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On inverse scattering at a fixed energy for potentials with a regular behaviour at infinity*

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Abstract
We study the inverse scattering problem for electric potentials and magnetic fields in $\mathbb{R}^d$, $d \geqslant 3$, that are asymptotic sums of homogeneous terms at infinity. The main result is that all these terms can be uniquely reconstructed from the singularities in the forward direction of the scattering amplitude at some positive energy.

1. Introduction

1.1.

There are different formulations of the inverse scattering problem for the multidimensional Schrödinger operator. The most complete and difficult problem is to find a one-to-one correspondence between potentials on $\mathbb{R}^d$ and the scattering data. For the solution of this problem see the paper by Faddeev [10] and also [26, 33]. A review and further references can be found in [22, 25]. Here, the difficult part is to find necessary and sufficient conditions on the scattering amplitude $f(\nu, \omega; \lambda)$, $\nu, \omega \in \mathbb{S}^{d-1}$, $\lambda > 0$, such that the corresponding potential is local.

A simpler problem of uniqueness and reconstruction of the potential from the scattering data has different settings. One of them is uniqueness and reconstruction from the high-energy limit of the scattering amplitude which was considered in [3, 9] (see also the books [6, 22] and the papers [7, 2]). In [9], it is shown that the first Born approximation is essentially the Fourier transform of the electric potential. In the time-dependent method of [2, 7], the x-ray or radon transforms of the electric and magnetic potentials are reconstructed from the high-energy limit.

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A similar problem of uniqueness and reconstruction of the electric and magnetic potentials from the scattering amplitude at a fixed positive energy was considered in [21] for potentials of compact support and in [8, 14, 24, 32] for potentials decaying exponentially at infinity. In contrast, for general short-range potentials the scattering matrix at a fixed positive energy does not uniquely determine the potential. Indeed, in [6] examples—in three dimensions—are given of non-trivial radial oscillating potentials with decay as $|x|^{-3/2}$ at infinity such that the corresponding scattering amplitude is identically zero at some positive energy. Moreover, in dimension 2 there are examples [11] of potentials with a regular decay as $|x|^{-2}$ at infinity that have zero scattering amplitude at some positive energy. Nevertheless, if two potentials have the same scattering amplitude at some positive energy and they coincide outside some ball, they necessarily coincide everywhere [34].

Actually, the same problem appears in different settings. Thus, it is proven in [12, 15, 35] that the scattering matrix at a fixed positive energy uniquely determines an exponentially decreasing perturbation of stratified media. As another example, we mention that the scattering matrix at a fixed quasi-energy uniquely determines (see [36]) time-periodic potentials that decay exponentially at spatial infinity.

1.2.

We consider the related (at least in spirit) problem of the unique reconstruction of the asymptotics of the electric and magnetic potentials if only the singularity of the scattering amplitude $f(\nu, \omega; \lambda)$ on the diagonal $\nu = \omega$ is known at some fixed energy $\lambda > 0$. This problem was already studied in [16–18], where the methods of microlocal analysis were heavily used. We mention also a related paper [23] where using the method of [7] the asymptotics of the electric potential is reconstructed if the scattering amplitude is known on some (perhaps arbitrarily small) interval of positive energies.

We show that this problem admits a quite elementary solution. It consists of two steps and both of them are in some sense well known. Let us explain them for the reconstruction of the leading terms of the electric V and magnetic A potentials. The first is the small angle approximation to the scattering amplitude $f(\nu, \omega; \lambda)$ (see, e.g., [20], section 127) given by the formula

$$f(\nu, \omega; \lambda) \approx \mathrm{e}^{-\pi \mathrm{i}(d-1)/4} k^{(d-1)/2} (2\pi)^{-(d-1)/2} \int_{\Pi_\omega} \mathrm{e}^{\mathrm{i}k\langle y, \omega-\nu\rangle} R(y, \omega; \lambda; \mathbf{V})\, \mathrm{d}y, \qquad k = \lambda^{1/2}, \tag{1.1}$$

where $\mathbf{V} = (V, A)$, Π_ω is the hyperplane orthogonal to ω,

$$R(y, \omega; \lambda; \mathbf{V}) = (2\mathrm{i}k)^{-1} \int_{-\infty}^{\infty} (V(y + t\omega) - 2k\langle \omega, A(y + t\omega)\rangle)\, \mathrm{d}t \tag{1.2}$$

and $\nu \to \omega$. The second is the classical inversion formula for the radon transform. Actually, for any d we always use the two-dimensional radon transform.

1.3.

Let us recall now the definition of the scattering amplitude. If the electric, $V(x)$, and magnetic, $A(x)$, potentials are real-valued functions satisfying the assumption

$$|V(x)| + |A(x)| + |\mathrm{div}\, A(x)| \leqslant C(1 + |x|)^{-\rho}, \tag{1.3}$$

where $\rho > (d+1)/2$, then the corresponding Schrödinger equation

$$(\mathrm{i}\nabla + A(x))^2\psi + V(x)\psi = \lambda\psi, \qquad \lambda = k^2 > 0, \tag{1.4}$$

has a solution with the asymptotics

$$\psi(x, \omega, \lambda) = \exp(\mathrm{i}k\langle\omega, x\rangle) + f|x|^{-(d-1)/2}\exp(\mathrm{i}k|x|) + o(|x|^{-(d-1)/2})$$

as $|x| \to \infty$. Moreover, such a solution is unique. The coefficient $f(\nu, \omega; \lambda)$ depends on the incident direction ω of the incoming plane wave $\exp(\mathrm{i}k\langle\omega, x\rangle)$, its energy λ and the direction $\nu = x|x|^{-1}$ of observation of the outgoing spherical wave $|x|^{-(d-1)/2}\exp(\mathrm{i}k|x|)$. The function $f(\nu, \omega; \lambda)$ is known as the scattering amplitude. In terms of the scattering amplitude, the scattering matrix $S(\lambda)$ is defined by the formula

$$(S(\lambda)u)(\omega) = u(\omega) + \mathrm{i}e^{\mathrm{i}\pi(d-3)/4}\lambda^{(d-1)/4}(2\pi)^{-(d-1)/2}\int_{\mathbb{S}^{d-1}} f(\nu, \omega; \lambda)u(\omega)\,\mathrm{d}\omega. \tag{1.5}$$

A discussion of these results can be found, for example, in [37]. In contrast, the scattering amplitude can be formally defined by equation (1.5) via the scattering matrix (see subsection 2.1 for its time-dependent definition) for all short-range potentials when $\rho > 1$ in (1.3). Moreover, if the assumption

$$|\partial^\alpha V(x)| + |\partial^\alpha A(x)| \leqslant C_\alpha(1+|x|)^{-\rho-|\alpha|}, \qquad \rho > 1, \tag{1.6}$$

is valid for all α, then, as shown in [1], $f(\nu, \omega; \lambda)$ is a C^∞-function of $\nu, \omega \in \mathbb{S}^{d-1}$ away from the diagonal $\nu = \omega$.

1.4.

The diagonal singularity of the scattering amplitude contains all information about the behaviour of the potentials $V(x)$ and $A(x)$ as $|x| \to \infty$. This allows one to recover their asymptotics at infinity.

