x) = \operatorname{grad}\phi(x),$$

where $\phi(x)$ satisfies the assumptions of Proposition 3.4. Then the wave operators $W_\pm(H, H_0)$ exist and

$$W_\pm(H,H_0) = e^{i\phi(x)}\mathcal{F}^*e^{-i\phi_0(\pm\xi)}\mathcal{F},$$

$$\mathbf{S}(H,H_0) = \mathcal{F}^*e^{i\phi_0(\xi)-i\phi_0(-\xi)}\mathcal{F}.$$

The corresponding SM does not depend on λ and

$$S(H,H_0) = e^{i\phi_0(\omega)-i\phi_0(-\omega)}.$$

Relations (3.15) or (3.16) show that the SM is not determined by the magnetic field $B(x) = \operatorname{curl} A(x)$ only although we have an explicit formula which connects the SM in different gauges. This seems to contradict the following mental experiment. Suppose that a quantum particle interacts with a magnetic field. Note that it is exactly a field but not a potential which can be created by our hands. However, to calculate the SM theoretically, we have to introduce a magnetic potential and then solve the Schrödinger equation. Thus, the SM depends on a potential. So it appears that a particle itself chooses a gauge convenient for it. There could be (at least) two possible explanations of this seeming contradiction. The first is that the scattering amplitude $s(\omega, \omega'; \lambda)$ cannot be measured experimentally although it is widely believed to be possible. From this point of view only the (differential) scattering cross section

$$\Sigma_{diff}(\omega, \omega', \lambda) = (2\pi)^{d-1}\lambda^{-(d-1)/2}|s(\omega, \omega'; \lambda)|^2, \quad \omega \neq \omega', \tag{3.17}$$

(ω' is an incident direction of a beam of particles and ω is a direction of observation) can be practically found which is compatible with (3.16). Another point of view is that experimental devices used for observation of a quantum particle are not harmless and fix some specific gauge.

On the other hand, for a given field, the SM is stable with respect to short-range perturbations of a potential. To be more precise, we have the following

Proposition 3.7. *Let the wave operators* $W_\pm(H, H_0)$ *exist. Suppose that* $\operatorname{curl} A(x) = \operatorname{curl} \tilde{A}(x)$ *and*

$$\tilde{A}(x) - A(x) = O(|x|^{-\rho}), \quad \rho > 1, \tag{3.18}$$

as $|x| \to \infty$. *Then the wave operators* $W_\pm(\tilde{H}, H_0)$ *also exist and the scattering operators and matrices for the pairs* H_0, H *and* $H_0, \tilde{H}$ *coincide.*

Proof. – According to Propositions 3.4 and 3.5, it suffices to show that $A(x)$ and $\tilde{A}(x)$ are related by equality (3.12) where the function $\phi(x)$ has a limit (which does not depend on $\hat{x}$) as $|x| \to \infty$. Let us define $\phi(x)$ as a curvilinear integral

$$\phi(x) = \int_{\Gamma_x} \langle \tilde{A}(y) - A(y), dy\rangle$$

taken between 0 and a variable point x. By the Stokes theorem, this integral does not depend on a choice of Γ_x which implies that equality (3.12) holds. Moreover, choosing Γ_x as the piece of straight line connecting 0 and $x = r\omega$, $\omega \in \mathbb{S}^{d-1}$, and using (3.18), we see that the limit of $\phi(r\omega)$ as $r \to \infty$ exists. It remains to show that this limit does not depend on $\omega \in \mathbb{S}^{d-1}$. Again by the Stokes theorem,

$$\phi(r\omega_2) - \phi(r\omega_1) = \int_{\mathbb{S}_r(\omega_1,\omega_2)} \langle \tilde{A}(y) - A(y), dy\rangle, \tag{3.19}$$

where $\mathbb{S}_r(\omega_1, \omega_2)$ is the arc of the circle centered at the origin and passing through the points $r\omega_1$ and $r\omega_2$. Condition (3.18) implies that integral (3.19) tends to 0 as $r \to \infty$. $\square$

4. Long-range Aharonov-Bohm effect

4.1. Let us first discuss the case $d = 2$. For a given magnetic field $B(x) = (0, 0, B(x))$, $x \in \mathbb{R}^2$, the magnetic potential $A^{(tr)}(x) = (A_1^{(tr)}(x), A_2^{(tr)}(x), 0)$ satisfying equation (1.1) (or (1.2)) and obeying the transversal gauge condition (1.9) can be constructed by the formulas

$$A_1^{(tr)}(x) = -x_2 \int_0^1 B(sx)sds, \quad A_2^{(tr)}(x) = x_1 \int_0^1 B(sx)sds. \tag{4.1}$$

If condition (1.10) is satisfied, then it follows from (4.1) that $A^{(tr)}(x)$ admits representation (3.3) where $A^{(\infty)}$ is a homogeneous function of degree -1 and $A^{(reg)}(x) = O(|x|^{-\rho})$ with $\rho = r - 1$ as $|x| \to \infty$. Indeed, $A^{(\infty)}$ is given by the formula

$$A^{(\infty)}(x) = a(\hat{x})(-x_2, x_1, 0)|x|^{-2}, \quad \hat{x} = x/|x|, \tag{4.2}$$

where

$$a(\hat{x}) = \int_0^\infty B(s\hat{x})sds \tag{4.3}$$

is a function on the unit circle and

$$A^{(reg)}(x) = |x|^{-2}(x_2, -x_1)\int_{|x|}^\infty B(s\hat{x})sds.$$

Moreover, if $B \in C^\infty(\mathbb{R}^d)$ and satisfies the condition

$$|\partial^\alpha B(x)| \leq C_\alpha(1+|x|)^{-r-|\alpha|}, \quad r > 2, \quad \forall\alpha, \tag{4.4}$$

then $A^{(tr)} \in C^\infty(\mathbb{R}^d)$, $a \in C^\infty(\mathbb{S})$ and estimates (3.5) hold for all α.

