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The Hausdorff dimension of the two-dimensional Edwards' random walk

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Abstract. We prove that the Hausdorff dimension of the two-dimensional Edwards' random walk is two with probability one. This result is in contradiction with the widely accepted conjecture that the Hausdorff dimension of a random walk is the inverse of the critical exponent ν; actually, ν is $\frac{3}{4}$ for the two-dimensional Edwards' random walk.

1. Introduction

The notion of Hausdorff dimension has in recent years become of increasing interest to physicists [1, 2]. It allows the precise characterisation of the dimensional properties of a set and enables one to distinguish between sets of equal topological dimension or of equal Lebesgue measure. In most of the physically relevant situations, it turns out that a direct computation of the Hausdorff dimension d_H is practically impossible. For this reason various physical characteristics, like critical exponents, have been used to try to estimate it. In this paper, we precisely investigate the use of such a critical exponent in the geometrical study of random walks.

The critical behaviour of geometrical objects such as random walks, random surfaces, polymers in solution, percolation clusters and so on, have been extensively studied of late. The large-scale behaviour of all these objects is governed by the exponent ν. For example, the root-mean-squared Euclidean end-to-end distance r for random walks of length L behaves asymptotically as $r \sim L^{\nu}$. Similarly, the mean gyration radius r of a triangulated random surface containing L triangles behaves as $r \sim L^{\nu}$. By asymptotic behaviour we mean that

$$\lim_{L \to \infty} \frac{\log r}{\log L} = \nu.$$

An old conjecture, probably due to des Cloizeaux (see [3]), asserts that, for random walks, the inverse of ν equals d_H; or, equivalently, that d_H is given by $1/\nu$. This conjecture is by now widely accepted and used for all kinds of random geometrical objects, but it has been rigorously verified only in the case of ordinary random walks (ORW); it has never been tested for more complicated objects because of the difficulty in estimating d_H in such cases. Heuristic arguments in favour of the conjecture are given in [4].

The first and most difficult step in computing the Hausdorff dimension of a random walk is to construct its continuous limit. Until now, this has been done only for the d-dimensional ORW and, more recently, for the two-dimensional Edwards' random

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walk [5-7]. We shall therefore concentrate on the two-dimensional Edwards' walk: we show that its Hausdorff dimension can be rigorously computed; using the existing results on the critical exponent ν, we are able to get some insight into the conjecture. The Edwards' random walk is, furthermore, of great physical interest because it provides a quite realistic model for polymers [8].

Roughly speaking, the Edwards' walk (EW)—known also as a weakly self-avoiding walk (WSAW) in the literature—is obtained by assigning to each ordinary random walk ω a weight of the form $\exp(-gN(\omega))$, where $N(\omega)$ is formally the number of self-intersections of the walk ω. For every $g>0$ (no matter how small it may be) the critical behaviour of the Edwards' walk is the same as that of the self-avoiding walk (SAW), i.e. the renormalisation group flow drives g to infinity. In particular, the critical exponent ν is expected to be $\frac{3}{4}$ in two dimensions [9]. The renormalisation group computation of the critical exponent ν is carried out in [10] and is very strongly supported by direct numerical simulations [11] of EW establishing that $\nu=\frac{3}{4}$ with very good accuracy.

According to the conjecture, one would thus expect that the Hausdorff dimension, d_{H}, of the Edwards' random walk is $\frac{4}{3}$. Using the known construction of the continuous limit of Edwards' walks, we shall however prove the following theorem.

Theorem. The Hausdorff dimension of the two-dimensional Edwards' path is almost surely two.

This result is quite surprising since we know that Edwards' paths differ qualitatively from the Brownian ones in the sense that almost every Edwards' path can be decomposed into a Brownian one plus a strictly non-vanishing square-integrable perturbation [12]. In spite of this qualitative difference, the Edwards' and Brownian paths have the same dimension. Moreover, this result seemingly contradicts the conjecture.

This paper is organised as follows. In § 2 we recall the mathematical definition of the Hausdorff dimension. In § 3 we define the Edwards' model and briefly review its construction in the continuum. The theorem is proved in § 4. In § 5 we discuss our result and compare it with the conjecture; we state some related open problems.

