RANDOM WALKS IN RANDOM ENVIRONMENT ON TREES AND MULTIPLICATIVE CHAOS[1]

Mikhail MENSHIKOV[a] and Dimitri PETRITIS[b]

a. Department of Mathematical Sciences, University of Durham
South Road, Durham DH1 3LE, United Kingdom, Mikhail.Menshikov@durham.ac.uk

b. Institut de Recherche Mathématique, Université de Rennes I and CNRS UMR 6625
Campus de Beaulieu, 35042 Rennes Cedex, France, petritis@maths.univ-rennes1.fr

9 March 2001

Abstract: We study random walks in a random environment on a regular, rooted, coloured tree. The asymptotic behaviour of the walks is classified for ergodicity/recurrence/transience in terms of the geometric properties of the matrix describing the random environment. A related problem, with only one type of vertices and quite stringent conditions on the transition probabilities but on general trees has been considered previously in the literature [17]. In the presentation we give here, we restrict the study of the process on a regular graph instead of the irregular graph used in [17]. The close connection between various problems on random walks in random environment and the so called multiplicative chaos martingale is underlined by showing that the classification of the random walk problem can be drawn by the corresponding classification for the multiplicative chaos, at least for those situations where both problems have been solved by independent methods. The chaos counterpart of the problem we considered here has not yet been solved. The results we obtain for the random walk problem localise the position of the critical point. We hope that the additional conditions needed for the chaos problem to have non trivial solutions will be the same as the ones needed for the random walk to be null recurrent.

[1] *1991 Mathematics Subject Classification:* 60J10, 60K20
Key words and phrases: Markov chain, trees, random environment, recurrence criteria, multiplicative chaos.

1 Introduction

1.1 Notations

Let d be a fixed non-negative integer. We consider the rooted regular tree of order d, *i.e.* a connected graph without loops with a denumerable set of vertices $\mathbb{V}$ and a denumerable set of non oriented edges $\mathbb{A}(\mathbb{V})$. There is a distinguished vertex called the root that has degree d; all other vertices have degree $d+1$. Vertices are completely determined by giving their genealogical history from their common ancestor, the root; hence they are bijectively indexed by the set of sequences of arbitrary length over an alphabet of d letters. We use the same symbol for the indexing set so that $\mathbb{V} = \cup_{n=0}^{\infty}\mathbb{V}_n$ with $\mathbb{V}_0 = \{\emptyset\}$ and $\mathbb{V}_n = \{v = (v_1, \dots, v_n) : v_i \in \{1, \dots, d\}, i = 1 \dots n\}$ for $n \geq 1$. For every $v \in \mathbb{V}$, we denote $|v|$ the length of the path from v to the root *i.e.* the number of edges encountered. For $v \in \mathbb{V}$ and $k \leq |v|$ we denote by $v|_k$ the truncation of the sequence v to its k first elements, *i.e.* if $v = (v_1 \dots v_n) \in \mathbb{V}_n$ and $k \leq n$, then $v|_k = (v_1 \dots v_k) \in \mathbb{V}_k$; the symbol $v|_k$ must not be confused therefore with v_k, representing the letter appearing at the k-th position of the sequence. For $0 \leq k < \ell \leq |v|$ we denote $v|_\ell^k$ the subsequence of length $\ell - k$ defined by $v|_\ell^k = (v_{k+1}, \dots, v_\ell)$. If $u \in \mathbb{V}$, we write $u \leq v$ if $|u| \leq |v|$ and $v = (u_1, \dots u_{|u|}, v_{|u|+1}, \dots v_{|v|})$ *i.e.* if u is the initial sequence of v; we write $u < v$ when $u \leq v$ and $|u| < |v|$. Similarly for every sequence u and any letter $\ell \in \{1 \dots d\}$, the sequence $u\ell$ will have length $|u|+1$ and last letter ℓ.

Edges are unordered pairs $\langle u, v\rangle$ of adjacent vertices u and v. Now if u and v are the end vertices of an edge, either $u \leq v$ with $|u|+1 = |v|$ or $v \leq u$ with $|v|+1 = |u|$. In both cases, there is a vertex, uniquely defined, that is the most remote from the root among the two end vertices of the edge. Since every vertex has an unique ancestor, every edge is uniquely defined by its most remote vertex. Hence, every vertex $v \in \mathbb{V}\setminus\{\emptyset\}$ defines an edge $a(v) = \langle\, v|_{|v|-1}, v\,\rangle$. Edges are thus also indexed by the set $\mathbb{V}$, more precisely by $\mathring{\mathbb{V}} = \mathbb{V} \setminus \{\emptyset\}$ and we denote $a(v)$ the edge defined by v; therefore $\mathbb{A}(\mathbb{V}) \simeq \mathring{\mathbb{V}}$. Since the set of vertices $\mathbb{V}$ uniquely determines the set of edges, we use by abuse of notation the symbol $\mathbb{V}$ to denote also the tree.

If $u, v \in \mathbb{V}$ and $u < v$, we denote $[u; v]$ the (unique) path from u to v *i.e.* the collection of edges $(a_1, a_2, \dots)$ with $a_j \equiv a(v|_{|u|+j})$, for $j = 1, \dots, |v|$. For every $u \in \mathbb{V}$, the symbol $[u; u]$ denotes an empty set of edges. If u and v are not comparable vertices, *i.e.* neither $u \leq v$ nor $v \leq u$ holds, although there is a canonical way to define the path $[u; v]$, this definition is not necessary in the present paper and hence omitted. We write simply $[v]$ to denote the path joining the root to v, namely $[\emptyset; v]$.

At every edge a we assign a number $\xi_a \in [0, \infty[$ in some specific manner. This specification differs from model to model and since various models are considered here, we don't wish to be more explicit about these variables at the present level. Mind however that for most of the models to be considered, the numbers $(\xi_a)_{a\in\mathbb{A}(\mathbb{V})}$ are random variables neither necessarily independent nor necessarily equi-distributed. For the time being, we only assume that we dispose of a specific collection $(\xi_a)_{a\in\mathbb{A}}$, called the *edge-environment.*

1.2 Multiplicative chaos

Let $(\mathbb{V}, (\xi_a)_{a\in\mathbb{A}(\mathbb{V})})$ be a given tree and a given edge environment. For $u, v \in \mathbb{V}$, with $u < v$ we denote

$$\xi[u;v] = \prod_{a\in[u;v]} \xi_a$$

the product of environment values encountered on the path of edges from u to v; the symbol $\xi[v]$ is defined to mean $\xi[\emptyset; v]$ and $\xi[v; v]$ — as a product over an empty set — is consistently defined to be 1. It is not necessary for the purpose of the present article to define the value of $\xi[u; v]$ when u and v are not comparable.

For every $u \in \mathbb{V}$, we consider the process $Y_n(u)_{n\in\mathbb{N}}$ defined by $Y_0(u) = 1$ and

$$Y_n(u) = \sum_{v\in\mathbb{V}_{n+|u|}} \xi[u;v] = \sum_{v\in\mathbb{V}_{n+|u|}} \prod_{a\in[u;v]} \xi_a,$$

for $n \geq 1$. This process is known as the *multiplicative chaos process* (at least when $(\xi_a)_{a\in\mathbb{A}}$ are genuine random variables). Notice that even when $(\xi_a)_{a\in\mathbb{A}}$ is a family of independent random variables, the random variables $\xi[u; v]$ are not independent for v scanning the set $\mathbb{V}_{n+|u|}$. Hence the asymptotic behaviour of $Y_n(u)$ when $n \to \infty$ is far from trivial and it is studied for several particular cases of dependences of the family (ξ_a) in an extensive literature; see for instance [9, 5, 8, 2, 12, 13].

The study of the asymptotics of the process (Y_n) — in the case when it is a genuine random process, *i.e.* when the variables ξ_a, for $a \in \mathbb{A}$ are genuine random variables — is done by various techniques. One such technique is by remarking that for $n \geq 1$, the process can be written as

$$\begin{aligned} Y_n(u) &= \sum_{v\in\mathbb{V}_{n+|u|}} \xi[u;v] = \sum_{v\in\mathbb{V}_{n+|u|}} \xi[u; v|_{|u|+1}]\,\xi[v|_{|u|+1}; v] \\ &= \sum_{w\in\mathbb{V}_{|u|+1}} \xi[u;w] \sum_{v\in\mathbb{V}_{|u|+n}} \xi[w;v] \\ &= \sum_{w\in\mathbb{V}_{|u|+1}} \xi[u;w]\,Y_{n-1}(w). \end{aligned}$$

If the limit $\lim_{n\to\infty} Y_n(u) \stackrel{\mathrm{d}}{=} Y(u)$ exists in distribution for all $u \in \mathbb{V}$ then it must verify the functional equation

$$Y(u) \stackrel{\mathrm{d}}{=} \sum_{w\in\mathbb{V}_{|u|+1}} \xi[u;w]\,Y(w). \quad (1)$$

The process $(Y_n(u))_n$ and the corresponding functional equation (1) are thoroughly studied in the literature for some particular choices of dependencies of the family (ξ_a).

A second technique of study of the asymptotics is by martingale analysis. We have, as a matter of fact,

$$\begin{aligned} Y_n(u) &= \sum_{v\in\mathbb{V}_{n+|u|}} \xi[u;v] \\ &= \sum_{v\in\mathbb{V}_{n+|u|}} \prod_{a\in[u;v]} \xi_a \\ &= \sum_{v_1,\dots,v_n\in\{1,\dots,d\}} \xi_{a(uv_1)}\xi_{a(uv_1v_2)}\cdots\xi_{a(uv_1v_2\dots v_n)} \end{aligned}$$

Now, if for any fixed $u\in\mathbb{V}$, $(\mathcal{F}_n)$ denotes the natural filtration $\mathcal{F}_k=\sigma(\xi_{a(uv_1\dots v_k)}, v_i\in\{1,\dots,d\}, i=1,\dots,k\}$ for $k\in\mathbb{N}$, we have

$$\mathbb{E}(Y_n(u)|\mathcal{F}_{n-1}) = \sum_{v_1,\dots,v_n\in\{1,\dots,d\}} \xi_{a(uv_1)}\cdots\xi_{a(uv_1\dots v_{n-1})}\mathbb{E}(\xi_{a(uv_1v_2\dots v_n)})$$

and in the special case where the distribution of $\xi_{a(uv_1v_2\dots v_n)}$ depends solely on v_n, the previous formula simplifies into

$$\mathbb{E}(Y_n(u)|\mathcal{F}_{n-1}) = Y_{n-1}(u)\sum_{v_n=1}^{d}\mathbb{E}(\xi_{a(uv_1v_2\dots v_n)}).$$

In this special case and provided that $\sum_{v_n=1}^{d}\mathbb{E}(\xi_{a(uv_1v_2\dots v_n)})$ is less, equal, or more than 1, the process is a non-negative supermartingale, martingale or submartingale. It will be shown in section 5.1 that it is enough to consider the martingale case since by suitable renormalisation of the random variables (ξ_a) and (Y_n) we can always limit ourselves in the study of martingales.

Although the process (Y_n) is thoroughly studied, the closely related process

$$Z_n(u) = \sum_{k=0}^{n} Y_k(u) \quad \text{for} \quad n\geq 0$$

does not seem — to the best of our knowledge — to have attracted much attention. However, if we are interested in connections between multiplicative chaos and random walks in random environment on a tree, it is this latter process that naturally appears in both subjects.

The relevant question that can be addressed for the multiplicative chaos problem is whether the functional equation (1) has a non trivial solution. The precise statement of this question is model-dependent and some instances of it will be examined below, in subsection 1.4.

1.3 Nearest neighbours random walk on a tree in an inhomogeneous environment

To every vertex $u = (u_1, \ldots, u_{|u|}) \in \mathring{\mathbb{V}}$ are assigned $d+1$ numbers $(p_{u,0}, p_{u,1}, \ldots, p_{u,d})$ with $p_{u,0} > 0$, $p_{u,i} \geq 0$ $\forall i = 1, \ldots, d$ and $\sum_{i=0}^{d} p_{u,i} = 1$. To $u \in \mathbb{V}_0 = \{\emptyset\}$ are assigned only d numbers $(p_{\emptyset,1}, \ldots p_{\emptyset,d})$ with $p_{\emptyset,i} \geq 0$ $\forall i = 1, \ldots, d$ and $\sum_{i=1}^{d} p_{\emptyset,1} = 1$. Most often, these numbers will be random variables with some specific dependence properties that will be defined later. These numbers stand for transition probabilities of a reversible Markov chain $(X_n)_{n\in\mathbb{N}}$ on the tree verifying for $|u| \geq 1$

$$P_{u,v} = \mathbb{P}(X_{n+1} = v | X_n = u) = \begin{cases} p_{u,0} & \text{if } v = u|_{|u|-1} \\ p_{u,v_{|v|}} & \text{if } u = v|_{|v|-1} \\ 0 & \text{otherwise.} \end{cases}$$

For $u = (\emptyset)$ we have the slightly modified transition probabilities

$$P_{\emptyset,v} = \mathbb{P}(X_{n+1} = v | X_n = (\emptyset)) = \begin{cases} p_{\emptyset,v_1} & \text{if } v \in \mathbb{V}_1 \\ 0 & \text{othrewise.} \end{cases}$$

For $u \in \mathbb{V}$ with $|u| \geq 2$ we consider the edge $a(u) = \langle\, u|_{|u|-1}, u \,\rangle$ and attach to this edge the variable

$$\xi_{a(u)} = \frac{p_{u|_{|u|-1}, u_{|u|}}}{p_{u|_{|u|-1}, 0}} \in [0, \infty[.$$

For $u \in \mathbb{V}_1$ we attach $\xi_{a(u)} = p_{\emptyset,u_1}$. Notice that we have not required irreducibility for the Markov chain $(X_n)_{n\in\mathbb{N}}$. The condition $0 \leq \xi_{a(u)} < \infty$ only excludes vanishing of the probability of transition towards the root on every active branch of the tree. When the variables (ξ_a) are genuine random variables, we require also some mild integrability conditions on the random variables in the form $\mathbb{E}\xi_{a(v)} \log^+(\xi_{a(v)}) < \infty$, where for all $z > 0$, $\log^+(z) = \max(0, \log(z))$. On can easily check the validity of the following

Lemma 1.1 *For every $v \in \mathbb{V}$ define the variable*

$$\pi[v] = \begin{cases} \pi[\emptyset]\, \xi[v] \frac{1}{p_{v,0}} & \text{if } v \in \mathring{\mathbb{V}} \\ \pi[\emptyset] & \text{if } v = (\emptyset), \end{cases}$$

with $\pi[\emptyset]$ an arbitrary constant. Then $\pi[v]$ verifies the stationarity condition

$$\sum_{v\in\mathbb{V}} \pi[v] P_{v,v'} = \pi[v'], \quad \forall v' \in \mathbb{V}.$$

We assume that the variable $(p_{v,0})^{-1}$ is well behaved uniformly in v. To avoid technicalities, we assume that

$$\mathbb{E}((p_{v,0})^{-1}) < \infty. \tag{2}$$

Then, apart the irrelevant factor $\frac{1}{p_{v,0}}$ the expression for the invariant measure $\pi[v]$ involves the product $\xi[v]$ of variables along the edges of the path form $\emptyset$ to v as it was the case in the expression of multiplicative chaos. The form of the invariant measure established in lemma 1.1 has been already established in [17], where the problem of random walk in a random environment on an inhomogeneous tree was studied in a particular case of dependence of the random variables (ξ_a). However, the close analogy with multiplicative chaos, although reminiscent of considerations in [16], does not seem to have been exploited.

