On random walks in random environment on trees and their relationship with multiplicative chaos

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ABSTRACT: *The purpose of this paper is to report on recent results concerning random walks in a random environment on monochromatic and coloured trees and their relationship with multiplicative chaos. The proofs are omitted since they are extensively given elsewhere [12]. It is worth noticing that for the random walk on monochromatic tree the results we give were previously known [11]; we provide however a totally new proof, based solely on multiplicative chaos results, that allows to relax some stringent conditions on independence properties of the random transition probabilities. For the random walk on a coloured tree the results are new; the classification of the asymptotic behaviour of the random walk allows to obtain some hints for the classification of the yet unsolved corresponding multiplicative chaos problem.*

1 Notation

Let d be a fixed non-negative integer. We consider the rooted regular tree of order d, *i.e.* a connected graph without loops with a denumerable set of vertices $\mathbb{V}$ and a denumerable set of non oriented edges $\mathbb{A}(\mathbb{V})$. There is a distinguished vertex called the root that has degree d; all other vertices have degree $d+1$. Vertices are completely determined by giving their genealogical history from their common ancestor, the root; hence they are bijectively indexed by the set of sequences of arbitrary length over an alphabet of d letters. We use the same symbol for the indexing set so that $\mathbb{V} = \cup_{n=0}^{\infty} \mathbb{V}_n$ with $\mathbb{V}_0 = \{\emptyset\}$ and $\mathbb{V}_n = \{v = (v_1, \ldots, v_n) : v_i \in \{1, \ldots, d\}, i = 1 \ldots n\}$ for $n \geq 1$. For every $v \in \mathbb{V}$, we denote $|v|$ the length of the path from v to the root *i.e.* the number of edges encountered. For $v \in \mathbb{V}$ and $k \leq |v|$ we denote by $v|_k$ the truncation of the sequence v to its k first elements, *i.e.* if $v = (v_1 \ldots v_n) \in \mathbb{V}_n$ and $k \leq n$, then $v|_k = (v_1 \ldots v_k) \in \mathbb{V}_k$; the symbol $v|_k$ must not be confused therefore with v_k, representing the letter appearing at the k-th position of the sequence. For $0 \leq k < \ell \leq |v|$ we denote $v|_\ell^k$ the subsequence of length $\ell - k$ defined by $v|_\ell^k = (v_{k+1}, \ldots, v_\ell)$. If $u \in \mathbb{V}$, we write $u \leq v$ if $|u| \leq |v|$ and $v = (u_1, \ldots u_{|u|}, v_{|u|+1}, \ldots v_{|v|})$ *i.e.* if u is the initial sequence of v; we write $u < v$ when $u \leq v$ and $|u| < |v|$. Similarly for every sequence u and any letter $\ell \in \{1 \ldots d\}$, the sequence $u\ell$ will have length $|u| + 1$ and last letter ℓ.

Edges are unordered pairs $\langle u, v \rangle$ of adjacent vertices u and v. Since every vertex has an unique ancestor, every edge is uniquely defined by its most remote vertex. Hence, every vertex $v \in \mathbb{V} \setminus \{\emptyset\}$ defines an edge $a(v) = \langle\, v|_{|\,v\,|-1}, v \,\rangle$. Edges are thus also indexed by the set $\mathbb{V}$, more precisely by $\mathring{\mathbb{V}} = \mathbb{V} \setminus \{\emptyset\}$ and we denote $a(v)$ the edge defined by v; therefore $\mathbb{A}(\mathbb{V}) \simeq \mathring{\mathbb{V}}$.

If $u, v \in \mathbb{V}$ and $u < v$, we denote $[u; v]$ the (unique) path from u to v *i.e.* the collection of edges $(a_1, a_2, \ldots)$ with $a_j \equiv a(v|_{|u|+j})$, for $j = 1, \ldots, |v|$. For every $u \in \mathbb{V}$, the symbol $[u; u]$ denotes an empty set of edges. If u and v are not comparable vertices, *i.e.* neither $u \leq v$ nor $v \leq u$ holds, although there is a canonical way to define the path $[u; v]$, this definition is not necessary in the present paper and hence omitted. We write simply $[v]$ to denote the path joining the root to v, namely $[\emptyset; v]$.

