MARKOV CHAINS IN A WEDGE WITH EXCITABLE BOUNDARIES[1]

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Abstract

We consider two models of Markov chains with unbounded jumps. In the first model the chain evolves in a quadrant with boundaries having internal structure; when the chain is in the interior of the quadrant, it moves as a standard Markov chain without drift. When it touches the boundary, it can spend some random time in internal — invisible — degrees of freedom of the boundary before it emerges again in the quadrant. The second model deals with a Markov chain — again without drift — evolving in two adjacent quadrants with excitable boundaries and interface with some invisible degrees of freedom. We give, for both models, conditions for transience, recurrence, ergodicity, existence and non existence of moments of passage times that are expressed in terms of simple geometrical properties of the wedge, the covariance matrix of the chain and its average drifts on the boundaries, by using martingale estimates coming from Lyapunov functions.

1 Introduction

1.1 Motivation

Random or ballistic motion in wedges with reflecting boundaries has been thoroughly studied lately [9, 7, 2, 4, 5, 6] both for its intrinsic mathematical interest and for its use in modelling storage systems or queueing networks. In [2], the Lyapunov function method, developped in [4], was used to study the recurrence properties and $\mathbb{L}^p$ integrability of

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recurrence times for Markov chains in a quadrant with reflecting boundaries.

Here, two similar problems with boundaries that can be internally excited are studied along the same lines. The first problem studies the behaviour of a Markov chain in a quadrant with excitable boundaries. The second problem deals with a Markov chain evolving into two adjacent quadrants communicating through a permeable and excitable interface and having excitable boundaries. These Markov chains allow for a more faithful modelling of storage or queuing systems but may also be of some interest in the problem of optimal shielding of nuclear reactors.

We are interested in the *critical régime* for these chains, that is to say in the régime where the mean drift is zero on the whole space but the boundaries.

In order not to excessively burden this introductory section, only the first model is introduced here and the corresponding results presented. The presentation of the second model and its results are postponed in section 5 after terminology and results for the first model are established. The first model is a genaralisation of the model studied in [2]. Its exposition is reproduced here first because it is actually interesting to show that the internal degrees of freedom can be "forgotten" provided some embedding is performed and second because it allows to introduce the terminology, establish the intermediate technical results and render the second model more understandable.

Both models can also be formulated in continuous space and time generalising thus the results of [6]. This continuous time formulation is not considered here however.

1.2 Notations and definitions

Le $\mathbb{X}$ be the set of pairs of positive integers, *i.e.* integers in the quadrant

$$\mathbb{X} = \{\mathbf{x} = (x_1, x_2) \in \mathbb{Z}^2 : x_1 \geq 0, x_2 \geq 0\} = (\mathbb{Z}^+)^2,$$

that will be called *the space* in the sequel. We define

$$\partial_1\mathbb{X} = \{\mathbf{x} = (x_1, 0); x_1 \in \mathbb{Z}^+\} \subseteq \mathbb{X}$$

and

$$\partial_2\mathbb{X} = \{\mathbf{x} = (0, x_2); x_2 \in \mathbb{Z}^+\} \subseteq \mathbb{X}.$$

The set $\partial\mathbb{X} = \partial_1\mathbb{X} \cup \partial_2\mathbb{X}$ will be called *the boundaries* and $\overset{\circ}{\mathbb{X}} = \mathbb{X} \setminus \partial\mathbb{X}$ *the interior* of the space. By abuse of notation, we use the same symbol, $\mathbb{X}$, to denote the quadrant $(\mathbb{R}^+)^2$ when we refer to functions defined on the space because we are interested eventually only on the values these functions take on the points of the integer lattice. Thus, $f \in C^2(\mathbb{X}, \mathbb{R})$, for instance, will mean a twice continuously differentiable real function f on $(\mathbb{R}^+)^2$.

For some strictly positive integer $L > 0$ and every $\mathbf{x} \in \mathbb{X}$, denote by

$$\mathbb{A}_{\mathbf{x}} = \begin{cases} \{0, \cdots, L\} & \text{if} \quad \mathbf{x} \in \partial\mathbb{X} \\ \{0\} & \text{if} \quad \mathbf{x} \in \overset{\circ}{\mathbb{X}}. \end{cases}$$

A state of the system will be parametrised by $u = (\mathbf{x}; \alpha) = (x_1, x_2; \alpha)$ where $\mathbf{x} = (x_1, x_2) \in \mathbb{X}$ and, for this particular $\mathbf{x}$, the coordinate $\alpha \in \mathbb{A}_{\mathbf{x}}$. The set of all these states,

$$\mathbb{U} = \{u = (\mathbf{x}; \alpha) : \mathbf{x} \in \mathbb{X}; \alpha \in \mathbb{A}_{\mathbf{x}}\},$$

will be called *the configuration space*; that is to say $\mathbb{U}$ is a trivial fibre bundle with base $\mathbb{X}$ and fibre $\mathbb{A}_{\mathbf{x}}$ over each $\mathbf{x} \in \mathbb{X}$. A more intuitive way of seeing this construction is to say that on the boundaries $\partial\mathbb{X}$ there is some internal degrees of freedom which are absent from the interior, $\mathring{\mathbb{X}}$, of the space; we may think of these degrees of freedom as colours.

Definition 1.1 For $u = (x_1, x_2; \alpha) \in \mathbb{U}$, we call *canonical projections* $\mathbf{X}$ and A, the mappings

$$\mathbf{X} : \mathbb{U} \to \mathbb{X} \text{ and } A : \mathbb{U} \to \mathbb{A}$$

defined by

$$u \mapsto \mathbf{X}(u) \equiv (x_1, x_2) \in \mathbb{X} \text{ and } u \mapsto A(u) \equiv \alpha \in \mathbb{A}.$$

We denote as usual $X_1(u) = x_1$ and $X_2(u) = x_2$ the horizontal and vertical components of the spatial projection.

1.3 Definition and properties of the Markov chain

We consider a discrete-time, time-homogeneous, irreducible, aperiodic, $\mathbb{U}$-valued Markov chain, $\xi = (\xi_n)_{n\in\mathbb{N}}$, defined by the stochastic transition matrix

$$\mathbf{P} = (P_{u,u'})_{u,u'\in\mathbb{U}},$$

with

$$P_{u,u'} \equiv P_{(x_1,x_2;\alpha),(x_1',x_2';\alpha')} = \mathbb{P}(\xi_{n+1} = (x_1', x_2'; \alpha') | \xi_n = (x_1, x_2; \alpha)).$$

Remark: It is evident that equivalently to our fibre bundle description, we can extend the configuration space into $\mathbb{X} \times \mathbb{A}$, where $\mathbb{A} = \{0, \cdots, L\}$, by requiring

$$P_{(x_1,x_2;\alpha),(x_1',x_2';\alpha')} = 0 \text{ if } (x_1', x_2') \in \mathring{\mathbb{X}} \text{ and } \alpha' \neq 0.$$

The transition probabilities of the Markov chain have various properties.

Definition 1.2 [Lower boundedness of jumps (LBJ)] We say the chain has *the LBJ property* if

1. For jumps starting in the interior of the space, $\mathbf{x} \in \overset{\circ}{\mathbb{X}}$, the transition probability vanishes for westbound or southbound jumps of length more than 1 and for entering the boundary through colours different from 0, *i.e.*

$$P_{(x_1,x_2;\alpha),(x_1',x_2';\alpha')} = \begin{cases} 0 & \text{if} \quad (x_1 \geq 1, x_2 \geq 1) \;\text{ and }\; x_1' - x_1 < -1 \\ 0 & \text{if} \quad (x_1 \geq 1, x_2 \geq 1) \;\text{ and }\; x_2' - x_2 < -1 \\ 0 & \text{if} \quad (x_1 = 1, x_1' = 0, x_2 \neq 0, \alpha' \neq 0) \;\text{ or }\; (x_2 = 1, x_2' = 0, x_1 \neq 0, \alpha' \neq 0). \end{cases}$$

2. For jumps starting on the boundaries, westbound (horizontal) or southbound (vertical) jumps are uniformly bounded, *i.e.* there exists $K_{\text{LBJ}} > 0$ such that

$$P_{(x_1,0;\alpha),(x_1',x_2';\alpha')} = 0 \;\text{ if }\; x_1' - x_1 < -K_{\text{LBJ}}, \text{for } \;\mathbf{x} \in \partial_1\mathbb{X}, x_1 > K_{\text{LBJ}},$$

   and

$$P_{(0,x_2;\alpha),(x_1',x_2';\alpha')} = 0 \;\text{ if }\; x_2' - x_2 < -K_{\text{LBJ}}, \text{for } \;\mathbf{x} \in \partial_2\mathbb{X}, x_2 > K_{\text{LBJ}}.$$

Remark: Notice that the condition of westbound or southbound jumps bounded by 1 in the internal space is important and cannot be relaxed in our approach.

Definition 1.3 [Partial spatial homogeneity (PSH)] The Markov chain has *the PSH property* if for $\alpha, \alpha' \in \mathbb{A}$, there exist functions $p_{\alpha\alpha'} : \mathbb{Z} \to [0,1]$, $q_{\alpha\alpha'} : \mathbb{Z} \to [0,1]$, $r^{(1)}_{\alpha\alpha'} : \mathbb{Z}^+ \times \mathbb{Z} \to [0,1]$, $r^{(2)}_{\alpha\alpha'} : \mathbb{Z} \times \mathbb{Z}^+ \to [0,1]$, and $r_{0,0} : \mathbb{Z} \times \mathbb{Z} \to [0,1]$ such that the stochastic matrix $P_{u,u'}$ can be decomposed into

$$P_{(x_1,x_2;\alpha),(x_1',x_2';\alpha')} = \begin{cases} p_{\alpha\alpha'}(x_1' - x_1) & \text{if } \;\mathbf{x}, \mathbf{x}' \in \partial_1\mathbb{X}, x_1 > K_{\text{LBJ}} \\ q_{\alpha\alpha'}(x_2' - x_2) & \text{if } \;\mathbf{x}, \mathbf{x}' \in \partial_2\mathbb{X}, x_2 > K_{\text{LBJ}} \\ r^{(1)}_{\alpha 0}(x_1' - x_1, x_2' - x_2) & \text{if } \;\mathbf{x} \in \partial_1\mathbb{X}, \mathbf{x}' \in \overset{\circ}{\mathbb{X}}, \alpha' = 0 \\ r^{(2)}_{\alpha 0}(x_1' - x_1, x_2' - x_2) & \text{if } \;\mathbf{x} \in \partial_2\mathbb{X}, \mathbf{x}' \in \overset{\circ}{\mathbb{X}}, \alpha' = 0 \\ r_{00}(x_1' - x_1, x_2' - x_2) & \text{if } \;\mathbf{x} \in \overset{\circ}{\mathbb{X}} \\ 0 & \text{otherwise.} \end{cases}$$

Remark: The existence of function $r_{00}(\cdot,\cdot)$ means complete spatial homogeneity in the interior of the quadrant, the existence of functions $p_{\alpha\alpha'}(\cdot)$ and $q_{\alpha\alpha'}(\cdot)$ means partial horizontal (resp. vertical) homogeneity on the boundary $\partial_1\mathbb{X}$ (resp. $\partial_2\mathbb{X}$) far from the origin (*i.e.* for $\|x\| > K_{\text{LBJ}}$.)

Definition 1.4 [Moment boundedness (γ-MB)] We say the Markov chain has *the γ-MB property* if there exist a $\gamma > 2$ and a constant $K_{\text{MB}} = K_{\text{MB}}(\gamma) > 0$ such that, for every $u \in \mathbb{U}$,

$$\mathbb{E}(\|\mathbf{X}(\xi_{n+1}) - \mathbf{X}(\xi_n)\|^\gamma | \xi_n = u) < K_{\text{MB}},$$

where $\|\cdot\|$ denotes the Euclidean norm in $\mathbb{R}^2$.

Definition 1.5 [Zero drift property (ZD)] We say that the chain has *zero drift* in the interior of the space if, for $u = (x_1, x_2; 0)$ and $(x_1, x_2) \in \overset{\circ}{\mathbb{X}}$,

$$\mathbb{E}(X_1(\xi_{n+1}) - X_1(\xi_n))|\xi_n = u) = \mathbb{E}(X_2(\xi_{n+1}) - X_2(\xi_n))|\xi_n = u) = 0.$$

We define finally conditional second moments

$$\begin{aligned}\lambda_1 &= \mathbb{E}(X_1(\xi_{n+1}) - X_1(\xi_n))^2|\xi_n = u) \\ &= \sum_{(x'_1,x'_2)\in\mathbb{X}} r_{00}(x'_1 - x_1, x'_2 - x_2)(x'_1 - x_1)^2 \geq 0;\end{aligned}$$

$$\begin{aligned}\lambda_2 &= \mathbb{E}(X_2(\xi_{n+1}) - X_2(\xi_n))^2|\xi_n = u) \\ &= \sum_{(x'_1,x'_2)\in\mathbb{X}} r_{00}(x'_1 - x_1, x'_2 - x_2)(x'_2 - x_2)^2 \geq 0;\end{aligned}$$

and

$$\begin{aligned}\kappa &= \mathbb{E}(X_1(\xi_{n+1}) - X_1(\xi_n))(X_2(\xi_{n+1}) - X_2(\xi_n))|\xi_n = u) \\ &= \sum_{(x'_1,x'_2)\in\mathbb{X}} r_{00}(x'_1 - x_1, x'_2 - x_2)(x'_1 - x_1)(x'_2 - x_2) \in \mathbb{R}.\end{aligned}$$

Definition 1.6 [Positive definiteness of the covariance (PDC)] We say the Markov chain has *the PDC property* if the matrix $\begin{pmatrix} \lambda_1 & \kappa \\ \kappa & \lambda_2 \end{pmatrix}$ is positive definite, *i.e.* $\lambda_1\lambda_2 - \kappa^2 \geq 0$.