2. Hausdorff and spherical dimensions

The Hausdorff dimension is an old, well defined mathematical notion which goes back to Hausdorff [13]. Most of its mathematical properties were established by Besicovitch [14]. It has the peculiarity of not necessarily being an integer. Furthermore, two sets with the same topological dimension may have different Hausdorff dimensions. For example, a Brownian path has topological dimension one, like an ordinary curve. But it has Hausdorff dimension two, which means, intuitively, that in two dimensions it fills the plane.

Let us now recall some mathematical definitions (see [15] for more details). Let A be a subset of $\mathbb{R}^n$. A δ cover of A is a collection $\{U_i\}$ of subsets of $\mathbb{R}^n$ such that

(i) $\quad A\subset\bigcup U_i$

(ii) $\quad |U_i|\equiv\sup\{|x-y|: x, y\in U_i\}<\delta \qquad \forall i.$

Let s be a non-negative number. Define the Hausdorff measure $\mathscr{H}^s$ by

$$\mathscr{H}^s(A) = \lim_{\delta \to 0} \left(\inf \sum_{i=1}^{\infty} |U_i|^s \right) \tag{1}$$

where the infimum is over all δ covers of A ($\mathscr{H}^s$ is actually not a measure but only an outer measure). This limit always exists but may be infinite. The Hausdorff dimension d_H is then defined as

$$d_H(A) = \inf\{s\colon \mathscr{H}^s(A) = 0\} = \sup\{s\colon \mathscr{H}^s(A) = \infty\}. \tag{2}$$

Otherwise stated, $d_H(A)$ is the unique value for which

$$s < d_H(A) \Rightarrow \mathscr{H}^s(A) = \infty \qquad s > d_H(A) \Rightarrow \mathscr{H}^s(A) = 0. \tag{3}$$

Notice that $\mathscr{H}^{d_H(A)}(A)$ may be zero, finite or infinite.

It is worth noticing that the collection $\{U_i\}$ entering in the δ cover must be a collection of general subsets. However, it can be proven [16] that the collection can be restricted to convex sets, or open sets or closed sets always yielding the same ($\delta \to 0$ limit) Hausdorff measure $\mathscr{H}^s$. If the collection $\{U_i\}$ is restricted to balls then the limit measure is in general different; we denote it by $\mathscr{S}^s$ and call it the spherical Hausdorff measure. The first example where a discrepancy between the two measures occurs is explicitly constructed in [14]. In the following, we reserve the symbol d_H to denote the Hausdorff dimension, obtained by using $\mathscr{H}^s$, and the symbol S_H to denote the spherical Hausdorff dimension obtained by using $\mathscr{S}^s$. The merit of S_H is that this dimension is more easy to handle and in fact intuitive arguments always use coverings by balls. It is therefore interesting to check whether d_H and S_H coincide in the case of random walks. Finally, note that d_H and S_H are defined for subsets of $\mathbb{R}^n$. So when speaking about the Hausdorff dimension of a random walk, what is actually meant is the dimension of its continuous limit.

3. The Edwards' model

As explained above, the first step in computing the Hausdorff or spherical dimensions of a random walk is to construct its continuous limit. Let us briefly recall the construction of the Edwards' probability measure in two dimensions.

One would like to define on the Wiener space $\Omega = C([0, T], \mathbb{R}^2)$, with $0 < T < \infty$, a probability measure of the form

$$d\mu(\omega) = Z^{-1} \exp(-gN(\omega))\, dP(\omega) \tag{4}$$

where g is a positive constant, Z is a normalisation constant and dP denotes the Wiener measure. $N(\omega)$ represents the number of self-intersections of the Brownian path ω, and is formally defined as

$$N(\omega) = \int_0^T \int_0^T ds\, dt\, \delta(\omega(t) - \omega(s)) \tag{5}$$

where δ denotes the Dirac measure at the origin.