Since $\operatorname{curl} A^{(tr)}(x) = O(|x|^{-r})$ and $\operatorname{curl} A^{(reg)}(x) = O(|x|^{-r})$, it follows from representation (3.3) that $\operatorname{curl} A^{(\infty)}(x) = O(|x|^{-r})$ where $r > 2$. On the other hand, $\operatorname{curl} A^{(\infty)}(x)$ is a homogeneous function of degree -2 so that necessarily

$$\operatorname{curl} A^{(\infty)}(x) = 0, \quad x \neq 0. \tag{4.5}$$

The same arguments (or representation (4.2)) show that the transversal condition (3.4) is satisfied. Thus, the potential $A^{(tr)}(x)$ satisfies Assumption 3.2.

In view of the equalities (1.3) and (4.2), the total magnetic flux equals

$$\Phi = \int_{\mathbb{S}} a(\psi)d\psi. \tag{4.6}$$

Recall that $\omega^{(\pm)}$ is obtained from $\omega \in \mathbb{S}$ by rotation at the angle $\pm\pi/2$ in the positive (counter-clockwise) direction. Set

$$f(\omega) = \int_{\mathbb{S}(\omega^{(-)},\omega^{(+)})} a(\psi)d\psi, \quad \omega \in \mathbb{S}, \tag{4.7}$$

where the integral is taken in the positive direction over the half-circle between the points $\omega^{(-)}$ and $\omega^{(+)}$. Then for any $\omega \in \mathbb{S}$

$$f(\omega) + f(-\omega) = \Phi. \tag{4.8}$$

Comparing formulas (4.3) and (4.7), we can express the function $f(\omega)$ in terms of the magnetic field

$$f(\omega) = \int_{\langle x,\omega\rangle \geq 0} B(x)dx. \tag{4.9}$$

In its turn, integral (3.6) can be expressed in terms of the function $f(\omega)$.

Lemma 4.1. *For all $\omega \in \mathbb{S}^{d-1}$, we have that*

$$I(\omega, \omega^{(\pm)}) = \pm f(\omega). \tag{4.10}$$

Proof. – By virtue of (3.8), it suffices to consider the case of the upper sign. Since

$$< (-\omega_2 - t\omega_2^{(+)}, \omega_1 + t\omega_1^{(+)}), (\omega_1^{(+)}, \omega_2^{(+)}) >= \omega_1\omega_2^{(+)} - \omega_2\omega_1^{(+)} = 1,$$

we have that for potentials (4.2)

$$I(\omega, \omega^{(+)}) = \int_{-\infty}^\infty a\Big(\frac{\omega + t\omega^{(+)}}{\sqrt{t^2+1}}\Big)\frac{dt}{t^2+1}.$$

Making the change of variables $t = \tan\psi$, we get formula (4.10). $\square$

We denote by $S^{(tr)}(\lambda)$ the SM corresponding to the potential $A^{(tr)}$. Given Theorem 3.3 the following two assertions are immediate consequences of Propositions 2.1 and 2.2 (see also Remark 2.3).

Theorem 4.2. *Let $d = 2$ and let condition* (4.4) *be satisfied. Define the function $f(\omega)$ by formula* (4.9) *and set $\gamma_+ = \max f(\omega)$, $\gamma_- = \min f(\omega)$. Then for all $\lambda > 0$ relation*

$$\sigma_{ess}(S^{(tr)}(\lambda)) = [\exp(i\gamma_-), \exp(i\gamma_+)] \cup [\exp(-i\gamma_+), \exp(-i\gamma_-)] \tag{4.11}$$

holds if $\gamma_+ - \gamma_- < 2\pi$, and $\sigma_{ess}(S^{(tr)}(\lambda))$ covers the whole unit circle $\mathbb{T} \subset \mathbb{C}$ if $\gamma_+ - \gamma_- \geq 2\pi$.

Proof. – Let us apply Proposition 2.1 to the PDO S_0 with principal symbol (3.9). It follows from formula (4.10) that in this case

$$p(\omega, \omega^{(\pm)}) = e^{\pm if(\omega^{(\mp)})}. \tag{4.12}$$

Therefore the images of the functions $p(\omega, \omega^{(+)})$ and $p(\omega, \omega^{(-)})$ coincide with the first and the second arcs in (4.11), respectively, if $\gamma_+ - \gamma_- < 2\pi$. If $\gamma_+ - \gamma_- \geq 2\pi$, then each of these images covers the whole unit circle. So it remains to take into account that according to representation (3.10) the essential spectra of the operators $S^{(tr)}(\lambda)$ and S_0 are the same. □

Theorem 4.3. *Let the assumptions of Theorem* 4.2 *hold. Set*

$$s_0(\omega, \omega') = e^{i(f(\omega^{(-)}) - f(\omega^{(+)}))/2} \Big(\cos(\Phi/2)\delta(\omega, \omega') + \pi^{-1} \sin(\Phi/2) \mathrm{P.V.} \frac{\mathrm{sgn}\{\omega, \omega'\}}{|\omega - \omega'|} \Big). \tag{4.13}$$

Then for an arbitrary $\lambda > 0$

$$|s^{(tr)}(\omega, \omega'; \lambda) - s_0(\omega, \omega')| \leq C(\lambda) |\omega - \omega'|^{-3+r_0}.$$

Here $r_0 = r$ if $r < 3$ and r_0 is an arbitrary number smaller than 3 *if $r \geq 3$.*

Proof. – Now we apply Proposition 2.2 to the PDO S_0 with principal symbol (3.9). Comparing formulas (2.3) and (4.12), we see that in the case considered

$$p^{(av)}(\omega) = 2^{-1}(e^{if(\omega^{(-)})} + e^{-if(\omega^{(+)})}), \quad p^{(s)}(\omega) = (2\pi i)^{-1}(e^{if(\omega^{(-)})} - e^{-if(\omega^{(+)})}).$$

Using identity (4.8), we find that formula (2.3) yields expression (4.13) for the singular part of kernel of operator S_0. The "regular" part of its kernel is $O(|\omega - \omega'|^{-\varepsilon})$ as $|\omega - \omega'| \to 0$ for any $\varepsilon > 0$. Kernel of the operator $S_p(\lambda)$ in (3.10) satisfies the same estimate if $r \geq 3$ and it is $O(|\omega - \omega'|^{-3+r})$ if $r < 3$. □

Corollary 4.4. *The diagonal singularity of the scattering cross section* (3.17) *is given by the formula*

$$\Sigma_{diff}(\omega, \omega'; \lambda) = 2\pi^{-1}\lambda^{-1/2} \sin^2(\Phi/2) \, |\omega - \omega'|^{-2} + O(|\omega - \omega'|^{-4+r_0}). \tag{4.14}$$