1.4 Drifts and linear transformation of the lattice

In order to state our main results, two additional notions are needed. Consider $\mathbb{U}^{(1)}$ and $\mathbb{U}^{(2)}$ the two configuration spaces whose spatial components are respectively the upper half plane $\mathbb{X}^{(1)} = \mathbb{Z} \times \mathbb{Z}^+$ and the right half plane $\mathbb{X}^{(2)} = \mathbb{Z}^+ \times \mathbb{Z}$. The boundaries of these spaces are respectively $\partial\mathbb{X}^{(1)} = \partial_1\mathbb{X}^{(1)} = \mathbb{Z}$ and $\partial\mathbb{X}^{(2)} = \partial_2\mathbb{X}^{(2)} = \mathbb{Z}$. Consider now two modified Markov chains $(\xi_n^{(1)})_{n\in\mathbb{N}}$ and $(\xi_n^{(2)})_{n\in\mathbb{N}}$ evolving respectively in $\mathbb{U}^{(1)}$ and $\mathbb{U}^{(2)}$ and having complete, rather than just partial, homogeneity in the horizontal, respectively the vertical, directions. Therefore, the transtion matrix for $(\xi_n^{(1)})_{n\in\mathbb{N}}$ is

$$P^{(1)}_{(x_1,x_2;\alpha),(x'_1,x'_2;\alpha')} = \begin{cases} p_{\alpha\alpha'}(x'_1 - x_1) & \text{if } \mathbf{x}, \mathbf{x}' \in \partial_1\mathbb{X}^{(1)} \\ r^{(1)}_{\alpha 0}(x'_1 - x_1, x'_2 - x_2) & \text{if } \mathbf{x} \in \partial\mathbb{X}^{(1)}, \mathbf{x}' \in \overset{\circ}{\mathbb{X}}{}^{(1)} \text{ and } \alpha' = 0 \\ r_{00}(x'_1 - x_1, x'_2 - x_2) & \text{if } \mathbf{x} \in \overset{\circ}{\mathbb{X}}{}^{(1)} \text{ and } \alpha = \alpha' = 0 \\ 0 & \text{otherwise,} \end{cases}$$

and similarly for the $(\xi_n^{(2)})_{n\in\mathbb{N}}$ chain

$$P^{(2)}_{(x_1,x_2;\alpha),(x_1',x_2';\alpha')} = \begin{cases} q_{\alpha\alpha'}(x_1'-x_1) & \text{if } \mathbf{x},\mathbf{x}' \in \partial_2\mathbb{X}^{(2)} \\ r^{(2)}_{\alpha 0}(x_1'-x_1, x_2'-x_2) & \text{if } \mathbf{x}\in\partial\mathbb{X}^{(2)}, \mathbf{x}' \in \overset{\circ}{\mathbb{X}}{}^{(2)} \quad \text{and} \quad \alpha'=0 \\ r_{00}(x_1'-x_1, x_2'-x_2) & \text{if } \mathbf{x}\in \overset{\circ}{\mathbb{X}}{}^{(2)} \quad \text{and} \quad \alpha=\alpha'=0 \\ 0 & \text{otherwise.} \end{cases}$$

Definition 1.7 Use the symbol $\sharp$ to denote void or (1) or (2). For points $\mathbf{x} \in \partial\mathbb{X}^\sharp$ on the boundaries of the spaces $\mathbb{X}^\sharp$, we define the *effective jump matrices* $\mathbf{Q}^\sharp = (Q^\sharp_{\alpha,\alpha'})_{\alpha,\alpha'\in\mathbb{A}}$ between internal states by

$$Q^\sharp_{\alpha,\alpha'} = \sum_{\mathbf{x}'\in\mathbb{X}} P^\sharp_{(\mathbf{x};\alpha),(\mathbf{x}';\alpha')}.$$

Obviously, $\mathbf{Q}^\sharp$ are stochastic matrices over the *finite* internal space $\mathbb{A}$, inheriting irreducibility and aperiodicity properties of the initial transition matrix $\mathbf{P}^\sharp$. Hence they admit unique invariant probabilities $\pi^\sharp$ over $\mathbb{A}$, all of them being ergodic.

Definition 1.8 For the previously defined chains $(\xi_n)_{n\in\mathbb{N}}$, $(\xi_n^{(1)})_{n\in\mathbb{N}}$, and $(\xi_n^{(2)})_{n\in\mathbb{N}}$, collectively denoted $(\xi_n^\sharp)_{n\in\mathbb{N}}$, and every $u \in \mathbb{U}^\sharp$, define the *drift* of the chains by

$$\mathbf{m}^\sharp(u) = \sum_{u'\in\mathbb{U}^\sharp} P^\sharp_{u,u'}\left(\mathbf{X}(u') - \mathbf{X}(u)\right).$$

Remark: It is worth noticing that for $u = (\mathbf{x},\alpha)$, with $\mathbf{x} \in \overset{\circ}{\mathbb{X}}{}^\sharp$, the drift $\mathbf{m}^\sharp(u) = 0$ vanishes due to the zero drift assumptions valid in the interior of the space. Hence the only interesting values of the drifts are on the boundaries. For $u = (\mathbf{x},\alpha)$, with $\mathbf{x} \in \partial\mathbb{X}^{(2)}$, due to the vertical homogeneity of the chain $\xi^{(2)}$, the drift depends only on α, i.e.

$$\mathbf{m}^{(2)}(0, x_2; \alpha) \equiv \mathbf{m}^{(2)}(\alpha).$$

Similarly, $\mathbf{m}^{(1)}(x_1, 0; \alpha) \equiv \mathbf{m}^{(1)}(\alpha)$. Therefore, the drifts on each boundary are functions only of the colour.

Definition 1.9 Let $\pi^\flat$ be the stationary probability of $\mathbf{Q}^\flat$, for $\flat = 1, 2$. Define the *average drift on the boundaries* by

$$\overline{\mathbf{m}}^\flat = \sum_{\alpha\in\mathbb{A}} \pi^\flat(\alpha)\mathbf{m}^\flat(\alpha).$$

The vectors $\overline{\mathbf{m}}^\flat$ play a crucial *rôle* in determining the asymptotic behaviour of the chain in the quadrant. For $f \in C^2(\mathbb{X};\mathbb{R})$, we define the *time evolution generator* L as the differential operator given by

$$L = \left(\frac{\partial}{\partial x_1} \ \frac{\partial}{\partial x_2}\right)\begin{pmatrix} \lambda_1 & \kappa \\ \kappa & \lambda_2 \end{pmatrix}\begin{pmatrix} \frac{\partial}{\partial x_1} \\ \frac{\partial}{\partial x_2} \end{pmatrix}.$$

For quadratic functions $f : \mathbb{X} \to \mathbb{R}$, we have obviously

$$\mathbb{E}(f(\mathbf{X}(\xi_{n+1})) - f(\mathbf{X}(\xi_n)))|\mathbf{X}(\xi_n) = \mathbf{x}) = (Lf)(\mathbf{x}).$$

We shall see later that for a suitable class of functions, this is valid up to a remainder term which is negligible for large $\|\mathbf{x}\|$ for all the functions of the class. We are seeking for a linear transformation $\Phi : (\mathbb{R}^+)^2 \to (\mathbb{R}^+)^2$ such that $L(f \circ \Phi) = \nabla f \circ \Phi$ for every function $f \in C^2(\mathbb{X})$. Now, Φ is a 2×2 matrix contravariantly transforming $\mathbf{x} \in \mathbb{X}$ vectors into $\mathbf{y} \in \mathbb{Y}$ vectors, *i.e.* $\begin{pmatrix} y_1 \\ y_2 \end{pmatrix} = \Phi \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$. Consequently, the derivatives are transformed covariantly $(\frac{\partial}{\partial y_1} \ \frac{\partial}{\partial y_2})\Phi = (\frac{\partial}{\partial x_1} \ \frac{\partial}{\partial x_2})$. Hence the operator equation $L(f \circ \Phi) = \nabla f \circ \Phi$ is equivalent to the matrix equation

$$\Phi^T \Phi = \begin{pmatrix} \lambda_1 & \kappa \\ \kappa & \lambda_2 \end{pmatrix}^{-1} = \frac{1}{\lambda_1 \lambda_2 - \kappa^2} \begin{pmatrix} \lambda_2 & -\kappa \\ -\kappa & \lambda_1 \end{pmatrix}.$$

This matrix equation has an one dimensional manifold of solutions (4 unknown matrix elements of Φ, 3 equations). A possible choice is:

$$\Phi = \frac{1}{\sqrt{\lambda_1 \lambda_2 - \kappa^2}} \begin{pmatrix} \sqrt{\lambda_2} & -\frac{\kappa}{\sqrt{\lambda_2}} \\ 0 & \sqrt{\lambda_1 - \frac{\kappa^2}{\lambda_2}} \end{pmatrix};$$

but this choice is made only to depict the example of figure 1 below. The relevant quantities ψ and $\psi^\flat$ are defined intrinsically in terms of invariant quantities of the matrix Φ such as its determinant, etc.

Of course, the state space before transformation was the lattice

$$\mathbb{X} = \{\mathbf{x} \in (\mathbb{R}^+)^2 : \mathbf{x} = x_1 \mathbf{e}_1 + x_2 \mathbf{e}_2, \ \text{ with } \ (x_1, x_2) \in (\mathbb{Z}^+)^2\} \simeq (\mathbb{Z}^+)^2.$$

After transformation, it remains a lattice, namely

$$\mathbb{Y} = \{\mathbf{y} \in (\mathbb{R}^+)^2 : \mathbf{y} = x_1 \mathbf{f}_1 + x_2 \mathbf{f}_2, \ \text{ with } \ (x_1, x_2) \in (\mathbb{Z}^+)^2\} \simeq (\mathbb{Z}^+)^2,$$

where $\mathbf{f}_1 = \Phi \mathbf{e}_1$ and $\mathbf{f}_2 = \Phi \mathbf{e}_2$. We shall abusively use the same symbol $\mathbb{Y}$ to denote the whole wedge when refering to functions defined on that space, as we did for $\mathbb{X}$.

Obviously this linear transformation is not orthogonal; the fundamental cell of the lattice is not the unit square but the parallelogram defined by the vectors $\mathbf{f}_1$ and $\mathbf{f}_2$. Hence the quadrant $\mathbb{X}$ with a right angle at its summit, will be squeezed after transformation into a wedge $\mathbb{Y}$ with an angle $\psi = \arccos(-\frac{\kappa}{\sqrt{\lambda_1 \lambda_2}}) \in [0, \pi]$ at its summit. Moreover, the tranformation Φ has determinant $\frac{1}{\sqrt{\lambda_1 \lambda_2 - \kappa^2}}$ so that the volume of the unit cell is squeezed by that factor.

The drifts $\overline{\mathbf{m}}^\flat$ form angles $\phi^\flat$ with the normals to the boundaries $\partial \mathbb{X}^\flat$, for $\flat = 1, 2$, *i.e.*

$$\overline{\mathbf{m}}^1 = \|\overline{\mathbf{m}}^1\| \begin{pmatrix} -\sin \phi^{(1)} \\ \cos \phi^{(1)} \end{pmatrix}, \quad [\overline{\mathbf{m}}^2 = \|\overline{\mathbf{m}}^2\| \begin{pmatrix} \cos \phi^{(2)} \\ -\sin \phi^{(2)} \end{pmatrix}.$$

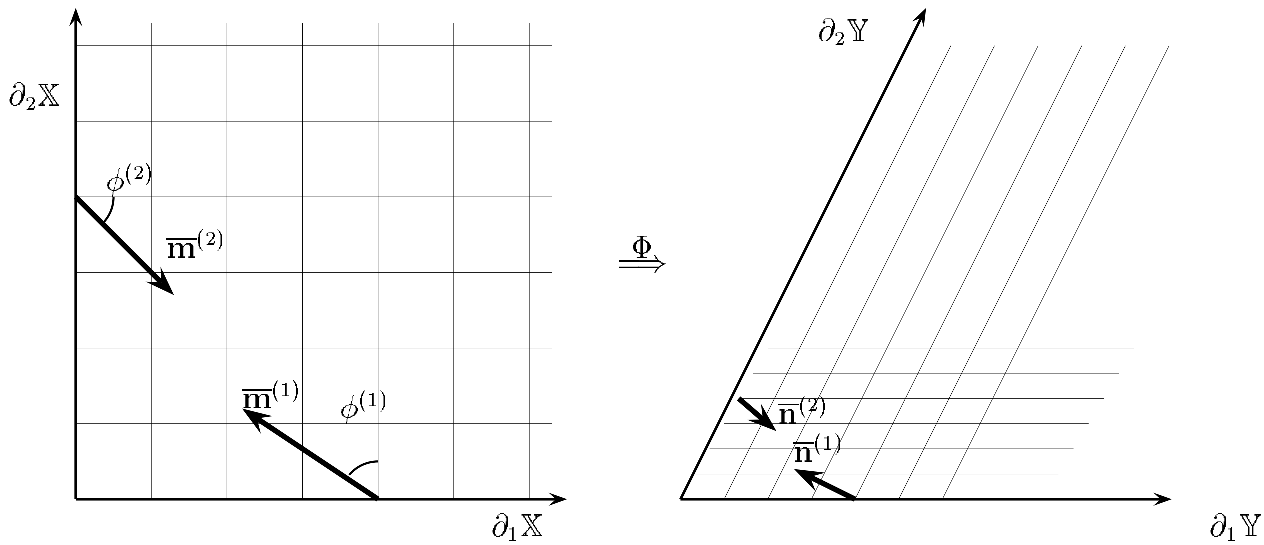

Figure 1: The transformation of the geometry by Φ, calculated for the example values $\lambda_1 = 4$, $\lambda_2 = 9$, and $\kappa = -3$. In that case, $\psi = \pi/3$ and the $\mathbb{Y}$-lattice points read $\mathbf{y} = m\mathbf{f}_1 + n\mathbf{f}_2$, for $(m, n) \in (\mathbb{Z}^+)^2$, with $\mathbf{f}_1 = (1/\sqrt{3}, 0)$ and $\mathbf{f}_2 = (2/3\sqrt{3}, 1/6)$. The angles formed by the $\overline{\mathbf{m}}^{(1)}$ and $\overline{\mathbf{m}}^{(2)}$ with the normals to the boundaries $\partial_1\mathbb{X}$ and $\partial_2\mathbb{X}$ respectively are depicted. In the example shown both these angles are positive. In order not to excessively burden the figure, the angles $\psi^{(1)}$ and $\psi^{(2)}$ are not shown on this figure. They are defined again as the angles formed by the vetors $\overline{\mathbf{n}}^{(1)}$ and $\overline{\mathbf{n}}^{(2)}$ with the normals to the boundaries $\partial_1\mathbb{Y}$ and $\partial_2\mathbb{Y}$. They are positive if the vectors point towards the vertex of the wedge and negative otherwise.