Indeed, let us consider first the leading terms of the asymptotics of $V(x)$ and $A(x)$ when formulae (1.1) and (1.2) can be used. Given the leading singularity of the scattering amplitude on the diagonal, we can recover from (1.1) the function $R(y, \omega; \lambda)$ as the inverse Fourier transform of this singularity in the hyperplane Π_ω. If $A = 0$, then, using (1.2), we can reconstruct the asymptotics of $V(x)$ by the inversion formula for the radon transform. In the general case, we take into account that the integrals

$$R_{\mathrm{e}}(y, \omega; V) = \int_{-\infty}^{\infty} V(y + t\omega)\,\mathrm{d}t, \qquad \omega \in \mathbb{S}^{d-1}, \quad y \in \Pi_\omega, \tag{1.7}$$

and

$$R_{\mathrm{m}}(y, \omega; A) = \int_{-\infty}^{\infty} \langle\omega, A(y + t\omega)\rangle\,\mathrm{d}t, \qquad \omega \in \mathbb{S}^{d-1}, \quad y \in \Pi_\omega, \tag{1.8}$$

are, respectively, even and odd functions of $\omega \in \mathbb{S}^{d-1}$, which determine both these integrals separately. This allows us to reconstruct, as before, the electric potential V, but it is of course impossible to find the magnetic potential A from the integral (1.8). Nevertheless, we show that this integral determines the magnetic field $F = \operatorname{curl} A$. One cannot of course do better because the scattering matrix, as well as the integral (1.8), is invariant with respect to gauge transformations (see subsection 2.2).

Assuming that the potentials are asymptotically homogeneous, we actually use, for all $d \geqslant 3$, the inversion of the radon transform in two-dimensional planes. This is important for the reconstruction of the magnetic field. If $d = 2$, then the integral (1.7) can be equal to zero for all $y \neq 0$ for a non-trivial homogeneous function V (see counter-example 4.4). Therefore, in dimension 2 one cannot hope to recover the asymptotics of a potential from the forward singularity of the scattering amplitude.

Our approach extends to the complete asympotic expansion. Assume that an electric potential

$$V(x) \simeq \sum_{j=1}^{\infty} V_j(x) \tag{1.9}$$

and a magnetic field

$$F(x) \simeq \sum_{j=1}^{\infty} F_j(x) \tag{1.10}$$

are asymptotic sums of homogeneous functions (see subsection 2.3 for precise definitions) of orders $-\rho_j^{(e)}$ and $-r_j^{(m)}$, respectively, where $1 < \rho_1^{(e)} < \rho_2^{(e)} < \cdots$ and $2 < r_1^{(m)} < r_2^{(m)} < \cdots$. Suppose that all singularities in the forward direction of the scattering amplitude at some positive energy are known. We recover all functions $V_j(x)$ and $F_j(x)$ in a recursive way using at every step the inversion of the two-dimensional radon transform. In particular, our results imply that all asymptotic coefficients $V_j(x)$ and $F_j(x)$ are uniquely determined by these scattering data.

Compared to papers [17, 18], we do not assume that the degrees of homogeneity $\rho_j^{(e)} = 1 + j, r_j^{(m)} = 2 + j$. In contrast, we recover these numbers from the singularity of the scattering amplitude. In particular, the assumption $\rho^{(e)} \geqslant 2, r^{(m)} \geqslant 3$ imposed in [17, 18] turns out to be unnecessary. Another difference of our work is that we show that both sets of functions V_j and F_j can be obtained from the singularities of the scattering amplitude at only one energy (not at two as required in [17, 18]).

We proceed from a complete description of the diagonal singularity of the scattering amplitude. Such explicit formulae were obtained in the papers [30] and [38, 39]. These formulae generalize (1.2) but formula (1.1) remains the same. This shows that actually it is natural to regard the scattering matrix as a pseudodifferential rather than an integral operator. This point of view was probably first explicitly adopted in [4] where the asymptotics of the eigenvalues of the scattering matrix was found.

Under some minor additional *a priori* assumptions, our results combined with those of [34] show that the electric potential (not only its asymptotic expansion at infinity) is uniquely determined by the scattering amplitude at some energy (see theorem 4.6). This result implies that the uniqueness theorem does not require the super-power decay of the potential at infinity. Of course, it does not contradict the examples of [6] because we exclude oscillations of the potential at infinity and those of [11] because we assume that $d > 2$.

The description of the diagonal singularities of the scattering amplitude is given in section 3. The (inverse) problem of the reconstruction of the asymptotic expansions of V and F from these singularities is considered in section 4. The main result is formulated as theorem 4.2.

Finally, we note that our approach extends naturally to long-range potentials.

The inverse scattering for the Schrödinger equation has many important applications in atomic, molecular and nuclear physics. Also, the inverse scattering problem at a fixed energy for the Schrödinger equation is actually the same as the inverse scattering problem at a fixed frequency for the wave equation with a variable speed. This problem has many important applications in applied science and in engineering. For the applications of inverse scattering for the Schrödinger equation and the wave equation see [27].

2. Basic notions

2.1.

Here we recall some basic definitions of scattering theory. We consider the Schrödinger operator

$$H = (\mathrm{i}\nabla + A(x))^2 + V(x), \tag{2.1}$$

with electric $V(x)$ and magnetic $A(x) = (A^{(1)}(x), \ldots, A^{(d)}(x))$ potentials in the space $L^2(\mathbb{R}^d)$ where $d \geqslant 2$. Under the condition that V, A as well as the divergence div A of A are real and bounded functions, the Hamiltonian H is well defined and self-adjoint on the Sobolev class $\mathsf{H}^2(\mathbb{R}^d)$. Let us denote by $H_0 = -\Delta$ the 'free' Hamiltonian corresponding to the case $V = 0$ and $A = 0$. Under assumption (1.3) where $\rho > 1$, the positive spectrum of the operator H is absolutely continuous (and its negative spectrum consists of eigenvalues) and the wave operators

$$W_\pm = \text{s-}\lim_{t\to\pm\infty} \mathrm{e}^{\mathrm{i}tH}\,\mathrm{e}^{-\mathrm{i}tH_0} \tag{2.2}$$

exist and enjoy the intertwining property $HW_\pm = W_\pm H_0$ (see, for example, [19] or [28]). It follows that the scattering operator

$$\mathbf{S} = W_+^* W_- \tag{2.3}$$

commutes with H_0 and is unitary.

We are now able to define the scattering matrix. Let $\mathbb{S}^{d-1}$ be the unit sphere in $\mathbb{R}^d$, $\mathbb{R}^+ = (0, \infty)$ and let $L^2(\mathbb{R}^+, L^2(\mathbb{S}^{d-1}))$ be the L^2-space of functions defined on $\mathbb{R}^+$ with values in $L^2(\mathbb{S}^{d-1})$. Define the unitary mapping

$$\mathcal{F} : L^2(\mathbb{R}^d) \to L^2(\mathbb{R}^+, L^2(\mathbb{S}^{d-1}))$$

by the equation

$$(\mathcal{F}u)(\omega; \lambda) = 2^{-1/2}\lambda^{(d-2)/4}\hat{u}(\lambda^{1/2}\omega), \qquad \text{where} \quad \hat{u}(\xi) = (2\pi)^{-d/2}\int_{\mathbb{R}^d} \mathrm{e}^{-\mathrm{i}\langle x,\xi\rangle}u(x)\,\mathrm{d}x$$

is the Fourier transform of u and the spectral parameter λ plays the role of the energy of a quantum particle. Then, $(\mathcal{F}H_0u)(\lambda) = \lambda(\mathcal{F}u)(\lambda)$ and

$$(\mathcal{F}\mathbf{S}u)(\lambda) = S(\lambda)(\mathcal{F}u)(\lambda).$$

The unitary operator $S(\lambda) : L^2(\mathbb{S}^{d-1}) \to L^2(\mathbb{S}^{d-1})$ is known as the scattering matrix at energy λ. If assumption (1.6) is valid for all α, then, as shown in [1], $S(\lambda)$ is an integral operator with C^∞-kernel $s(\nu, \omega; \lambda)$ away from the diagonal $\nu = \omega$. Moreover, if $\rho > (d+1)/2$ in (1.3), then $S(\lambda)$ is related to the scattering amplitude introduced in section 1 by formula (1.5). For these results see, for example, [37].