Thus, the singular part S_0 of the SM $S^{(tr)}(\lambda)$ is the integral operator in $L_2(\mathbb{S})$ with kernel (4.13). Up to the phase factor, it is determined by the magnetic flux Φ only (and does not depend on λ). We see that in the dimension two even for magnetic fields of compact support with $\Phi \notin 2\pi\mathbb{Z}$, the SM contains the singular integral operator and the forward singularity (4.14) of the scattering cross section is stronger than for short-range magnetic potentials where it is $O(|\omega - \omega'|^{-4+2\rho})$. On the contrary, if $\Phi \in 2\pi\mathbb{Z}$, then according to (4.8), (4.13) the operator S_0 acts as multiplication by the function $e^{if(\omega^{(-)})}$. As we shall see in the next section, this situation is typical for dimensions $d \geq 3$. Note also that if $B(x)$ is an even function, that is $B(x) = B(-x)$, then, again by (4.8), $f(\omega) = \Phi/2$ for all $\omega \in \mathbb{S}$, and hence the first factor in the right-hand side of (4.13) equals 1. In this case $\sigma_{ess}(S(\lambda))$ consists of the two points $e^{i\Phi/2}$ and $e^{-i\Phi/2}$. Of course in the case $a(\omega) = -\alpha$ formula (4.13) coincides, up to smooth terms, with formula (1.6) if the natural parametrization of the unit circle $\mathbb{S}$ by points $\theta \in [0, 2\pi)$ is used.

As a concrete example, let us consider the field

$$B(x) = B_0(r) + B_1(r)\langle q, \hat{x}\rangle, \quad r = |x|,$$

where B_0 and B_1 are C^∞-functions with compact supports and $q \in \mathbb{R}^2$ is some given vector. It follows from formula (4.3) that in this case

$$a(\hat{x}) = -\alpha + \langle p, \hat{x}\rangle, \quad \alpha \in \mathbb{R}, \quad p \in \mathbb{R}^2,$$

where

$$\alpha = -\int_0^\infty B_0(r) r dr, \quad p = q \int_0^\infty B_1(r) r dr.$$

Clearly, $\Phi = -2\pi\alpha$. Let us calculate function (4.7). For an arbitrary $\omega \in \mathbb{S}$, let φ be the angle between ω and p, and let θ be the angle between $\hat{x}$ and p. Then $a(\hat{x}) = -\alpha + |p| \cos\theta$ and

$$f(\omega) = -\pi\alpha + |p| \int_{\varphi-\pi/2}^{\varphi+\pi/2} \cos\theta d\theta = -\pi\alpha + 2|p|\cos\varphi = -\pi\alpha + 2\langle p, \omega\rangle.$$

Therefore the conclusion of Theorem 4.2 is true with $\gamma_+ = -\pi\alpha + 2|p|$ and $\gamma_- = -\pi\alpha - 2|p|$. In particular, if $2|p| \geq \pi$, then $\sigma_{ess}(S^{(tr)}(\lambda)) = \mathbb{T}$. On the contrary, $\sigma_{ess}(S^{(tr)}(\lambda))$ consists of the two points $\exp(\pi i\alpha)$ and $\exp(-\pi i\alpha)$ if $p = 0$. The phase factor in (4.13) equals $\exp(2i\langle p, \omega^{(-)}\rangle)$.

Actually, the results above do not require that the potential satisfy transversal condition (1.9). The next result follows again from Theorem 3.3 combined with Propositions 2.1 and 2.2.

Theorem 4.5. *Suppose that $A \in C^\infty(\mathbb{R}^2)$ admits representation* (3.3) *where $A^{(\infty)}$ is function* (4.2) *with $a \in C^\infty(\mathbb{S})$ and $A^{(reg)}$ satisfies estimates* (3.5). *Let f be function* (4.7). *Then all conclusions of Theorems* 4.2 *and* 4.3 *remain true for the SM corresponding to the potential A.*

Let us discuss this result from the point of view of gauge transformations. Let two potentials $A(x)$ and $\tilde{A}(x)$ satisfy the assumptions of Theorem 4.5, and let Φ and $\tilde{\Phi}$ be the corresponding magnetic fluxes. If they are related by equality (3.12) where $\phi(x)$ satisfies the conditions of Proposition 3.4, then $\tilde{\Phi} = \Phi$,

$$\tilde{a}(\omega) = a(\omega) + \phi_0'(\omega) \tag{4.15}$$

and hence according to (4.7)

$$\tilde{f}(\omega^{(-)}) - \tilde{f}(\omega^{(+)}) = f(\omega^{(-)}) - f(\omega^{(+)}) + 2(\phi_0(\omega) - \phi_0(-\omega)).$$

It follows from (4.13) that singular parts of the corresponding SM are connected by the equality

$$\tilde{s}_0(\omega, \omega') = e^{i\phi_0(\omega) - i\phi_0(-\omega)} s_0(\omega, \omega'), \tag{4.16}$$

which agrees with exact formula (3.16) for scattering amplitudes.

Conversely, if $\tilde{\Phi} = \Phi$, then the function

$$\phi_0(\omega) = \int_{\mathbb{S}(\omega_0,\omega)} (\tilde{a}(\psi) - a(\psi))d\psi \tag{4.17}$$

(the point $\omega_0 \in \mathbb{S}$ is arbitrary but fixed) is correctly defined on the unit circle and equality (4.15) is satisfied. Set $\phi(x) = \eta(|x|)\phi_0(\hat{x})$ where $\eta \in C^\infty$, $\eta(r) = 0$ in a neighbourhood of zero and $\eta(r) = 1$ for large r. It follows from (4.15) that equality (3.12) holds, up to a term $A_{sr}(x)$ satisfying estimates (3.5), that is, $\tilde{A} = A + \operatorname{grad}\phi + A_{sr}$. Therefore the SM $S(\lambda)$ and $\tilde{S}(\lambda)$ for the Schrödinger operators with magnetic potentials A and $\tilde{A} - A_{sr}$ are related by equality (3.15) and hence their singular parts are related by equality (4.16). This implies that if (4.13) is verified for $\tilde{A} - A_{sr}$, then it is also true for A. Thus, for a given Φ, it suffices to prove Theorem 4.3 only for one function a satisfying (4.6) (but for all short range terms $A^{(reg)}$). We can choose $a(\omega) = (2\pi)^{-1}\Phi$, which reduces the proof of Theorem 4.3 to the case of a constant function a. In particular, if $\Phi = 0$, then the problem reduces to the short range case. The same is true with respect to Theorem 4.2 for even functions $a(\omega)$ only. Then function (4.17) is also even so that, by virtue of (3.15), the SM $\tilde{S}(\lambda)$ and $S(\lambda)$ are unitarily equivalent.