The relevant questions that can be addressed in the context of random walk in inhomogeneous environments concern ergodicity or non-ergodicity, recurrence or transience that arise for almost all realisations of the variables (ξ_a).

Remark: Notice that since our chains are periodic of period 2, we use the term *ergodic* to mean positive recurrent chains (or else admitting a stationary probability distribution π) and not in the sense that $\mathbb{P}(X_n = v | X_0 = u)$ converges towards $\pi(v)$; as a matter of fact this conditional probability does not admit a limit when $n \to \infty$, it is only the subsequence of even times that converges towards this limit, odd terms being always 0.

1.4 Models covered by the present formalism

We present below a unified treatment of both the multiplicative chaos process and the random walk problem stating in the same theorem the asymptotic behaviour of the limiting chaos process and of the random walk.

1.4.1 A toy model: the random walk on $\mathbb{N}$

The ordinary random walk on $\mathbb{N}$ with reflection at the origin is a Markov chain with transition matrix

$$\mathbb{P}(X_{n+1} = y | X_n = x) = \begin{cases} p & \text{if } \ y = x-1, x \in \mathbb{N}^* \\ q & \text{if } \ y = x+1, x \in \mathbb{N}^* \\ 1 & \text{if } \ y = 1, x = 0 \\ 0 & \text{otherwise.} \end{cases}$$

This problem can be rephrased in terms of the tree formalism by considering the process not on vertices but on generations. This implies that all vertices $v \in \mathbb{V}_n$ of the n^{th} generation, with $n \in \mathbb{N}^*$, are identified. Transition probability from any vertex $v \in \mathbb{V}_n$, with $n \in \mathbb{N}$, to the next generation reads $q = p_{v,1} + \ldots + p_{v,d}$ and to the previous generation (for $n \neq 0$) reads $p = p_{v,0}$. The edges $k \in \mathbb{N}^*$ of the lattice are thus assigned the variables $\frac{p_{k,1}+\cdots+p_{k,d}}{p_{k,0}} = \frac{q}{p} = \lambda$. Therefore the problem can be fitted in the previous formalism in the following way. Let $\boldsymbol{\eta} = (\eta_1, \ldots, \eta_d)$, with $\eta_i = p_{v,i}/p_{v,0}$, be a vector of non-negative constants with $\lambda = \eta_1 + \ldots + \eta_d$. Choose for every $v \in \overset{\circ}{\mathbb{V}}$, $\xi_v = \eta_{v_{|v|}}$. If we recall the definition of variables ξ in terms of transition probabilities, this model deals with a random walk on a regular tree and in a constant environment. The study of this model is straightforward since $Y_n(u)$ does not depend on $u \in \mathbb{V}$ and hence the functional equation (1) degenerates to the single recurrence

$$Y_n = \lambda Y_{n-1}$$

for $n \geq 1$, having a non trivial solution (*i.e.* different from 0 and ∞) if and only if $\lambda = 1$.

The asymptotic behaviour of the random walk $(X_n)_n$ can be established by studying the induced process on tree generations. Anyway we have the following straightforward results for this model.

Proposition 1.2 • *If* $\lambda < 1$ *then* $\lim_{n\to\infty} Y_n = 0$, $\lim_{n\to\infty} Z_n = (1-\lambda)^{-1} < \infty$ *and the random walk* $(X_n)_n$ *is almost surely ergodic.*

- *If* $\lambda = 1$ *then* $Y_n = 1$, *for all* $n \in \mathbb{N}$, $\lim_{n\to\infty} Z_n = \infty$ *and the random walk* $(X_n)_n$ *is almost surely null recurrent.*

- *If* $\lambda > 1$ *then* $\lim_{n\to\infty} Y_n = \infty$, $\lim_{n\to\infty} Z_n = \infty$ *and the random walk* $(X_n)_n$ *is almost surely transient.*

Notice that for the random walk case the significance of the parameter λ is the ratio of total outgoing (out of the root) probability on ingoing (towards the root) probability and since the induced process on the generations is an ordinary simple random walk on $\mathbb{N}$ having probability ratio λ the classification for this model is clear.

1.4.2 Random walk in random environment on $\mathbb{N}$

To define the random walk in random environment on $\mathbb{N}$, we first fix a sequence of independent and identically distributed random bi-dimensional vectors $(p_x, q_x)_{x\in\mathbb{N}^*}$, with $0 \leq p_x, q_x \leq 1$ and $p_x + q_x = 1$ for all $x \in \mathbb{N}^*$ and the special values $q_0 = 1 - p_0 = 1$ for $x = 0$. Notice that although the vectors $(p_x, q_x)_{x\in\mathbb{N}}$, for $x \in \mathbb{N}$, are independent, their components *cannot* be independent since they must verify the condition $p_x + q_x = 1$. The process is defined as a Markov chain having transition matrix

$$\mathbb{P}(X_{n+1} = y | X_n = x) = \begin{cases} p_x & \text{if } \ y = x-1, x \in \mathbb{N}^* \\ q_x & \text{if } \ y = x+1, x \in \mathbb{N} \\ 0 & \text{otherwise.} \end{cases}$$

Again this model can be viewed as a special case of the random walk in random environment on a regular tree by identifying the vertices of each generation $\mathbb{V}_n$ of the tree and setting $q_n = p_{n,1} + \ldots + p_{n,d}$ and $p_n = p_{n,0}$ for $n \in \mathbb{N}^*$. Assigning then to the edges of the lattice $\mathbb{N}$ the ratio $\frac{q_n}{p_n}$, for $n \in \mathbb{N}^*$, the problem can be rephrased as follows. Let $\boldsymbol{\eta} = (\eta_1, \ldots, \eta_d)$, with $\eta_i \overset{\mathrm{d}}{=} p_{n,i}/p_{n,0}$ for $n \in \mathbb{N}^*$, be a random vector of non-negative entries and of given law. Although we allow the support of $(\eta_i)_{i=1\dots d}$ distributions to extend up to 0, to avoid unnecessary complications we exclude the possibility of taking the value 0 with strictly positive probability. Consider a family $(\boldsymbol{\zeta}^{(n)})_{n\in\mathbb{N}}$ of independent copies of the vector $\boldsymbol{\eta}$ and for every $v \in \mathbb{V}_n$ such that $v_n = j \in \{1, \dots, d\}$ we assign the *same* variable $\zeta_j^{(n)}$ so that the random variable ξ_v attached to the edge $a(v)$ reads in the present case $\xi_v = \zeta_{v_{|v|}}^{(|v|)}$. Obviously ξ_v and $\xi_{v'}$ are independent if $|v| \neq |v'|$ but they are the same random variables if $|v| = |v'|$ and $v_{|v|} = v'_{|v'|}$. Now we can identify edges carrying the same random variable. This identification changes the vertex set of the graph into $\mathbb{N}$; the original edge set is changed into a multi-edge set (with multiplicity d) between subsequent vertices of $\mathbb{N}$. Recalling the significance of ξ as transition probabilities multiple edges can be merged; the d edges between vertices k and $k+1$ of $\mathbb{N}$ can be replaced by a single edge carrying the random variable $\theta_k = \sum_{j=1}^d \zeta_j^{(k)}$.

Another way to grasp the previous identifications is to consider the multiplicative chaos process attached to this model

$$\begin{aligned} Y_n &= \sum_{v\in\mathbb{V}_n} \xi[v] \\ &= \sum_{v_1,\dots,v_n\in\{1,\dots,d\}} \zeta_{v_1}^{(1)} \dots \zeta_{v_n}^{(n)} \\ &= \theta_1 Y'_{n-1} = \theta_1 \dots \theta_n, \end{aligned}$$

where $\theta_k = \sum_{j=1}^d \zeta_j^{(k)}$ for $k \in \mathbb{N}$ and $(\theta_k)_{k\in\mathbb{N}}$ is an independent and identically distributed sequence. The process $(Y'_n)_{n\in\mathbb{N}}$, in the last equation, is a process that can be chosen independent of $(Y_n)_{n\in\mathbb{N}}$ having the same distribution with it; this leads to the functional equation

$$Y_n \stackrel{\mathrm{d}}{=} \theta_1 \, Y'_{n-1}.$$

An obvious necessary (but not sufficient) condition for the equation satisfied by the limit Y_∞ — if the latter exists — to have a non-trivial solution is $\mathbb{E}\theta_1 = 1$. The precise reason for requiring integrable solutions appears in section 5.1. Under this condition, $(Y_n)_n$ becomes a non-negative martingale and this guarantees the almost sure existence of the limit Y_∞. However, for non constant independent and identically distributed variables $(\theta_k)_k$ the limit Y_∞ is almost surely 0 by Kakutani's theorem (see theorem 14.12 in [22] for instance). If $\mathbb{E}\theta_1 \neq 1$, then we can repeat the same arguments by considering not the process $(Y_n)_n$ but the process $(\widetilde{Y}_n)_n$ constructed with variables $(\widetilde{\theta}_k)_k = \theta_k/\mathbb{E}\theta_1$. Since $Y_n = (\mathbb{E}\theta_1)^n \widetilde{Y}_n$, the asymptotic behaviour of Y_n is recovered from that of $\widetilde{Y}_n$.

Hence the problem of existence of non-trivial solutions for multiplicative chaos process has a quite trivial answer for this model.

For establishing the asymptotic behaviour of the random walk in a random environment process it is necessary to study the process $(Z_n)_{n\in\mathbb{N}}$ with

$$Z_n = 1 + \theta_1 + \theta_1\theta_2 + \dots + \theta_1\theta_2 \dots \theta_n.$$

The importance of this process was already pointed out in [11] where asymptotic estimates of (Z_n) using probabilistic methods are obtained. Analytic methods using Mellin transform were used in [4] and martingale techniques based on Lyapunov's functions in [3]. The classification for the random walk in random environment problem is given in the following

Theorem 1.3 *[21] Suppose that* $\mathbb{E}|\log\theta_1| < \infty$ *and denote* $\lambda = \mathbb{E}\log\theta_1$.

1. *If* $\lambda > 0$ *then the random walk in random environment is almost surely transient.*
2. *If* $\lambda = 0$ *then the random walk in random environment is almost surely null recurrent.*
3. *If* $\lambda < 0$ *then the random walk in random environment is almost surely ergodic.*

Moreover, the multiplicative chaos equation has no non trivial solution.

1.4.3 Random walk in a random environment on a regular tree

At every vertex $v \in \mathring{\mathbb{V}}$ is assigned a $(d+1)$-dimensional random vector with positive components $(p_{v,0}, \ldots, p_{v,d})$ verifying $\sum_{j=0}^{d} p_{v,j} = 1$. For the vertex $v = \emptyset$, the corresponding random vector is d-dimensional and its components verify $\sum_{j=1}^{d} p_{\emptyset,j} = 1$. These random vectors are independent for different v's and, for $v \in \mathring{\mathbb{V}}$ they have the same distribution. The random walk in the random environment defined by these random vectors on the regular tree is the Markov chain with transition matrix

$$\mathbb{P}(X_{n+1} = v | X_n = u) = \begin{cases} p_{u,j} & \text{if } u \in \mathbb{V} \text{ and } v = uj, j = 1, \ldots, d \\ p_{u,0} & \text{if } u \in \mathring{\mathbb{V}} \text{ and } v = u|_{|u|-1} \\ 0 & \text{otherwise.} \end{cases}$$

Notice again that although the random vectors $(p_{v,0}, \ldots, p_{v,d})$ for $v \in \mathring{\mathbb{V}}$ are independent, their components *cannot* be independent since they satisfy $\sum_{j=0}^{d} p_{v,j} = 1$. Assigning to every edge the ratio of outwards over inwards probabilities, the problem can now be rephrased in a way that it fits the general formalism. Let $\boldsymbol{\eta} = (\eta_1, \ldots, \eta_d)$, be a vector of non-negative random variables η_i, $i = 1, \ldots, d$, having the same distribution with $p_{v,i}/p_{v,0}$, for $v \in \mathring{\mathbb{V}}$, with not necessarily independent nor identically distributed components. We assume the law of the random vector is explicitly known with $\mathbb{E}\eta_i < \infty$ and $\mathbb{E}\eta_i \log^+ \eta_i < \infty$, $\forall i = 1, \ldots, d$. Moreover, to avoid technicalities we assume that although the support of the random variables η_i extends up to 0, their law has no atom at 0.

To the edge $a(v)$, having most remote vertex $v \in \mathring{\mathbb{V}}$, we assign the random variable $\xi_{a(v)}$ having the same distribution as $\eta_{v_{|v|}}$; the variables $\xi_{a(v)}$ and $\xi_{a(v')}$ are independent if

$v|_{|v|-1} \neq v'|_{|v'|-1}$. Notice that if the components of the random vector η are not independent, the variables $\xi_{a(v)}$ and $\xi_{a(v')}$ with $|v| = |v'|$ and $v|_{|v|-1} = v'|_{|v'|-1}$ are not independent either.

When $\boldsymbol{\eta}$ has independent and identically distributed components, the process (X_n) has been studied in the context of multiplicative chaos and the existence of non trivial solutions of the functional equation (1) is established in [9, 8, 5]. In a context of statistical mechanics this functional equation is also studied in [2, 19] and in a more general branching process in [7, 12]. For $\boldsymbol{\eta}$ with components having a general joint distribution, the process (Y_n) has been studied in [5] and when the dimension d of the vector is also a random variable having a general joint distribution with the components of η has been studied in [13] (see [14] for a recent survey of the state of the art on the subject).