2.2.

It follows from (2.2) that the operators $\mathbf{S}$ and $S(\lambda)$ are invariant under short-range gauge transformations, $A \mapsto A - \nabla\phi$, where $\phi \in C^\infty(\mathbb{R}^d)$ and $\partial^\alpha\phi(x) = O(|x|^{-\varrho-|\alpha|})$ for all α and some $\varrho > 0$ as $|x| \to \infty$. Therefore, it is natural to associate the scattering matrix $S(\lambda)$ directly with the magnetic field $F(x)$. However, since the definition of $S(\lambda)$ was given in terms of the magnetic potential $A(x)$, we have to discuss here the relation between magnetic fields and potentials. Below, we use freely a three-dimensional notation for the curl and the divergence keeping in mind that in the general case A is a 1-form and F is a 2-form. By definition, $F(x) = \operatorname{curl} A(x)$, that is in terms of components

$$F^{(ij)}(x) = \partial_i A^{(j)}(x) - \partial_j A^{(i)}(x). \tag{2.4}$$

In contrast, the magnetic potential can be reconstructed from the magnetic field $F(x)$ such that $\operatorname{div} F(x) = 0$ only up to arbitrary gauge transformations. It is not convenient to work in the standard transversal gauge $\langle x, A_{\mathrm{tr}}(x)\rangle = 0$ since even for magnetic fields of compact support the potential $A_{\mathrm{tr}}(x)$ decays only as $|x|^{-1}$ at infinity.

Therefore, we use the procedure suggested in [40] of construction of the short-range magnetic potential for an arbitrary magnetic field satisfying the condition

$$|\partial^\alpha F(x)| \leqslant C_\alpha (1+|x|)^{-r-|\alpha|}, \qquad r > 2, \quad \forall \alpha. \tag{2.5}$$

Let us introduce the auxiliary potentials

$$A^{(i)}_{\mathrm{reg}}(x) = \int_1^\infty s \sum_{j=1}^d F^{(ij)}(sx)x_j \, \mathrm{d}s, \qquad A^{(i)}_\infty(x) = -\int_0^\infty s \sum_{j=1}^d F^{(ij)}(sx)x_j \, \mathrm{d}s. \tag{2.6}$$

Note that A_∞ is a homogeneous function of order -1, $\operatorname{curl} A_\infty(x) = 0$ for $x \neq 0$ and $A_{\mathrm{tr}} = A_{\mathrm{reg}} + A_\infty$. Next we define the function $U(x)$ for $x \neq 0$ as a curvilinear integral

$$U(x) = \int_{\Gamma_{x_0,x}} \langle A_\infty(y), \mathrm{d}y\rangle \tag{2.7}$$

taken between some fixed point $x_0 \neq 0$ and a variable point x. It is required that $0 \notin \Gamma_{x_0,x}$, so that, in view of the Stokes theorem, for $d \geqslant 3$ the function $U(x)$ does not depend on the choice of a contour $\Gamma_{x_0,x}$ and $\operatorname{grad} U(x) = A_\infty(x)$. Let us now choose an arbitrary function $\eta \in C^\infty(\mathbb{R}^d)$ such that $\eta(x) = 0$ in a neighbourhood of zero, $\eta(x) = 1$ for, say, $|x| \geqslant 1$ and set

$$A(x) = A_{\mathrm{reg}}(x) + (1-\eta(x))A_\infty(x) - U(x)\operatorname{grad}\eta(x). \tag{2.8}$$

Then, $\operatorname{curl} A(x) = F(x)$, $A \in C^\infty(\mathbb{R}^d)$ if $F \in C^\infty(\mathbb{R}^d)$ and $A(x) = A_{\mathrm{reg}}(x)$ for $|x| \geqslant 2$. Therefore, it follows from definition (2.6) that under assumption (2.5) $A(x)$ satisfies estimates (1.6) with $\rho = r - 1$.

Below, we always associate with a magnetic field F the magnetic potential A by formulae (2.6)–(2.8) and then construct the scattering matrix $S(\lambda)$ in terms of the Schrödinger operator (2.1). If another short-range potential $\tilde{A}$ satisfies the relation $\operatorname{curl} \tilde{A}(x) = F(x)$, then necessarily the scattering matrices corresponding to potentials A and $\tilde{A}$ coincide. This allows us to speak about the scattering matrix $S(\lambda)$ corresponding to the magnetic field F.

2.3.

Let us introduce some notation. We denote by $\dot{\mathcal{S}}^{-\rho} = \dot{\mathcal{S}}^{-\rho}(\mathbb{R}^d)$ the set of $C^\infty(\mathbb{R}^d\backslash\{0\})$-functions $f(x)$ such that $\partial^\alpha f(x) = O(|x|^{-\rho-|\alpha|})$ as $|x| \to \infty$ for all α. Then, the standard Hörmander class is defined as $\mathcal{S}^{-\rho}(\mathbb{R}^d) := C^\infty(\mathbb{R}^d) \cap \dot{\mathcal{S}}^{-\rho}(\mathbb{R}^d)$. An important example of functions from the class $\dot{\mathcal{S}}^{-\rho}$ are homogeneous functions $f \in C^\infty(\mathbb{R}^d\backslash\{0\})$ of order $-\rho$ such that $f(tx) = t^{-\rho} f(x)$ for all $x \in \mathbb{R}^d$, $x \neq 0$, and $t > 0$.

Let functions $f_j \in \dot{\mathcal{S}}^{-\rho_j}$ where $\rho_j \to \infty$ (but the condition $\rho_j < \rho_{j+1}$ is not required). The notation

$$f(x) \simeq \sum_{j=1}^\infty f_j(x) \tag{2.9}$$

means that, for any N, the remainder

$$f - \sum_{j=1}^N f_j \in \dot{\mathcal{S}}^{-\rho}, \qquad \text{where} \quad \rho = \min_{j \geqslant N+1} \rho_j. \tag{2.10}$$

In particular, if the sum (2.9) consists of a finite number N of terms, then the inclusion (2.10) should be satisfied for all ρ. A function $f \in C^\infty$ is determined by its asymptotic expansion (2.9) up to a term from the Schwarz class $\mathcal{S} = \mathcal{S}^{-\infty}$.

It follows from formulae (2.4) and (2.6)–(2.8) that the magnetic field $F(x)$ is homogeneous of order $-r^{(\mathrm{m})} < -2$ if and only if the magnetic potential $A(x)$ is homogeneous of order $-\rho^{(\mathrm{m})} = -r^{(\mathrm{m})} + 1 < -1$. Therefore, $A(x)$ is an asymptotic sum

$$A(x) \simeq \sum_{j=1}^{\infty} A_j(x) \tag{2.11}$$

of homogeneous functions $A_j(x)$ of orders $-\rho_j^{(\mathrm{m})}$ if and only if the representation (1.10) holds for $F(x)$ with homogeneous functions $F_j(x)$ of orders $-r_j^{(\mathrm{m})}$. Here, the coefficients $F_j(x)$ and $A_j(x)$ are related by formulae (2.4) and (2.6)–(2.8); in particular, $\rho_j^{(\mathrm{m})} = r_j^{(\mathrm{m})} - 1$.