4.2. Here we consider arbitrary magnetic potentials $A(x)$, $x \in \mathbb{R}^3$, with Coulomb decay at infinity satisfying, at least asymptotically, the transversal gauge condition. For such potentials, the magnetic field $B(x) = \operatorname{curl} A(x)$ decays, in general, as $|x|^{-2}$ at infinity, so that assumption (1.10) is not satisfied. We shall show that in this case the SM contains a singular integral operator and hence the long-range A-B effect occurs.

The next two results extend Theorems 4.2 and 4.3 to the case $d = 3$. They follow again from Theorem 3.3 and Propositions 2.1 and 2.2 applied to the PDO with symbol (3.9).

Theorem 4.6. *Suppose that $d = 3$. Let Assumption 3.2 be satisfied, and let $I(x, \xi)$ be integral (3.6). Then $\sigma_{ess}(S(\lambda))$ coincides with the image of the function $I(\psi, \omega)$ for all $\psi, \omega \in \mathbb{S}^2$ such that $\langle\psi, \omega\rangle = 0$.*

Theorem 4.7. *Under the assumptions of Theorem 4.6 define the functions*

$$p^{(av)}(\omega) = (2\pi)^{-1} \int_{\mathbb{S}_\omega} \exp(iI(\psi, \omega))d\psi, \quad \mathbb{S}_\omega = \mathbb{S}^2 \cap \Pi_\omega, \tag{4.18}$$

$$q(\omega, \tau) = -(2\pi)^{-2} \int_{\mathbb{S}_\omega} (\exp(iI(\psi, \omega)) - p^{(av)}(\omega))(\langle\psi, \tau\rangle - i0)^{-2}d\psi, \tag{4.19}$$

and set

$$s_0(\omega, \omega') = p^{(av)}(\omega)\delta(\omega, \omega') + \mathrm{P.V.}q(\omega, \omega' - \omega). \tag{4.20}$$

Then for an arbitrary $\lambda > 0$ the scattering amplitude satisfies the estimate

$$|s(\omega, \omega'; \lambda) - s_0(\omega, \omega')| \le C(\lambda)|\omega - \omega'|^{-3+\rho_0},$$

where $\rho_0 = \rho$ if $\rho \in (1, 2)$ and $\rho_0 = 2$ if $\rho \ge 2$.

Corollary 4.8. *If $\omega \ne \omega'$ but $\omega - \omega' \to 0$, then*

$$s(\omega, \omega'; \lambda) = q(\omega, \omega' - \omega) + O(|\omega - \omega'|^{-3+\rho_0}).$$

Corollary 4.9. *If* $\omega \to \omega'$, *then*

$$\Sigma_{diff}(\omega, \omega'; \lambda) = (2\pi)^2 \lambda^{-1} |q(\omega, \omega' - \omega)|^2 + O(|\omega - \omega'|^{-5+\rho_0}).$$

Note that the order of singularity $|\omega - \omega'|^{-4}$ here is the same as for electric Coulomb potentials. We emphasize also that singular part (4.20) of the SM does not depend on λ.

It follows from equality (4.19) that if the function $I(x, \omega)$ does not depend on x, then $q(\omega, \tau) = 0$ so that the singular integral operator disappears in (4.20). We shall see in the next section that this situation really occurs if the magnetic field satisfies condition (1.10).

Nevertheless $q(\omega, \tau)$ is non-trivial in the general case. Let us consider (see [16], for details) two concrete examples of potentials $A^{(\infty)}(x)$ homogeneous of degree -1 and satisfying transversal condition (3.4). The corresponding fields $\operatorname{curl} A(x)$ decay only as $|x|^{-2}$ at infinity.

We define the first of these potentials by the equation

$$A^{(\infty)}(x) = |x|^{-3}(\alpha_1 x_2 x_3, \alpha_2 x_3 x_1, \alpha_3 x_1 x_2), \quad x = (x_1, x_2, x_3) \in \mathbb{R}^3,$$

where α_j are constants and $\alpha_1 + \alpha_2 + \alpha_3 = 0$. An easy calculation shows that function (3.6) equals

$$I(x, \omega) = 2|x|^{-2}(\alpha_1 \omega_1 x_2 x_3 + \alpha_2 \omega_2 x_3 x_1 + \alpha_3 \omega_3 x_1 x_2).$$

Since this function depends on λ, it cannot be expected that $q(\omega, \tau) = 0$.

Actually, the functions $p^{(av)}(\omega)$ and $q(\omega, \tau)$ can be calculated explicitly. For an arbitrary $\omega = (\omega_1, \omega_2, \omega_3)$, the coordinates of an arbitrary point $x = (x_1, x_2, x_3) \in \mathbb{S}_\omega = \mathbb{S}^2 \cap \Pi_\omega$ can be written, for some $\theta \in [0, 2\pi)$, as

$$\left.\begin{aligned} x_1 &= -(\omega_1^2 + \omega_2^2)^{-1/2}(\omega_2 \cos\theta + \omega_1 \omega_3 \sin\theta), \\ x_2 &= (\omega_1^2 + \omega_2^2)^{-1/2}(\omega_1 \cos\theta - \omega_2 \omega_3 \sin\theta), \quad x_3 = (\omega_1^2 + \omega_2^2)^{1/2} \sin\theta. \end{aligned}\right\} \tag{4.21}$$