The results are expressed in terms of the functions

$$f(x) = \mathbb{E}\,(\sum_{i=1}^{d} \eta_i^x), x \in \mathbb{R}^+ \quad \text{and} \quad g(x) = \log f(x),$$

and of the parameter $\lambda = \inf_{x\in[0,1]} f(x)$.

The equation (1) reduces asymptotically into the form

$$Y[\emptyset] \stackrel{\mathrm{d}}{=} \sum_{v\in\mathbb{V}_1} \eta_{v_1} Y'[v] \tag{3}$$

where $(Y'[v])_{v\in\mathbb{V}_1}$ are mutually independent variables and independent of (η_i) each having the same distribution as $Y[\emptyset] = \lim_{n\to\infty} Y_n[\emptyset]$. Since only equality in distribution is required in equation (3), the variables $Y[\emptyset]$ and $Y'[v]$ can be chosen independent.

The random walk problem in a random environment on a tree has been first considered in several papers by Lyons, Pemantle, and Peres (see [16, 17, 18] for instance.) In particular in [17] Lyons and Pemantle studied the case of more general trees than the ones considered here, namely trees whose degree is not constant but have merely a constant asymptotic branching ratio. On the other hand, these authors make more stringent assumptions on the distribution of the random variables. We provide here proofs based on results on multiplicative chaos, totally independent from the proofs given by these authors. We state here our result for the case of a regular tree (constant degree) but with *arbitrary distribution* for the components of the random vector η_i *i.e.* with neither independent nor identically distributed components. It is worth noticing that although we limit ourselves in the case of regular trees, this limitation is not so crucial. As a matter of fact, the recent results on multiplicative chaos allow the treatment of random branching numbers and not to excessively burden the present paper, this will be considered in a subsequent article.

Theorem 1.4 *Let* $\lambda = \inf_{x\in[0,1]} f(x)$ *and* $x_0 \in [0,1]$ *be such that* $f(x_0) = \lambda$. *Then*

1. *If $\lambda < 1$, then almost surely the random walk is ergodic and $Z_\infty < \infty$.*

2. *If $\lambda > 1$, then almost surely the random walk is transient, $Y_\infty = \infty$, and $Z_\infty = \infty$.*

3. *If $\lambda = 1$ and moreover $f'(1) < 0$, then almost surely $0 < Y_\infty < \infty$, $Z_\infty = \infty$, and the random walk is null-recurrent.*

Assertions 1. and 2. in this theorem can be viewed as special cases of the corresponding assertions of theorem 1.6. However, we shall prove this theorem independently, using a new method, because we wish to demonstrate how techniques developed in the context of multiplicative chaos can be used in random walk problems. Moreover, this result is interesting in its own because it is a generalisation of the result in [17], valid for random variables that need not be either independent or identically distributed. Although this result is presented as a single theorem, it follows from several partial theorems on multiplicative chaos stated and/or proved in section 5.1.

1.4.4 Random strings in a random environment

The problem in its generality is considered in [3], where non reversible Markov chains on the tree $\mathbb{V}$ are considered and general conditions for transience/null recurrence/ergodicity are given. Here we limit ourselves to a particular case of string that can be fitted in the present formalism. To describe the problem of random strings in a random environment, the underlying tree must be enlarged to distinguish the d children of every vertex; this can be done by adding a colour index, chosen without replacement from the set $\{1, \ldots, d\}$, to each child. For instance, a natural way to do so is by assigning the colour $i \in \{1, \ldots, d\}$ to the vertex $v \in \mathring{\mathbb{V}}$ if $v_{|v|} = i$. The root is assigned an arbitrary colour $\alpha \in \{1, \ldots, d\}$. Consequently, every edge $a(v)$ with $v \in \mathring{\mathbb{V}}$ is assigned the bicolour $(ij) \in \{1, \ldots, d\}^2$, where $i = v_{|v|-1}$ and $j = v_{|v|}$. Introduce a sequence of independent and identically distributed random $d(d+1)$-dimensional vectors with positive components $(p_{k,i,j}, i = 1, \ldots, d; j = 0, \ldots, d)_{k \in \mathbb{N}^*}$ such that $\sum_{j=0}^{d} p_{k,i,j} = 1, \forall i \in \{1, \ldots, d\}, \forall k \in \mathbb{N}^*$. For $k = 0$ we have the special assignement $(p_{0,\alpha,j}, j = 1, \ldots, d)$ verifying $\sum_{j=1}^{d} p_{0,\alpha,j} = 1$, where α is the colour arbitrarily assigned to the root. Notice again that although the vectors are independent for different k's, their components *cannot* be independent. The random string in the random environment defined by these random vectors is the Markov chain with transition matrix

$$\mathbb{P}(X_{n+1} = v | X_n = u) = \begin{cases} p_{k,i,j} & \text{if } |u| = k \in \mathbb{N}, u_{|u|} = i, v = uj, j \in \{1, \ldots, d\} \\ p_{k,i,0} & \text{if } |u| = k \in \mathbb{N}^*, u_{|u|} = i, v = u|_{|u|-1} \\ 0 & \text{otherwise.} \end{cases}$$

Passing to the ratio of outwards over inwards probabilities, the model is rephrased as follows. Let $\boldsymbol{\eta} = \begin{pmatrix} \eta_{11} & \cdots & \eta_{1d} \\ \vdots & & \vdots \\ \eta_{d1} & \cdots & \eta_{dd} \end{pmatrix}$ be a random matrix of given law. More specifically,

$\eta_{ij}\stackrel{\mathrm{d}}{=}p_{k,i,j}/p_{k,i,0}$ for $k \in \mathbb{N}^*$. We do not require that the entries of the matrix $\boldsymbol{\eta}$ are independent or distributed according to the same law. We only require that $\eta_{ij} \geq 0$ and $\mathbb{E}\eta_{ij} < \infty \ \ \forall ij$.

To every edge of the $\mathbb{V}$ tree (except the one attached to the root) defined by its most remote from the root vertex $v \in \mathbb{V}_n$ for some $n \geq 1$, we assign, the random variable ξ_v that is the *same* independent copy of η_{ij} for all $v \in \mathbb{V}_n$ with $v_n = j$ and $v_{n-1} = i$. The random variables ξ_v and $\xi_{v'}$ are independent if $|v| \neq |v'|$. On the contrary the tree $\mathbb{V}$ is considered as coloured, each one of the d children of a given vertex is assigned a number in $\{1, \ldots, d\}$ and all the random variables of the n^{th} generation among vertices of types i and j, with $i,\ j \in \{1, \ldots, d\}$ are all the *same* independent copy of η_{ij}. (This case corresponds to the particular choice $q_{ij}^{(n)} = 0$ in the notation of [3].)

Denote by $\boldsymbol{e}_\alpha = \begin{pmatrix} 0 \\ \vdots \\ 1 \\ \vdots \\ 0 \end{pmatrix}$ the unit vector of $\mathbb{R}^d$ on the α direction, by $\boldsymbol{e}_\alpha^T$ its transposed, and by $\boldsymbol{e} = \sum_{\alpha=1}^d \boldsymbol{e}_\alpha$ the vector whose all components are equal to 1. Let $(\mathbf{A}_n)_{n\in\mathbb{N}}$ be a sequence of independent identically distributed $d \times d$ matrices; every $\mathbf{A}_n$ being an independent copy of $\boldsymbol{\eta}$. Suppose moreover that we arbitrarily assign colour $\alpha \in \{1 \ldots d\}$ to the root. Then it is easily shown that

$$Y_n^{(\alpha)} = \boldsymbol{e}_\alpha^T \mathbf{A}_1 \ldots \mathbf{A}_n \boldsymbol{e}.$$

We recall here the classification criteria for the random walk in a random environment as obtained in [3].

Theorem 1.5 *Assume that $\mathbf{A}_1$ is almost surely invertible, $\mathbb{E}\log^+\|\mathbf{A}\| < \infty$, and $\mathbb{E}\log^+\|\mathbf{A}^{-1}\| < \infty$. Let λ be the largest Lyapunov exponent of the sequence $(\mathbf{A}_n)$.*

1. *If $\lambda > 0$ the random walk in random environment is almost surely transient,*
2. *if $\lambda < 0$ the random walk in random environment is almost surely ergodic,*
3. *if $\lambda = 0$ and no finite union of proper subspaces of $\mathbb{R}^d$ is almost surely stable by $\mathbf{A}_1$ then the random walk in random environment is almost surely recurrent.*

These results allow to study the asymptotic behaviour of the corresponding chaos processes Y_n and Z_n. The classification obtained above permit the classification of the chaos processes as well.

1.4.5 Random walk in a random environment on a coloured tree

Here again, the tree is coloured to distinguish the various children of every vertex and the root is assigned an arbitrary colour $\alpha \in \{1, \dots, d\}$ as it was the case in the previous subsection. The main difference is that here various families of random $(d+1)$-dimensional vectors are assigned at the vertices of a given generation, these families having not necessarily the same distribution. To be more precise, for every $v \in \overset{\circ}{\mathbb{V}}$, assign the $(d+1)$-dimensional vector with positive components $(p_{v,v_{|v|},j}, j \in \{0, \dots, d\})$, verifying $\sum_{j=0}^{d} p_{v,v_{|v|},j} = 1$ for every $v \in \overset{\circ}{\mathbb{V}}$. These vectors are independent for different $v \in \overset{\circ}{\mathbb{V}}$ but not equidistributed, their distribution depending on the colour $v_{|v|} \in \{1, \dots, d\}$. For the root $v = \emptyset$, we have the special assignment $(p_{\emptyset,\alpha,j}, j \in \{1, \dots, d\})$ with $\sum_{j=1}^{d} p_{\emptyset,\alpha,j} = 1$. The random walk in this random environment on the regurar tree is the Markov chain with transition matrix

$$\mathbb{P}(X_{n+1} = v | X_n = u) = \begin{cases} p_{u,u_{|u|},j} & \text{if } u \in \mathbb{V}, v = uj, j \in \{1, \dots, d\} \\ p_{u,u_{|u|},0} & \text{if } u \in \overset{\circ}{\mathbb{V}}, v = u|_{|u|-1} \\ 0 & \text{otherwise.} \end{cases}$$

Passing to the edge-indexed ratio of outwards over inwards probabilities, the model can be rephrased to fit the present formalism. Let

$$\boldsymbol{\eta} = \begin{pmatrix} \eta_{11} & \cdots & \eta_{1d} \\ \vdots & & \\ \eta_{d1} & \dots & \eta_{dd} \end{pmatrix}$$

be a matrix of non-negative random elements of known joint distribution. The matrix elements are not necessarily independent. As it was the case in the previous subsection, each one of the d children of a vertex of the tree carries a specific colour in $\{1, \dots, d\}$, distinct from the colour of all its siblings. More specifically, $\eta_{ij} \overset{\mathrm{d}}{=} p_{u,u_{|u|},j} / p_{u,u_{|u|},0}$ for $u \in \overset{\circ}{\mathbb{V}}$.

The root is arbitrarily assigned a colour $\alpha \in \{1, \dots, d\}$. Every edge of the tree, indexed by its most remote vertex $v \in \overset{\circ}{\mathbb{V}}$, carries thus a bicolour-type (i, j) with $v_{|v|-1} = i$ and $v_{|v|} = j$; it is assigned a random variable ξ_v that is a copy of the random variable η_{ij}. The random variables ξ_v, $\xi_{v'}$ are independent if $v|_{|v|-1} \neq v'|_{|v'|-1}$.

Thus, contrary to the previous model where all random variables attached to a given generation of the tree are the same, here the random variables attached to children of different parents (*i.e.* non siblings) are independent.

It is evident that the process Y_n will depend on the colour α attached to the root. To recall this, we denote $Y_n^{(\alpha)}$ for the process starting from an α-coloured root.

Then

$$Y_n^{(\alpha)}[\emptyset] = \sum_{v \in \mathbb{V}_n} \xi_{v|_0 v|_1} \dots \xi_{v|_{n-1} v} \quad (4)$$

$$= \sum_{v \in \mathbb{V}_1} \xi_{\alpha v_1} Y_{n-1}^{(v_1)}[v], \quad (5)$$

where $Y^{(v_1)}_{n-1}[v]$ are mutually independent random variables for different v_1 and independent of the random variables $\xi_{\alpha v_1}$. If the sequence $(Y^{(\alpha)}_n[\emptyset])_n$ has a limit (in probability) when $n \to \infty$, then $\lim_{n\to\infty} Y^{(\alpha)}_n[\emptyset]$ will depend only on α; similarly, for every fixed value of α, the $\lim_{n\to\infty} Y^{(v_1)}_{n-1}[v]$ will depend only on v_1. However, for two different initial colours α and α', the $\lim_{n\to\infty} Y^{(v_1)}_{n-1}[v]$ gives rise to two independent copies, with the same distribution, which we denote respectively $Y^{(\alpha,v_1)}$ and $Y^{(\alpha',v_1)}$ to distinguish them. Therefore, if the limit $Y^{(\alpha)}_\infty(u)$ exists $\forall \alpha \in \{1,\dots,d\}$ and $\forall u \in \mathbb{V}$ it must verify the functional equation

$$Y^{(\alpha)} \stackrel{\mathrm{d}}{=} \sum_{\beta=1}^{d} \eta_{\alpha\beta} Y^{(\alpha,\beta)} \quad \forall \alpha \in \{1,\dots,d\}, \tag{6}$$

where $Y^{(\alpha,\beta)}$ are mutually independent random variables and independent of the random variables $\eta_{\alpha\beta}$. For every $\alpha,\beta \in \{1,\dots,d\}$ the random variables $Y^{(\alpha,\beta)}$ and $Y^{(\beta)}$ have the same law; moreover, since equation (6) holds only in distribution, they can be chosen independent. A very particular case of the equation (6) has been considered in [1]. To the extend of our knowledge, the functional equation (6) in this generality has not been studied in the literature. Restricting ourselves in the case of integrable solutions and taking expectations from both sides, it is evident that a necessary condition for the existence of non-trivial solution is that 1 is an eigenvalue of the matrix $\mathbb{E}\eta$. This condition seems far from being sufficient. Some conjectures about this functional equation are formulated in the last section of this paper.