Adding terms which are equal to zero, we can assume, without loss of generality, that the numbers $\rho_j^{(\mathrm{e})}$ and $r_j^{(\mathrm{m})}$ in (1.9) and (1.10) are related by the equality $\rho_j^{(\mathrm{e})} = r_j^{(\mathrm{m})} - 1$. Anyway, by the reconstruction procedure, equality of some terms to zero can never be excluded. Set $\mathbf{V}(x) = (V(x), A(x))$. Then, expansions (1.9) and (1.10) are equivalent to the expansion

$$\mathbf{V}(x) \simeq \sum_{j=1}^{\infty} \mathbf{V}_j(x) \tag{2.12}$$

in homogeneous functions $\mathbf{V}_j(x) = (V_j(x), A_j(x))$ of order $-\rho_j$, where $\rho_j = \rho_j^{(\mathrm{m})} = \rho_j^{(\mathrm{e})}$.

3. The direct problem

3.1.

Following [38, 39], we give in this section a complete description of the diagonal singularity of the integral kernel $s(\nu, \omega; \lambda)$ of the scattering matrix $S(\lambda)$ for a fixed energy $\lambda > 0$. The dimension $d \geqslant 2$ is arbitrary. Our goal is to construct, for all N, an explicit function $s^{(N)}$ such that the difference

$$s - s^{(N)} \in C^{p(N)}(L^2(\mathbb{S}^{d-1}) \times L^2(\mathbb{S}^{d-1})), \qquad \text{where} \quad \lim_{N\to\infty} p(N) = \infty. \tag{3.1}$$

Since s is a C^∞-function away from the diagonal $\nu = \omega$, it suffices to construct $s^{(N)}$ in a neighbourhood of the diagonal only. This construction is given in terms of special approximate solutions of the Schrödinger equation (1.4). Let us set

$$u_\pm^{(N)}(x, \xi) = \mathrm{e}^{\mathrm{i}\langle x, \xi\rangle + \mathrm{i}\Phi_\pm(x,\hat\xi)} \mathbf{b}_\pm^{(N)}(x, \xi), \qquad \xi = |\xi|\hat\xi \in \mathbb{R}^d, \tag{3.2}$$

where the phase

$$\Phi_\pm(x, \hat\xi) = \mp \int_0^\infty \langle \hat\xi, A(x \pm t\hat\xi)\rangle \, \mathrm{d}t \tag{3.3}$$

and

$$\mathbf{b}_\pm^{(N)}(x, \xi) = \sum_{n=0}^{N} (2\mathrm{i}|\xi|)^{-n} b_\pm^{(n)}(x, \hat\xi). \tag{3.4}$$

The coefficients $b_\pm^{(n)}$ are defined here by the recurrent formulae $b_\pm^{(0)}(x, \hat\xi) = 1$ and

$$b_\pm^{(n+1)}(x, \hat\xi) = \mp \int_0^\infty f_\pm^{(n)}(x \pm t\hat\xi, \hat\xi)\, \mathrm{d}t, \tag{3.5}$$

where

$$f_\pm^{(n)} = 2\mathrm{i}\big\langle A - \nabla\Phi_\pm, \nabla b_\pm^{(n)}\big\rangle - \Delta b_\pm^{(n)} + (|\nabla\Phi_\pm|^2 - 2\langle A, \nabla\Phi_\pm\rangle + V + |A|^2 + \mathrm{i}\,\mathrm{div}\, A - \mathrm{i}\Delta\Phi_\pm)b_\pm^{(n)}. \tag{3.6}$$

One can arrive to these expressions for the coefficients $b_\pm^{(n)}$ by plugging (3.2) and (3.3) into the Schrödinger equation (1.4) where $\lambda = |\xi|^2$. This yields the transport equation for the function $\mathbf{b}_\pm^{(N)}(x, \xi)$. Seeking it in the form (3.4) and equating coefficients at the same powers of $(2\mathrm{i}|\xi|)^{-n}$, we obtain recurrent equations for the functions $b_\pm^{(n)}$. They are solved by formulae (3.5) and (3.6). This is a standard construction of approximate solutions of the Schrödinger equation going back in the context of the present paper to [5]. It is important that the remainder arising when function (3.2) is plugged into the Schrödinger equation satisfies, for larger N, better and better estimates as $|x| \to \infty$ everywhere except the forward direction $\hat{x} = \hat{\xi}$ for the function $u_+^{(N)}(x, \xi)$ and except the back direction $\hat{x} = -\hat{\xi}$ for the function $u_-^{(N)}(x, \xi)$. At the same time, the estimates of the remainder improve as the energy $\lambda \to \infty$.

A complete description of the diagonal singularity of the function $s(\nu, \omega; \lambda)$ is given in the following theorem obtained in [38, 39].

Theorem 3.1. *Let estimates (1.6) hold for all α. Let $\omega_0 \in \mathbb{S}^{d-1}$ be an arbitrary point, Π_{ω_0} be the plane orthogonal to ω_0 and $\Omega = \Omega(\omega_0, \delta) \subset \mathbb{S}^{n-1}$ be determined by the condition $\langle\omega, \omega_0\rangle > \delta > 0$. Set $x = \omega_0 z + y$, $y \in \Pi_{\omega_0}$,*

$$h_\pm^{(N)}(x, \xi) = \mathrm{e}^{\mathrm{i}\Phi_\pm(x,\xi)}\mathbf{b}_\pm^{(N)}(x, \xi), \tag{3.7}$$

$$\begin{aligned}\mathbf{a}^{(N)}(y, \nu, \omega; \lambda) &= 2^{-1}\langle\nu + \omega, \omega_0\rangle\overline{h_+(y, k\nu)}h_-(y, k\omega) \\ &\quad + (2\mathrm{i}k)^{-1}[\overline{h_+(y, k\nu)}(\partial_z h_-)(y, k\omega) - \overline{(\partial_z h_+)(y, k\nu)}h_-(y, k\omega) \\ &\quad - 2\mathrm{i}\langle A(y), \omega_0\rangle\overline{h_+(y, k\nu)}h_-(y, k\omega)], \qquad h_\pm = h_\pm^{(N)}, \quad k = \lambda^{1/2},\end{aligned} \tag{3.8}$$

and define

$$s^{(N)}(\nu, \omega; \lambda) = (2\pi)^{-d+1}k^{d-1}\int_{\Pi_{\omega_0}} \mathrm{e}^{\mathrm{i}k\langle y,\omega-\nu\rangle}\mathbf{a}^{(N)}(y, \nu, \omega; \lambda)\,\mathrm{d}y. \tag{3.9}$$

Then, relation (3.1) holds.

It can be also shown that the $C^{p(N)}$-norm of the kernel (3.1) is $O(\lambda^{-q(N)})$ as $\lambda \to \infty$ where $q(N) \to \infty$ for $N \to \infty$. Thus, all singularities of $s(\nu, \omega; \lambda)$ both for high energies and in smoothness are described by the explicit formulae (3.8) and (3.9).

Note that the amplitude (3.8) is invariant under a gauge transformation $A \mapsto A - \nabla\phi$, $h_\pm \mapsto \mathrm{e}^{-\mathrm{i}\phi}h_\pm$, where $\phi \in C^\infty$ is an arbitrary function.

Formula (3.9) gives the singular part $s^{(N)}(\nu, \omega; \lambda)$ of the kernel $s(\nu, \omega; \lambda)$ for $\nu, \omega \in \Omega(\omega_0, \delta)$. Since $\omega_0 \in \mathbb{S}^{d-1}$ is arbitrary, this determines the function $s^{(N)}(\nu, \omega; \lambda)$ for all $\nu, \omega \in \mathbb{S}^{d-1}$.

3.2.