Set

$$\mathcal{A}(\omega) = (\omega_1^2 + \omega_2^2)^{-1}\Big(4\alpha_3^2 \omega_1^2 \omega_2^2 \omega_3^2 + (\alpha_1(\omega_1^2 - \omega_2^2 \omega_3^2) - \alpha_2(\omega_2^2 - \omega_1^2 \omega_3^2))^2\Big)^{1/2}$$

and define the angle $\theta_0(\omega)$ by the equations

$$\left.\begin{aligned} \sin\theta_0(\omega) &= -2\alpha_3 \omega_1 \omega_2 \omega_3 (\omega_1^2 + \omega_2^2)^{-1} \mathcal{A}(\omega)^{-1}, \\ \cos\theta_0(\omega) &= (\alpha_1(\omega_1^2 - \omega_2^2 \omega_3^2) - \alpha_2(\omega_2^2 - \omega_1^2 \omega_3^2))(\omega_1^2 + \omega_2^2)^{-1} \mathcal{A}(\omega)^{-1}. \end{aligned}\right\}$$

Then

$$p^{(av)}(\omega) = (2\pi)^{-1} \int_0^{2\pi} \cos(\mathcal{A}(\omega) \sin\theta) d\theta$$

and

$$q(\omega, \tau) = -(2\pi)^{-2} \int_0^{2\pi} \Big(e^{i\mathcal{A}(\omega)\sin(2\theta + \theta_0(\omega))} - p^{(av)}(\omega)\Big)(\langle x(\theta), \tau\rangle - i0)^{-2} d\theta$$

where $x(\theta)$ is defined by formulas (4.21).

As another example, we choose a modification of the A-B potential

$$A^{(\infty)}(x) = -\alpha |x|^{-2}(-x_2, x_1, 0).$$

In this case

$$I(x, \omega) = \pi\alpha |x|^{-1}(\omega_1 x_2 - \omega_2 x_1) = \pi\alpha(1 - \omega_3^2)^{1/2} \cos\theta$$

if x, ω and θ are related by equalities (4.21). Plugging this expression into (4.18) and (4.19), we obtain explicit representations for the functions $p^{(av)}(\omega) = p^{(av)}(\omega_3)$ and $q(\omega, \tau) = q(\omega_3, \tau)$:

$$p^{(av)}(\omega_3) = (2\pi)^{-1} \int_0^{2\pi} \cos(\pi\alpha(1-\omega_3^2)^{1/2}\cos\theta) d\theta,$$

$$q(\omega_3, \tau) = -(2\pi)^{-2} \int_0^{2\pi} \Big(e^{i\pi\alpha(1-\omega_3^2)^{1/2}\cos\theta} - p^{(av)}(\omega) \Big) (\langle x(\theta), \tau\rangle - i0)^{-2} d\theta.$$

In both these examples the SM contain singular integral operators and hence the long-range A-B effect occurs.

5. There is no long-range Aharonov-Bohm effect in dimension three

5.1. In this section we suppose that the dimension $d = 3$. Our results remain true for all $d \geq 3$ but not for $d = 2$. Let $B(x) = (B_1(x), B_2(x), B_3(x))$ be a magnetic field such that $\operatorname{div} B(x) = 0$. Recall that a magnetic potential $A^{(tr)}(x) = (A_1^{(tr)}(x), A_2^{(tr)}(x), A_3^{(tr)}(x))$ satisfying equation (1.1) and the transversal gauge condition (1.9) is constructed by the formula

$$A_1^{(tr)}(x) = \int_0^1 \Big(B_2(sx)x_3 - B_3(sx)x_2 \Big) s ds. \tag{5.1}$$

Expressions for components $A_2^{(tr)}(x)$ and $A_3^{(tr)}(x)$ are obtained by cyclic permutations of indices in (5.1). If estimate (1.10) is satisfied, then $A^{(tr)}(x)$ admits the representation (3.3) where

$$A_1^{(\infty)}(x) = |x|^{-2} \int_0^\infty \Big(B_2(s\hat{x})x_3 - B_3(s\hat{x})x_2 \Big) s ds, \tag{5.2}$$

$$A_1^{(reg)}(x) = -|x|^{-2} \int_{|x|}^\infty \Big(B_2(s\hat{x})x_3 - B_3(s\hat{x})x_2 \Big) s ds. \tag{5.3}$$

Thus, $A^{(\infty)}(x)$ is a homogeneous function of degree -1 and $A^{(reg)}(x) = O(|x|^{-\rho})$ with $\rho = r - 1 > 1$ as $|x| \to \infty$. Quite similarly to the two-dimensional case (see subsection 4.1), we have that $A^{(\infty)}(x)$ satisfies equations (3.4) and (4.5).

Given a magnetic field $B(x)$ obeying condition (1.10), we construct now a magnetic potential $A(x)$ satisfying equation (1.1) and estimate (1.7). We proceed from the magnetic potential $A^{(tr)}$ in the transversal gauge. Let $A^{(\infty)}$ be function (5.2). We define the function $U(x)$ for $x \neq 0$ as a curvilinear integral

$$U(x) = \int_{\Gamma_{x_0,x}} \langle A^{(\infty)}(y), dy \rangle \tag{5.4}$$

taken between some fixed point $x_0 \neq 0$ and a variable point x. It is required that $0 \notin \Gamma_{x_0,x}$, so that, in view of (4.5) and the Stokes theorem, $U(x)$ does not depend on a choice of a contour $\Gamma_{x_0,x}$. Here it is used that the set $\mathbb{R}^3 \setminus \{0\}$ (and $\mathbb{R}^d \setminus \{0\}$ for all $d \geq 3$) is simply connected. Clearly,

$$A^{(\infty)}(x) = \operatorname{grad} U(x). \tag{5.5}$$

Moreover, the function $U(x)$ is homogeneous of degree 0. Indeed, if $x_2 = \gamma x_1$, $\gamma > 1$, then we can choose $\Gamma_{x_0,x_2} = \Gamma_{x_0,x_1} \cup (x_1, x_2)$ where (x_1, x_2) is the piece of straight line connecting x_1 and x_2. If $y \in (x_1, x_2)$, then according to (3.4)

$\langle A^{(\infty)}(y), dy\rangle = 0$. Hence $U(x_1) = U(x_2)$. We use definition (5.4) away from some neighbourhood of the point $x = 0$ and extend $U(x)$ as a differentiable function to all $\mathbb{R}^3$. For example, we can choose some numbers $R_2 > R_1 > 0$ and a function $\eta \in C^\infty(\mathbb{R}^3)$ such that $\eta(x) = 1$ for $|x| \geq R_2$, $\eta(x) = 0$ for $|x| \leq R_1$ and then replace $U(x)$ by $\eta(x)U(x)$. Let us now set

$$\begin{aligned} A(x) &= A^{(tr)}(x) - \operatorname{grad}(\eta(x)U(x)) \\ &= A^{(reg)}(x) + (1-\eta(x))A^{(\infty)}(x) - U(x)\operatorname{grad}\eta(x), \end{aligned} \tag{5.6}$$

so that $A(x) = A^{(reg)}(x)$ for $|x| \geq R_2$ and $A(x) = A^{(tr)}(x)$ for $|x| \leq R_1$. Thus, we have the following result.