After application of the transformation Φ. the transformed vectors $\overline{\mathbf{n}}^\flat = \Phi\overline{\mathbf{m}}^\flat$ form angles $\psi^\flat$ with the normals to the boundaries $\partial\curvearrowright^\flat$ given by

$$\begin{aligned}
\psi^{(1)} &= \arccos \frac{\sin\psi}{\sqrt{1 + \frac{\lambda_2}{\lambda_1}\tan^2\phi^{(1)}}};\\
\psi^{(2)} &= \arccos \frac{\sin\psi}{\sqrt{1 + \frac{\lambda_1}{\lambda_2}\tan^2\phi^{(2)}}}.
\end{aligned}$$

In figure 1 we display the action of Φ on the wedges, their boundaries, and to the lattice points. We denote abusively $\mathbb{V} = \Phi(\mathbb{U})$ the configuration space for the Markov chain with Φ-transformed spatial component.

1.5 Statement of the main results

We can now state our main theorems. For all these results we assume that the chain $(\xi_n)_{n\in\mathbb{N}}$ verifies the following assumptions:

A1 The chain is irreducible.

A2 The chain is aperiodic.

A3 The chain verifies the conditions LBJ, γ-MB for some $\gamma > 2$, PSH, and PDC.

A4 The chain has zero drift in the interior of the wedge.

A5 The chain starts deterministically at $\xi_0 = (\mathbf{x}, 0)$ with $\|\mathbf{x}\| > K$ for some $K > 0$.

Denote
$$\tau_K = \inf\{n \geq 0 : \|\mathbf{X}(\xi_n)\| \leq K\},$$
and
$$\chi = \frac{\psi^{(1)} + \psi^{(2)}}{\psi},$$
where $\psi^{(1)}$ and $\psi^{(2)}$ are the angles of the average drifts with the normals to the boundaries after the linear transformation Φ. Although the results concern primarily the chain $(\xi_n)_{n\in\mathbb{N}}$, they are more easily formulated in terms of the geometric characteristics in the setting transformed by Φ.

Theorem 1.10 *Let $(\xi_n)_{n\in\mathbb{N}}$ be a Markov chain verifying A1–A5. If $\chi < 0$ then the chain is transient. If $\chi > 0$ then the chain is recurrent.*

Remark: The case $\chi = 0$ is also interesting but it is quite tricky as it was shown for a related model studied in [3]. This case necessitates considering Lyapunov fucntions in $\log\log$ form and will be not studied here.

Theorem 1.11 *Let $(\xi_n)_{n\in\mathbb{N}}$ be a Markov chain verifying A1–A5. Let $p_0 \in]0, \min(\chi,\gamma)/2[$. Then for every $p < p_0$, there exists a constant $C > 0$ such that*
$$\mathbb{E}[\tau_K^p] \leq C\|\mathbf{x}\|^{2p_0}.$$
If moreover, $\min(\chi,\gamma) > 2$ *then*
$$\mathbb{E}[\tau_K^p] \leq C\|\mathbf{x}\|^{2p}.$$

Theorem 1.12 *Let $(\xi_n)_{n\in\mathbb{N}}$ be a Markov chain verifying A1–A5. If $\chi < \gamma$, then for every $p > \frac{\chi}{2}$, $\mathbb{E}\tau_K^p = \infty$.*

Remark: We study here phenomena that are profoundly due to some martingale properties and should persist even when $\gamma = \infty$ (existence of arbitrary moments of the jumps). However, our methods do not allow to prove divergence of moments of passage times if $\chi \geq \gamma$.

The proofs are quite technical and go through various steps as explained in subsequent Sections. We describe briefly the main steps of the proof here. First, we show in Section 2, that the average drifts on the boundaries have a very intuitive interpretation in terms of induced drifts for embedded chains, *i.e.* chains observed only on the moments when they evolve in space but for which the evolution in internal space is "forgotten".

After these preliminary steps, the problem is finally led to a form that completely fits the case studied in [2]. In Section 3, we obtain some preliminary results concerning smooth functions f in the wedge; we show that for smooth functions it is possible to reduce the study of the asymptotic behaviour of the conditional expectation of the increment $\mathbb{E}(f(\xi_{n+1}) - f(\xi_n)|\xi_n = (\mathbf{x};\alpha))$, for $\mathbf{x} \in \overset{\circ}{\mathbb{X}}$, into the study of its linear approximation, the remainder being negligible for large $\|\mathbf{x}\|$.

In the Section 4 finally, we show explicitly that it is possible to construct a Lyapunov function all over the wedge and its boundaries transforming the chain into a supermartingale. Using then the results of [2], we can conclude. A closely related model of chains evolving in two adjacent wedges is introduced and studied in Section 5.

2 Embedding

The scope of this Section is to show that the model with internal degrees of freedom can be regarded as a model without internal degrees provided the time spent in the internal space is "erased".

2.1 Sequence of entrance-exit times and embedded chains

For every possible value of the symbol $\sharp$, *i.e.* void or (1) or (2), we define the *embedded chain*, $(\zeta_n^\sharp)_{n\in\mathbb{N}}$, for the original chain $(\xi_n^\sharp)_{n\in\mathbb{N}}$ as the random process coinciding with $\xi_n^\sharp$ when the original chain has internal degree $\alpha = 0$ and whose instants of evolution are erased when $\xi^\sharp$ evolves in the internal space $\mathbb{A} \setminus \{0\}$. More specifically, for every value of $\sharp$ choose $\xi_0^\sharp$ such that $\mathbf{X}(\xi_0^\sharp) \in \overset{\circ}{\mathbb{X}}{}^\sharp$ and define the sequence of entrance-exit random times (the dependence on $\sharp$ is not indicated on the random times) by

$$
\begin{aligned}
\tau_0 &= 0 \\
\sigma_1 &= \inf\{n \geq \tau_0 : \mathbf{X}(\xi_n^\sharp) \in \partial\mathbb{X}^\sharp\} \\
\tau_1 &= \inf\{n > \sigma_1 : A(\xi_n^\sharp) = 0\} \\
&\vdots \\
\sigma_k &= \inf\{n \geq \tau_{k-1} : \mathbf{X}(\xi_n^\sharp) \in \partial\mathbb{X}^\sharp\} \\
\tau_k &= \inf\{n > \sigma_k : A(\xi_n^\sharp) = 0\} \\
&\vdots
\end{aligned}
$$

Notice that since westbound or southbound jumps are only of magnitude -1, the above sequence is well defined. Erase now time instants between $]\sigma_i, \tau_i]$, for $i = 1, 2, \ldots$, and relabel time so that at moment τ_k, the new time reads $(\sigma_1 - \tau_0) + (\sigma_2 - \tau_1) + \ldots + (\sigma_k - \tau_{k-1})$. Define now a process $\zeta^\sharp = (\zeta^\sharp_n)_{n \in \mathbb{N}}$ with configuration space $\mathbb{X}$ by

$$\begin{aligned} \zeta^\sharp_{\tau_0 + i} &= \zeta^\sharp_i = \xi^\sharp_i \quad \text{for} \quad i = 0, \ldots, \sigma_1, \\ \zeta^\sharp_{(\sigma_1 - \tau_0) + (\sigma_2 - \tau_1) + \ldots + (\sigma_k - \tau_{k-1}) + 1 + i} &= \xi^\sharp_{\tau_k + i}, \quad \text{for} \ \ k = 1, 2, 3, \ldots \ \text{ and } \ i = 0, \ldots, (\sigma_{k+1} - \tau_k), \end{aligned}$$

and transition matrix

$$\tilde{P}^\sharp_{\mathbf{x},\mathbf{x}'} = \mathbb{P}(\zeta^\sharp_{n+1} = \mathbf{x}' | \zeta^\sharp_n = \mathbf{x}), \quad \text{for} \ \ \mathbf{x}, \mathbf{x}' \in \mathbb{X}.$$

2.2 Drifts

Definition 2.1 We define the *induced drift* for the embedded chain $(\zeta^\sharp_n)_{n \in \mathbb{N}}$ by

$$\tilde{\mathbf{m}}^\sharp(\mathbf{x}) = \sum_{\mathbf{x}' \in \mathbb{X}^\sharp} \tilde{P}^\sharp{}_{\mathbf{x},\mathbf{x}'}(\mathbf{x}' - \mathbf{x}), \quad \text{for} \ \ \mathbf{x} \in \mathbb{X}^\sharp,$$

where the symbol $\sharp$ stands for void or (1) or (2).

Due to the fact that for $\mathbf{x} \in \overset{\circ}{\mathbb{X}}$, the southbound or westbound jumps are bounded by -1, we have equality between drifts and induced drifts, *i.e.*

$$\tilde{\mathbf{m}}^\sharp(\mathbf{x}) = \mathbf{m}^\sharp(\mathbf{x}; \alpha = 0) = 0 \ \text{ for } \ \mathbf{x} \in \overset{\circ}{\mathbb{X}}.$$

Hence the only points where the determination of $\tilde{\mathbf{m}}^\sharp$ is not trivial are boundary points. Now homogeneity properties of the chains $\xi^\sharp$ are inherited by embedded chains $\zeta^\sharp$. Hence, for $\mathbf{x} \in \partial \mathbb{X}^{(2)}$ it happens that $\tilde{\mathbf{m}}^{(2)}(\mathbf{x}) \equiv \tilde{\mathbf{m}}^{(2)}(x_1, x_2) = \tilde{\mathbf{m}}^{(2)}(0, x_2)$ is in fact independent of x_2 due to the vertical homogeneity of the chain $\zeta^{(2)}$; therefore,

$$\tilde{\mathbf{m}}^{(2)}(\mathbf{x}) = \tilde{\mathbf{m}}^{(2)}, \forall \mathbf{x} \in \partial \mathbb{X}^{(2)},$$

and similarly

$$\tilde{\mathbf{m}}^{(1)}(\mathbf{x}) = \tilde{\mathbf{m}}^{(1)}, \forall \mathbf{x} \in \partial \mathbb{X}^{(1)}.$$

We need some additional definitions.

Definition 2.2 Let $u = (\mathbf{x}; \alpha)$ be a state of the boundary. For $\sharp$ taking one of the values void, (1), or (2), define *the exit time from the boundary*, the random time (its dependence on $\sharp$ is omitted)

$$T_u = \inf\{n \geq 1 : A(\xi^\sharp_n) = 0; \xi^\sharp_0 = u\}.$$

Now it is intuitevely clear that for $\mathbf{x}$ with $\|\mathbf{x}\|$ very large, the chains $\xi^{(1)}$ or $\xi^{(2)}$ differ very little from the initial chain ξ. This property is expected to be inherited from the embedded chains.

Definition 2.3 For $u = (\mathbf{x};\alpha)$ with $\mathbf{x} \in \partial\mathbb{X}^\flat$ a state of the boundary and $S \subseteq \mathbb{A}$ an arbitrary subset of $\{0,\cdots,L\}$, define the *restricted drift* by

$$\mathbf{m}^\sharp(u;S) = \sum_{\mathbf{x}'}\sum_{\alpha'\in S} P^\sharp_{(\mathbf{x};\alpha),(\mathbf{x}';\alpha')}(\mathbf{x}'-\mathbf{x}).$$

Obviously, $\mathbf{m}^\sharp(u;S) + \mathbf{m}^\sharp(u;S^c) = \mathbf{m}^\sharp(u)$.