Let's consider the random walk in random environment problem. Our main result is formulated in the following

Theorem 1.6 *Let* $\boldsymbol{m}(x) = \begin{pmatrix} \mathbb{E}(\eta^x_{11}) & \cdots & \mathbb{E}(\eta^x_{1d}) \\ \vdots & & \\ \mathbb{E}(\eta^x_{d1}) & \dots & \mathbb{E}(\eta^x_{dd}) \end{pmatrix}$ *for* $x \in [0,1]$. *Assume that the matrix* $\boldsymbol{m}(x)$ *is regular i.e. that for every* $x \in [0,1]$, *there exists some integer* $N = N(x)$ *such that* $(\boldsymbol{m}(x)^N)_{ij} > 0 \quad \forall i,j$. *Denote by* $\rho(x)$ *the largest eigenvalue of* $\boldsymbol{m}(x)$ *for* $x \in [0,1]$ *and* $\lambda = \inf_{x\in[0,1]} \rho(x)$.

1. *If* $\lambda < 1$ *the random walk is almost surely ergodic*
2. *If* $\lambda > 1$ *the random walk is almost surely transient.*

Remark: The previous result allows to study the asymptotic behaviour of the corresponding chaos process $Z_\infty = \lim Z_n$. It is immediate to show that almost surely $Z_\infty < \infty$ if $\lambda < 1$ and $Z_\infty = \infty$ if $\lambda > 1$.

The sections 2, 3, and 4 below are devoted to the proof of the theorem 1.6. In some places we used techniques already used in [17], either literally or in an extended version. In other places, some independent probabilistic techniques are used that beyond their interest for the random walk problem they illuminate aspects of the yet unsolved chaos problem.

2 Ergodicity of the random walk in random environment for the case $\lambda < 1$

The proof of ergodicity is much simpler than the proof of non ergodicity.

Lemma 2.1 *If $\rho(x) < 1$ for some $x \in [0,1]$ then $Z_\infty = \sum_{v \in \mathbb{V}} \xi[v] < \infty$ a.s.*

Proof: Fix first some $v \in \mathbb{V}_n$, $v = (v_1, \dots, v_n)$ and assign colour α at the root. Denote $\mathbb{E}_\alpha(\cdot)$ expectations for starting colour α. We have $\mathbb{E}_\alpha(\xi^x[v]) = m_{\alpha v_1}(x) m_{v_1 v_2}(x) \dots m_{v_{n-1} v_n}(x)$. Hence

$$\mathbb{E}_\alpha\Big(\sum_{v \in \mathbb{V}_n} \xi^x[v]\Big) = \boldsymbol{e}_\alpha \boldsymbol{m}^n(x) \boldsymbol{e},$$

where we recall that the vectors $\boldsymbol{e}_\alpha$ and $\boldsymbol{e}$ are already defined in section 1.4.4. If for some $x_0 \in [0,1]$, $\rho(x_0) < 1$ then $\sum_{n=1}^\infty \boldsymbol{m}^n(x_0)$ is a well defined $d \times d$ matrix whose all elements are finite. Hence, for this particular x_0, $\sum_{v \in \mathring{\mathbb{V}}} \mathbb{E}_\alpha \xi^{x_0}[v] < \infty$ and since $\xi^{x_0}[v] \geq 0$ this means that $\sum_v \xi^{x_0}[v] < \infty$ almost surely. The last majorisation means that there exists at most a *finite* subset $V \subseteq \mathbb{V}$ on which $\xi^{x_0}[v] > 1$ for $v \in V$. We have then: $\sum_{v \in \mathbb{V}} \xi[v] = \sum_{v \in V} \xi[v] + \sum_{v \in \mathbb{V} \setminus V} \xi[v]$ on $\mathbb{V}$. On $\mathbb{V} \setminus V$, $\xi^{x_0}[v] \geq \xi[v]$ for $x_0 \leq 1$ since $\xi[v] \leq 1$. Hence

$$\sum_{v \in \mathbb{V}} \xi[v] \leq \sum_{v \in V} \xi[v] + \sum_{v \in \mathbb{V} \setminus V} \xi^{x_0}[v]$$

both terms of the right hand side being finite. ♠

Corollary 2.2 *If $\lambda < 1$ the random walk in random environment is almost surely ergodic.*

Proof: It is enough to show that $\sum_{v \in \mathbb{V}} \pi[v] < \infty$ since then, choosing the arbitrary normalising value $\pi[\emptyset]$, we can guarantee that $\sum_{v \in \mathbb{V}} \pi[v] = 1$ showing that the random walk admits an invariant probability. Now $\sum_{v \in \mathbb{V}} \pi[v] = \pi[\emptyset] + \sum_{n \in \mathbb{N}^*} \sum_{v \in \mathbb{V}_n} \pi[v]$. For $x_0 \in]0,1[$ such that $\rho(x_0) < 1$, we have

$$\begin{aligned}
\mathbb{E}_\alpha\Big(\sum_{v \in \mathbb{V}} \pi[v]^{x_0}\Big) &= \mathbb{E}_\alpha(\pi[\emptyset]^{x_0}) + \sum_{n \in \mathbb{N}^*} \sum_{v \in \mathbb{V}_n} \mathbb{E}_\alpha(\pi[\emptyset]^{x_0} \xi[v]^{x_0} p_{v,0}^{-x_0}) \\
&= \mathbb{E}_\alpha(\pi[\emptyset]^{x_0}) + \sum_{n \in \mathbb{N}^*} \sum_{v \in \mathbb{V}_n} \mathbb{E}_\alpha(\pi[\emptyset]^{x_0} \xi[v]^{x_0}) \mathbb{E}(p_{v,0}^{-x_0}),
\end{aligned}$$

because $p_{v,0}$ is independent of $\xi[v]$ for all $v \in \overset{\circ}{\mathbb{V}}$. Now for $n \in \mathbb{N}^*$, $v \in \mathbb{V}_n$, and $x_0 < 1$, we have $1 \leq \frac{1}{p_{v,0}^{x_0}} \leq \frac{1}{p_{v,0}}$. Therefore, $\mathbb{E}(\frac{1}{p_{v,0}^{x_0}}) \leq \max_{v \in \mathbb{V}_n} \mathbb{E}(\frac{1}{p_{v,0}}) = \max_{i=1,\dots,d} \mathbb{E}(\frac{1}{p_{v'i,0}})$ for $v' = v|_{|v|-1}$ since the random variables $p_{v,0}$ and $p_{u,0}$ are independent and identically distributed for $u, v \in \mathbb{V}_n$ with $u|_{|u|-1} \neq v|_{|v|-1}$. Hence by using condition (2) and the integrability condition $\mathbb{E}_\alpha(\sum_{v \in \mathbb{V}} \pi[v]^{x_0}) < \infty$, the proof is completed by the same arguments as those used in the proof of the previous lemma. ♠

3 Technical results

3.1 Colouring

Our results can be expressed more easily by introducing a systematic way to deal with the colouring of the tree.

We call *colouring function* a map $c : \mathbb{V} \to C$, where C is a finite discrete set, *the colour set*. We shall use in the sequel the colouring function with colour set $\{1, \dots, d\}$ defined by

$$c(u) = \begin{cases} u_{|u|} & \text{if } u \in \overset{\circ}{\mathbb{V}} \\ \alpha \in \{1, \dots, d\} & \text{if } u = (\emptyset). \end{cases}$$

Thus the colouring of the tree introduces a natural ordering among vertices that are siblings; the root vertex that is assigned some arbitrarily prescribed colour α. If some ambiguity appears, we use subscripts or superscripts to indicate the root colour, *e.g.* $\mathbb{E}_\alpha(\cdot)$ for expectations, $Y_n^{(\alpha)}$ for random variables, etc.

For $i, j \in \{1, \dots, d\}$, we denote

$$N_{ij}[u; v] = \sum_{k=|u|}^{|v|-1} \mathbb{1}_{(ij)}(c(v|_k), c(v|_{k+1}))$$

the number of edges in the path $[u; v]$ with *chromatic type* (ij). If $u = (\emptyset)$, we use as usual the simplified notation $N_{ij}[v]$. Using the independence of the random variables encountered on the edges of every path $[u; v]$, we remark that for every $x \in [0, 1]$,

$$\begin{aligned} \mathbb{E}_\alpha(\xi^x[u; v]) &= m_{11}(x)^{N_{11}[u;v]} \dots m_{dd}(x)^{N_{dd}[u;v]} \\ &= \prod_{i=1}^{d} \prod_{j=1}^{d} m_{ij}(x)^{N_{ij}[u;v]}, \end{aligned}$$

where $m_{ij}(x)$, for $i, j = 1, \dots, d$ are the matrix elements of the matrix $\boldsymbol{m}(x)$.

Notations 3.1 Let $\boldsymbol{\beta} = (\beta_{ij})_{i,j,\in\{1,\dots,d\}}$ be a $d\times d$ matrix with $\beta_{ij} \in [0,1]$ and $\sum_{i,j=1}^d \beta_{ij} = 1$. Let $x \in [0,1]$. We introduce the following functions

$$\begin{aligned}\psi(\boldsymbol{\beta}) &= \prod_{i=1}^d \prod_{k=1}^d \left(\frac{\sum_{j=1}^d \beta_{ij}}{\beta_{ik}} \right)^{\beta_{ik}}, \\ \chi(x,\boldsymbol{\beta}) &= \prod_{i=1}^d \prod_{j=1}^d m_{ij}(x)^{\beta_{ij}}, \text{and} \\ \phi(x,\boldsymbol{\beta}) &= \psi(\boldsymbol{\beta})\chi(x,\boldsymbol{\beta}),\end{aligned}$$

with the convention $0^0 = 1$.

Lemma 3.2 *Let $\boldsymbol{n} = (n_{i,j})_{i,j\in\{1,\dots,d\}}$ be a $d\times d$ matrix of non-negative integers verifying $\sum_{j=1}^d (n_{ij} - n_{ji}) = \delta_{i,\alpha}$ for all $i \in \{1,\dots,d\}$. We denote $n_i = \sum_{j=1}^d n_{ij}$ and $n = \sum_{i=1}^d n_i$. Let $\mathbb{V}(\boldsymbol{n}) = \{v \in \mathbb{V}_n : N_{ij}[v] = n_{ij}\}$ and $K(\boldsymbol{n}) = \operatorname{card} \mathbb{V}(\boldsymbol{n})$. Then*

$$K(\boldsymbol{n}) = C_{n_1}^{(n_{11},\dots,n_{1d})} \dots C_{n_d}^{(n_{d1},\dots,n_{dd})},$$

where

$$C_{a_1+\dots+a_d}^{(a_1,\dots,a_d)} = \frac{(a_1+\dots+a_d)!}{a_1!\dots a_d!},$$

for non-negative integers $a_1,\dots,a_d$

Proof: To describe a path $[v]$ with $N_{ij}[v] = n_{ij}$ for all i and j, it is enough to choose among the n_i vertices of colour i their children of colour 1 (there are n_{i1} of them), ... , their children of colour d (there are n_{id} of them.) Hence for every colour i, there are $C_{n_i}^{(n_{i1},\dots,n_{id})}$ choices and the choice for every i is independent from the choice for every $j \neq i$. ♠

Remark: Notice that the numbers (n_{ij}) must verify certain constraints to be admissible as values of $N_{ij}[v]$ for some path $[v]$. Since each edge has a source, the relation $n_i = \sum_{j=1}^d n_{ij}$ holds, and since each vertex — but the root — of colour i is reached from exactly one parent, of colour j, the relation $n_i = \sum_{j=1}^d n_{ji} + \delta_{i\alpha}$ holds, so that

$$\forall i, \sum_{j=1}^d (n_{ij} - n_{ji}) = \delta_{i,\alpha}, \tag{7}$$

and

$$\sum_{i,j=1}^d n_{ij} = n. \tag{8}$$

Therefore, the matrix $\boldsymbol{n}$ uniquely determines the level n by equation (8) and we have $\mathbb{V}(\boldsymbol{n}) \subset \mathbb{V}_n$ with $n = \sum_{i,j=1}^d n_{ij}$.

It is immediate to show by independence that

$$\sum_{v \in \mathbb{V}(\boldsymbol{n})} \mathbb{E}_\alpha(\xi^x[v]) = K(\boldsymbol{n}) m_{11}(x)^{n_{11}} \dots m_{dd}(x)^{n_{dd}}.$$

Notice that $\mathbb{V}(\boldsymbol{n}) \subset V_n$ is a *finite* set of vertices. A related notion is the *infinite* subtree $\mathbb{U}(\boldsymbol{n}) \subset \mathbb{V}$ constructed with skeleton pattern $\boldsymbol{n}$ in the following way. Each $v \in \mathbb{V}(\boldsymbol{n})$ defines a path $[v]$ from $(\emptyset)$ to v, so that vertices $v \in \mathbb{V}(\boldsymbol{n})$ can be viewed as leaves of a finite tree. Now each such leaf can be considered as the root of a new finite subtree composed only by paths of the type $\mathbb{V}(\boldsymbol{n})$. This procedure can be repeated *ad infinitum* and it gives rise to the infinite subtree $\mathbb{U}(\boldsymbol{n})$.

3.2 Directional estimates

Let $\boldsymbol{\beta}$ be a $d \times d$ matrix with $\beta_{ij} \geq 0$, $\sum_{i,j=1}^d \beta_{ij} = 1$, and $\sum_{j=1}^d \beta_{ij} = \sum_{j=1}^d \beta_{ji}$. For every $n \in \mathbb{N}$, denote $\nu_{ij}(n) = \lfloor \beta_{ij} n \rfloor$.

Lemma 3.3 *For large n, we have $K(\boldsymbol{\nu}(n))^{1/n} = \psi(\boldsymbol{\beta})(1 + o(1))$.*

Proof: Remark that $\lim_{n \to \infty} \frac{\nu_{ij}(n)}{n} = \beta_{ij}$. The proof is completed by applying Stirling's formula to the expression for $K(\boldsymbol{\nu}(n))$ obtained in lemma 3.2 and noticing that the matrix $\boldsymbol{\nu}(n)$ verifies constraints (7) and (8). ♠

Remark: The previous matrix $\boldsymbol{\beta}$ must be thought as defining a main direction on the tree. In the sequel, we shall call *direction* $\boldsymbol{\beta}$ a matrix $\boldsymbol{\beta}$ with positive elements such that $\sum_{i,j=1}^d \beta_{ij} = 1$ and verifying the constraint $\sum_{j=1}^d (\beta_{ij} - \beta_{ji}) = 0$.