Proposition 5.1. *Suppose that* $\operatorname{div} B(x) = 0$ *and that condition* (1.10) *holds. Let the magnetic potential* $A(x)$ *be defined by formula* (5.6) *where* $A^{(\infty)}(x)$, $A^{(reg)}(x)$ *and* $U(x)$ *are functions* (5.2), (5.3) *and* (5.4), *respectively. Then* $A(x)$ *satisfies equation* (1.1) *and estimate* (1.7). *Moreover,* $A(x)$ *has compact support if* $B(x)$ *has compact support.*

In the case of magnetic fields $B(x)$ with compact supports our construction is close to that of [3]. By the proof of Proposition 5.1 we could have proceeded from the magnetic potential $A^{(c)}(x)$ satisfying the Coulomb gauge condition $\operatorname{div} A^{(c)}(x) = 0$. This is however less convenient.

Suppose that a magnetic field is supported by some ball $\mathbb{B}$, $\mathbb{B}'$ is a slightly larger ball and a direct interaction of quantum particles with this field is excluded by the Dirichlet boundary condition on $\partial\mathbb{B}'$. Proposition 5.1 shows that we can choose a magnetic potential supported by $\mathbb{B}'$ so that scattering in this case is trivial. On the other hand, if a magnetic field is supported by some torus $\mathbf{T}$ and the Dirichlet boundary condition is put on the boundary of a slightly larger torus $\mathbf{T}'$, then the Stokes theorem does not allow us to find a potential supported by $\mathbf{T}'$ (provided the magnetic flux through a section of $\mathbf{T}$ is not zero). Therefore scattering in this case is non-trivial although it is of short-range nature.

Let us denote by $\mathcal{A}(B)$ the class of magnetic potentials satisfying equation (1.1) and estimate (1.7) for some $\rho > 1$. This class is non-empty according to Proposition 5.1. If $A \in \mathcal{A}(B)$, then, for an arbitrary function $\phi(x)$ such that $\operatorname{grad}\phi(x) = O(|x|^{-\rho})$, potential (3.12) also belongs to this class. According to Proposition 3.7 the SM for the pair $H_0 = -\Delta$, $H = (i\nabla + A(x))^2$ does not depend on the choice of $A \in \mathcal{A}(B)$ and, thus, is determined by the magnetic field $B(x)$ only. We say that this SM $S(\lambda) = S(\lambda; B)$ is the SM for the field $B(x)$.

Comparing Propositions 3.1 and 5.1, we arrive at the following result.

Theorem 5.2. *Let a magnetic field* $B(x)$ *be such that* $\operatorname{div} B(x) = 0$ *and condition* (1.10) *holds, and let a magnetic potential* $A \in \mathcal{A}(B)$. *Then the wave operators for the pair* $H_0 = -\Delta$, $H = (i\nabla + A(x))^2$ *exist, are unitary and the SM* $S(\lambda)$ *for the magnetic field* $B(x)$ *is a unitary operator for all* $\lambda > 0$. *The operator* $T(\lambda) = S(\lambda) - I$ *is compact and it belongs to the trace class if* $r > 4$. *If* $r > 4 + n$, $n = 0, 1, 2, \ldots$, *then* $T(\lambda)$ *is integral operator with kernel from the class* $C^n(\mathbb{S}^2 \times \mathbb{S}^2)$. *If condition* (4.4) *holds for some* $r \in (2, 4)$ *and all multi-indices* α, *then the operator* $T(\lambda)$ *has integral kernel which is a* C^∞*-function away from the diagonal* $\omega = \omega'$ *and is bounded by* $C(\lambda)|\omega - \omega'|^{-4+r}$ *as* $\omega' \to \omega$.

Corollary 5.3. *If estimate* (1.10) *is satisfied for* $r > 4$, *then* $\Sigma_{diff}(\omega, \omega'; \lambda)$ *is a bounded function of* $\omega, \omega' \in \mathbb{S}^2$. *If condition* (4.4) *is satisfied for some* $r \in (2, 4)$

and all multi-indices α, *then*

$$\Sigma_{diff}(\omega,\omega';\lambda)=O(|\omega-\omega'|^{-8+2r})\quad\text{as}\quad \omega\to\omega'.$$

Using the first formula (5.6) and applying Proposition 3.5 to the function $\phi(x)=U(x)$, we can also describe the structure of the SM in the trasversal gauge.

Proposition 5.4. *Suppose that a magnetic field* $B(x)$ *satisfies the assumptions of Theorem* 5.2. *Let* $S^{(tr)}(\lambda)$ *be the SM for the pair* $H_0=-\Delta$, $H=(i\nabla+A^{(tr)}(x))^2$. *Set*

$$u(\omega)=U(\omega)-U(-\omega)$$

where the function $U(x)$ *is defined by formula* (5.4). *Denote by* S_0 *the operator of multiplication by the function* $\exp(iu(\omega))$. *Then all the results of Theorem* 5.2 *about the operator* $T(\lambda)$ *are true for the operator* $T^{(tr)}(\lambda)=S^{(tr)}(\lambda)-S_0$.

If a magnetic field $B(x)$ satisfies assumption (4.4), then, similarly to the two-dimensional case (see subsection 4.1), Proposition 5.4 can be deduced from Theorem 3.3 and Proposition 2.2. Such approach was used in [8]. Indeed, if $\operatorname{curl} A(x)=o(|x|^{-2})$ as $|x|\to\infty$, then necessarily condition (4.5) is satisfied and hence function (5.4) is correctly defined. In this case the function $I(x,\omega)$ does not depend on x and equals $u(\omega)$. Actually, it follows from (5.5) that

$$\begin{aligned}I(x,\omega)&=\int_{-\infty}^{\infty}\langle\operatorname{grad}U(x+t\omega),\omega\rangle dt=\lim_{T\to\infty}\int_{-T}^{T}\frac{d}{dt}U(x+t\omega)dt\\&=\lim_{T\to\infty}(U(x+T\omega)-U(x-T\omega))=U(\omega)-U(-\omega).\end{aligned}\tag{5.7}$$

Therefore function (4.19) equals zero so that the singular integral operator disappears in (4.20).