Proposition 2.4 *Let $\mathbf{Q}^\flat = (Q^\flat_{\alpha,\alpha'})$ be the stochastic matrix of effective jumps within the internal space given in definition 1.7, $\pi^\flat$ be its unique stationary probability, and $\tilde{\mathbf{m}}^\flat$ the corresponding induced drifts. Then the induced drift and the average drift on the boundary coincide, i.e.*

$$\pi^\flat(0)\tilde{\mathbf{m}}^\flat = \overline{\mathbf{m}}^\flat \equiv \sum_{\alpha\in\mathbb{A}} \pi^\flat(\alpha)\mathbf{m}^\flat(\alpha) \quad \textit{for} \quad \flat = 1,2.$$

Proof: For an arbitrary $u = (\mathbf{x};\alpha)$ with $\mathbf{x} \in \partial\mathbb{X}^\flat$, we have

$$\tilde{\mathbf{m}}^\flat = \sum_{\mathbf{x}'\in\mathbb{X}} \tilde{P}^\flat_{\mathbf{x},\mathbf{x}'}(\mathbf{x}'-\mathbf{x}) = \sum_{\mathbf{x}'\in\mathbb{X}}(\mathbf{x}'-\mathbf{x})\mathbb{P}_u\left(\xi^\flat_{T_u} = (x_1',x_2';0)\right).$$

Conditioning on the first move, decompose

$$\begin{aligned}\mathbb{P}_u\left(\xi^\flat_{T_u} = (x_1',x_2';0)\right) &= \mathbb{P}_u\left(\xi^\flat_{T_u} = (x_1',x_2';0); T_u = 1\right) + \mathbb{P}_u\left(\xi^\flat_{T_u} = (x_1',x_2';0); T_u > 1\right)\\ &= P^\flat_{(0,x_2;\alpha),(x_1',x_2';0)} + \sum_{\mathbf{x}''\in\partial\mathbb{X}^\flat,\alpha''\neq 0} P^\flat_{(0,x_2;0),(0,x_2'';\alpha'')}\mathbb{P}_{u''}\left(\xi^\flat_{T_{u''}} = (x_1',x_2';0)\right).\end{aligned}$$

Now,

$$\begin{aligned}\tilde{\mathbf{m}}^\flat &= \sum_{\mathbf{x}'\in\mathbb{X}}(\mathbf{x}'-\mathbf{x})P_{(\mathbf{x};\alpha),(\mathbf{x}';0)}\\ &+ \sum_{\mathbf{x}''\in\partial\mathbb{X}^\flat,\alpha''\neq 0}\sum_{\mathbf{x}'\in\mathbb{X}^\flat}[(\mathbf{x}'-\mathbf{x}'')+(\mathbf{x}''-\mathbf{x})]P^\flat_{u,u''}\mathbb{P}_{u''}\left(\xi^\flat_{T_{u''}} = (\mathbf{x}';0)\right)\\ &= \mathbf{m}^\flat(\alpha,\{A=0\})\\ &\quad + \sum_{\mathbf{x}''\in\partial\mathbb{X}^\flat,\alpha''\neq 0}(\mathbf{x}''-\mathbf{x})P^\flat_{u,u''}\sum_{\mathbf{x}'\in\mathbb{X}^\flat}\mathbb{P}_{u''}\left(\xi^\flat_{T_{u''}} = (\mathbf{x}';0)\right)\\ &\quad + \sum_{\mathbf{x}''\in\partial\mathbb{X}^\flat,\alpha''\neq 0}P^\flat_{u,u''}\sum_{\mathbf{x}'\in\mathbb{X}^\flat}\mathbb{P}_{u''}\left(\xi^\flat_{T_{u''}} = (\mathbf{x}';0)\right)(\mathbf{x}'-\mathbf{x}'')\\ &= \mathrm{I}+\mathrm{II}+\mathrm{III}.\end{aligned}$$

Since the internal space $\mathbb{A}$ is finite and the chain $\xi^\sharp$ is irreducible and aperiodic,

$$\sum_{\mathbf{x}'\in\mathbb{X}^\flat} \mathbb{P}_{u''}\left(\xi^\flat_{T_{u''}} = (\mathbf{x}';0)\right) = 1,$$

so that the term II in the sum above yields $\mathbf{m}^\flat(\alpha, \{A \neq 0\})$. To handle the term III, specify for the moment $\flat = 2$. The case $\flat = 1$ is treated analogously. Denoting $\mathbf{e}_1$ and $\mathbf{e}_2$ the unit vectors of $\mathbb{Z}^2$, this term reads

$$\mathrm{III} = \sum_{x_2'',\alpha''\neq 0}\sum_{x_1',x_2'}[x_1'\mathbf{e}_1 + (x_2' - x_2'')\mathbf{e}_2]P^\flat_{(0,x_2;\alpha),(x_1'',x_2'';\alpha'')} \times \mathbb{P}_{(0,x_2'';\alpha'')}\left(\xi_{T_{u''}} = (x_1', x_2';0)\right).$$

Change the dummy summation variable $x_2' \in \mathbb{Z}$ into $x_2' - x_2''$ and use the vertical homogeneity of the $\xi^{(2)}$ chain to write this term as

$$\begin{aligned}
\mathrm{III} &= \sum_{x_2'',\alpha''\neq 0} P_{(0,x_2;\alpha),(0,x_2'';\alpha'')} \sum_{\mathbf{x}'} \mathbb{P}_{(0,0;\alpha'')}\left(\xi^{(2)}_{T_{(0,0;\alpha'')}} = (\mathbf{x}';0)\right)\mathbf{x}' \\
&= \sum_{\alpha''\neq 0} Q^{(2)}_{\alpha,\alpha''}\tilde{\mathbf{m}}^{(2)} \\
&= \tilde{\mathbf{m}}^{(2)} - Q_{\alpha,0}\tilde{\mathbf{m}}^{(2)}.
\end{aligned}$$

In general, we get

$$\mathrm{III} = \tilde{\mathbf{m}}^\flat - Q_{\alpha,0}\tilde{\mathbf{m}}^\flat.$$

Hence,

$$\tilde{\mathbf{m}}^\flat = \mathbf{m}^\flat(\alpha;\{A = 0\}) + \mathbf{m}^\flat(\alpha;\{A \neq 0\}) + \tilde{\mathbf{m}}^\flat - Q^\flat_{\alpha,0}\tilde{\mathbf{m}}^\flat$$

or in other words

$$Q^\flat_{\alpha,0}\tilde{\mathbf{m}}^\flat = \mathbf{m}^\flat(\alpha).$$

Multiplying both sides by $\pi^\flat(\alpha)$, summing over α, and using the fact that $\pi^\flat$ is a stationary measure for $Q^\flat$, we get

$$\overline{\mathbf{m}}^\flat = \sum_{\alpha\in\mathbb{A}} \pi^\flat(\alpha)\mathbf{m}^\flat(\alpha) = \pi^\flat(0)\tilde{\mathbf{m}}^\flat.$$

Using ergodicity, we can affirm that $\tilde{\mathbf{m}}^\flat$ can vanish if, and only if, $\overline{\mathbf{m}}^\flat$ vanishes. □

Remark: The meaning of this proposition is that the model with internal degrees of freedom $\alpha \in \{0, \ldots, L\}$ behaves as a model without internal degrees of freedom, provided that the boundary drifts depending on the colour are replaced by a weighted average, the weight of each colour being the probability that the system is on that colour. The proof of the main theorems can also be formulated in terms of embedded quantities $\tilde{\mathbf{m}}$; however such proofs should be less formal and therefore more difficult to check since they lie on intuitive ideas.

3 Harmonic functions in the wedge

3.1 Controlling jumps of smooth functions

Let $(\xi_n)_{n\in\mathbb{N}}$ be the previously defined Markov chain in $\mathbb{U}$ and $(\eta_n)_{n\in\mathbb{N}}$ be its image under the linear transformation $\Phi : \mathbb{X} \to \mathbb{Y}$. Denote by $\boldsymbol{\theta}_{n+1} = \mathbf{Y}(\eta_{n+1}) - \mathbf{Y}(\eta_n)$, where $\mathbf{Y}(\cdot)$ is the canonical projection $\mathbf{Y} : \mathbb{V} \to \mathbb{Y}$ for the transformed chain, the increment of the tranformed chain at time n. It is obvious that $(\eta_n)_{n\in\mathbb{N}}$ is $\mathbb{V}$-valued and has normalised covariance matrix. We shall study the conditional expectation of the increment for some smooth function f, namely

$$\mathbb{E}(f(\mathbf{Y}(\eta_{n+1})) - f(\mathbf{Y}(\eta_n))|\mathbf{Y}(\eta_n) = \mathbf{y}) = \mathbb{E}(f(\mathbf{y} + \boldsymbol{\theta}_{n+1}) - f(\mathbf{y})|\mathbf{Y}(\eta_n) = \mathbf{y})$$

by using Taylor expansion

$$\begin{aligned} f(\mathbf{y}+\boldsymbol{\theta}) &= f(\mathbf{y}) + (\nabla, \boldsymbol{\theta}) f(\mathbf{y}) + R_1(f; \mathbf{y}, \boldsymbol{\theta}) \\ &= f(\mathbf{y}) + (\nabla, \boldsymbol{\theta}) f(\mathbf{y}) + \frac{1}{2!}(\nabla, \boldsymbol{\theta})^2 f(\mathbf{y}) + R_2(f; \mathbf{y}, \boldsymbol{\theta}). \end{aligned}$$

The purpose of this section is to show that for sufficiently smooth functions $f \in C^3(\mathbb{Y}; \mathbb{R})$ and having some homogeneous growth properties, we can prove that the conditional expectations of the remainder terms are negligible with respect to the previous terms. More precisely we have the following

Proposition 3.1 *Let $f \in C^3(\mathbb{Y}; \mathbb{R})$ be such that for $k = 0, 1, 2, 3$, every $l \in \{0, \dots, k\}$, and for sufficiently large $\|\mathbf{y}\|$, there exist positive constants $D_k \geq 0$ and a real constant $\rho \in \mathbb{R}$ such that the following bound holds:*

$$\left| \frac{\partial^k f(\mathbf{y})}{\partial y_1^l \partial y_2^{k-l}} \right| \leq D_k \|\mathbf{y}\|^{\rho-k}.$$

If $\rho \in]-\infty, \gamma]$, there exist positive constants $a_1, a_2, \overline{a}_1$, and $\overline{a}_2$ such that

1. $|\mathbb{E}((\nabla, \boldsymbol{\theta}_{n+1}) f(\mathbf{y})|\mathbf{Y}(\eta_n) = \mathbf{y})| \leq a_1 \|\mathbf{y}\|^{\rho-1}$,
2. $|\mathbb{E}((\nabla, \boldsymbol{\theta}_{n+1})^2 f(\mathbf{y})|\mathbf{Y}(\eta_n) = \mathbf{y})| \leq a_2 \|\mathbf{y}\|^{\rho-2}$,
3. $|\mathbb{E}(R_1(f; \mathbf{y}, \boldsymbol{\theta}_{n+1})|\mathbf{Y}(\eta_n) = \mathbf{y})| \leq \overline{a}_1 \|\mathbf{y}\|^{\rho-2}$, *and*
4. $|\mathbb{E}(R_2(f; \mathbf{y}, \boldsymbol{\theta}_{n+1})|\mathbf{Y}(\eta_n) = \mathbf{y})| \leq \overline{a}_2 \|\mathbf{y}\|^{\rho-\gamma}$

Proof: It follows from lemma 5 of [2]. □

3.2 Controlling jumps of harmonic functions

The harmonic requirement $\left(\frac{\partial^2}{\partial y_1^2}+\frac{\partial^2}{\partial y_2^2}\right)h(y_1,y_2) = 0$ in the transformed wedge $\mathbb{Y}$ for a function $h : \mathbb{Y}\to\mathbb{R}$ is equivalent in writing h as a linear combination of elementary harmonic functions of order β,

$$\begin{aligned} h_\beta(y_1,y_2) &= (y_1^2+y_2^2)^{\beta/2}\cos(\beta\arctan\frac{y_2}{y_1}-\beta_1) \\ &= r^\beta\cos(\beta\omega-\beta_1), \end{aligned}$$

for $\beta,\beta_1\in\mathbb{R}$, where (r,ω) are the polar coordinates of the point $\mathbf{y}=(y_1,y_2)$. Negative values of β give rise to ill-defined functions near the summit of the wedge but this difficulty is irrelevant for us since we are interested only in the large $\|\mathbf{y}\|$ behaviour of such functions. Harmonic functions of order $\beta\le\gamma$ verify the smoothness and homogeneous growth requirements of the previous proposition 3.1, with $\rho=\beta$, so that we have for the conditional expectation of the increments can be estimated as

$$\begin{aligned} \mathbb{E}(h_\beta(\mathbf{Y}(\eta_{n+1}))-h_\beta(\mathbf{Y}(\eta_n))|\eta_n=(\mathbf{y},\alpha)) &= (\nabla h_\beta(\mathbf{y}),\mathbb{E}(\boldsymbol{\theta}_{n+1}|\eta_n=(\mathbf{y},\alpha))+\mathcal{O}(\|\mathbf{y}\|^{\beta-2}) \\ &= (\nabla h_\beta(\mathbf{y}),\mathbb{E}((\nabla h_\beta(\mathbf{y}),|\eta_n=(\mathbf{y},\alpha)) \\ &\quad +\frac{1}{2!}\mathbb{E}((\nabla h_\beta(\mathbf{y}),(\nabla h_\beta(\mathbf{y}))^2|\eta_n=(\mathbf{y},\alpha))+\mathcal{O}(\|\mathbf{y}\|^{\beta-\gamma}), \end{aligned}$$

where, denoting (r,ω) the polar coordinates of $\mathbf{y}$,

$$\begin{aligned} \nabla h_\beta(\mathbf{y}) &= \beta\|\mathbf{y}\|^{\beta-2}\begin{pmatrix} y_1\cos(\beta\arctan\frac{y_2}{y_1}-\beta_1)+y_2\sin(\beta\arctan\frac{y_2}{y_1}-\beta_1) \\ y_2\cos(\beta\arctan\frac{y_2}{y_1}-\beta_1)-y_1\sin(\beta\arctan\frac{y_2}{y_1}-\beta_1)\end{pmatrix}. \\ &= \beta r^{\beta-1}\begin{pmatrix}\cos((\beta-1)\omega-\beta_1) \\ -\sin((\beta-1)\omega-\beta_1)\end{pmatrix}. \end{aligned}$$

Obviously, $\|\nabla h_\beta(\mathbf{y})\|=\mathcal{O}(\|\mathbf{y}\|^{\beta-1})$, hence the dominant part in the conditional expectation of the increments is due to the gradient part. It is therefore a matter of straightforward estimates ot show the following

Lemma 3.2 *Let $g(\mathbf{y})=h_\beta(\mathbf{y})+ah_{\beta-1}(\mathbf{y})$, with $h_\beta(\mathbf{y})=(y_1^2+y_2^2)^{\beta/2}\cos(\beta\arctan\frac{y_2}{y_1}-\beta_1)$ and $h_{\beta-1}(\mathbf{y})=(y_1^2+y_2^2)^{(\beta-1)/2}\cos((\beta-1)\arctan\frac{y_2}{y_1}-\delta_1)$, where β_1 and δ_1 are arbitrary. Then, $f(\mathbf{y})\equiv g(\mathbf{y})^s$, for $s\beta<\gamma$, satisfies the conditions of the proposition 3.1 with $\rho=s\beta$.*

4 Lyapunov functions and proof of the main results

4.1 Criteria based on the Lyapunov function method

In order to study the asymptotic behaviour and the recurrence properties of Markov chains, the Lyapunov function method proves very useful [4, 2]. Let us remind the main results of this method.