Lemma 3.4 *For every direction $\boldsymbol{\beta}$, there exists $x_{\boldsymbol{\beta}} \in [0,1]$ such that*

$$\min_{x \in [0,1]} \phi(x, \boldsymbol{\beta}) = \phi(x_{\boldsymbol{\beta}}, \boldsymbol{\beta}).$$

Proof: Since, for every i, j, the distribution of the random variables ξ_{ij} has not an atom at 0, it follows that $\mathbb{P}(\xi_{ij} > 0) = 1$. Therefore, the function $\log m_{ij}(x) = \log \mathbb{E}(\xi_{ij}^x) = \log \mathbb{E}(\exp(x \log \xi_{ij}))$, defined on $\mathbb{R}$, takes values in $]-\infty, \infty]$. Now by Hölder's inequality, it is convex and since ξ_{ij} is not a trivial random variable (not almost surely a constant) it is strictly convex. Hence, the set $\{x \in \mathbb{R} : \log m_{ij}(x) < \infty\}$ is an interval of $\mathbb{R}$ containing 0 and 1 since $\log m_{ij}(0) = 0$ and $\log m_{ij}(1) = \log \mathbb{E}(\xi_{ij}) < \infty$ by the assumed integrability of ξ_{ij}. Hence $m_{ij}(x)$ is well defined on $[0,1]$ and by standard arguments, infinitely differentiable in $]0,1[$.

Moreover, for $x \in]0,1[$, $(\log m_{ij})'(x) = \frac{\mathbb{E}(\xi_{ij}^x \log \xi_{ij})}{m_{ij}(x)} = \frac{m'_{ij}(x)}{m_{ij}(x)}$. Now $\log \phi(x,\boldsymbol{\beta}) = \log \psi(\boldsymbol{\beta}) + \sum_{i,j} \beta_{ij} \log m_{ij}(x)$; hence $\frac{\partial}{\partial x} \log \phi(x,\boldsymbol{\beta}) = \sum_{i,j} \beta_{ij} \frac{m'_{ij}(x)}{m_{ij}(x)}$ and either this derivative vanishes for some $x_{\boldsymbol{\beta}} \in]0,1[$ — and this will be a minimum because of convexity — or it never vanishes in $]0,1[$ and the minimum will be attained at one of the borders of $[0,1]$. Hence the minimum always exists in $[0,1]$.

Moreover, for non trivial ξ_{ij}, *i.e.* not identically equal to a constant for all i,j, the minimum is unique. ♠

Proposition 3.5 *Let $\boldsymbol{\beta}$ be a direction on the tree. Let $x_{\boldsymbol{\beta}} = \arg\min_{x\in[0,1]} \phi(x,\boldsymbol{\beta})$. Denote by $\rho(x_{\boldsymbol{\beta}})$ the largest eigenvalue of the matrix $\boldsymbol{m}(x_{\boldsymbol{\beta}})$ (assumed to be regular) and $\boldsymbol{l}(x_{\boldsymbol{\beta}})$ — respectively $\boldsymbol{r}(x_{\boldsymbol{\beta}})$ — the left — respectively right — eigenvector associated with the largest eigenvalue. On the space of $d \times d$ direction matrices, introduce a mapping*

$$S : \boldsymbol{\beta} \mapsto \boldsymbol{\beta}' = (\beta'_{ij})_{i,j=1,\dots,d}$$

by

$$\beta'_{ij} = b l_i(x_{\boldsymbol{\beta}}) m_{ij}(x_{\boldsymbol{\beta}}) r_j(x_{\boldsymbol{\beta}}).$$

Then

1. *the constant b can be chosen so that $\boldsymbol{\beta}'$ is also a direction matrix,*
2. *$\beta'_{ij} = 0$ if, and only if, $m_{ij}(x_{\boldsymbol{\beta}}) = 0$, and*
3. *the mapping S has a fixed point.*

Proof:

1. Remark that $\beta'_{ij} \geq 0$ and since $\boldsymbol{r}$ is a right eigenvector, $\sum_j \beta'_{ij} = b l_i(x_{\boldsymbol{\beta}}) \rho(x_{\boldsymbol{\beta}}) r_i(x_{\boldsymbol{\beta}})$, we can chose $b = (\rho(x_{\boldsymbol{\beta}}) \langle \boldsymbol{l}(x_{\boldsymbol{\beta}}) \mid \boldsymbol{r}(x_{\boldsymbol{\beta}}) \rangle)^{-1}$, where $\langle \cdot \mid \cdot \rangle$ is the scalar product in $\mathbb{R}^d$, to guarantee $\sum_{ij} \beta'_{ij} = 1$. Similarly, since $\boldsymbol{l}$ is the left eigenvector, we get $\sum_j \beta'_{ji} = b l_i(x_{\boldsymbol{\beta}}) \rho(x_{\boldsymbol{\beta}}) r_i(x_{\boldsymbol{\beta}})$, fulfilling thus the constraint $\sum_j (\beta'_{ij} - \beta'_{ji}) = 0$. Therefore, $\boldsymbol{\beta}'$ is also a direction.
2. Since $\boldsymbol{m}(x)$ is a regular matrix, *i.e.* there exists N such that $\boldsymbol{m}(x)^N$ has all its elements strictly positive, the eigenvectors corresponding to the maximal eigenvalue have no zero components.
3. Let H be the set of direction matrices, viewed as a subset of the finite dimensional linear space $\mathbb{R}^{d^2}$. Then H is compact and convex and the map $S : H \to H$ is continuous. Therefore, by Brower's theorem S has (at least) a fixed point.

♠

Proposition 3.6 *If for every direction $\boldsymbol{\beta}$, $\inf_{x\in[0,1]} \phi(x,\boldsymbol{\beta}) \leq C$ for some constant C, then $\exists x_0 \in [0,1]$ such that $\rho(x_0) \leq C$.*

Proof: Let $\overline{\boldsymbol{\beta}}$ be a fixed point of the map S. Then $\inf_{x\in[0,1]} \phi(x,\overline{\boldsymbol{\beta}}) = \phi(y,\overline{\boldsymbol{\beta}}) \leq C$. Compute now $\phi(y,\overline{\boldsymbol{\beta}})$ by using the fact that $\boldsymbol{l}$ and $\boldsymbol{r}$ are left and right eigenvectors of $\boldsymbol{m}$.

$$\begin{aligned}\phi(y,\overline{\boldsymbol{\beta}}) &= \left(\frac{bl_1(m_{11}r_1+\ldots+m_{1d}r_d)}{bl_1 m_{11} r_1} m_{11}\right)^{\overline{\beta}_{11}} \cdots \left(\frac{bl_d(m_{d1}r_1+\ldots+m_{dd}r_d)}{bl_d m_{dd} r_d} m_{dd}\right)^{\overline{\beta}_{dd}} \\ &= \rho(y)^{\sum \overline{\beta}_{ij}} (\frac{r_1}{r_1})^{\overline{\beta_{11}}} (\frac{r_1}{r_2})^{\overline{\beta_{12}}} \cdots (\frac{r_1}{r_d})^{\overline{\beta_{1d}}} \cdots (\frac{r_d}{r_1})^{\overline{\beta_{d1}}} \cdots (\frac{r_d}{r_d})^{\overline{\beta_{dd}}}\end{aligned}$$

and since $\sum_{ij} \overline{\beta}_{ij} = 1$ and $\sum_j (\overline{\beta}_{ij} - \overline{\beta}_{ji}) = 0$, we have finally that

$$\phi(y,\overline{\boldsymbol{\beta}}) = \rho(y) \leq C.$$

♠

Corollary 3.7 *If $\inf_{x\in[0,1]} \rho(x) > 1$, then there exists a direction $\overline{\boldsymbol{\beta}}$ such that $\inf_{x\in[0,1]} \phi(x,\overline{\boldsymbol{\beta}}) > 1$.*

Proof: Immediate from the previous proposition. ♠

Lemma 3.8 *If $\inf_{x\in[0,1]} \rho(x) > 1$, then there exists a direction $\hat{\boldsymbol{\beta}}$ with rational coefficients such that $\inf_{x\in[0,1]} \phi(x,\hat{\boldsymbol{\beta}}) > 1$*

Proof: From the previous corollary, there exists a direction $\overline{\boldsymbol{\beta}}$ (with real coefficients) such that $\inf_{x\in[0,1]} \phi(x,\overline{\boldsymbol{\beta}}) > 1$. Now the function $f(\boldsymbol{\beta}) = \inf_{x\in[0,1]} \phi(x,\boldsymbol{\beta})$ is continuous. Hence, for arbitrary directions $\overline{\beta}$ and $\hat{\beta}$ there exists C' such that $|f(\overline{\boldsymbol{\beta}}) - f(\hat{\boldsymbol{\beta}})| \leq C' \|\overline{\boldsymbol{\beta}} - \hat{\boldsymbol{\beta}}\|$ and since every real can be approximated arbitrarily well by rationals, the lemma follows. ♠

3.3 Subtrees, branching and Chernoff-Cramér bound

Let $\hat{\boldsymbol{\beta}}$ be a direction with rational coefficients. Since all its matrix elements are rational, there exist integers γ, depending on $\hat{\boldsymbol{\beta}}$, such that all the products $\gamma\hat{\beta}_{ij}$, for $i,j = 1,\ldots,d$ are integer-valued. It is enough to choose the smallest such integer,

$$\gamma = \inf\{n \in \mathbb{N} : \lfloor \hat{\beta}_{ij} n \rfloor = \hat{\beta}_{ij} n = \nu_{ij} \in \mathbb{N}, \forall i,j \in \{1,\ldots,d\}\}.$$

Consider the infinite subtree $\mathbb{U}_\gamma(\boldsymbol{\nu})$ of $\mathbb{V}$ composed only from those paths that between levels $k\gamma$ and $(k+1)\gamma$, with $k \in \mathbb{N}$, have prescribed number $\nu_{ij} = \hat{\beta}_{ij}\gamma \in \mathbb{N}$ of edges of type ij. This subtree — viewed as a subtree of $\mathbb{V}$ — has a branching ratio (see [15] for definition of branching ration) $\mathrm{br}(\mathbb{U}_\gamma(\boldsymbol{\nu})) = \psi(\hat{\boldsymbol{\beta}})$, therefore, viewed as a tree on its own, it will have branching ratio $\psi(\hat{\boldsymbol{\beta}})^\gamma$. We give below an extension of a technical result of [17] that yields the Chernoff-Cramér bound in our case.

Proposition 3.9 *Suppose* $\lambda = \inf_{x\in[0,1]} \rho(x) > 1$. *Then there exists a rational direction* $\hat{\boldsymbol{\beta}}$, *integers* k *and* γ *in* $\mathbb{N}^*$, *and a real* $y \in]0,1]$ *such that*

$$\mathbb{P}(\xi[u;v] > y^{\gamma k}) > \left(\frac{1}{y\psi(\hat{\boldsymbol{\beta}})}\right)^{\gamma k},$$

for all $u, v \in \mathbb{U}_\gamma(\boldsymbol{\nu})$ *with* $u < v$, $|v| - |u| = k\gamma$, *and* $\nu_{ij} = \hat{\beta}_{ij}\gamma = \lfloor \hat{\beta}_{ij}\gamma \rfloor \in \mathbb{N}$.

Proof: Since $\lambda > 1$, by lemma 3.8 there exists a rational direction $\hat{\boldsymbol{\beta}}$ such that $\inf_{x\in[0,1]} \phi(x, \hat{\boldsymbol{\beta}}) > 1$. Choose γ the smallest integer such that all the products $\hat{\beta}_{ij}\gamma$ are integer-valued for all i, j. This choice of γ fixes all the integers $\nu_{ij} = \hat{\beta}_{ij}\gamma = \lfloor \hat{\beta}_{ij}\gamma \rfloor$ for $i, j = 1, \ldots, d$. Therefore the subtree $\mathbb{U}_\gamma(\boldsymbol{\nu})$ is well defined. For some $k \in \mathbb{N}$, whose value will be fixed later, let $u, v \in \mathbb{U}_\gamma(\boldsymbol{\nu})$ with $u < v$ and $|v| - |u| = k\gamma$. The family of the random variables $(\xi_a)_{a\in[u;v]}$ is independent. Moreover, the product $\prod_{a\in[u;v]} \xi_a = \xi[u;v]$ can be written as $\xi[u;v] = \prod_{l=1}^{k} A_l$, with $A_l = \prod_{a\in[v|_{|u|+(l-1)\gamma}; v|_{|u|+l\gamma}]} \xi_a$. The family $(A_l)_{l=1,\ldots,k}$ is independent and identically distributed. Let $\tau(x) = \mathbb{E}(A_1^x)$ and $\alpha = \mathbb{E}(\log A_1)$ and denote $I(x^*)$ the Legendre transform of $\tau(x)$, *i.e.* $I(x^*) = \sup_{x\in[0,1]}(xx^* - \log\tau(x))$ for $x^* \in \mathbb{R}$. Then the sequence (S_n) with $S_n = \sum_{l=1}^{n} \log(A_l)$ satisfies a large deviation principle with rate function I, *i.e.*

$$\liminf_n \frac{1}{n} \log \mathbb{P}(S_n > n\theta) \geq -\inf_{x^*>\theta} I(x^*).$$

In other words, for every $\epsilon > 0$ we can find an integer N such that $\frac{1}{n}\log\mathbb{P}(S_n > n\theta) \geq -\inf_{x^*>\theta} I(x^*) - \epsilon$ for all $n \geq N$. Choose $k \geq N$. Then for this sufficiently large k, we have

$$\mathbb{P}(\prod_{l=1}^{k} A_l > (\exp(\theta))^k) > \exp(\inf_{x^*>\theta} I(x^*))^{-k}.$$

This bound is non-trivial if $\theta > \alpha$ and then $\inf_{x^*>\theta} I(x^*) = I(\theta)$. Writing, in that case, $\log\hat{y} = \theta$, we compute explicitly $\exp(I(\log\hat{y})) = \frac{\hat{y}}{\hat{y}^{1-x_0}\mathbb{E}A_1^{x_0}}$ where x_0 is the position where the function $x\log\hat{y} - \log\tau(x)$ attains its minimum (as a function of x). Writing $\hat{y} = \overline{y}^\gamma$, we get finally

$$\mathbb{P}(\xi[u;v] > \overline{y}^{k\gamma}) > \frac{\overline{y}^{k\gamma(1-x_0)}(\mathbb{E}A_1^{x_0})^k}{\overline{y}^{k\gamma}} = \left(\frac{\overline{y}^{1-x_0}\chi(x_0, \hat{\boldsymbol{\beta}})}{\overline{y}}\right)^{k\gamma} = \left(\frac{\overline{y}^{1-x_0}\chi(x_0, \hat{\boldsymbol{\beta}})\psi(\hat{\boldsymbol{\beta}})}{\overline{y}}\right)^{k\gamma}.$$

By Fenchel's equality (see [17]),

$$\sup_{\overline{y}\in]0,1]}\inf_{x\geq 0}\overline{y}^{1-x}\chi(x,\hat{\boldsymbol{\beta}}) = \inf_{x\in[0,1]}\chi(x,\hat{\boldsymbol{\beta}}).$$

Denoting y the value of $\overline{y}\in]0,1]$ attaining the infimum in the left hand side of Fenchel's equality, the previous inequality becomes

$$\mathbb{P}(\xi[u;v] > y^{k\gamma}) > \left(\frac{\inf_{x\in[0,1]}\chi(x,\hat{\boldsymbol{\beta}})\psi(\hat{\boldsymbol{\beta}})}{y\psi(\hat{\boldsymbol{\beta}})}\right)^{k\gamma} = \left(\frac{\inf_{x\in[0,1]}\phi(x,\hat{\boldsymbol{\beta}})}{y\psi(\hat{\boldsymbol{\beta}})}\right)^{k\gamma} \geq (y\psi(\hat{\boldsymbol{\beta}}))^{-k\gamma}.$$

♠

4 Proof of non-ergodicity and transience

It will be shown in subsection 4.2, that in the case $\lambda > 1$ the random walk is transient; this result is based on some non-probabilistic estimates coming from the analogy between reversible Markov chains and electrical networks. We start however by proving under the same conditions, in subsection 4.1, a seemingly weaker result, guaranteeing only non-ergodicity. The reason is that this method is purely probabilistic and contains some information that can be used for the yet unsolved multiplicative chaos problem for the random walk in random environment on a coloured tree.