In the dimension 2 the construction above works if (and only if) the total magnetic flux Φ is zero. Indeed, in this case

$$\int_{|x|=R}\langle A^{(\infty)}(x),dx\rangle=0$$

for any $R>0$ so that function (5.4) is again correctly defined. Then Proposition 5.1 for potential (5.6) and Theorem 5.2 for the SM remain true. The only difference is that under assumption (4.4) the integral kernel of the operator $T(\lambda)$ is $O(|\omega-\omega'|^{-3+r})$ as $\omega-\omega'\to 0$.

5.2. As a concrete example, let us consider a toroidal solenoid $\mathbf{T}$ in the space $\mathbb{R}^3$ symmetric with respect to rotations around the x_3-axis (which does not intersect $\mathbf{T}$). Suppose (which looks quite realistic) that a magnetic field

$$B(x_1,x_2,x_3)=-\alpha(x_1^2+x_2^2)^{-1}(-x_2,x_1,0),\quad \alpha=\text{const},$$

inside of $\mathbf{T}$ and is zero outside. Then $\operatorname{div}B(x)=0$ and the current $\operatorname{curl}B(x)=0$ if $x\notin\partial\mathbf{T}$. Of course, Theorem 5.2 applies to this field and hence $S(\lambda)-I$ is integral operator with kernel from the class $C^\infty(\mathbb{S}^2\times\mathbb{S}^2)$.

Let us illustrate our construction on this example. First, we construct the potential $A^{(tr)}(x)$ by formula (5.1). We assume that the section $\mathbf{S}$ of $\mathbf{T}$, for example, by the half-plane $x_2=0$, $x_1\geq 0$ is strictly convex and has a smooth boundary $\partial\mathbf{S}$ but is not necessarily a disc. Let the half-line L_z, $z\in\mathbb{R}$, consist of points $s(1+z^2)^{-1/2}(1,0,z)$ for all $s\in\mathbb{R}_+$. Denote by z_1 and z_2 the values of z for which L_z is tangent to $\partial\mathbf{S}$ and, for $z\in[z_1,z_2]$, denote by $\varkappa_\pm(z)$, $\varkappa_+(z)\geq\varkappa_-(z)$, the values of s for which L_z intersects $\mathbf{S}$. For $x=(x_1,x_2,x_3)$, set $z=z(x)=x_3(x_1^2+x_2^2)^{-1/2}$. Taking into account the rotational symmetry, we see that a point $sx\in\mathbf{T}$ if and only if $s|x|(1+z^2)^{-1/2}(1,0,z)\in\mathbf{S}$ or $\varkappa_-(z)\leq s|x|\leq\varkappa_+(z)$. Thus, integral (5.1) equals zero (and hence $A^{(tr)}(x)=0$) if $z(x)\notin(z_1,z_2)$ and it is actually taken over the set $[0,1]\cap[\varkappa_-(z)|x|^{-1},\varkappa_+(z)|x|^{-1}]$ if $z(x)\in(z_1,z_2)$. Therefore $A^{(tr)}(x)=0$ if $|x|\leq\varkappa_-(z)$,

$$\begin{aligned} A_1^{(tr)}(x) &= -\alpha x_1x_3(x_1^2+x_2^2)^{-1}\int_{\varkappa_-(z)/|x|}^1 ds \\ &= -\alpha x_1x_3(x_1^2+x_2^2)^{-1}(1-\varkappa_-(z)/|x|) =: A_1^{(0)}(x) \end{aligned} \tag{5.8}$$

if $\varkappa_-(z)\leq|x|\leq\varkappa_+(z)$ and

$$\begin{aligned} A_1^{(tr)}(x) &= -\alpha x_1x_3(x_1^2+x_2^2)^{-1}\int_{\varkappa_-(z)/|x|}^{\varkappa_+(z)/|x|} ds \\ &= -\alpha x_1x_3(x_1^2+x_2^2)^{-1}|x|^{-1}(\varkappa_+(z)-\varkappa_-(z)) \end{aligned} \tag{5.9}$$

if $|x|\geq\varkappa_+(z)$. Components $A_2^{(tr)}(x)$ and $A_3^{(tr)}(x)$ can be found quite similarly. In particular,

$$\left.\begin{aligned} A_1^{(\infty)}(x) &= x_1x_3(x_1^2+x_2^2)^{-1}|x|^{-1}g(z), \\ A_2^{(\infty)}(x) &= x_2x_3(x_1^2+x_2^2)^{-1}|x|^{-1}g(z), \\ A_3^{(\infty)}(x) &= -|x|^{-1}g(z), \end{aligned}\right\} \tag{5.10}$$

where

$$g(z)=-\alpha(\varkappa_+(z)-\varkappa_-(z)).$$

Clearly, $g(z)$ is a continuous function, $\pm g(z)>0$ if $\mp\alpha>0$ for $z\in(z_1,z_2)$ and $g(z)=0$ for $z\notin(z_1,z_2)$.

Let $\mathbf{K}$ be the cone in $\mathbb{R}^3$ where $z(x)\in[z_1,z_2]$. Then $\mathbf{T}\subset\mathbf{K}$, and $\mathbf{T}$ and $\mathbf{K}$ are tangent to each other. The internal (external) part of $\mathbf{K}\setminus\mathbf{T}$ will be denoted $\mathbf{K}_{int}$ ($\mathbf{K}_{ext}$). Of course $A^{(tr)}(x)=0$ if $x\notin\mathbf{K}$. It follows from (5.8), (5.9) that

$$\left.\begin{aligned} A^{(tr)}(x) &= 0, & x&\in\mathbf{K}_{int}, \\ A^{(tr)}(x) &= A^{(0)}(x), & x&\in\mathbf{T}, \\ A^{(tr)}(x) &= A^{(\infty)}(x), & x&\in\mathbf{K}_{ext}. \end{aligned}\right\}$$