At every edge a we assign a number $\xi_a \in [0, \infty[$ in some specific manner. This specification differs from model to model and since various models are considered here, we don't wish to be more explicit about these variables at the present level. Mind however that the numbers $(\xi_a)_{a \in \mathbb{A}(\mathbb{V})}$ are random variables neither necessarily independent nor necessarily equi-distributed. For the time being, we only assume that we dispose of a specific collection $(\xi_a)_{a \in \mathbb{A}}$, called the *edge-environment.*

2 Multiplicative chaos

Let $(\mathbb{V}, (\xi_a)_{a \in \mathbb{A}(\mathbb{V})})$ be a given tree and a given edge environment. For $u, v \in \mathbb{V}$, with $u < v$ we denote

$$\xi[u; v] = \prod_{a \in [u; v]} \xi_a$$

the product of environment values encountered on the path of edges from u to v; the symbol $\xi[v]$ is defined to mean $\xi[\emptyset; v]$ and $\xi[v; v]$ — as a product over an empty set — is consistently defined to be 1. It is not necessary for the purpose of the present article to define the value of $\xi[u; v]$ when u and v are not comparable.

For every $u \in \mathbb{V}$, we consider the process $Y_n(u)_{n \in \mathbb{N}}$ defined by $Y_0(u) = 1$ and

$$Y_n(u) = \sum_{v \in \mathbb{V}_{n+|u|}; v > u} \xi[u; v] = \sum_{v \in \mathbb{V}_{n+|u|}; v > u} \prod_{a \in [u; v]} \xi_a,$$

for $n \geq 1$. This process is known as the *multiplicative chaos process.* Notice that even when $(\xi_a)_{a \in \mathbb{A}}$ is a family of independent random variables, the random variables $\xi[u; v]$ are not independent for v scanning the set $\mathbb{V}_{n+|u|}$. Hence the asymptotic behaviour of $Y_n(u)$ when $n \to \infty$ is far from trivial and it is studied for several particular cases of dependences of the family (ξ_a) in an extensive literature; see for instance [7, 5, 6, 3, 8, 9, 14, 2, 10].

The study of the asymptotics of the process (Y_n) is done by various techniques:

1. If the limit $\lim_{n \to \infty} Y_n(u) \stackrel{\mathrm{d}}{=} Y(u)$ exists in distribution for all $u \in \mathbb{V}$ then it must verify the functional equation

$$Y(u) \stackrel{\mathrm{d}}{=} \sum_{w \in \mathbb{V}_{|u|+1}; w > u} \xi[u; w] \, Y(w). \tag{1}$$

   The process $(Y_n(u))_n$ and the corresponding functional equation (1) are thoroughly studied in the literature for some particular choices of dependencies of the family (ξ_a).

2. A second technique of study of the asymptotics is by martingale analysis. If for any fixed $u \in \mathbb{V}$, $(\mathcal{F}_n^{(u)})$ denotes the natural filtration $\mathcal{F}_k^{(u)} = \sigma(\xi_{a(uv_1 \ldots v_k)}, v_i \in \{1, \ldots, d\}, i = 1, \ldots, k\}$ for $k \in \mathbb{N}$, we have

$$\mathbb{E}(Y_n(u) | \mathcal{F}_{n-1}^{(u)}) = \sum_{v_1, \ldots, v_n \in \{1, \ldots, d\}} \xi_{a(uv_1)} \cdots \xi_{a(uv_1 \ldots v_{n-1})} \mathbb{E}(\xi_{a(uv_1 v_2 \ldots v_n)} | \mathcal{F}_{n-1}^{(u)})$$

and in the special case where the distribution of $\xi_{a(uv_1v_2\dots v_n)}$ depends solely on v_n and the random variables are independent for different generations, the previous formula simplifies into

$$\mathbb{E}(Y_n(u)|\mathcal{F}^{(u)}_{n-1}) = Y_{n-1}(u) \sum_{v_n=1}^{d} \mathbb{E}(\xi_{a(uv_1v_2\dots v_n)}).$$