4.1 Non-ergodicity for the random walk in random environment in the case $\lambda > 1$

Inspired from the result 2.1.7 of [6] we prove the following

Lemma 4.1 *Let $(X_n)_{n\in\mathbb{N}}$ be a real-valued process on the filtred space $(\Omega,\mathcal{F},(\mathcal{F})_{n\in\mathbb{N}},\mathbb{P})$, adapted to the filtration $(\mathcal{F}_n)_{n\in\mathbb{N}}$, with $X_0 = x$, some constant, having uniformly bounded increments Y_n, i.e. there exists $a > 0$, such that $|Y_n| = |X_n - X_{n-1}| \leq a$ for all n. Suppose moreover that $(X_n)_{n\in\mathbb{N}}$, with $X_n = \sum_{k=1}^n Y_k$, is a strong submartingale i.e. there exists $\epsilon > 0$ such that $\mathbb{E}(Y_n|\mathcal{F}_{n-1}) \geq \epsilon$ almost surely for all n. Let $\tau(\delta) = \inf\{n \geq 1 : X_n < x + n\delta\}$. Then, there exists some $\delta_1 = \delta_1(\epsilon) > 0$ such that for all $\delta < \delta_1$, $\mathbb{P}(\tau(\delta) = \infty) > 0$.*

Proof: Write $X_n = x + \sum_{k=1}^n Y_k$. Then

$$\begin{aligned} \mathbb{P}(X_n < x + \delta n) &= \mathbb{P}(-\sum_{k=1}^n Y_k > -\delta n) \\ &= \mathbb{P}(\exp(-h\sum_{k=1}^n Y_k) > \exp(-h\delta n)) \quad \text{for} \quad h \geq 0 \\ &\leq \exp(h\delta n)\mathbb{E}(\exp(-h\sum_{k=1}^n Y_k)) \\ &= \exp(h\delta n)\mathbb{E}(\prod_{k=1}^n \mathbb{E}(\exp(-hY_k)|\mathcal{F}_{k-1})). \end{aligned}$$

Chose $h \leq 1/a$ and use the inequality $\exp(x) \leq 1 + x + 3x^2/2$ valid for $|x| < 1$ to bound the conditional expectation $\mathbb{E}(\exp(-hY_k)|\mathcal{F}_{k-1})) \leq 1 - h\epsilon + \frac{3h^2a^2}{2} \equiv \exp(-\delta_2(h))$. Choosing $h \in]0, \frac{2\epsilon}{3a^2}[$ we set $\delta_2 > 0$. Hence $\mathbb{P}(X_n < x + \delta n) \leq \exp(n(h\delta - \delta_2(h)))$ and choosing $\delta < \delta_1 = \delta_2(h)/h$ we get a bound that is exponentially small for large n, that is $\mathbb{P}(X_n < x + \delta n) \leq p_n$ with $p_n = \exp(-n\delta_3)$ with $\delta_3 > 0$. Now since $\sum_{n\in\mathbb{N}} p_n$ converges, the events $A_n = \{X_n < x + \delta n\}$ are realised for a finite number of indices, *i.e.* $\forall\gamma \in]0, 1[$ we can find $m = m(\gamma) \in \mathbb{N}$ such that $\mathbb{P}(\cap_{n=m}^\infty A_n^c) > \gamma$ and consequently $\mathbb{P}(\tau(\delta) = \infty) > 0$. ♠

We shall now prove that when $\lambda > 1$, not only $\mathbb{E}Z_n \to \infty$ but also $Z_n \to \infty$ almost surely, where

$$Z_n = \sum_{k=0}^n \sum_{v\in\mathbb{V}_k} \xi[v] = \sum_{k=0}^n Y_k,$$

and this result is enough to prove non-ergodicity for the random walk in random environment since it shows that the invariant measure $\pi[v]$ cannot be normalised.

Theorem 4.2 *If $\lambda > 1$, then $Z_n \to \infty$ almost surely.*

Remark: This result in conjunction with the result concerning ergodicity localises the critical point for the chaos functional equation to $\lambda = 1$.

Proof of the theorem: From lemma 3.8, since $\lambda > 1$, there exists a direction with rational coefficients $\hat{\boldsymbol{\beta}}$ such that $\inf_{x\in[0,1]} \phi(x, \hat{\boldsymbol{\beta}}) > 1$. Choose some integer γ large enough so that $\nu_{ij} = \hat{\beta}_{ij}\gamma = \lfloor\hat{\beta}_{ij}\gamma\rfloor \in \mathbb{N}$ for all i and j and such that all the integers ν_{ij} are so large that Stirling's formula applies.

Consider the infinite subtree $\mathbb{U}_\gamma(\boldsymbol{\nu})$ having prescribed number of ij-type edges between levels that are at distances $l\gamma$ and $(l+1)\gamma$ from the root. Additionally, since $\lambda > 1$, by proposition 3.9, it is possible to choose $y \in]0,1]$ and $k \in \mathbb{N}$ so that for all $u, v \in \mathbb{U}_\gamma(\boldsymbol{\nu})$ with $u < v$ and $|u| = lk\gamma$ and $|v| = (l+1)k\gamma$ for some $l \in \mathbb{N}$, the Chernoff-Cramér bound

$$\mathbb{P}(\xi[u;v] > y^{k\gamma}) > \left(\frac{1}{y\psi(\hat{\boldsymbol{\beta}})}\right)^{k\gamma}$$

applies. We shall construct a minorating process $(\tilde{Y}_n)_{n\in\mathbb{N}}$ of $(Y_n)_{n\in\mathbb{N}}$ such that the corresponding sum process $\tilde{Z}_n = \sum_{k=0}^n \tilde{Y}_k \to \infty$.

For every $u \in \mathbb{U}_\gamma(\boldsymbol{\nu})$ define the random set

$$D(u) = \{v \in \mathbb{U}_\gamma(\boldsymbol{\nu}) : v > u; |v| - |u| = k\gamma; \xi[u;v] > y^{k\gamma}\}.$$

We have

$$y^{k\gamma}\mathbb{E}|D(u)| = y^{k\gamma} \sum_{\substack{v\in\mathbb{U}_\gamma(\nu)\\ v>u;|v|-|u|=\gamma k}} \mathbb{P}(\xi[u;v] > y^{k\gamma}) > 1.$$

We shall proceed recursively. Let $\tilde{Y}_0 = 1$. Define $C_1 = B_1 = D(\emptyset)$ and let $B_1' \subseteq B_1$ be such that $1 < \sum_{v\in B_1'} y^{|v|} < 2$. Now either $B_1' = \emptyset$ and then we define $\tau = 1$ and stop the process $\tilde{Y}$ or $B_1' \neq \emptyset$ and then we set $\tau > 1$, $B_1'' = B_1 \setminus B_1'$ and $\tilde{Y}_1 = \sum_{v\in B_1} y^{|v|}$. On the set $\{\tau > 1\}$, we have $|\tilde{Y}_1 - \tilde{Y}_0| \leq y^{k\gamma}d^{k\gamma} + 1 < 2y^{d\gamma}d^{k\gamma}$.

Suppose that $\tau > n$ and the sequences $(\tilde{Y}_n), (C_n), (B_n), (B_n'), (B_n'')$ have been constructed up to time n. Define $C_{n+1} = \cup_{u\in B_n'} D(u)$, $B_{n+1} = C_{n+1} \cup B_n''$ and let $B_{n+1}' \subseteq B_{n+1}$ be an arbitrary subset of B_{n+1} chosen so that $1 < \sum_{v\in B_{n+1}'} y^{|v|} < 2$. Now, either $B_{n+1}' = \emptyset$ and then $\tau = n+1$ and the process is stopped, or else $B_{n+1}' \neq \emptyset$ and then $\tau > n+1$, $B_{n+1}'' = B_{n+1} \setminus B_{n+1}'$, $C_{n+1} = \cup_{v\in B_n'} D(v)$, and $\tilde{Y}_{n+1} = \sum_{v\in B_{n+1}} y^{|v|}$. The increments of the process $(\tilde{Y}_n)$ are bounded since

$$\begin{aligned}
|\tilde{Y}_{n+1} - \tilde{Y}_n| &= \left| \sum_{v\in C_{n+1}} y^{|v|} + \sum_{v\in B_n''} y^{|v|} - \sum_{v\in B_n'} y^{|v|} - \sum_{v\in B_n''} y^{|v|} \right| \\
&= \left| \sum_{v\in C_{n+1}} y^{|v|} - \sum_{v\in B_n'} y^{|v|} \right| \\
&= \sum_{v\in B_n'} y^{|v|}(y^{k\gamma}|D(v)| - 1) \\
&\leq 2y^{k\gamma}d^{k\gamma}.
\end{aligned}$$

On the other hand, the process is a strong submartingale since the conditional increments with respect to the natural filtration verify

$$\mathbb{E}(\tilde{Y}_{n+1} - \tilde{Y}_n|\mathcal{F}_n) = \sum_{v\in B_n'} y^{|v|}|\mathbb{E}(y^{k\gamma}|D(v)| - 1|\mathcal{F}_n) > \epsilon,$$

by virtue of the previous induction step and of the Chernoff-Cramér bound, valid because $\lambda > 1$.

Hence, $(\tilde{Y}_l)_{l\in\mathbb{N}}$ is a strong submartingale with bounded jumps and by the previous lemma there exists some index L and some positive δ such that $\tilde{Y}_l > 1 + \delta l$ for all $l \geq L$, showing that $\tilde{Y}_l \to \infty$ with some strictly positive probability. A fortiori, the same conclusion holds for the processes (Y_n) and (Z_n). Now the event $\{Z_n \to \infty\}$ is a tail-measurable event and the random variables $(\xi_a)_{a\in\mathbb{A}(\mathbb{V})}$ are independent for different generations. Hence by the 0-1 law, $\mathbb{P}(Z_n \to \infty) = 1$. ♠

4.2 Transience for the random walk in random environment for the case $\lambda > 1$

To prove transience, we need some not probabilistic methods based on electric networks analogy (see [10] for instance). This is possible since we are dealing with reversible Markov chains.

Theorem 4.3 *If $\lambda > 1$, the random walk in random environment is almost surely transient.*

Proof: After the long preparatory work on directional estimates, the proof of transience is essentially reduced to an appropriate extension of the Chernoff-Cramér bound. Recall that a cutset is a finite set, C, of vertices not including the root vertex, such that any path from the root vertex to the boundary of the tree intersects the set C at exactly one vertex. It is enough to show, as it was the case in [17], that there exists $w \in]0,1[$ such that for every cutset C, there exists some $\epsilon > 0$ such that $\sum_{v\in C} w^{|v|}\xi[v] > \epsilon$. We only sketch the proof since — using the extension of the Chernoff-Cramér bound established in proposition 3.9 — it follows the same lines as the proof of item *i)* of theorem 1 in [17]. Let $\hat{\boldsymbol{\beta}}, \gamma, k$, and y be the parametres determined in proposition 3.9. Fix some sufficiently small $\epsilon > 0$ and consider the random subgraph $\mathbb{W}$ of $\mathbb{U}_\gamma(\boldsymbol{\nu})$ whose paths $[u; v]$ with $u, v \in \mathbb{U}_\gamma(\boldsymbol{\nu})$, $u < v$, and $|v| - |u| = k\gamma$ have been erased if either $\xi[u; v] < y^{k\gamma}$ holds or $A_l < \epsilon$ for some $l = 1, \ldots, k$ holds, where $A_l = \xi[v|_{|u|+(l-1)\gamma}; v|_{|u|+l\gamma}]$. Therefore, by proposition 3.9, edges remain with probability $p > (y\psi(\hat{\boldsymbol{\beta}}))^{-k\gamma}$. Choose $w \in](y\psi(\hat{\boldsymbol{\beta}})p^{1/k\gamma})^{-1}, 1[$. Since $p\psi(\hat{\boldsymbol{\beta}})^{k\gamma} > 1/(wy)^{k\gamma} > 1$ by percolation arguments there is a subtree $\mathbb{W}_{k\gamma}$ of $\mathbb{V}$ having branching bigger than $1/(wy)^{k\gamma} > 1$. Then we conclude as in [17]. ♠

5 Some results on random walks in random environments stemming from multiplicative chaos and vice-versa

5.1 A multiplicative chaos approach of the model on a regular tree

We use here the results on multiplicative chaos established in a series of papers [5, 12, 13] to study the behaviour of the random walk in random environment on a regular tree by providing independent proofs of those stated in [17], stemming solely from results on multiplicative chaos.