According to (3.9), it is natural to regard the scattering matrix $S(\lambda)$ as a pseudodifferential operator (on the unit sphere) determined by its amplitude. For our purposes, it is convenient to reformulate theorem 3.1 in terms of asymptotic series. First, replace $\mathrm{e}^{\mathrm{i}\Phi_\pm(x,\xi)}$ in (3.7) by its Taylor expansion. Then, rearrange the representation (3.8) collecting together terms of the same power with respect to $\mathbf{V} = (V, A)$. Thus, the amplitude $\mathbf{a}(y, \nu, \omega; \lambda)$ of the pseudodifferential operator $S(\lambda)$ admits the expansion into the asymptotic series

$$\mathbf{a}(y, \nu, \omega; \lambda) \simeq \sum_{n=0}^{\infty}\mathbf{a}_n(y, \nu, \omega; \lambda),$$

where $\mathbf{a}_0(\nu,\omega;\lambda) = 2^{-1}\langle \nu+\omega, \omega_0\rangle$, $\mathbf{a}_1$ depends linearly on $\mathbf{V}$, $\mathbf{a}_2$ depends quadratically on $\mathbf{V}$, etc. It follows from formulae (3.5) and (3.6) that $\mathbf{a}_n \in \mathcal{S}^{-(\rho-1)n}$ for all n and, moreover,

$$2\mathrm{i}k\mathbf{a}_1(y,\nu,\omega;\lambda) = 2^{-1}\langle \nu+\omega,\omega_0\rangle \int_0^\infty (V(y+t\nu)+V(y-t\omega) - 2k\langle \nu, A(y+t\nu)\rangle - 2k\langle \omega, A(y-t\omega)\rangle)\,\mathrm{d}t, \tag{3.10}$$

up to a term from the class $\mathcal{S}^{-\rho}$.

Applications to the inverse problem require that the pseudodifferential operator $S(\lambda)$ be determined by its symbol. Let us recall briefly the standard procedure (see, e.g., [29, 31]) of the passage in local coordinates from the amplitude to the corresponding (right) symbol. Let us consider the orthogonal projection p of Ω on the hyperplane Π_ω and set

$$a(y,\omega;\lambda) \simeq \sum_\alpha \alpha!^{-1}(\mathrm{i}k)^{-|\alpha|}\partial_y^\alpha\partial_\eta^\alpha \mathbf{a}(y,t(\eta),\omega;\lambda)|_{\eta=p(\omega)}, \qquad y\in\Pi_\omega, \tag{3.11}$$

where the inverse mapping $t = p^{-1} : p(\Omega) \to \Omega$ is given by the formula $t(\eta) = (\eta,(1-|\eta|^2)^{1/2})$. Note that here we have taken into account the factor $-k$ in the phase in (3.9) which differs our definition from the standard terminology. Then,

$$s(\nu,\omega;\lambda) = (2\pi)^{-d+1}k^{d-1}\int_{\Pi_\omega} \mathrm{e}^{-\mathrm{i}k\langle y,\nu\rangle} a(y,\omega;\lambda)\,\mathrm{d}y, \tag{3.12}$$

so that the scattering matrix $S(\lambda)$ can be regarded as a pseudodifferential operator on $\mathbb{S}^{d-1}$ with right symbol $a(y,\omega;\lambda)$. This leads us to the following conclusion.

Theorem 3.2. *Let estimates (1.6) hold for all α. Then, the scattering matrix $S(\lambda)$ is a pseudodifferential operator on the unit sphere. Its right symbol $a(y,\omega;\lambda)$ admits the expansion into the asymptotic sum*

$$a(y,\omega;\lambda) \simeq 1 + \sum_{n=1}^\infty a_n(y,\omega;\lambda), \qquad \omega\in\mathbb{S}^{d-1}, \quad y\in\Pi_\omega, \tag{3.13}$$

where a_1 depends linearly on $\mathbf{V}$, a_2 depends quadratically on $\mathbf{V}$, etc. Explicit expressions for all functions a_n are obtained by putting formulae (3.8) and (3.11) together. In particular, we have that

$$a_n \in \mathcal{S}^{-(\rho-1)n} \tag{3.14}$$

and, moreover,

$$\tilde{a}_1 := a_1 - R \in \mathcal{S}^{-\rho}, \tag{3.15}$$

where the function R is defined by formula (1.2). Thus,

$$a - 1 - R \in \mathcal{S}^{-\rho+1-\varepsilon}, \qquad \varepsilon = \min\{\rho-1, 1\}, \tag{3.16}$$

so that $R(y,\omega;\lambda)$ is the principal symbol of the pseudodifferential operator $S(\lambda) - I$.

It is important that, for all n, the values of $a_n(y,\omega;\lambda)$, $y\in\Pi_\omega$, $y\neq 0$, are determined only by the values of $\mathbf{V}(x)$ and its derivatives on the line $x = \omega + ty$, $t\in\mathbb{R}$, which does not pass through the origin.

Note that the symbol $a(y,\omega;\lambda)$ can be recovered from the kernel $s(\nu,\omega;\lambda)$ by the inversion of the Fourier transform (3.12):

$$a(y,\omega;\lambda) = \int_{\Pi_\omega} \mathrm{e}^{\mathrm{i}k\langle y,\eta\rangle} s(t(\eta),\omega;\lambda)\gamma(\eta)\,\mathrm{d}\eta, \qquad y\in\Pi_\omega, \tag{3.17}$$

where $\gamma \in C_0^\infty(\mathbb{R}^{d-1})$ is an arbitrary function such that $\gamma(\eta) = 1$ in some neighbourhood of zero and $\gamma(\eta) = 0$ for, say, $|\eta| \geqslant 1/2$.

Let a be the function constructed in theorem 3.2. Let us denote by T the mapping that sends $\mathbf{V}$ into the function $a - 1$. Of course, it is defined up to a symbol from the class $\mathcal{S}^{-\infty}$. Thus, we put

$$T(y, \omega; \lambda; \mathbf{V}) = a(y, \omega; \lambda) - 1. \tag{3.18}$$

Moreover, we distinguish the linear part R of T and set

$$Q(y, \omega; \lambda; \mathbf{V}) = T(y, \omega; \lambda; \mathbf{V}) - R(y, \omega; \lambda; \mathbf{V}), \tag{3.19}$$

so that

$$Q(y, \omega; \lambda; \mathbf{V}) = \tilde{a}_1(y, \omega; \lambda) + \sum_{n \geqslant 2} a_n(y, \omega; \lambda).$$

We need the following simple property of the mapping Q.

Proposition 3.3. *Suppose that $\mathbf{V}^{(j)} \in \mathcal{S}^{-\rho^{(j)}}$, $j = 1, 2$, where $\rho^{(2)} > \rho^{(1)} > 1$. Then,*

$$Q(\mathbf{V}^{(1)} + \mathbf{V}^{(2)}) - Q(\mathbf{V}^{(1)}) \in \mathcal{S}^{-\rho^{(2)}+1-\varepsilon}, \qquad \varepsilon = \min\{\rho^{(1)} - 1, 1\} > 0.$$

Indeed, it follows from (3.15) that

$$\tilde{a}_1(\mathbf{V}^{(1)} + \mathbf{V}^{(2)}) - \tilde{a}_1(\mathbf{V}^{(1)}) = \tilde{a}_1(\mathbf{V}^{(2)}) \in \mathcal{S}^{-\rho^{(2)}}.$$

To consider the functionals $a_n(\mathbf{V})$ for $n \geqslant 2$, let us temporarily write them as $a_n(\mathbf{V}) = \mathsf{a}_n(\mathbf{V}, \mathbf{V}, \ldots, \mathbf{V})$ where $\mathsf{a}_n(\mathbf{V}^{(1)}, \mathbf{V}^{(2)}, \ldots, \mathbf{V}^{(n)})$ is linear with respect to its arguments $\mathbf{V}^{(1)}, \mathbf{V}^{(2)}, \ldots, \mathbf{V}^{(n)}$. If $\mathbf{V}^{(j)} \in \mathcal{S}^{-\rho^{(j)}}$ for all $j = 1, \ldots, n$, then

$$\mathsf{a}_n(\mathbf{V}^{(1)}, \ldots, \mathbf{V}^{(n)}) \in \mathcal{S}^{-\rho^{(1)}-\cdots-\rho^{(n)}+n}.$$

This property generalizes (3.14). Therefore,

$$a_2(\mathbf{V}^{(1)} + \mathbf{V}^{(2)}) - a_2(\mathbf{V}^{(1)}) = \mathsf{a}_2(\mathbf{V}^{(1)}, \mathbf{V}^{(2)}) + \mathsf{a}_2(\mathbf{V}^{(2)}, \mathbf{V}^{(1)}) + a_2(\mathbf{V}^{(2)})$$

and the right-hand side belongs to the class $\mathcal{S}^{-\rho^{(1)}-\rho^{(2)}+2}$. The terms $a_n(\mathbf{V})$ for $n > 2$ can be considered quite similarly.