4.1.1 Criteria for recurrence/transience

Proposition 4.1 *The Markov chain $(\xi_n)_{n\in\mathbb{N}}$, taking values in some space $\mathbb{U}$ is recurrent if, and only if, there exists a function $f:\mathbb{U}\to\mathbb{R}^+$ and a finite set $\Lambda\subset\mathbb{U}$ such that*

1. $\mathbb{E}(f(\xi_{n+1})-f(\xi_n)|\xi_n=u)\le 0$ *for all* $u\notin\Lambda$,
2. *the set* $\mathbb{U}_r=\{u\in\mathbb{U}:f(u)\le r\}$ *is finite.*

Proposition 4.2 *The Markov chain $(\xi_n)_{n\in\mathbb{N}}$, taking values in some space $\mathbb{U}$ is transient if, and only if, there exists a function $f:\mathbb{U}\to\mathbb{R}^+$ and a set $\Lambda\subset\mathbb{U}$ such that*

1. $\mathbb{E}(f(\xi_{n+1})-f(\xi_n)|\xi_n=u)\le 0$ *for all* $u\notin\Lambda$,
2. *there exists* $v\notin K$ *such that* $f(v)<\inf_{u\in\Lambda}f(u)$.

These propositions can be found in [1, 4, 8].

4.1.2 Criterion for ergodicity

Theorem 4.3 (Foster) *A Markov chain $(\xi_n)_{n\in\mathbb{N}}$ on $\mathbb{U}$ is ergodic if, and only if, there exists a function $f:\mathbb{U}\to\mathbb{R}^+$, an $\epsilon>0$, and a finite set $\Lambda\subset\mathbb{U}$ such that*

1. *the transformed sequence has finite conditional expectation*
$$\mathbb{E}(f(\xi_{n+1})|\xi_n=u)<\infty,\quad \text{for}\ \ u\in\Lambda,$$
2. *the sequence $(f(\xi_n))_{n\in\mathbb{N}}$ is a strong supermartingale outside Λ,* i.e.
$$\mathbb{E}(f(\xi_{n+1})-f(\xi_n)|\xi_n=u)\le -\epsilon\ \ \text{for}\ \ u\notin\Lambda.$$

4.1.3 Sufficient condition for the existence of moments of the passage time

Theorem 4.4 (Aspandiiarov, Iasnogorodski, Menshikov) *Let $(\Omega,\mathcal{F},(\mathcal{F}_n),\mathbb{P})$ be a filtred probability space, $(X_n)_n$ a real-valued adapted process such that $X_0=x$, with $x>K$ for some positive K. Let $\tau_K=\inf\{n\ge 0: X_n\le K\}$, Assume that there exist positive constants $\lambda>0$ and $p_0>0$ such that, for all n, we have $\mathbb{E}|X_n|^{2p_0}<\infty$ and*

$$\mathbb{E}(X_{n+1}^{2p_0}-X_n^{2p_0})|\mathcal{F}_n)\le -\lambda X_{n+1}^{2p_0-2}\quad \text{on}\ \ \{\tau_K>n\}.$$

Then, for all $p<p_0$, there exists a constant $c=c(\lambda,p,p_0)$ such that, for all initial conditions x (with $\mathbb{P}(X_0=x)=1$) verifying $x>K$, we have

$$\mathbb{E}\tau_K^p\le cx^{2p_0}.$$

4.1.4 Sufficient condition for non-existence of moments of the passage time

Theorem 4.5 (Aspandiiarov, Iasnogorodski, Menshikov) *Suppose $(X_n)_{n\in\mathbb{N}}$ and $(Y_n)_{n\in\mathbb{N}}$ are adapted processes to some filtration $(\mathcal{F}_n)_{n\in\mathbb{N}}$ and take values in an unbounded subset of $\mathbb{R}^+$. Let K be some large constant and define $\sigma_K = \inf\{n \geq 1 : X_n \leq K\}$ and $\tau_K = \inf\{n \geq 1 : Y_n \leq K\}$. Suppose that $Y_0 = y > K$ and that there exist positive constants c_1, c_2, and B, such that*

1. *$KB < X_0 = x \leq By$,*
2. *for all n, $X_n \leq BY_n$,*
3. *$\mathbb{E}(Y_{n+1}^2 - Y_n^2|\mathcal{F}_n) \geq -c_1$ on $\{\tau_K > n\}$, and*
4. *for some $r > 1$, $\mathbb{E}(Y_{n+1}^{2r} - Y_n^{2r}|\mathcal{F}_n) \leq c_2 Y_n^{2r-2}$ on $\{\tau_K > n\}$.*

If for some positive p_0, the process $(X^{2p_0}_{n\wedge\sigma_{KB}})_n$ is a submartingale, then for all $p > p_0$, the p-th moment does not exist, i.e. $\mathbb{E}\tau_K^p = \infty$.

4.2 Construction of Lyapunov function for the wedge problem

We are seeking for a function $f : \mathbb{V} \to \mathbb{R}^+$ such that $f(\eta_n)$ is a supermartingale. Let $h_\beta : \mathbb{Y} \to \mathbb{R}^+$ be a positive harmonic function over $\mathbb{Y}$ of order β.

Since every harmonic function of order β can be written as

$$h_\beta(y_1, y_2) = (y_1^2 + y_2^2)^{\beta/2} \cos(\beta \arctan \frac{y_2}{y_1} - \beta_1),$$

the positivity condition implies certain limits on the range of parametres β and β_1. In order to specify both these parametres we note $h_{\beta,\beta_1}(y_1, y_2)$ for the previous function in the sequel.

Theorem 4.6 *Let $v = (\mathbf{y}, \alpha) \in \mathbb{V}$, where $\mathbf{y} \in \partial_\flat \mathbb{Y}$ is a boundary point. Suppose that for some $\epsilon > 0$*

$$(\nabla h_{\beta,\beta_1}(\mathbf{y}), \vec{\mathbf{n}}^\flat) < -\epsilon, for \ \flat = 1, 2.$$

Then, there exists a function $f : \mathbb{V} \to \mathbb{R}$ with $f \in C^3(\mathbb{V}, \mathbb{R}^+)$ and a constant $K > 0$ such that for $\|\mathbf{y}\|$ sufficiently large,

1. *the sequence $(f(\eta_n))_n$ is a strong supermartingale near the boundary i.e.*

$$D(v) \equiv \mathbb{E}(f(\eta_{n+1}) - f(\eta_n)|\eta_n = v) < -K\epsilon\|\mathbf{y}\|^{\beta-1}, for \ \mathbf{y} \in \partial\mathbb{Y},$$

2. *the sequence* $(f(\eta_n))_n$ *is almost a martingale, in the interior space i.e. for* $v = (\mathbf{y}, 0)$, *with* $\mathbf{y} \in \mathring{\mathbb{Y}}$,
$$\mathbb{E}(f(\eta_{n+1}) - f(\eta_n)|\eta_n = v) = \mathcal{O}(\|\mathbf{y}\|^{\beta-2}).$$

To prove this theorem, we need a lemma from elementary linear algebra.

Lemma 4.7 *Let* $(Q_{\alpha,\alpha'})$ *be the stochastic matrix of an ergodic Markov chain on the finite space* $\mathbb{A}$ *and* $(\pi_\alpha)_\alpha$ *its invariant probability. Let* $(\nu_\alpha)_\alpha$ *be a given vector on* $\mathbb{A}$. *Then the set of linear inequalities for the variables* $(c_\alpha)_\alpha$
$$\nu_\alpha - c_\alpha + \sum_{\alpha' \in \mathbb{A}} Q_{\alpha,\alpha'} c_{\alpha'} < -\epsilon, \quad \alpha \in \mathbb{A},$$
defines a non-void subset of $\mathbb{R}^{\mathbb{A}}$ *if, and only if,* $\sum_{\alpha \in \mathbb{A}} \pi_\alpha \nu_\alpha < -\epsilon$.

Proof of theorem 4.6: Define a function $f : \mathbb{V} \to \mathbb{R}$ by
$$f(v) = \begin{cases} h_{\beta,\beta_1}(\mathbf{y}) + c_0 h_{\beta-1,\delta_1}(\mathbf{y}) & \text{if } v = (\mathbf{y}, 0), \mathbf{y} \in \mathring{\mathbb{Y}} \\ h_{\beta,\beta_1}(\mathbf{y}) + a_\alpha h_{\beta-1,\delta_1}(\mathbf{y}) & \text{if } v = (\mathbf{y}, \alpha), \mathbf{y} \in \partial_1 \mathbb{Y} \\ h_{\beta,\beta_1}(\mathbf{y}) + c_\alpha h_{\beta-1,\delta_1}(\mathbf{y}) & \text{if } v = (\mathbf{y}, \alpha), \mathbf{y} \in \partial_2 \mathbb{Y}. \end{cases}$$

Then for $v = (\mathbf{y}, \alpha)$ with $y \in \partial_2 \mathbb{Y}$, we compute the conditional increment $D(v) = \mathbb{E}(f(\eta_{n+1}) - f(\eta_n)|\eta_n = v)$. For definiteness, we consider the case $\mathbf{y} \in \partial_2 \mathbb{Y}$. The computation for the case $\mathbf{y} \in \partial_1 \mathbb{Y}$ is carried out along the same lines.
$$\begin{aligned} D(v) &= P_{v,(\mathbf{y}';0)}[h_\beta(\mathbf{y}') + c_0 h_{\beta-1}(\mathbf{y}') - h_\beta(\mathbf{y}) - c_\alpha h_{\beta-1}(\mathbf{y})] \\ &+ \sum_{\mathbf{y}' \in \partial_2 \mathbb{Y};\alpha'} P_{v,(\mathbf{y}';\alpha)}[h_\beta(\mathbf{y}') + c_{\alpha'} h_{\beta-1}(\mathbf{y}') - h_\beta(\mathbf{y}) - c_\alpha h_{\beta-1}(\mathbf{y})]. \end{aligned}$$

Use now the smoothness of harmonic functions $h_{\beta-1,\beta_1}$ and h_{β,β_1} to write
$$D(v) = (\nabla h_{\beta,\beta_1}(\mathbf{y}), \mathbf{n}^{(2)}(\alpha)) + \sum_{\alpha'} Q_{\alpha,\alpha'}(c_{\alpha'} - c_\alpha) h_{\beta-1,\delta_1}(\mathbf{y}) + r_1$$
with $\|\nabla h_{\beta,\beta_1}(\mathbf{y})\| = \mathcal{O}(\|\mathbf{y}\|^{\beta-1})$, $|h_{\beta-1,\delta_1}(\mathbf{y})\| = \mathcal{O}(\|\mathbf{y}\|^{\beta-1})$, and $|r_1| = \mathcal{O}(\|\mathbf{y}\|^{\beta-2})$. Hence, we can write
$$\frac{D(v)}{\|\mathbf{y}\|^{\beta-1}} = \left(\frac{\nabla h_{\beta,\beta_1}(\mathbf{y})}{\|\mathbf{y}\|^{\beta-1}}, \mathbf{n}^{(2)}(\alpha)\right) + C \sum_{\alpha' \in \mathbb{A}} Q_{\alpha,\alpha'}(c_{\alpha'} - c_\alpha) + \mathcal{O}(\|\mathbf{y}\|^{-1}).$$

But now, for very large fixed $\|\mathbf{y}\|$, we are in the situation of the previous lemma; we can always choose constants $(c_\alpha)_\alpha$ so that $\frac{D(v)}{\|\mathbf{y}\|^{\beta-1}} < -\epsilon$ for every α provided that
$$\left(\frac{\nabla h_{\beta,\beta_1}(\mathbf{y})}{\|\mathbf{y}\|^{\beta-1}}, \overline{\mathbf{n}}^{(2)}\right) < -\epsilon.$$

In a completely analogous way, we determine constants $(a_\alpha)_\alpha$ so that $\frac{D(v)}{\|\mathbf{y}\|^{\beta-1}} < -\epsilon$ for every α provided that

$$\left(\frac{\nabla h_{\beta,\beta_1}(\mathbf{y})}{\|\mathbf{y}\|^{\beta-1}}, \overline{\mathbf{n}}^{(1)}\right) < -\epsilon.$$

It remains to show that $(f(\eta_n))_n$ is an almost martingale in the interior of the space $\overset{\circ}{\mathbb{Y}}$ when compared with the boundary terms. For that it is enough to see that $\mathbf{n}(\mathbf{y}, 0) = 0$ for $\mathbf{y} \in \overset{\circ}{\mathbb{Y}}$, thanks to the zero drift property in the interior of the space. Hence the dominant part of the conditional increment is just $\mathcal{O}(\|\mathbf{y}\|^{\beta-2})$. □

Remark: Notice that the part due to the contribution of the function $h_{\beta-1,\delta_1}$ is subdominent and need not be positive. It is only the function $f(v)$ that is required to be positive for large $\|\mathbf{y}\|$. This allows to choose the parametre δ_1 totally freely.