Let us first remind some basic results on multiplicative chaos. For the problem of the random walk in random environment on a regular tree, the multiplicative chaos process reads

$$Y_n[\emptyset] \simeq Y_n = \sum_{v_1,\dots,v_n=1}^{d} \xi_{a(v_1)}\xi_{a(v_1v_2)}\cdots\xi_{a(v_1\dots v_n)}.$$

Denoting for every $n \in \mathbb{N}$ by $\mathcal{F}_n = \sigma(\xi_{a(v)}, v \in \cup_{k=0}^{n}\mathbb{V}_k)$, we have that

$$\begin{aligned}\mathbb{E}(Y_n|\mathcal{F}_{n-1}) &= \sum_{v_1,\dots,v_{n-1}=1}^{d} \xi_{a(v_1)}\xi_{a(v_1v_2)}\cdots\xi_{a(v_1\dots v_{n-1}n)} \sum_{v_n=1}^{d} \mathbb{E}\xi_{a(v_1\dots v_n)} \\ &= \sum_{v_1,\dots,v_{n-1}=1}^{d} \xi_{a(v_1)}\xi_{a(v_1v_2)}\cdots\xi_{a(v_1\dots v_{n-1}n)} \sum_{j=1}^{d} \mathbb{E}\eta_j \\ &= Y_{n-1}f(1),\end{aligned}$$

where we recall that $f(x) = \sum_{i=1}^{d} \mathbb{E}\eta_i^x$ and $g(x) = \log f(x)$. Now, if $f(1) = 1$, the process (Y_n) is a non-negative martingale converging almost surely to a random variable $Y_\infty \in \mathbb{R}^+$. The interesting and highly non-trivial question is whether this convergence holds also in $\mathcal{L}^1$.

Remark on the other hand that we can write $Y_n[\emptyset] = \sum_{v\in\mathbb{V}_1} \xi_{a(v)}Y'_{n-1}[v]$ where $Y'_{n-1}[v]$ are independent from the $(\xi_{a(v)})_v$. If the limit in distribution when $n \to \infty$ exists, this gives rise to the functional equation

$$Y \overset{\mathrm{d}}{=} \sum_{j=1}^{d} \eta_i Y'_i, \tag{9}$$

where Y and Y'_i have the same law for all $i = 1, \dots, d$ and can be chosen independent. If we denote by μ the common distribution of the random variables Y'_i and $\hat{\mu}$ its Laplace transform, the functional equation can be seen as a mapping, T, from the space of Laplace transforms of probability measures into itself, reading

$$T\hat{\mu}(s) = \mathbb{E}\left(\prod_{i=1}^{d} \hat{\mu}(\eta_i s)\right).$$

We use the same symbol, T, to denote the induced mapping on the space of probability measures $\mathcal{M}_1^+([0,\infty[)$. Denote $\mathsf{NonTrivFix(T)} = \{\mu \in \mathcal{M}_1^+([0,\infty[) : T\mu = \mu \text{ and } \mu \neq \delta_0\}$.

There are several known results on the non-trivial fixed points of T provided some additional moment conditions hold.

Theorem 5.1 (Durrett and Liggett [5]) *Assume that for some $\delta > 1$, $\mathbb{E}\eta_i^\delta < \infty$ for all $i \in \{1, \ldots, d\}$. Then NonTrivFix(T) $\neq \emptyset$ if and only if for some $\alpha \in]0,1]$ we have $g(\alpha) = 0$ and $g'(\alpha) \leq 0$. If $g(1) = 0$ and $g'(1) < 0$, then every $\mu \in$ NonTrivFix(T) has β moments with $\beta > 1$ if and only if $g(\beta) < 0$.*

Liu substantially improved these results in [12, 13] by both weakening the moment condition and by allowing random branching *i.e.* d becoming a random variable having some known joint distribution with the random variables $(\eta_i)_i$. We state his result in the special case of interest for us here, namely the case where d is a constant and where we have assumed that $\mathbb{P}(\eta_i > 0) = 1$ for all i. Notice however that the case of random d allows the treatment of general trees with random branching, covering — and even extending — the ones considered in [17]. This study is postponed to a subsequent paper.

Theorem 5.2 (Liu [12]) *Assume that $\mathbb{E}[(\sum_{i=1}^d \eta_i) \log^+(\sum_{i=1}^d \eta_i)] < \infty$, where $\log^+ z = \max(0, \log z)$. Then NonTrivFix(T) $\neq \emptyset$ if and only if $g(1) = 0$ and $g'(1) < 0$. Moreover, the solutions of the functional equation have finite first moment and there is a unique such probability measure having first moment equal to 1.*

It is enough to consider the chaos process (Y_n) only in the case $f(1) = 1$. In the other situations, by renormalising the random variables we always construct some process satisfying $f(1) = 1$. For this case, (Y_n) is a non-negative martingale verifying $\mathbb{E}(Y_n) = 1$ for all n and converging thus almost surely to a limit Y_∞. By Fatou's lemma, $\mathbb{E}(Y_\infty) \leq \mathbb{E}(Y_n) = 1$. Thus the limit of the process (Y_n) is always integrable. The question is whether $\mathbb{E}(Y_\infty) = 1$. On the other hand, we can seek for solutions of the functional equation (9). For such solutions to be interpreted as limits of the martingale (Y_n), they must be integrable. By Liu's theorem, we know the conditions of existence of non-trivial integrable solutions to the functional equation. The following theorem guarantees that then the corresponding martingale is uniformly integrable.

Theorem 5.3 (Kahane and Peyrière [9]) *With the same notation as above, the following conditions are equivalent:*

1. $\mathbb{E}(Y_\infty) = 1$

2. $\mathbb{E}(Y_\infty) > 0$

3. the functional equation (9) has an integrable solution Y *such that* $\mathbb{E}(Y) = 1$.

Proof: Obviously 1. implies 2. which in turn implies 3. Suppose that 3. is true and let Y be a solution of the functional equation with $\mathbb{E}(Y) = 1$. Then, there exists a sequence of random variables $(\zeta_{a(v_1,\ldots,v_n)})$ with $n \in \mathbb{N}^*$ and $v_i \in \{1, \ldots, d\}$ for $i \leq n$ such that for any choice of $(v_1, \ldots, v_{n-1})$ the random vector $(\zeta_{a(v_1,\ldots,v_n)})_{v_n=1,\ldots,d}$ has the same distribution as the vector $(\eta_{v_n})_{v_n=1,\ldots,d}$ and a sequence of random variables $(Y_{a(v_1,\ldots,v_n)})$ independent of the random variables ζ and having the same distribution as Y, such that for all n

$$Y = \sum_{v_1,\ldots,v_n=1}^{d} \zeta_{a(v_1 v_2)} \zeta_{a(v_1 v_2 v_3)} \cdots \zeta_{a(v_1 \ldots v_n)} Y_{a(v_1 \ldots v_n)}.$$

In fact the last equation coincides with the original functional equation (9) if $n = 1$ and then by induction, supposing the equation true up to order n, if the functional equation is applied to the random variable $Y_{a(v_1,\ldots,v_n)}$ yields

$$Y_{a(v_1,\ldots,v_n)} = \sum_{v_{n+1}=1}^{d} \zeta_{a(v_1 \ldots v_{n+1})} Y_{a(v_1,\ldots,v_{n+1})}.$$

Denote $\mathcal{F}_n = \sigma(\zeta_{a(v)}), v \in \cup_{k=0}^{n} \mathbb{V}_k)$. Then obviously

$$Y_n = \mathbb{E}(Y|\mathcal{F}_n).$$

Therefore the martingale (Y_n) is closed and hence uniformly integrable. Subsequently, $\mathbb{E}(Y_\infty) = 1$ and the theorem is proved. ♠

We are now able to give the proof of the theorem 1.4 that will be split into three different *régimes.*

Theorem 5.4 *Let* $\lambda = \inf_{x\in[0,1]} f(x)$. *If* $\lambda < 1$, *then, almost surely,* $Z_\infty < \infty$ *and the random walk is ergodic.*

Proof: Since $\lambda < 1$, the qualitative behaviour of the function f can only be in one of the three possibilities depicted in figure 1.

In all these cases, there is an $\alpha \in]0,1[$ such that $g(\alpha) = 0$ and $g'(\alpha) < 0$. Introduce now the renormalised random variables $\tilde{\eta}_i = \eta_i^\alpha$ for this value of $\alpha \in]0,1[$. We denote with a tilde all the objects relative to the new random variables $\tilde{\eta}_i$, like $\tilde{\xi}[v]$, $\tilde{Y}_n$, $\tilde{f}$, or $\tilde{g}$, defined in the same way as the corresponding objects without tilde for the random variables η_i. We compute for instance $\tilde{f}(x) = \sum_{i=1}^{d} \tilde{\eta}_i^x = f(\alpha x)$ and $\tilde{g}(x) = g(\alpha x)$.

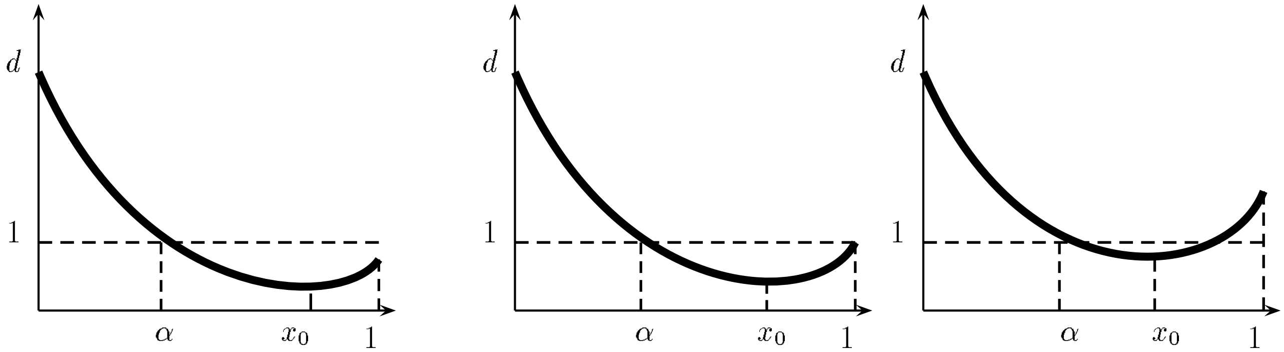

Figure 1: The generic behaviour of the function f in the subcritical case $\lambda < 1$.

Introduce now the variables $\hat{\eta}_i = \frac{\tilde{\eta}_i^t}{\tilde{f}(t)}$ for some value of t that will be determined later. All the quantities relative to the new variables $\hat{\eta}_i$ will now be distinguished by the caret symbol ˆ . We have

$$\hat{g}(y) = g(\alpha t y) - y g(\alpha t)$$

and

$$\hat{g}'(y) = \alpha t g'(\alpha t y) - g(\alpha t),$$

establishing thus that $\hat{g}(1) = 0$ for all possible choices of t.

Let β_0 be the abscissa of the point of the graph of g that lies also on the straight line from the origin tangent to the graph (see figure 2).

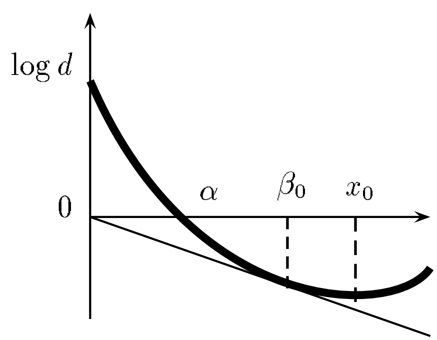

Figure 2: The graph of the fucntion $g(x) = \log f(x)$ in the case $\lambda < 1$. The straight line from the origin tangent to the graph of g touches the graph at a point with abscissa $\beta_0 \in]\alpha, x_0[$.

Choose now $t = \beta/\alpha$ for some $\beta \in]\alpha, \beta_0[$. For that choice of t, we have $g'(1) = \beta g'(\beta) - g(\beta) < 0$ and $\tilde{f}(t) = f(\beta) < f(\alpha) < 1$.

Since now $\hat{g}(1) = 0$ and $\hat{g}'(1) < 0$ the process $(\hat{Y}_n)_n$ with

$$\hat{Y}_n = \sum_{v \in \mathbb{V}_n} \hat{\xi}[v] = \frac{1}{f(\beta)^n} \sum_{v \in \mathbb{V}_n} \hat{\xi}[v],$$

converges in $\mathcal{L}^1$ to a random variable $\hat{Y}_\infty \in \mathbb{R}^+$ by theorem 5.2. Since $f(\beta) < 1$ and $\hat{Y}_\infty < \infty$ almost surely, then

$$\sum_n \sum_{v \in \mathbb{V}_n} \xi^\beta[v] = \sum_n f(\beta)^n \hat{Y}_n < \infty \text{ almost surely.}$$

Hence we conclude as in lemma 2.1 that $Z_\infty < \infty$ almost surely. Therefore the invariant measure is normalisable and the random walk is ergodic. ♠

Theorem 5.5 *If $\lambda = \inf_{x \in [0,1]} f(x) = 1$ and $\sum_{i=1}^d \mathbb{E}(\eta_i \log \eta_i) < 0$ then, almost surely, $0 < Y_\infty < \infty$, $Z_\infty = \infty$, and the random walk is null-recurrent.*

Proof: Denote by x_0 the unique point of the interval $]0, 1]$ attaining the infimum, *i.e.* $f(x_0) = \lambda = 1$. By the strict convexity of the function $g(x) = \log f(x)$, we can have only three possibilities, depicted generically for the function f on the figure 3 below; similar figures can be drawn for $g = \log f$.