Although the process (Y_n) is thoroughly studied, the closely related process

$$Z_n(u) = \sum_{k=0}^{n} Y_k(u) \quad \text{for} \quad n \geq 0$$

does not seem — to the best of our knowledge — to have attracted much attention. However, if we are interested in connections between multiplicative chaos and random walks in random environment on a tree, it is this latter process that naturally appears in both subjects.

3 Nearest neighbours random walk on a tree in an inhomogeneous environment

To every vertex $u = (u_1, \dots, u_{|u|}) \in \mathring{\mathbb{V}}$ are assigned $d+1$ numbers $(p_{u,0}, p_{u,1}, \dots, p_{u,d})$ with $p_{u,0} > 0$, $p_{u,i} \geq 0$ $\forall i = 1, \dots, d$ and $\sum_{i=0}^{d} p_{u,i} = 1$. To $u \in \mathbb{V}_0 = \{\emptyset\}$ are assigned only d numbers $(p_{\emptyset,1}, \dots p_{\emptyset,d})$ with $p_{\emptyset,i} \geq 0$ $\forall i = 1, \dots, d$ and $\sum_{i=1}^{d} p_{\emptyset,1} = 1$. These numbers will be random variables with some specific dependence properties that will be defined later. These numbers stand for transition probabilities of a reversible Markov chain $(X_n)_{n\in\mathbb{N}}$ on the tree verifying for $|u| \geq 1$

$$P_{u,v} = \mathbb{P}(X_{n+1} = v|X_n = u) = \begin{cases} p_{u,0} & \text{if} \quad v = u|_{|u|-1} \\ p_{u,v_{|v|}} & \text{if} \quad u = v|_{|v|-1} \\ 0 & \text{otherwise.} \end{cases}$$

For $u = (\emptyset)$ we have the slightly modified transition probabilities

$$P_{\emptyset,v} = \mathbb{P}(X_{n+1} = v|X_n = (\emptyset)) = \begin{cases} p_{\emptyset,v_1} & \text{if} \quad v \in \mathbb{V}_1 \\ 0 & \text{othrewise.} \end{cases}$$

For $u \in \mathbb{V}$ with $|u| \geq 2$ we consider the edge $a(u) = \langle\, u|_{|u|-1}, u \,\rangle$ and attach to this edge the variable

$$\xi_{a(u)} = \frac{p_{u|_{|u|-1}, u_{|u|}}}{p_{u|_{|u|-1},0}} \in [0, \infty[.$$

For $u \in \mathbb{V}_1$ we attach $\xi_{a(u)} = p_{\emptyset,u_1}$. One can easily check the validity of the following

Lemma 3.1 *For every $v \in \mathbb{V}$ define the variable*

$$\pi[v] = \begin{cases} \pi[\emptyset]\, \xi[v] \frac{1}{p_{v,0}} & \text{if } v \in \mathring{\mathbb{V}} \\ \pi[\emptyset] & \text{if } v = (\emptyset), \end{cases}$$

with $\pi[\emptyset]$ an arbitrary constant. Then $\pi[v]$ verifies the stationarity condition

$$\sum_{v \in \mathbb{V}} \pi[v] P_{v,v'} = \pi[v'], \quad \forall v' \in \mathbb{V}.$$

To avoid technical difficulties, we assume that

$$\mathbb{E}((p_{v,0})^{-1}) < \infty.$$

Then, apart the factor $\frac{1}{p_{v,0}}$, the expression for the invariant measure $\pi[v]$ involves the product $\xi[v]$ of variables along the edges of the path form $\emptyset$ to v as was the case in the expression of multiplicative chaos.