Remark: The condition $\left(\frac{\nabla h_{\beta,\beta_1}(\mathbf{y})}{\|\mathbf{y}\|^{\beta-1}}, \overline{\mathbf{n}}^\flat\right) < -\epsilon$ has a very simple geometric meaning. It states that if the vector $\overline{\mathbf{n}}^\flat$ points in the interior of the level set of the function f near the $\partial_\flat\mathbb{Y}$ boundary, then $(f(\eta_n))_n$ is a strong supermartingale for large $\|\mathbf{y}\|$.

4.3 Recurrence and ergodic properties of the chain

We are now able to prove the main results of this paper.

Proof of the theorem 1.10: First examine the case where $\chi = \frac{\psi^1+\psi^2}{\psi} > 0$. Choose $\beta_1 < \psi_1$ and $\beta_2 < \psi_2$ so that $0 < \beta = \frac{\beta_1+\beta_2}{\psi} < \chi \equiv \frac{\psi_1+\psi_2}{\psi}$. Choose some p_0 with $0 < p_0 \leq \frac{\min(\beta,\gamma)}{2}$ and consider the functions

$$f(v) = \begin{cases} h_{\beta,\beta_1}(\mathbf{y}) + c_0 h_{\beta-1,\beta_1}(\mathbf{y}) & \text{if} \quad \mathbf{y} \in \overset{\circ}{\mathbb{Y}} \\ h_{\beta,\beta_1}(\mathbf{y}) + c_\alpha h_{\beta-1,\beta_1}(\mathbf{y}) & \text{if} \quad \mathbf{y} \in \partial_2\mathbb{Y} \\ h_{\beta,\beta_1}(\mathbf{y}) + a_\alpha h_{\beta-1,\beta_1}(\mathbf{y}) & \text{if} \quad \mathbf{y} \in \partial_1\mathbb{Y}, \end{cases}$$

and $g(v) = f^{2p_0/\beta}(v)$.

Near the boundary, it is enough to consider the first order Taylor expansion of the conditional increment. In the interior of the space, we need to continue up to the second order. In fact, near the boundary,

$$D(v) = \frac{2p_0}{\beta} f^{\frac{2p_0}{\beta}-1}(v)(\nabla f(v), \mathbb{E}(\boldsymbol{\theta}_{n+1}|\eta_n = v) + \mathbb{E}(R_1(f^{2p_0/\beta}; \mathbf{y}, \boldsymbol{\theta}_{n+1})).$$

Now, the choice $\beta_1 < \psi_1$ and $\beta_2 < \psi_2$ guarantees that $(\frac{\nabla f(v)}{\|\nabla f(v)\|}, \mathbb{E}(\boldsymbol{\theta}_{n+1}|\eta_n = v)) \leq -C\epsilon$, for $\mathbf{y}$ large near $\partial_\flat\mathbb{Y}$ so that $D(v) < 0$. In the interior of the space,

$$\begin{aligned} D(v) &= \frac{2p_0}{\beta} f^{\frac{2p_0}{\beta}-1}(v)(\nabla f(v), \mathbb{E}(\boldsymbol{\theta}_{n+1}|\eta_n = v)) + \frac{2p_0}{2\beta}(\frac{2p_0}{\beta} - 1) f^{\frac{2p_0}{\beta}-1} \mathbb{E}((\nabla f(v), \boldsymbol{\theta}_{n+1})^2|\eta_n = v)) \\ &\quad + \mathbb{E}(R_2(f^{2p_0/\beta}; \mathbf{y}, \boldsymbol{\theta}_{n+1}). \end{aligned}$$

Now the first order term in the Taylor expansion vanishes due to the zero drift condition in the interior of the space while the expectation of the square of the scalar product in the second order term is positive and of the order $\mathcal{O}(\|\mathbf{y}\|^{2p_0-2})$. Moreover, the numerical factor $(\frac{2p_0}{\beta}-1)$ is negative so that the $(g(\eta_n))_n$ is a supermartingale in the whole space. Finaly the set $\mathbb{V}_K = \{v \in \mathbb{V} : g(v) \leq K\}$ is finite and use of the theorem **??** allows the completion of the proof of recurrence for the case of positive p_0.

For the case of $\chi < 0$, choose β_1 and β_2 so that $\frac{\psi^1+\psi^2}{\psi} \equiv \chi < \frac{\beta_1+\beta_2}{\psi} \equiv \beta < 0$ $p_0 < 0$ with $|p_0| \leq \frac{\min(|\beta|,\gamma)}{2}$. Consider again the same functions $f(v)$ and $g(v) = f^{2p_0/\beta}(v)$. Therefore the conditional increments near the boundaries and in the interior are given by the same formula. Now the fact that $\beta < 0$ makes the gradient ∇f point towards the summit of the wedge. Therefore the choice $\psi^1 < \beta_1$ and $\psi^2 < \beta_2$ guarantees that near the boundary the scalar product $(\frac{\nabla f(v)}{\|\nabla f(v)\|}, \mathbb{E}(\boldsymbol{\theta}_{n+1}|\eta_n = v)) \leq -C\epsilon$ and since $2p_0/\beta$ is always positive, the conditional increment $D(v) < 0$. In the interior of the space, again the first term of the Taylor expansion vanishes due to the zero drift condition and the second term is negative because $0 < \frac{2p_0}{\beta} \leq 1$. On the other hand for large $\mathbf{y}$ $g(v) = \mathcal{O}(\|\mathbf{y}\|^{2p_0})$ with $p_0 < 0$, hence g is bounded. Therefore the transience is proved. □

Proof of theorem 1.11: The proof follows exactly the lines of [2]. It is enough to show that $f^s(v)$, with $s > 0$ and

$$f(v) = \begin{cases} h_{\beta,\beta_1}(\mathbf{y}) + c_0 h_{\beta-1,\delta_1}(\mathbf{y}) & \text{if } v = (\mathbf{y},0), \mathbf{y} \in \mathring{\mathbb{Y}} \\ h_{\beta,\beta_1}(\mathbf{y}) + a_\alpha h_{\beta-1,\delta_1}(\mathbf{y}) & \text{if } v = (\mathbf{y},\alpha), \mathbf{y} \in \partial_1\mathbb{Y} \\ h_{\beta,\beta_1}(\mathbf{y}) + c_\alpha h_{\beta-1,\delta_1}(\mathbf{y}) & \text{if } v = (\mathbf{y},\alpha), \mathbf{y} \in \partial_2\mathbb{Y}. \end{cases}$$

still fits the conditions of applicability of lemma 4.6, so that

$$\mathbb{E}[f^s(\eta_{n+1}) - f^s(\eta_n)|\eta_n] \leq -\lambda\|\mathbf{y}(\eta_n)\|^{s-2} \quad \text{on} \quad \{\tau_K > n\}.$$

Choose $\beta_1 < \psi_1$ and $\beta_2 < \psi_2$, $\beta = \frac{\beta_1+\beta_2}{\psi}$ and $p_0 < \beta/2$. Then the Lyapunov function f above, is such that f^s, with $s = 1/\beta$, satisfies the conditions of proposition 3.1. Following the lines of the proof of theorem 6 of [2] it is then possible to show that the process $X_n = f^{1/\beta}(\eta_n)$ verifies

$$\mathbb{E}[X_{n+1}^{2p_0} - X_n^{2p_0}|\mathcal{F}_n] \leq -\lambda X_n^{2p_0-2} \quad \text{on} \quad \{\tau_K > n\}.$$

This remark allows to conclude. □

Corollary 4.8 *If ψ_1 and ψ_2 belong to $]-\pi/2,\pi/2[$ and $\frac{\psi_1+\psi_2}{\psi} > 2$, the chain is ergodic.*

Proof of theorem 1.12: We use the criterion of non-existence 4.5. Let $\chi = \frac{\psi_1+\psi_2}{\psi}$ be determined by the geometry and choose parametres $\beta_1 > \psi_1$, $\beta_2 > \psi_2$, $\beta = \frac{\beta_1+\beta_2}{\psi} > \chi$. Consider functions $g_{\delta,\delta_1} = h_{\delta,\delta_1} + ch_{\delta-1,\delta_1}$ and processes $X_n = g^{1/\beta}_{\beta,\beta_1}(\eta_n)$ and $y_n = g^{1/\beta}_{\chi,\psi_1}(\eta_n)$.

Observe that both processes X_n and Y_n satisfy conditions of applicability of the criterion 4.5, namely that

$$\mathbb{E}(Y_{n+1}^2 - Y_n^2|\mathcal{F}_n) = (\nabla f(\mathbf{y}), \mathbb{E}(\boldsymbol{\theta}_{n+1}|\mathcal{F}_n)) + r_1(f;\mathbf{y}),$$

with $f = g_{\chi,\psi_1}^{2/\chi}$. But f satisfies properties of proposition 3.1 with $\rho = 2\chi/\chi = 2$. Hence, $r_1(f;\mathbf{y}) = \mathcal{O}(\|\mathbf{y}\|^{\rho-2})$ and although $\|\nabla f(\mathbf{y})\| = \mathcal{O}(\|\mathbf{y}\|^{\rho-1})$, the dominant part of the scalar product is in fact strictly zero since we have chosen the parametres of the harmonic function in such a way as to give a gradient at the boundaries exactly perpendicular to the mean drifts. Hence, $\mathbb{E}(Y_{n+1}^2 - Y_n^2|\mathcal{F}_n) = \mathcal{O}(1)$ which means that it can be bounded from below by some constant.

Similarly, for $r > 1$ and $f = g_{\chi,\psi_1}^{2r/\chi}$

$$\mathbb{E}(Y_{n+1}^{2r} - Y_n^{2r}|\mathcal{F}_n) = (\nabla f(\mathbf{y}), \mathbb{E}(\boldsymbol{\theta}_{n+1}|\mathcal{F}_n)) + r_1(f;\mathbf{y}).$$

Again the function f satisfies the conditions of proposition 3.1 with $\rho = 2r\chi/\chi = 2r$ and the dominant part of the scalar product vanishes. To conclude, it is enough to prove that $X_n^{2p_0}$ is a submartingale for $p_0 = \chi/2$. But $X_n^{2p_0} = f(\eta_n)$ with $f = g_{\beta,\beta_1}^{2p_0/\beta}$ that satisfies conditions of proposition 3.1 with $\rho = 2p_0\beta/\beta = 2p_0$.

On the boundaries, it is easily established that $X_n^{2p_0}$ is a submartingale. In fact,

$$\mathbb{E}(X_{n+1}^{2p_0} - X_n^{2p_0}|\mathcal{F}_n) = (\nabla f(\mathbf{y}), \mathbb{E}(\boldsymbol{\theta}_{n+1}|\mathcal{F}_n)) + r_1(f;\mathbf{y}),$$

and $\nabla f = \frac{2p_0}{\beta} g_{\beta,\beta_1}^{\frac{2p_0}{\beta}-1}$. But $(\nabla g_{\beta,\beta_1}, \overline{\mathbf{n}}^{\flat}) > \epsilon$ and $\|\nabla f(\mathbf{y})\| = \mathcal{O}(\|\mathbf{y}\|^{2p_0-1})$. Hence the first term dominates the subleading remainder r_1. For $\mathbf{y}$ being in the interior space, it is enough to pursue the Taylor expansion up to second order because the drift vanishes in the interior. This yields

$$\begin{aligned}\mathbb{E}(f(\eta_{n+1}) - f(\eta_n)|\eta_n = (\mathbf{y},0)) &= \frac{2p_0}{\beta}(\frac{2p_0}{\beta} - 1) g_{\beta,\beta_1}^{\frac{2p_0}{\beta}-2}[(\frac{\partial g_{\beta,\beta_1}}{\partial y_1})^2 + \frac{\partial g_{\beta,\beta_1}}{\partial y_2})^2] \\ &\quad + 2\frac{2p_0}{\beta} g_{\beta,\beta_1}^{\frac{2p_0}{\beta}-1}[(\frac{\partial^2 g_{\beta,\beta_1}}{\partial y_1^2}) + \frac{\partial^2 g_{\beta,\beta_1}}{\partial y_2^2})] + r_2(g_{\beta,\beta_1}^{\frac{2p_0}{\beta}};\mathbf{y}).\end{aligned}$$

Now, $r_2(g_{\beta,\beta_1}^{\frac{2p_0}{\beta}};\mathbf{y}). = \mathcal{O}(\|\mathbf{y}\|^{2p_0-\gamma})$ while the other terms are of order $\mathcal{O}(\|\mathbf{y}\|^{2p_0-2})$. Hence the process is a submartingale provided that the numerical coefficient $\frac{2p_0}{\beta} - 1 > 0$ what happens for $2p_0 > \beta$. This remark ends the proof of the theorem. □

5 The model of a Markov chain in two adjacent wedges with permeable interface and excitable boundaries

5.1 Introduction of the model

In this section, the space $\mathbb{X}$ will be composed by two adjacent quadrants, *i.e.* $\mathbb{X} = \mathbb{Z} \times \mathbb{Z}^+ \equiv \mathbb{X}^- \cup \mathbb{X}^+$ with $\mathbb{X}^- = \mathbb{Z}^- \times \mathbb{Z}^+$, $\mathbb{X}^+ = \mathbb{Z}^+ \times \mathbb{Z}^+$, $\partial_1\mathbb{X} = \mathbb{Z}^- \cup \mathbb{Z}^+$, and $\partial_2\mathbb{X}^- = \partial_2\mathbb{X}^+ = \mathbb{Z}^+$.