Now formula (3.3) for $A^{(tr)}(x)$ implies that

$$\left.\begin{aligned} A^{(reg)}(x) &= -A^{(\infty)}(x), & x&\in\mathbf{K}_{int}, \\ A^{(reg)}(x) &= A^{(0)}(x)-A^{(\infty)}(x), & x&\in\mathbf{T}, \\ A^{(reg)}(x) &= 0, & x&\in\mathbf{K}_{ext}. \end{aligned}\right\} \tag{5.11}$$

Taking into account (5.10), we see that a function $U(x)$ satisfying (5.5) can be constructed by the explicit formula

$$U(x)=G(x_3(x_1^2+x_2^2)^{-1/2}), \tag{5.12}$$

where

$$G'(z)=-g(z)(z^2+1)^{-1/2}. \tag{5.13}$$

In particular, we see that $U(x)$ is a constant for $z(x)\notin(z_1,z_2)$. Since $U(x)$ is defined up to a constant, we can set $U(x)=0$ for $z(x)\leq z_1$. Then

$$U(x)=U_0=-\int_{-\infty}^{\infty}g(t)(t^2+1)^{-1/2}dt$$

for $z(x) \geq z_2$. It is easy to check that $-U_0$ equals the magnetic flux Φ_s through the section $\mathbf{S}$ of the solenoid $\mathbf{T}$. Indeed, let $\omega_0 = (0,0,1)$, $\langle x_0, \omega_0 \rangle = 0$, and let $|x_0|$ and R be sufficiently large. By the Stokes theorem, Φ_s equals the circulation of the potential $A^{(tr)}(x)$ over the closed contour formed by the four intervals $(-R\omega_0, R\omega_0)$, $(R\omega_0, R\omega_0 + x_0)$, $(R\omega_0 + x_0, -R\omega_0 + x_0)$ and $(-R\omega_0 + x_0, -R\omega_0)$. Remark that $A^{(tr)}(x) \neq 0$ only on the interval $(R\omega_0 + x_0, -R\omega_0 + x_0)$ where $A^{(tr)}(x) = A^{(\infty)}(x)$, so that

$$\Phi_s = -\int_{-R}^{R} \langle A^{(\infty)}(x_0 + t\omega_0), \omega_0 \rangle dt.$$

Passing to the limit $R \to \infty$ and using (3.6), we see that $\Phi_s = -I(x_0, \omega_0)$. Hence equality (5.7) implies that

$$\Phi_s = -U(\omega_0) = -U_0. \tag{5.14}$$

Suppose now that the number R_2 in the definition of the cut-off function $\eta(x)$ is chosen in such a way that the ball $|x| \leq R_2$ does not intersect $\mathbf{T}$. Let first $x \notin \mathbf{K}$ so that $A^{(tr)}(x) = 0$. Then the first formula (5.6) shows that $A(x) = 0$ if $z(x) \leq z_1$, and

$$A(x) = -U_0 \operatorname{grad} \eta(x), \quad x \notin \mathbf{K}, \quad z(x) \geq z_2.$$

If $x \in \mathbf{K}$, then according to the second formula (5.6) and equalities (5.11)

$$\begin{aligned} A(x) &= 0, & |x| &\leq R_1, \\ A(x) &= -\operatorname{grad}(\eta(x)U(x)), & R_1 \leq |x| &\leq R_2, \\ A(x) &= A^{(reg)}(x), & |x| &\geq R_2. \end{aligned}$$

In particular, $A(x) = 0$ if $x \in \mathbf{T}_{ext}$.

The function $g(z)$ can be calculated explicitly if $\mathbf{S}$ is a disc. Suppose that this disc has radius r, its center belongs to the x_3-axis and the distance from the center to the x_3-axis is l, $l > r$. Then the equation of $\partial\mathbf{T}$ is

$$((x_1^2 + x_2^2)^{1/2} - l)^2 + x_3^2 = r^2. \tag{5.15}$$

Setting here $x_2 = 0$, $x_3 = zx_1$, we obtain an equation for $x_1 = x_1(z)$. The roots of this equation yield us the numbers $(1+z^2)^{-1/2}\varkappa_\pm(z)$. Thus,

$$\varkappa_\pm(z) = (1+z^2)^{-1/2}(l \pm (r^2 - (l^2 - r^2)z^2)^{1/2})$$

and hence

$$g(z) = -2\alpha(1+z^2)^{-1/2}(r^2 - (l^2 - r^2)z^2)^{1/2}.$$

In particular, $-z_1 = z_2 = r(l^2 - r^2)^{-1/2}$ for this function.

Returning to the general case, we emphasize that a potential $A(x)$ satisfying the conclusions of Proposition 5.1 is highly non-unique. Actually, the gradient of an arbitrary short-range function can be added to $A(x)$. For example, in the case (5.15) the magnetic potential completely different from the one constructed above can be found in the book [1].

Let us finally calculate the SM $S^{(tr)}(\lambda)$. According to Proposition 5.4, up to an integral operator with C^∞-kernel, the SM $S^{(tr)}(\lambda)$ is the operator S_0 of multiplication by the function $\exp(iu(\omega))$, where by virtue of (5.12), (5.13)

$$u(\omega) = q(\omega_3(1-\omega_3^2)^{-1/2}), \quad \omega = (\omega_1, \omega_2, \omega_3) \in \mathbb{S}^2,$$

and

$$q(z) = G(z) - G(-z) = -\int_{-z}^{z} g(t)(t^2+1)^{-1/2} dt.$$

Clearly, $q(-z) = -q(z)$, $q(z)$ is an increasing (decreasing) function if $\alpha > 0$ ($\alpha < 0$) and it is a constant, $q(z) = q_0$, if $z \geq \max\{|z_1|, |z_2|\}$. It follows from (5.14) that

$$q_0 = u(\omega_0) = U(\omega_0) - U(-\omega_0) = -\Phi_s.$$

Thus, $u(\omega)$ depends only on the coordinate ω_3 and takes all the values between $-|\Phi_s|$ and $|\Phi_s|$. Therefore $\sigma_{ess}(S^{(tr)}(\lambda))$ coincides with the arc $[e^{-i|\Phi_s|}, e^{i|\Phi_s|}]$ if $|\Phi_s| < \pi$, and it covers the whole unit circle if $|\Phi_s| \geq \pi$.

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