3.3.

Let us specify theorem 3.2 for the perturbations considered in this paper. The following result is obtained by plugging (1.9) and (2.11) into expressions (3.8) and (3.11) for the coefficients $a_n(y, \omega; \lambda)$.

Theorem 3.4. *Suppose that an electric potential $V(x)$ and a magnetic field $F(x)$ are C^∞-functions and that they admit the asymptotic expansions (1.9) and (1.10) where $V_j(x)$ and $F_j(x)$ are homogeneous functions of orders $-\rho_j$ and $-r_j = -\rho_j - 1$, respectively, where $1 < \rho_1 < \rho_2 < \cdots$. Let the magnetic potential $A(x)$ be defined by equalities (2.6)–(2.8). Then, the symbol $(y, \omega; \lambda)$ of the PDO $S(\lambda)$ admits the expansion into the asymptotic sum*

$$a(y, \omega; \lambda) \simeq 1 + \sum_{n=1}^{\infty} \sum_{m=0}^{\infty} \sum_{j_1, j_2, \ldots, j_n} a_{n,m;j_1,j_2,\ldots,j_n}(y, \omega; \lambda), \tag{3.20}$$

where $j_k = 1, 2, \ldots$ for all $k = 1, \ldots, n$, $m = 0, 1, \ldots$, the functions $a_{n,m;j_1,j_2,\ldots,j_n}(y, \omega; \lambda)$ only depend on $\mathbf{V}_{j_1}, \mathbf{V}_{j_2}, \ldots, \mathbf{V}_{j_n}$ and are homogeneous of order

$$n - m - \rho_{j_1} - \rho_{j_2} - \cdots - \rho_{j_n}$$

in the variable y. In particular,

$$a_{1,0;j}(y,\omega;\lambda) = R(y,\omega;\lambda;\mathbf{V}_j) \tag{3.21}$$

is a homogeneous function of order $-\rho_j + 1$.

Corollary 3.5. *If, up to functions from the Schwarz class,* $V(x)$ *and* $F(x)$ *are homogeneous (for large* $|x|$*) functions of orders* $-\rho$ *and* $-r = -\rho - 1$*, then*

$$a(y,\omega;\lambda) \simeq 1 + \sum_{n=1}^{\infty}\sum_{m=0}^{\infty} a_{n,m}(y,\omega;\lambda),$$

where $a_{n,m}$ *is of order n in* $\mathbf{V}$ *(linear, quadratic, etc) and is a homogeneous function of order* $n - m - n\rho$ *of the variable y.*

4. The inverse problem

4.1.

Our approach to the inverse problem relies on the two-dimensional radon transform (see, e.g., [13]). Let us recall it briefly here. For $v \in \mathcal{S}^{-\rho}(\mathbb{R}^2)$, $\rho > 1$, the radon (or x-ray, which is the same as in dimension 2) transform is defined by the formula

$$r(\omega, y; v) = \int_{-\infty}^{\infty} v(\omega t + y)\,\mathrm{d}t, \qquad \omega \in \mathbb{S}, \quad \langle \omega, y\rangle = 0. \tag{4.1}$$

Obviously, $r(\omega, y) = r(-\omega, y)$. The Fourier transform $\hat{v}$ of v and hence the function v itself can be recovered in the following way. Let ω_ξ be one of the two unit vectors such that $\langle \omega_\xi, \xi\rangle = 0$. Then,

$$\hat{v}(\xi) = (2\pi)^{-1}\int_{-\infty}^{\infty} \mathrm{e}^{-\mathrm{i}|\xi|s} r(\omega_\xi, s\hat{\xi}; v)\,\mathrm{d}s, \qquad \hat{\xi} = \xi|\xi|^{-1}. \tag{4.2}$$

We apply this method for the reconstruction of a homogeneous function $V \in C^\infty(\mathbb{R}^d\backslash\{0\})$ of order $-\rho < -1$ from its x-ray transform (1.7) known for all $\omega \in \mathbb{S}^{d-1}$ and $y \in \Pi_\omega$, $y \neq 0$. For an arbitrary $x \in \mathbb{R}^d\backslash\{0\}$, consider some two-dimensional plane Λ_x orthogonal to x and, for $y \in \Lambda_x$, set $v_x(y) = V(x + y)$. Then, for all $\omega \in \Lambda_x$, $|\omega| = 1$, and all $y \in \Lambda_x$, $\langle\omega, y\rangle = 0$,

$$r(\omega, y; v_x) = R_{\mathrm{e}}(\omega, x + y; V). \tag{4.3}$$

Since $x + y \neq 0$, the function $v_x \in \mathcal{S}^{-\rho}(\mathbb{R}^2)$ so that we can recover the function v_x and, in particular, $v_x(0) = V(x)$ by formula (4.2). This procedure is used for the reconstruction of the asymptotics of the electric potentials.

In the magnetic case we are given only the integral (1.8) for all $\omega \in \mathbb{S}^{d-1}$ and $y \in \Pi_\omega$, $y \neq 0$. Since this integral is zero if $A(x) = \operatorname{grad}\phi(x)$ and $\phi(x) \to 0$ as $|x| \to \infty$, we cannot hope to recover A from this equation. Nevertheless, the corresponding magnetic field $F(x) = \operatorname{curl} A(x)$ can be recovered. Assume again that $A \in C^\infty(\mathbb{R}^d\backslash\{0\})$ is a homogeneous vector-valued function of order $-\rho < -1$. Let us consider one of the components of $F(x)$, for example, $F^{(12)}(x)$. We will first show how $F^{(12)}(x)$ can be reconstructed everywhere except the plane L_{12} where $x_3 = \cdots = x_d = 0$. Let $\omega = (\omega_1, \omega_2, 0, \ldots, 0)$ be any unit vector in the plane L_{12}, let $\nu = (-\omega_2, \omega_1, 0, \ldots, 0)$ be the unit vector obtained by rotating ω in the plane L_{12} by the angle $\pi/2$ in the counter-clockwise sense and let $\tilde{x} = (0, 0, x_3, \ldots, x_n) \neq 0$ be an arbitrary vector that is orthogonal to L_{12}. Set $f_{\tilde{x}}^{(12)}(y) = F^{(12)}(\tilde{x} + y)$ for $y \in L_{12}$. It is easy to see that

$$r\big(\omega, y; f_{\tilde{x}}^{(12)}\big) = -\partial R_{\mathrm{m}}(\omega, s\nu + \tilde{x}; A)/\partial s, \qquad y = s\nu. \tag{4.4}$$

Indeed, since $F^{(12)}$ is invariant under rotations in the plane L_{12}, it suffices to check (4.4) for the case $\omega_1 = 1, \omega_2 = 0$ when (4.4) reads as

$$\int_{-\infty}^{\infty} F^{(12)}(t, s, x_3, \ldots, x_n)\, \mathrm{d}t = -\frac{\partial}{\partial s} \int_{-\infty}^{\infty} A^{(1)}(t, s, x_3, \ldots, x_n)\, \mathrm{d}t.$$

For the proof of this relation, we only have to use the definition $F^{(12)} = \partial A^{(2)}/\partial t - \partial A^{(1)}/\partial s$ and the fact that the integral of $\partial A^{(2)}/\partial t$ is zero. Finding expression (4.4), we can recover $F^{(12)}(y + \tilde{x})$ for all $y \in L_{12}$ by formula (4.2). By virtue of the homogeneity of the function $F^{(12)}$, this yields $F^{(12)}(x)$ everywhere except the plane L_{12}. Thus, the magnetic field $F(x)$ is reconstructed for all x such that all coordinates $x_1, x_2, \ldots, x_d$ are not equal to zero. This condition can be achieved for any arbitrary $x \neq 0$ by a suitable choice of the coordinate system.