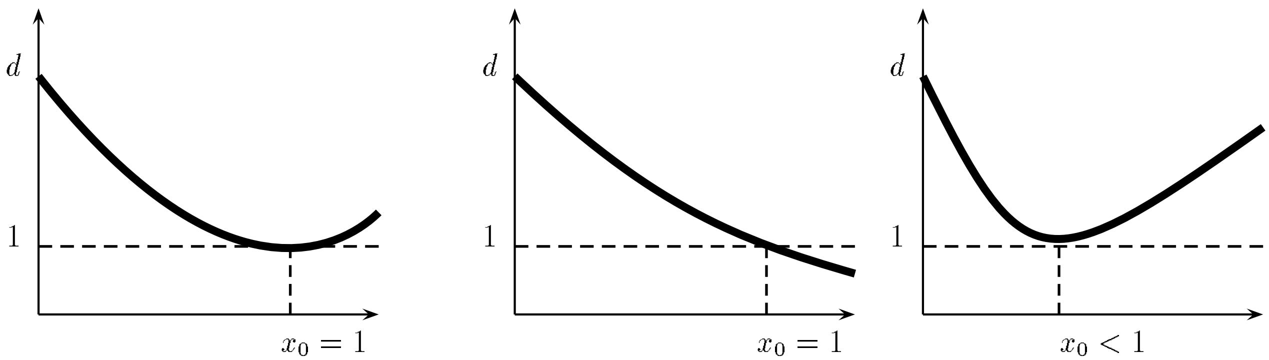

Figure 3: The generic behaviour of the function f in the critical case $\lambda = 1$. In a) $f(1) = 1$ and $f'(1) = 0$, in b) $f(1) = 1$ and $f'(1) < 0$, and in c) $f(x_0) = 1$ but $f(1) > 1$ and $f'(1) > 0$.

Cases a) and c) are excluded by the condition $\sum_{i=1}^d \mathbb{E}(\eta_i \log \eta_i) < 0$ so that it is enough to consider the case b) where $g(1) = 0$ and $g'(1) < 0$. Hence by theorem 5.2, the functional equation (9) has non-trivial integrable solutions and by theorem 5.3 the process $(Y_n)_n$ is an uniformly integrable martingale converging in $\mathcal{L}^1$ to the random variable Y_∞, *i.e.* $\lim_{n\to\infty} \mathbb{E}|Y_n - Y_\infty| = 0$ and subsequently the Cesàro mean, $\frac{1}{n}\sum_{k=0}^{n-1}(Y_k - Y_\infty) \to 0$. Therefore, $Z_n/n \to Y_\infty \in \mathbb{R}^+ \setminus \{0\}$ almost surely, establishing thus that $Z_\infty = \infty$ almost surely. This guarantees that the walk is not ergodic. On the other hand, since $Y_n \to Y_\infty \in \mathbb{R}^+ \setminus \{0\}$ almost surely, using the Nash-Williams criterion based on the electric circuit analogy (see corollary 4.3 of [15] for instance) the walk cannot be transient. Hence the walk is almost surely null-recurrent. ♠

Finally we treat the supercritical case $\lambda > 1$.

Theorem 5.6 *Let $\lambda = \inf_{x\in[0,1]} f(x)$. If $\lambda > 1$ then almost surely $Y_\infty = \infty$, $Z_\infty = \infty$, and the random walk is transient.*

Proof: Let $x_0 \in [0,1]$ be such that $f(x_0) = \inf_{x\in[0,1]} f(x) = \lambda > 1$. The qualitative behaviour of the function f has two possibilities depicted in figure (4) below.

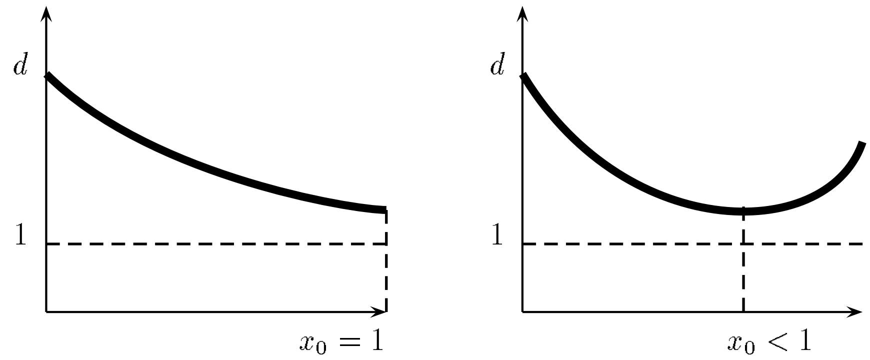

Figure 4: The generic behaviour of the function f in the supercritical case $\lambda > 1$.

We have omitted the totally trivial case where the infimum is attained at $x_0 = 0$, in which case $g(0) = \log d$ and $g'(0) > 0$ since then necessarily $\sum_{i=1}^d \eta_i \geq 1$ and consequently $Y_n \to \infty$ exponentially fast. In case a) where $x_0 = 1$, introduce the random variables $\tilde{\eta}_i = \frac{\eta_i}{f(1)}$ with corresponding function $\tilde{g}(x) = g(x) - xg(1)$. We have $\tilde{g}(1) = 0$ and $\tilde{g}'(1) = g'(1) - g(1) < 0$ since $g'(1) \leq 0$ and $g(1) > 0$. Thus by theorem 5.2 we have again the uniform integrability of the martingale $(\tilde{Y}_n)_n$ reading

$$\tilde{Y}_n = \frac{1}{\lambda^n} \sum_{v\in\mathbb{V}_n} \xi[v] = \frac{Y_n}{\lambda^n},$$

which converges in $\mathcal{L}^1$ to a random variable with $0 < \tilde{Y}_\infty < \infty$. Since $\lambda > 1$, necessarily $Y_n \to \infty$ and a fortiori $Z_\infty = \infty$ almost surely.

In case b) where $x_0 < 1$, we consider the random variables $\tilde{\eta}_i = \frac{\eta_i^{x_0}}{f(x_0)}$ with $\tilde{g}(x) = g(xx_0) - xg(x_0)$ and the random variables $\hat{\eta}_i = \frac{\tilde{\eta}_i^t}{\tilde{f}(t)}$ with $\hat{g}(y) = g(tyx_0) - y(g(tx_0)$ for some t that will be determined later. Obviously $\hat{g}(1) = 0$ for all choices of t. Compute $\hat{g}'(1) = tx_0 g'(tx_0) - g(tx_0)$ and define β_0 as the unique point in $]0,1]$ that is abscissa of the point where a straight line emanating from the origin is tangent to the graph of g (see figure 5).

Necessarily, $\beta_0 > x_0$. Choose an arbitrary $\beta \in]0, \beta_0[$ and let $t = \beta/x_0$. For that choice of t, $\hat{g}'(1) = \beta g'(\beta) - g(\beta)$ and since $\beta < \beta_0$, we have that $\hat{g}'(1) < 0$. Observe also that $\hat{\eta}_i = \frac{\eta_i^\beta}{f(\beta)}$. Hence the process $(\hat{Y}_n)_n$ with

$$\hat{Y}_n = \sum_{v\in\mathbb{V}_n} \frac{\xi^\beta[v]}{f(\beta)^n}$$

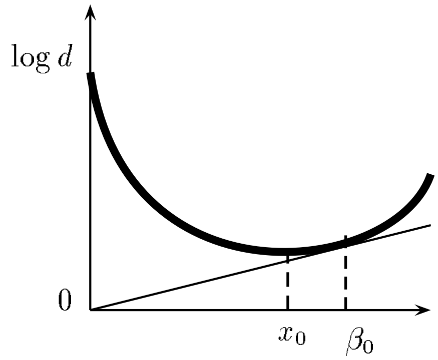

Figure 5: The graph of the fucntion $g(x) = \log f(x)$ in the case $\lambda > 1$. The straight line from the origin tangent to the graph of g touches the graph at a point with abscissa $\beta_0 > x_0$.

is a uniformly integrable martingale converging in $\mathcal{L}^1$ to an almost surely positive limit.

Denote $Y_n^{(\beta)} = \sum\limits_{v\in\mathbb{V}_n} \xi^\beta[v]$. Since $\frac{Y_n^{(\beta)}}{f(\beta)^n} \to \hat{Y}_\infty$ in $\mathcal{L}^1$ with $0 < \hat{Y}_\infty < \infty$ for all $\beta < \beta_0$, it follows that $\frac{1}{n}\log Y_n^{(\beta)} \to g(\beta)$ for all $\beta < \beta_0$.

Now by a standard argument used in statistical mechanics (subadditivity and law of large numbers) for all $\beta \in [0,1]$, $\liminf \frac{1}{n}\log Y_n^{(\beta)} = \sigma(\beta)$ where $\sigma(\beta)$ is a convex function. For $\beta < \beta_0$ the sequence has a limit (hence it coincides with its *limes infimum*) so that $\sigma(\beta) = g(\beta)$ for $\beta < \beta_0$. By convexity, the graph of $\sigma(\beta)$ for $\beta > \beta_0$ is bounded from below by the tangent of $g(\beta)$ at β_0 (this argument is used in the context of statistical mechanics in general [20] and that of disordered systems in particular in [2]). Since $g'(\beta_0) > 0$ and $g(\beta_0) > \lambda$, we have that $\sigma(\beta) \geq \lambda + (\beta - \beta_0)g'(\beta_0) > 0$ for all $\beta > \beta_0$. Hence $Y_n^{(\beta)} \to \infty$ for all values $\beta > \beta_0$ and in particular for $\beta = 1$ so that Y_n tends to ∞ exponentially fast and a fortiori $Z_\infty = \infty$. Since the process Y_n diverges exponentially fast, using again Nash-Williams criterion, for every $\epsilon > 0$ we can find some $w \in]0,1[$ such that for every cutset C, $\sum_{v\in C} w^{|v|}\xi[v] > \epsilon$ and thus the walk is transient. ♠

5.2 Some open problems on multiplicative chaos

The last subsection demonstrated the close relationship between results on multiplicative chaos and reversible Markov chains. In particular, the most difficult part for the Markov chain problem, namely the critical case $\lambda = 1$ becomes an immediate consequence of the theorem on the existence of non trivial solutions of the functional equation and the uniform integrability of the corresponding martingale, once the conditions for the existence of non trivial solutions are known. This analogy can even be extended on more general settings to include the case of random trees and of general distributions for the environment that correspond to situations much more general than the one considered in [17]. Actually, what plays an important *rôle* is the theorem (1) of [5] but this theorem is properly generalised by Liu [13] to include random number of variables d. Therefore, the treatment of random walks in general random environment on random trees becomes accessible by virtue of the results of Liu on multiplicative chaos.

We got conditions under which the chaos processes Y_n and Z_n tend to ∞ or remain finite according to the values of the parametre λ. The precise study of this classification gives rise to a multiplicative chaos functional equation of the type

$$Y^{(\alpha)} \stackrel{\mathrm{d}}{=} \sum_{\beta} \eta_{\alpha\beta} Y'^{(\alpha\beta)}$$

for which the conditions of existence of non trivial solutions are not known. In view of the results on the random walk problems it is expected that the classifying parametre in this problem is the largest eigenvalue of the matrix of moments $\boldsymbol{m}(x)$. This problem is actually under investigation. The above mentioned intuition is confirmed by some preliminary results, by the partial results of [1] and by physical intuition. As a matter of fact the random walk in a random environment can also be viewed as a physical system of spins in a quenched disorder. In the random string problem the quenching is quite stringent so that the Lyapunov's exponent appear. On the contrary, the random walk in random environment on the coloured tree behaves very much like a self-averaging problem.

Acknowledgments: M.M. wishes to thank Yasunari Higuchi for useful discussions and D.P. FAPESP for financial support and the Instituto de Matemática e Estatística, University of São Paulo for warm hospitality during the period when this paper was prepared.

References

[1] F. Ben Nasr. Mesures aléatoires de Mandelbrot associées à des substitutions. *C. Rendus Acad. Sciences (Paris)*, 304:253–258, 1987.

[2] P. Collet and F. Koukiou. Large deviations for multiplicative chaos. *Commun. Math. Phys.*, 147:329–342, 1992.

[3] F. Comets, M. Menshikov, and S. Popov. Lyapunov functions for random walks and strings in random environment. *Ann. Probab.*, 26:1433–1445, 1998.

[4] C. de Calan, J.-M. Luck, Th. Niewenhuizen, and D. Petritis. On the distribution of a random variable occuring in one-dimensional disordered systems. *Journal Stat. Phys.*, 18:501–523, 1985.

[5] R. Durrett and Th. Liggett. Fixed points of smoothing transformation. *Z. Wahrscheinlichkeitstheorie verw. Gebiete*, 64:275–301, 1983.

[6] G. Fayolle, V. Malyshev, and M. Menshikov. *Topics in the constructive theory of countable Markov chains.* Cambridge University Press, Cambridge, 1995.

[7] S. Graf, R. D. Mauldin, and S. C. Williams. The exact Hausdorff dimension in random recursive constructions. *Mem. Amer. Math. Soc.*, 71(381), 1988.

[8] Y. Guivarc'h. Sur une extension de la notion semi-stable. *Ann. Inst. H. Poicaré*, 26:261–285, 1990.

[9] J.-P. Kahane and J. Peyrière. Sur certaines martingales de Benoît Mandelbrot. *Adv. Math.*, 22:131–145, 1976.

[10] F. P. Kelly. *Reversibility and stochastic networks.* John Wiley, Chishester, 1979.

[11] H. Kesten, M.V. Kozlov, and F. Spitzer. A limit law for random walk in a random environment. *Compositio Math.*, 30:145–168, 1975.

[12] Q. Liu. Sur une équation fonctionnelle et ses applications: une extension du théorème de Kesten-Stigun concernant les processus de branchement. *Ann. Appl. Probab.*, 29:353–373, 1997.

[13] Q. Liu. Fixed points of a generalised smoothing transformation and applications to the branching random walk. *Ann. Appl. Probab.*, 30:85–112, 1998.

[14] Q. Liu. Processus de branchement, cascades multiplicatives et fractals aléatoires; habilitation thesis: Université de Rennes 1, 1999.

[15] R. Lyons. Random walk and percolation on trees. *Ann. Probab.*, 18:931–958, 1990.

[16] R. Lyons. Random walks, capacity, and percolation on trees. *Ann. Probab.*, 20:2043–2088, 1992.

[17] R. Lyons and R. Pemantle. Random walk in a random environment and first passage percolation on trees. *Ann. Probab.*, 20:125–136, 1991.

[18] R. Pemantle and Yu. Peres. Critical random walks in random environment on trees. *Ann. Probab.*, 23:105–140, 1995.

[19] D. Petritis. The zero-level set of the Brownian bridge as a model of disordered physical systems. *Probability theory and mathematical statistics*, 2:298–308, Mosklas, Vilnius, 1990.

[20] D. Ruelle. *Statistical mechanics.* W. A. Benjamin, Inc., London, 1974.

[21] F. Solomon. Random walk in a random environment. *Ann. Probab.*, 3:1–31, 1975.

[22] D. Williams. *Probability with martingales.* Cambridge University Press, Cambridge, 1993.