4 Models covered by the present formalism and main results

We present below a unified treatment of both the multiplicative chaos process and the random walk problem stating in the same theorem the asymptotic behaviour of the limiting chaos process and of the random walk. Several models fit the present formalism; by making appropriate identifications of random variables, the random walk in random environment on $\mathbb{N}$ or the problem of random strings in a random environment can be rephrased in the present language.

4.1 Random walk in a random environment on a regular tree

At every vertex $v \in \mathring{\mathbb{V}}$ is assigned a $(d+1)$-dimensional random vector with positive components $(p_{v,0}, \ldots, p_{v,d})$ verifying $\sum_{j=0}^{d} p_{v,j} = 1$. For the vertex $v = \emptyset$, the corresponding random vector is d-dimensional and its components verify $\sum_{j=1}^{d} p_{\emptyset,j} = 1$. These random vectors are independent for different v's and, for $v \in \mathring{\mathbb{V}}$ they have the same distribution. Let $\boldsymbol{\eta} = (\eta_1, \ldots, \eta_d)$, be a vector of non-negative random variables η_i, $i = 1, \ldots, d$, having the same distribution with $p_{v,i}/p_{v,0}$, for $v \in \mathring{\mathbb{V}}$, with not necessarily independent nor identically distributed components. We assume the law of the random vector is explicitly known with $\mathbb{E}\eta_i < \infty$ and $\mathbb{E}\eta_i \log^+ \eta_i < \infty$, $\forall i = 1, \ldots, d$. Moreover, to avoid technicalities we assume that although the support of the random variables η_i extends up to 0, their law has no atom at 0.

To the edge $a(v)$, having most remote vertex $v \in \mathring{\mathbb{V}}$, we assign the random variable $\xi_{a(v)}$ having the same distribution as $\eta_{v_{|v|}}$; the variables $\xi_{a(v)}$ and $\xi_{a(v')}$ are independent if $v|_{|v|-1} \neq v'|_{|v'|-1}$. Notice that if the components of the random vector η are not independent, the variables $\xi_{a(v)}$ and $\xi_{a(v')}$ with $|v| = |v'|$ and $v|_{|v|-1} = v'|_{|v'|-1}$ are not independent either.

The results are expressed in terms of the functions

$$f(x) = \mathbb{E}\,(\sum_{i=1}^{d} \eta_i^x), x \in \mathbb{R}^+ \quad \text{and} \quad g(x) = \log f(x),$$

and of the parameter $\lambda = \inf_{x\in[0,1]} f(x)$.

Theorem 4.1 *Let $\lambda = \inf_{x\in[0,1]} f(x)$ and $x_0 \in [0,1]$ be such that $f(x_0) = \lambda$. Then*

1. *If $\lambda < 1$, then almost surely the random walk is positive recurrent and $Z_\infty < \infty$.*
2. *If $\lambda > 1$, then almost surely the random walk is transient, $Y_\infty = \infty$, and $Z_\infty = \infty$.*
3. *If $\lambda = 1$ and moreover $f'(1) < 0$, then almost surely $0 < Y_\infty < \infty$, $Z_\infty = \infty$, and the random walk is null-recurrent.*

This theorem is already formulated, with some stringent conditions on the random variables, in [11]. In [12] a totally new proof of this result is provided, based on the multiplicative chaos results of [8].

4.2 Random walk in a random environment on a coloured tree

This problem is reminiscent of the problem on random strings in a random environement, studied in [4], where non reversible Markov chains on the tree $\mathbb{V}$ are considered and general conditions for transience/null recurrence/ergodicity are given in terms of Lyapuonov exponent of a product of matrices. To describe the problem of random strings in a random environment, we distinguish the d children of every vertex by assigning a colour index, chosen without replacement from the set $\{1,\ldots,d\}$, to each child. The root is assigned an arbitrary colour $\alpha \in \{1,\ldots,d\}$. Consequently, every edge $a(v)$ with $v \in \overset{\circ}{\mathbb{V}}$ is assigned the bicolour $(ij) \in \{1,\ldots,d\}^2$, where $i = v_{|v|-1}$ and $j = v_{|v|}$.