The internal space will be

$$\mathbb{A}_{\mathbf{x}} = \begin{cases} \{0,\dots,L\} & \text{if } x \in \partial\mathbb{X} \\ \{0\} & \text{if } x \in \mathbb{X}^+ \setminus \partial\mathbb{X} \\ \{L\} & \text{if } x \in \mathbb{X}^- \setminus \partial\mathbb{X}. \end{cases}$$

The configuration space $\mathbb{U}$ is constructed as the trivial fibre bundle with base $\mathbb{X}$ and fibre over $\mathbf{x} \in \mathbb{X}$ given by $\mathbb{A}_{\mathbf{x}}$ and decomposes into $\mathbb{U} = \mathbb{U}^- \cup \mathbb{U}^+$ with $\mathbb{U}^\pm$ being the configuration space over the quadrant $\mathbb{X}^\pm$ respectively. The major modification concerns the transition probabilities that verify now the modified LBJ conditions, namely

$$P_{x_1,x_2,0;x_1',x_2',\alpha'} = \begin{cases} 0 & \text{if } \ x_1 > 0 \ \text{ and } \ x_1' - x_1 < -1 \\ 0 & \text{if } \ x_2' - x_2 < -1 \\ 0 & \text{if } \ x_1 < 0 \ \text{ and } \ x_1' - x_1 > 1 \\ 0 & \text{if } \ x_1 = 1, x_1' = 0, x_2 > 0 \ \text{ and } \ \alpha' \neq 0 \\ 0 & \text{if } \ x_1 = -1, x_1' = 0, x_2 < 0 \ \text{ and } \ \alpha' \neq L \\ 0 & \text{if } \ x_2 = 1, x_2' = 0, x_1 > 0, \alpha' \neq 0 \ \text{ or } \ x_2 = 1, x_2' = 0, x_1 < 0, \alpha' \neq L. \end{cases}$$

Similarly the PSH condition reads in the present case:

Thus, in the present case, the entrance from the right quadrant into the interface can be only through $\alpha = 0$ state, conversely the entrance from the left quadrant into the interface can occur only through the $\alpha = L$ state. Obvious changes are introduced in the other properties of the transition probabilities. In particular, we impose

$$\mathbb{E}(\mathbf{X}(\xi_{n+1}) - \mathbf{X}(\xi_n) | \xi_n = (\mathbf{x}, 0)) = 0 \ \text{ if } \ \mathbf{x} \in \mathbb{X}^+ \setminus \partial\mathbb{X} \ \text{ or } \ \mathbf{x} \in \mathbb{X}^- \setminus \partial\mathbb{X}$$

and the conditional covariance matrices in the left/right space are positive definite, *i.e.* the matrices

$$\begin{pmatrix} \lambda_1^- & \kappa^- \\ \kappa^- & \lambda_2^+ \end{pmatrix} \text{ and } \begin{pmatrix} \lambda_1^+ & \kappa^+ \\ \kappa^+ & \lambda_2^+ \end{pmatrix}$$

are positive definite. Now left/right quadrants are transformed by linear transformations Φ^- and Φ^+ to get normalised covariance matrices. As we did in section 1.4, we denote $\mathbb{V}^- = \Phi^-(\mathbb{U}^-)$ and $\mathbb{V}^+ = \Phi^+(\mathbb{U}^+)$ the transformed sectors. These transformations change the two quadrants $\mathbb{U}^-$ and $\mathbb{U}^+$ with right angles at their summits into two wedges $\mathbb{V}^-$ and $\mathbb{V}^+$ with angles $\psi^- = \arccos(-\frac{\kappa^-}{\sqrt{\lambda_1^-\lambda_2^-}})$ and $\psi^+ = \arccos(-\frac{\kappa^+}{\sqrt{\lambda_1^+\lambda_2^+}})$ at their summits. We can again fix the parametres of the transformations Φ^- and Φ^+ so that one of the axes of each wedge $\mathbb{V}^-$ and $\mathbb{V}^+$ has a given direction. For instance, we can choose Φ^+ so that $\partial_1\mathbb{Y}^+$ remains parallel to the original axis $\partial_1\mathbb{X}^+$. This choice automatically places the axis $\partial_2\mathbb{Y}^+$ at angle ψ^+ from the previous. A natural choice for the axis $\partial_2\mathbb{Y}^-$ is at angle ψ^+ from $\partial_1\mathbb{X}^+$. This choice fixes $\partial_1\mathbb{Y}^-$ at angle $\psi^- + \psi^+$ from $\partial_1\mathbb{X}^+$.

Notice however that there is an additionnal difficulty here. For general values of the parametres $\lambda_1^-, \lambda_2^-, \kappa^-$ and $\lambda_1^+, \lambda_2^+, \kappa^+$ a lattice point of the form $x = (0, n) \in \partial_2\mathbb{X}$ is mapped to two different points $\Phi^-(x)$ and $\Phi^+(x)$ because the squeezing factors of the two quadrants are not the same in the left and right sectors. To be able to compare physical quantities, we decide to dilate the left sector by a global factor

$$\lambda = \frac{\sqrt{\lambda_2^- - \frac{\kappa^-}{\lambda_1^-}}}{\sqrt{\lambda_2^+ - \frac{\kappa^+}{\lambda_1^+}}},$$

so that lattice points on the interface have the same coordinates in the left and right sectors. Notice however that this dilation prevents the covariance matrix of the jumps of being the unit matrix on the left sector; it still remains diagonal but has the form $\lambda^2 \mathbb{1}$ instead. It is thus impossible to normalise simultaneously the covariance matrices on the left and right sectors keeping the coordinate systems commensurate with each other on the interface. The net effect of the λ-dilation is to multiply all vectors of the left sector by an overall λ factor. Notice that this keeps their direction unchanged.

We introduce, as we did in Section 1.4, transformed chains that are small perturbations of the chain ξ when observed far from the origin. These chains are $(\xi^{(2)})_n$, $(\xi_n^{(1,-)})_n$, and $(\xi_n^{(1,+)})_n$; they correspond to models with complete vertical homogeneity or complete horizontal homogeneity in the left or right quadrants. Therefore, the symbol $\sharp$ takes now four values, namely void, or (2), or $(1,-)$, or $(1,+)$. Chains $\xi^\sharp$ have transition matrices $\mathbf{P}^\sharp$. For $\mathbf{x} \in \partial\mathbb{X}^\sharp$, we introduce the induced transition matrix in the internal space

$$Q^\sharp_{\alpha,\alpha'} = \sum_{\mathbf{x}' \in \mathbb{X}^\sharp} P^\sharp_{(\mathbf{x},\alpha);(\mathbf{x}',\alpha')}.$$

Irreducibility and aperiodicity conditions of matrix $\mathbf{P}^\sharp$ are inherited by matrix $\mathbf{Q}^\sharp$ and since it is a stochastic matrix on a finite space, it admits an ergodic invariant probability $\pi^\sharp$. We extend also the notion of the restricted drift introduced in definition 2.3 to

$$\mathbf{m}^\sharp(u,S) = \sum_{u' \in S} P^\sharp_{u,u'}(\mathbf{X}(u') - \mathbf{X}(u)),$$

valid for every subset $S \subseteq \mathbb{U}^\sharp$. We shall be interested into two particular restricted drifts

$$\mathbf{m}^{(2)}(\alpha, \geq) = \sum_{x_1' \geq 0, x_2', \alpha'} P^{(2)}_{(x_1,x_2,\alpha);(x_1',x_2',\alpha')}(\mathbf{x}' - \mathbf{x}),$$

and

$$\mathbf{m}^{(2)}(\alpha, <) = \sum_{x_1' < 0, x_2', \alpha' = L} P^{(2)}_{(x_1,x_2,\alpha);(x_1',x_2',\alpha')}(\mathbf{x}' - \mathbf{x}),$$

together with the standard drifts $\mathbf{m}^{(1,-)}(\alpha)$ and $\mathbf{m}^{(1,+)}(\alpha)$.

Definition 5.1 Let $\pi^\flat$ be the stationary probability for the matrix $\mathbf{Q}^\flat$ for $\flat \in \{(2),(1,-),(1,+)\}$. Define the *average drifts* on the boundaries by

$$\overline{\mathbf{m}}^\flat = \sum_{\alpha \in \mathbb{A}} \pi^\flat_\alpha \mathbf{m}^\flat(\alpha),$$

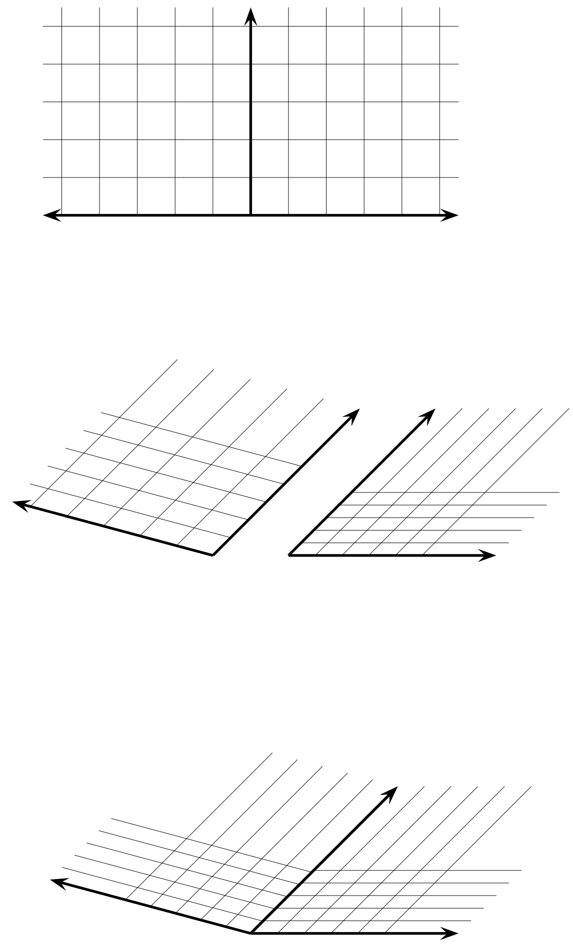

Figure 2: The transformation of the geometry by Φ^- and Φ^+ calculated for the example values $\lambda_1^- = 4/3$, $\lambda_2^- = 3$, $\kappa^- = 1$, $\lambda_1^+ = 4$, $\lambda_2^+ = 9$, and $\kappa^+ = -3\sqrt{2}$ gives rise to the wedges depicted above that remain incomensurate at the interface. Applying a global dilation of the left sector by λ, that reads $1/\sqrt{4+\sqrt{2}} \simeq 0.42977$ in this example, allows the gluing of the two coordinate systems.

and the *average restricted drifts*

$$\overline{\mathbf{m}}^{(2)}(<) = \sum_{\alpha \in \mathbb{A}} \pi^{\flat}_{\alpha} \mathbf{m}^{(2)}(\alpha, <),$$

and

$$\overline{\mathbf{m}}^{(2)}(\geq) = \sum_{\alpha \in \mathbb{A}} \pi^{\flat}_{\alpha} \mathbf{m}^{(2)}(\alpha, \geq).$$

We are now able to describe the geometric settings of the problem.

Define, for every $\beta \in \mathbb{R}$, the vectors

$$\mathbf{g}_{-}(\beta) = \mathrm{sgn}(\beta) \begin{pmatrix} \cos((\beta - 1)\psi_{-} - \psi_L) \\ -\sin((\beta - 1)\psi_{-} - \psi_L) \end{pmatrix}$$

and

$$\mathbf{g}_{+}(\beta) = \mathrm{sgn}(\beta) \begin{pmatrix} \cos((\beta - 1)\psi_{+} - \psi_0) \\ -\sin((\beta - 1)\psi_{+} - \psi_0) \end{pmatrix}.$$

Denote $c(\beta) = \frac{\cos(\beta\psi_{+} - \psi_0)}{\cos(\beta\psi_{-} - \psi_L)}$.

Definition 5.2 For every $\beta \in \mathbb{R}$, define

$$\overline{D}(\beta) = \lambda c(\beta)(\mathbf{g}_{-}(\beta), \overline{\mathbf{n}}^{(2)}(<)) + (\mathbf{g}_{+}(\beta), \overline{\mathbf{n}}^{(2)}(\geq)).$$

This quantity is called the *mean gain on the interface.*

Remark: Notice that $\overline{D}(\beta)$ is a purely geometrical quantity that can be computed in an elementary way out of the transition probabilities of the Markov chain.

Denote $\beta_{\min} = \max(\frac{-\pi/2+\psi_0}{\psi_+}, \frac{-\pi/2+\psi_L}{\psi_-})$ and $\beta_{\max} = \min(\frac{\pi/2+\psi_0}{\psi_+}, \frac{\pi/2+\psi_L}{\psi_-})$. Then $\beta_{\min} \leq 0 \leq \beta_{\max}$ and for all physical choices of the angles ψ_0, ψ_L, ψ_+, and ψ_-, strict inequalities hold.