Formula (4.4) can of course be rewritten in the invariant way. For example, in the case $d = 3$ it means that, for arbitrary $\omega, n \in \mathbb{S}^2$, $\langle \omega, n \rangle = 0$,

$$\int_{-\infty}^{\infty} \langle n, \operatorname{curl} A(t\omega + x) \rangle\, \mathrm{d}t = \left\langle \omega \wedge n, \nabla_x \int_{-\infty}^{\infty} \langle \omega, A(t\omega + x) \rangle\, \mathrm{d}t \right\rangle,$$

where the symbol '$\wedge$' means the vector product.

Finally, if only the combination (1.2) is known, then using that R_{e} and R_{m} are even and odd functions of $\omega \in \mathbb{S}^{d-1}$, respectively, we obtain that

$$R_{\mathrm{e}}(y, \omega; V) = \mathrm{i}k(R(y, \omega; \lambda; \mathbf{V}) + R(y, -\omega; \lambda; \mathbf{V})),$$
$$R_{\mathrm{m}}(y, \omega; A) = \mathrm{i}2^{-1}(R(y, -\omega; \lambda; \mathbf{V}) - R(y, \omega; \lambda; \mathbf{V})).$$

This allows us to reconstruct the functions V and F by the formulae given above.

In particular, we obtain

Proposition 4.1. *Let V and F be homogeneous functions on $\mathbb{R}^d$, $d \geqslant 3$, of orders $-\rho < -1$ and $-r = -\rho - 1$. If $R(y, \omega; \lambda; \mathbf{V}) = 0$ for all $\omega \in \mathbb{S}^{d-1}$ and $y \in \Pi_\omega$, $y \neq 0$, then $V(x) = F(x) = 0$.*

4.2.

Now we are in a position to reconstruct, under the assumptions of theorem 3.4, all asymptotic coefficients V_j and F_j in (1.9) and (1.10) from the kernel $s(\nu, \omega; \lambda)$ of the scattering matrix known, up to a $C^\infty(\mathbb{S}^{d-1} \times \mathbb{S}^{d-1})$-function, for some $\lambda > 0$ and all $\nu, \omega \in \mathbb{S}^{d-1}$. Thus, we assume that V and F admit the asymptotic expansions (1.9) and (1.10). Then, (2.12) holds but the functions $\mathbf{V}_j$, and, in particular, their orders of homogeneity $-\rho_j$, are not known. Let us define the function $a(y, \omega; \lambda)$ by formula (3.17), that is $a(y, \omega; \lambda)$ is the right symbol of the pseudodifferential operator $S(\lambda)$. This function admits an asymptotic expansion as in (3.13) where $a_0 = 1$ and a_n are homogeneous functions of order $-\mu_k$ and $0 < \mu_1 < \mu_2 < \cdots$. Our goal is to solve equation (3.18), where the function a is known with respect to $\mathbf{V}$. Of course, both sides of (3.18) are defined up to functions from the Schwarz class $\mathcal{S}$. Under the assumptions of theorem 3.4, the orders of homogeneity on the left- and right-hand sides of this equation, as well as the terms of the same order, should coincide.

Let us compare terms of the highest order in (3.18). Taking into account relation (3.16), we see that $\rho_1 = \mu_1 + 1$ and

$$R(y, \omega; \lambda; \mathbf{V}_1) = a_1(y, \omega; \lambda). \tag{4.5}$$

This allows us to reconstruct the coefficients V_1 of the electric potential and F_1 of the magnetic field by the formulae of the previous subsection. Then, we find A_1 by formulae (2.6)–(2.8).

Below, it is convenient to introduce the following notation. Suppose that some function f admits the expansion in the asymptotic series (2.9) of homogeneous functions. By taking sums of such functions, it cannot be excluded that some terms will be equal to zero. Therefore, we define $f^\sharp$ as the highest order homogeneous term f_k in (2.9) that is not identically zero. For example, $f^\sharp = 0$ if $f \in \mathcal{S}$.

Suppose now that we have found the coefficients $\mathbf{V}_k$ for all $k = 1, \dots, n-1$, $n \geqslant 2$. Let us reconstruct $\mathbf{V}_n$. We apply proposition 3.3 to the functions $\mathbf{V}^{(1)} = \sum_{j=1}^{n-1} \mathbf{V}_j$, $\mathbf{V}^{(2)} = \mathbf{V} - \mathbf{V}^{(1)}$ where $\rho^{(1)} = \rho_1$, $\rho^{(2)} = \rho_n$. This yields

$$Q(\mathbf{V}) - Q\left(\sum_{j=1}^{n-1} \mathbf{V}_j\right) \in \dot{\mathcal{S}}^{-\rho_n+1-\varepsilon}, \qquad \varepsilon > 0,$$

and thus, for an arbitrary ρ_n, this term can be neglected compared to $R(\mathbf{V}_n)$. All terms $R(\mathbf{V}_j)$, $j \geqslant n+1$, are also negligible compared to $R(\mathbf{V}_n)$. Therefore, rewriting equation (3.18) as

$$R(\mathbf{V}_n) + \sum_{j \geqslant n+1} R(\mathbf{V}_j) + \left(Q(\mathbf{V}) - Q\left(\sum_{j=1}^{n-1} \mathbf{V}_j\right)\right) = a - 1 - T\left(\sum_{j=1}^{n-1} \mathbf{V}_j\right)$$

and selecting terms of the highest order, we obtain that

$$R(\mathbf{V}_n) = \left(a - 1 - T\left(\sum_{j=1}^{n-1} \mathbf{V}_j\right)\right)^\sharp . \tag{4.6}$$

Having found $R(\mathbf{V}_n)$, we can, similarly to (4.5), recover the functions V_n, F_n and, consequently, using formulae (2.6)–(2.8) the corresponding A_n.

Thus, we have arrived at our main result.