Passing to the edge-indexed ratio of outwards over inwards probabilities, the model can be rephrased to fit the present formalism. Let

$$\boldsymbol{\eta} = \begin{pmatrix} \eta_{11} & \cdots & \eta_{1d} \\ \vdots & & \\ \eta_{d1} & \ldots & \eta_{dd} \end{pmatrix}$$

be a matrix of non-negative random elements of known joint distribution. The matrix elements are not necessarily independent.

Theorem 4.2 *Let $\boldsymbol{m}(x) = \begin{pmatrix} \mathbb{E}(\eta_{11}^x) & \cdots & \mathbb{E}(\eta_{1d}^x) \\ \vdots & & \\ \mathbb{E}(\eta_{d1}^x) & \ldots & \mathbb{E}(\eta_{dd}^x) \end{pmatrix}$ for $x \in [0,1]$. Assume that the matrix $\boldsymbol{m}(x)$ is regular i.e. there exists some integer N such that that for every $x \in [0,1]$, $(\boldsymbol{m}(x)^N)_{ij} > 0 \quad \forall i,j$. Denote by $\rho(x)$ the largest eigenvalue of $\boldsymbol{m}(x)$ for $x \in [0,1]$ and $\lambda = \inf_{x\in[0,1]} \rho(x)$.*

1. *If $\lambda < 1$ the random walk is almost surely positive recurrent and $Z_\infty < \infty$ almost surely*

2. *If $\lambda > 1$ the random walk is almost surely transient and $Y_\infty = \infty$ almost surely.*

5 Some open problems on multiplicative chaos and further development

We demonstrated the close relationship between results on multiplicative chaos and reversible Markov chains. In particular, the most difficult part for the Markov chain problem, namely the critical case $\lambda = 1$ becomes an immediate consequence of the theorem on the existence of non trivial solutions of the functional equation and the uniform integrability of the corresponding martingale, once the conditions for the existence of non trivial solutions are known. This analogy can even be extended on more general settings to include the case of random trees and of general distributions for the environment that correspond to situations much more general than the one considered in [11]. Actually, what plays an important *rôle* is the theorem (1) of [5] but this theorem is properly generalised by Liu [9] to include random number of variables d. Therefore, the treatment of random walks in general random environment on random trees becomes accessible by virtue of the results of Liu on multiplicative chaos.

We got conditions under which the chaos processes Y_n and Z_n tend to ∞ or remain finite according to the values of the parametre λ. The precise study of this classification gives rise to a multiplicative chaos functional equation of the type

$$Y^{(\alpha)} \stackrel{\mathrm{d}}{=} \sum_\beta \eta_{\alpha\beta} Y'^{(\alpha\beta)}$$

for which the conditions of existence of non trivial solutions are not known. In view of the results on the random walk problems it is expected that the classifying parametre in this problem is the largest eigenvalue of the matrix of moments $\boldsymbol{m}(x)$. This problem is actually under investigation. The above mentioned intuition is confirmed by some preliminary results, by the partial results of [1] and by physical intuition. As a matter of fact the random walk in a random environment can also be viewed as a physical system of spins in a quenched disorder. In the random string problem the quenching is quite stringent so that the Lyapunov's exponent appear. On the contrary, the random walk in random environment on the coloured tree behaves very much like a self-averaging problem.

Other random walk models on more general trees (multiplexed coloured trees) can also be introduced that involve matrix valued multiplicative chaos [13]. Again, classification of the random walk problem can be used as a hint for the classification of the chaos process.

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