Theorem 5.3 *Choose a $\beta_0 \in (\beta_{\min}, \beta_{\max})$ such that $\overline{D}(\beta_0) = 0$, with $\overline{D}(\beta_0^-) < 0$ and $\overline{D}(\beta_0^+) > 0$.*

1. *Such a β_0 always exists.*
2. *If $\beta_0 < 0$ then the Markov chain is transient.*
3. *If $\beta_0 > 0$ then the Markov chain is recurrent.*
4. *If $\beta_0 > 0$, let $p_0 = \frac{\min(\beta_0, \gamma)}{2}$. Then for every $p < p_0$ we have*

$$\mathbb{E}\tau_K^p \leq C\|\mathbf{x}\|^{2p_0}.$$

5. *Let* $p_0 = \frac{\min(\beta_0,\gamma)}{2}$. *If* $p_0 < \gamma/2$ *then for every* $p > p_0$,

$$\mathbb{E}\tau_K^p = \infty.$$

Corollary 5.4 *If* $2\psi_+ - \psi_0 > \pi/2$ *or* $2\psi_- - \psi_L > \pi/2$*, then the chain cannot be ergodic.*

Proof: If one of these conditions is verified, then $\beta_{\max} < 2$. Hence $p_0 \le \frac{\beta_0}{2} < 1$. Assertion 5 of the previous theorem establishes that $\mathbb{E}\tau_K = \infty$ preventing thus the chain from being ergodic. □

The proof of the theorem is based on the following proposition.

Proposition 5.5 *Let* $h_{\rho,\delta}(\mathbf{y}) = (y_1^2 + y_2^2)^{\rho/2}\cos(\rho \arctan\frac{y_2}{y_1} - \delta)$. *For* $v = (\mathbf{y};\alpha)$*, let* $f_\beta : \mathbb{V} \to \mathbb{R}$ *be defined by*

$$f(v) = \begin{cases} h_{\beta,\psi_0}(\mathbf{y}) + a_\alpha h_{\beta-1,\psi_0}(\mathbf{y}) & \text{if } \mathbf{y} \in \partial_1\mathbb{Y}^+ \\ h_{\beta,\psi_0}(\mathbf{y}) + a_0 h_{\beta-1,\psi_0}(\mathbf{y}) & \text{if } \mathbf{y} \in \overset{\circ}{\mathbb{Y}}{}^+ \\ h_{\beta,\psi_0}(\mathbf{y}) + b_\alpha h_{\beta-1,\psi_0}(\mathbf{y}) & \text{if } \mathbf{y} \in \partial_2\mathbb{V} \\ c(\beta)h_{\beta,\psi_L}(\mathbf{y}) + c_L\tilde{c}(\beta)h_{\beta-1,\psi_L}(\mathbf{y}) & \text{if } \mathbf{y} \in \overset{\circ}{\mathbb{Y}}{}^- \\ c(\beta)h_{\beta,\psi_L}(\mathbf{y}) + c_\alpha\tilde{c}(\beta)h_{\beta-1,\psi_L}(\mathbf{y}) & \text{if } \mathbf{y} \in \partial_1\mathbb{Y}^-. \end{cases}$$

If $\overline{D}(\beta) < -\epsilon$*, then it is always possible to choose the parametres* (a_α), (b_α), (c_α), $c(\beta)$*, and* $\tilde{c}(\beta)$ *in such a manner that* $D(v) = \mathbb{E}(f(\eta_{n+1}) - f(\eta_n)|\eta_n = v)$*, for large* $\|\mathbf{y}\|$,

- *is a strong supermartingale near the boundaries, i.e.*

$$D(v) \le -K\epsilon\|\mathbf{y}\|^{\beta-1},$$

- *is an "almost martingale" (with respect to its values near the boundaries) i.e. for* $\mathbf{y} \in \overset{\circ}{\mathbb{Y}}{}^- \cup \overset{\circ}{\mathbb{Y}}{}^+$,

$$|D(v)| = \mathcal{O}(\|\mathbf{y}\|^{\beta-2}).$$

Proof: The proof follows the same lines as the proof of theorem 4.6 in the first model. Let us check the supermartingale condition only on $\partial_2\mathbb{Y}$. The other cases are treated in a completely identical manner. We remark first that we can always choose the set of parametres so that $a_0 = b_0$ and $b_L = c_L$ since the vectors (a_α), (b_α), and (c_α) are defined only modulo a global multiplicative factor.

For $v \in \partial_2\mathbb{V}^{(2)}$ and $\|\mathbf{y}\|$ large enough, compute

$$\begin{aligned} D(v) &= \sum_{v' \in \mathbb{V}^{(2)}} P^{(2)}_{vv'}(f(v') - f(v)) \\ &= \sum_{\alpha',\mathbf{y}' \in \mathbb{Y}^{(2,+)}} P^{(2)}_{vv'}(f(v') - f(v)) + \sum_{\alpha'=L,\mathbf{y}' \in \mathbb{Y}^{(2,-)}} P^{(2)}_{vv'}(f(v') - f(v)) \\ &= D_1 + D_2. \end{aligned}$$

Now,

$$
\begin{aligned}
D_1 &= \sum_{\alpha',\mathbf{y}'\in\mathbb{Y}^{(2,+)}} P^{(2)}_{vv'}(h_{\beta,\psi_0}(\mathbf{y}')+b_{\alpha'}h_{\beta-1,\psi_0}(\mathbf{y}') \\
&\quad -h_{\beta,\psi_0}(\mathbf{y})+b_\alpha h_{\beta-1,\psi_0}(\mathbf{y})) \\
&= \sum_{\alpha',\mathbf{y}'\in\mathbb{Y}^{(2,+)}} P^{(2)}_{vv'}[(\nabla h_{\beta,\psi_0}(\mathbf{y}),\mathbf{y}'-\mathbf{y})+R_1(h_{\beta,\psi_0},\mathbf{y},\mathbf{y}'-\mathbf{y}) \\
&\quad +b_{\alpha'}(\nabla h_{\beta-1,\psi_0}(\mathbf{y}),\mathbf{y}'-\mathbf{y})+b_{\alpha'}R_1(h_{\beta-1,\psi_0},\mathbf{y},\mathbf{y}'-\mathbf{y}) \\
&\quad +(b_{\alpha'}-b_\alpha)h_{\beta-1,\psi_0}(\mathbf{y})],
\end{aligned}
$$

where $R_1(f,\cdot,\cdot)$ is the first order remainder of the Taylor expansion of f that can be shown to be subleading along the same lines as for the one wedge problem. Similarly,

$$
\begin{aligned}
D_2 &= \sum_{\alpha'=L,\mathbf{y}'\in\mathbb{Y}^{(2,-)}} P^{(2)}_{vv'}[c(\beta)h_{\beta,\psi_L}(\mathbf{y}')+b_L\tilde{c}(\beta)h_{\beta-1,\psi_L}(\mathbf{y}') \\
&\quad -h_{\beta,\psi_0}(\mathbf{y})+b_\alpha h_{\beta-1,\psi_0}(\mathbf{y})] \\
&= \sum_{\alpha'=L,\mathbf{y}'\in\mathbb{Y}^{(2,-)}} P^{(2)}_{vv'}[c(\beta)((h_{\beta,\psi_L}(\mathbf{y}')-h_{\beta,\psi_L}(\mathbf{y})) \\
&\quad +c(\beta)h_{\beta,\psi_L}(\mathbf{y})-h_{\beta,\psi_0}(\mathbf{y})) \\
&\quad +b_L\tilde{c}(\beta)(h_{\beta-1,\psi_L}(\mathbf{y}')-h_{\beta-1,\psi_L}(\mathbf{y})) \\
&\quad +b_L\tilde{c}(\beta)(h_{\beta-1,\psi_L}(\mathbf{y})-h_{\beta-1,\psi_0}(\mathbf{y}) \\
&\quad +(b_L-b_\alpha)h_{\beta-1,\psi_0}(\mathbf{y})] \\
&= \sum_{\alpha'=L,\mathbf{y}'\in\mathbb{Y}^{(2,-)}} P^{(2)}_{vv'}[c(\beta)(\nabla h_{\beta,\psi_L}(\mathbf{y}),\mathbf{y}'-\mathbf{y})+c(\beta)R_1(h_{\beta,\psi_L},\mathbf{y},\mathbf{y}'-\mathbf{y}) \\
&\quad +c(\beta)h_{\beta,\psi_L}(\mathbf{y})-h_{\beta,\psi_0}(\mathbf{y}) \\
&\quad +b_L\tilde{c}(\beta)((\nabla h_{\beta-1,\psi_L}(\mathbf{y}),\mathbf{y}'-\mathbf{y})+\tilde{c}(\beta)R_1(h_{\beta-1,\psi_L},\mathbf{y},\mathbf{y}'-\mathbf{y}) \\
&\quad +b_L(\tilde{c}(\beta)h_{\beta-1,\psi_L}(\mathbf{y})-h_{\beta-1,\psi_0}(\mathbf{y})) \\
&\quad +(b_L-b_\alpha)h_{\beta-1,\psi_0}(\mathbf{y})].
\end{aligned}
$$

Now choose $c(\beta)=\frac{h_{\beta,\psi_0}(\mathbf{y})}{h_{\beta,\psi_L}(\mathbf{y})}$ for $\mathbf{y}\in\partial_2\mathbb{Y}$. Hence, $c(\beta)=\frac{\cos(\beta\psi_+-\psi_0)}{\cos(\beta\psi_--\psi_L)}$. This choice guarantees that the second line in the last expression for D_2 vanishes. Similarly, we choose $\tilde{c}(\beta)=\frac{h_{\beta-1,\psi_0}(\mathbf{y})}{h_{\beta-1,\psi_L}(\mathbf{y})}=\frac{\cos((\beta-1)\psi_+-\psi_0)}{\cos((\beta-1)\psi_--\psi_L)}$ that guarantees vanshing of the penultimate line. Therefore, D_2 reads, in leading order $\mathcal{O}(\|\mathbf{y}\|^{\beta-1})$,

$$
D_2=\sum_{\alpha'=L,\mathbf{y}'\in\mathbb{Y}^{(2,-)}} P^{(2)}_{vv'}[c(\beta)(\nabla h_{\beta,\psi_L}(\mathbf{y}),\mathbf{y}'-\mathbf{y})+(b_L-b_\alpha)h_{\beta-1,\psi_0}(\mathbf{y})]+\mathcal{O}(\|\mathbf{y}\|^{\beta-2}).
$$

Recollecting terms, we get

$$
\begin{aligned}
D(v) &= D_1+D_2 \\
&= (\nabla h_{\beta,\psi_0}(\mathbf{y}),\mathbf{n}^{(2)}(\alpha;\geq))+c(\beta)(\nabla h_{\beta,\psi_L}(\mathbf{y}),\lambda\mathbf{n}^{(2)}(\alpha;<)) \\
&\quad +h_{\beta-1,\psi_0}(\mathbf{y})\sum_{\alpha'\in\mathbb{A}} Q^{(2)}_{\alpha,\alpha'}(b_{\alpha'}-b_\alpha)+\mathcal{O}(\|\mathbf{y}\|^{\beta-2}).
\end{aligned}
$$

Now we can determine parametres (b_α) so that $D(v) \leq -K\epsilon\|\mathbf{y}\|^{\beta-1}$ for every α, provided that the compatibility condition holds, i.e.

$$\lambda c(\beta)(\mathbf{g}_-(\beta), \overline{\mathbf{n}}^{(2)}(<)) + (\mathbf{g}_+(\beta), \overline{\mathbf{n}}^{(2)}(\geq)) < -\epsilon,$$

since $|\frac{h_{\beta-1,\psi_0}}{\|\nabla h_{\beta,\psi_0}\|}| = \mathcal{O}(1)$.

The behaviour on the other boundaries is treated in exactly the same lines as for the one wedge case. As for the "almost martingale" property, it is enough to remark that both $h_{\beta-1,\delta}$ and $h_{\beta,\delta}$ are harmonic functions. □

Proof of theorem 5.3:

1. The existence of such a β_0 is obtained by the continuity of $\overline{D}(\beta)$ and by explicit estimates of $\overline{D}(\beta)$ near the edges of the interval $(\beta_{\min}, \beta_{\max})$.
2. If $\beta_0 < 0$, then for every β_1 with $\beta_{\min} < \beta_1 < \beta_0$, we can construct a supermartingale on the whole space. Moreover, since $\beta_0 < 0$, the leading contribution of the Lyapunov's function is in $\|\mathbf{y}\|^{\beta_1}$ that remains bounded outside a finite set near the origin. We conclude therefore transience.
3. If $\beta_0 > 0$, then for every β_1 with $0 < \beta_1 < \beta_0$, we can construct a supermartingale for large $\|\mathbf{y}\|$. On the other hand, the leading contribution to the Lyapunov's function is in $\|\mathbf{y}\|^{\beta_0}$. Therefore, its level sets are compact and we conclude recurrence.
4. If $\beta_0 > 0$, then for every p with $0 < p < p_0 = \frac{\min(\beta,\gamma)}{2}$, we can construct a strict supermartingale, as in the one wedge problem, verifying condition

   $$\mathbb{E}(X_{n+1}^{2p} - X_n^{2p}|\mathcal{F}_n) \leq -\epsilon X_n^{2p-2}$$

   which proves finiteness of the p-th moment of passage time.
5. Finally, for $p > p_0$, if $p_0 < \gamma/2$, we can construct processes X_n and Y_n as in the one wedge problem, verifying the criterion of the non-existence of moments.

□

6 Conclusion and open problems

We have established that the ergodic properties of the chains are determined by simple geometrical characteristics that are derived form the Lyapunov's function method and this in spite of the complication steming from the internal degrees of freedom.

The results of the second model can be extended with only computational complications to wedges with an arbitrary number of sectors provided that the gluing of sectors does not result to a problem without external boundaries. The treatment for geometries without external boundaries remains open for the moment.

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