Theorem 4.2. *Suppose that an electric potential $V(x)$ and a magnetic field $F(x)$ are C^∞-functions on $\mathbb{R}^d$, $d \geqslant 3$, and that they admit the asymptotic expansions (1.9) and (1.10) where $V_j(x)$ and $F_j(x)$ are homogeneous functions of orders $-\rho_j$ and $-r_j = -\rho_j - 1$, respectively, where $1 < \rho_1 < \rho_2 < \cdots$. Let the magnetic potential $A(x)$ be defined by equalities (2.6)–(2.8). Then, the scattering data consisting of the kernel $s(\omega, \omega'; \lambda)$ of the scattering matrix at a fixed positive energy in a neighbourhood of the diagonal $\nu = \omega$ uniquely determine each one of $V_j(x)$ and $F_j(x)$. Moreover, the functions $V_1(x)$ and $F_1(x)$ can be reconstructed from formula (4.5) and the functions $V_j(x)$ and $F_j(x)$ for $j \geqslant 2$ can be recursively reconstructed from formula (4.6).*

Corollary 4.3. *If kernel of the operator $S(\lambda) - I$ belongs, for some $\lambda > 0$, to the space $C^\infty(\mathbb{S}^{d-1} \times \mathbb{S}^{d-1})$, then $V \in \mathcal{S}(\mathbb{R}^d)$ and $A \in \mathcal{S}(\mathbb{R}^d)$. More generally, let $V^{(1)}$, $F^{(1)}$ and $V^{(2)}$, $F^{(2)}$ satisfy the assumptions of theorem 4.2. Suppose that $s^{(1)}(\lambda) - s^{(2)}(\lambda) \in C^\infty(\mathbb{S}^{d-1} \times \mathbb{S}^{d-1})$ for some $\lambda > 0$. Then, $V^{(1)} - V^{(2)}$ and $F^{(1)} - F^{(2)}$ belong to the Schwarz class $\mathcal{S}(\mathbb{R}^d)$.*

Suppose that, for some $n \geqslant 2$,

$$a - 1 - T\left(\sum_{j=1}^{n-1} \mathbf{V}_j\right) \in \mathcal{S}(\Pi_\omega)$$

for all $\omega \in \mathbb{S}^{d-1}$. Then, it follows from (4.6) that $V_j = F_j = 0$ for all $j \geqslant n$.

We emphasize that, unlike $\rho_1 = \mu_1 + 1$, other orders $-\rho_n$ are not determined by the sequence $\mu_1, \mu_2, \ldots$ only, in particular, because a cancellation of different terms on the right-hand side of (4.6) might occur. For example, in

$$R(\mathbf{V}_2) = (a - 1 - T(\mathbf{V}_1))^\sharp = (a - 1 - a_1 - Q(\mathbf{V}_1))^\sharp, \tag{4.7}$$

a cancellation of terms in $a-1-a_1$ and $Q(\mathbf{V}_1)$ cannot be excluded. Nevertheless, $-\rho_2$ should be smaller than the maximal order of these two series. The order of $a - 1 - a_1$ equals $-\mu_2$ and the order of $Q(\mathbf{V}_1)$ equals or is smaller than $-\min\{\rho_1, 2\rho_1 - 2\}$. This yields the bound

$$\rho_2 \geqslant \min\{\mu_2, \rho_1, 2\rho_1 - 2\} = \min\{\mu_2, \mu_1 + 1, 2\mu_1\}.$$

4.3.

We give now a counter-example that shows that in dimension 2 the argument above does not work. Let us first calculate the radon transform of the function $V(x) = v(\hat{x})|x|^{-2}$ where $\hat{x} = x|x|^{-1} \in \mathbb{S}$. Making the change of variables $t = |y|\tau$, $\tau = \tan s$, we see that

$$\int_{-\infty}^{\infty} V(y + t\omega)\,\mathrm{d}t = |y|^{-1} \int_{-\infty}^{\infty} v\left(\frac{\hat{y} + \tau\omega}{\sqrt{\tau^2 + 1}}\right) \frac{\mathrm{d}\tau}{\tau^2 + 1} = |y|^{-1} \int_{-\pi/2}^{\pi/2} v(\cos s\,\hat{y} + \sin s\,\omega)\,\mathrm{d}s$$
$$= |y|^{-1} \int_{\mathbb{S}^{\pm}(\omega)} v(\theta)\,\mathrm{d}\theta, \qquad \langle\omega, y\rangle = 0, \quad y \neq 0, \quad \text{if} \quad \hat{y} \in \mathbb{S}^{\pm}(\omega), \tag{4.8}$$

where $\mathbb{S}^+(\omega)$ ($\mathbb{S}^-(\omega)$) is the half-circle passed from the point $-\omega$ to ω in the counter-clockwise (clockwise) direction.

It follows that the radon transform of V is equal to zero for all $\omega \in \mathbb{S}$, $\langle\omega, y\rangle = 0$, $y \neq 0$, if and only if

$$\int_{\mathbb{S}^+(\omega)} v(\theta)\,\mathrm{d}\theta = 0$$

for all $\omega \in \mathbb{S}$. It is easy to see that this condition is equivalent to two conditions

$$v(\omega) = v(-\omega) \qquad \text{and} \qquad \int_{\mathbb{S}} v(\theta)\,\mathrm{d}\theta = 0. \tag{4.9}$$

For example, they are satisfied if

$$v(\omega) = 2\langle\omega, \omega_0\rangle^2 - 1,$$

where $\omega_0 \in \mathbb{S}$ is some fixed vector.

Counter-example 4.4. Let $d = 2$, let $A = 0$ and let the electric potential V be a C^∞-function such that $V(x) = v(\hat{x})|x|^{-2}$ for sufficiently large $|x|$. Assume that the function v satisfies conditions (4.9) but is not identically zero. Then, the radon transform (4.8) of $v(\hat{x})|x|^{-2}$ equals zero for all $\omega \in \mathbb{S}$, $\langle\omega, y\rangle = 0$, $y \neq 0$, although v does not. According to theorem 3.2, in this case $S(\lambda) - I$ is the PDO from the class $\mathcal{S}^{-2}$.

We emphasize that in this example the leading singularity of the scattering amplitude disappears for all energies $\lambda > 0$. The examples of [11] are much stronger in the sense that the scattering amplitude (not only its leading singularity) equals zero. On the other hand, this is true for one energy only. It is interesting that both the examples above and of [11] are two dimensional and the potentials decay at infinity as homogeneous functions of order -2.

4.4.

In this subsection we assume that the magnetic field is zero and that $d \geqslant 3$. According to corollary 4.3, the electric potential is determined by the scattering matrix $S(\lambda)$ for some $\lambda > 0$ up to a term from the Schwarz class. Let us show now that under stronger *a priori* assumptions this remainder can be removed. To that end, we need the following result which is a particular case of theorem 1 of [34].

Theorem 4.5. *Let the potentials $V^{(1)}$ and $V^{(2)}$ on $\mathbb{R}^d$, $d \geqslant 3$, satisfy estimates (1.3) with $\rho > d$ and let $V^{(1)}(x) = V^{(2)}(x)$ for sufficiently large $|x|$. Suppose that the corresponding scattering matrices $S^{(1)}(\lambda) = S^{(2)}(\lambda)$ for some $\lambda > 0$. Then, $V^{(1)}(x) = V^{(2)}(x)$ for all $x \in \mathbb{R}^d$.*

Combining theorems 4.2 and 4.5, we obtain a new uniqueness result.

Theorem 4.6. *Let C^∞-potentials $V^{(k)}$, on $\mathbb{R}^d$, $d \geqslant 3$, $k = 1, 2$, equal (finite) sums of homogeneous functions of order $-\rho_j^{(k)}$ for sufficiently large $|x|$. Let $\rho_j^{(k)} > d$ for $k = 1, 2$ and all j. Suppose that the corresponding scattering matrices $S^{(1)}(\lambda) = S^{(2)}(\lambda)$ for some $\lambda > 0$. Then, $V^{(1)}(x) = V^{(2)}(x)$ for all $x \in \mathbb{R}^d$.*

Indeed, it follows from corollary 4.3 that $V^{(1)}(x) = V^{(2)}(x)$ up to a function from the Schwarz class. By our *a priori* assumption, this function has compact support. Then, it follows from theorem 4.5 that, actually, this function is zero.

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Endnotes

(1) Author: Please provide subsection headings in this paper.