Expression (5) is only formal because $N(\omega)$ is divergent. It is, however, possible to render it finite by a procedure known as Varadhan renormalisation [5]. One can show that the renormalised quantity

$$N_r(\omega) = \lim_{\varepsilon \to 0} \int_0^T \int_0^T ds\, dt\, \{\delta_\varepsilon(\omega(s) - \omega(t)) - E_P[\delta_\varepsilon(\omega(s) - \omega(t))]\} \tag{6}$$

is well defined and finite. $E_P[\cdot]$ means the expectation with respect to the Wiener measure P and δ_ε is a regularisation of δ with $\lim_{\varepsilon\to 0}\delta_\varepsilon = \delta$ in the sense of weak convergence.

Remark. Actually, Varadhan studied the Brownian bridge; his proof extends without difficulty to the case of Brownian motion. Le Gall reformulated (for Brownian motion) the principal results of Varadhan, using the formalism of intersection local time [6].

The renormalised quantity $N_r(\omega)$ has an acceptable meaning as the renormalised number of self-intersections. One has to note that, because of the renormalisation procedure, $N_r(\omega)$ is no longer positive definite. However, the probability that $N_r(\omega)$ is smaller than a negative constant can be shown to fall exponentially fast, i.e.

$$P(\{\omega: N_r(\omega) \leqslant -a,\, a>0\}) \sim \exp(-\pi a/8)/a^2.$$

Thus, it is possible to prove that $\exp(-gN_r(\omega)) \in L^1(\mathrm{d}P)$ for every finite g. We can therefore define μ by its Radon–Nikodým derivative with respect to the Wiener measure

$$\frac{\mathrm{d}\mu}{\mathrm{d}P}(\omega) = Z(g)^{-1}\exp(-gN_r(\omega)) \tag{7}$$

where $Z(g) = E_P[\exp(-gN_r(\omega))]$. Note that the L^1 integrability for every finite g of the Radon–Nikodým derivative is equivalent to L^p integrability for every finite p.

4. Proof of the theorem

Before computing the Hausdorff and spherical dimensions of Edwards' paths, we need to prove the following simple lemma.

Lemma. For every α with $0<\alpha<\frac{1}{2}$, the Edwards' paths are μ-almost surely Hölder continuous of order α.

Proof. Using Hölder's inequality and the integrability of $\exp(-gN_r(\omega))$, we prove that for every $p<\infty$ we have

$$E_\mu(|\omega(t+h)-\omega(t)|^p) \leqslant C|h|^{p/2} \tag{8}$$

for a positive constant C. Choose $p>2$ and use the Kolmogorov–Čentsov theorem (see, e.g., [17]). It follows that ω is μ-almost surely Hölder continuous of order α for every $\alpha \in (0, \frac{1}{2})$. □

The following definition formalises the familiar notion of electrostatic energy of a charge distribution in a form useful in potential theory. Let ρ be a mass distribution on $\mathbb{R}^2$ (i.e. a measure with finite total mass). The integral

$$I_t(\rho) = \int\int \frac{\mathrm{d}\rho(x)\,\mathrm{d}\rho(y)}{|x-y|^t} \tag{9}$$

is called the *t energy* of ρ.

We shall now use a standard result which relates geometric measure theory to potential theory; we state it without proof.

Proposition. Let A be a Souslin subset of $\mathbb{R}^n$. If there exists some mass distribution ρ supported by A and whose t energy $I_t(\rho)$ is finite for some t, then $d_H(A) \geq t$.

The proof of this proposition is originally due to Frostman [18] and Erdős and Gillis [19]. A more recent exposition can be found in [20].

We are now able to prove the theorem of § 1.

Proof of the theorem. Let $A = \omega([0, T])$. It is clear that $d_H(A) \leq 2$ since A is a subset of $\mathbb{R}^2$ (it would, however, be easy to prove this upper bound, using the lemma above). It remains to prove that $d_H(A) \geq 2$. We choose as mass distribution on $\mathbb{R}^2$ the occupation measure ρ defined for every $B \subset \mathbb{R}^2$ by

$$\rho(B) = T^{-1}\lambda\{t: 0 \leq t \leq T, \omega(t) \in B\} \tag{10}$$

where λ denotes the one-dimensional Lebesgue measure. ρ is supported (by construction) by the path ω. Using Hölder's inequality, the fact that $\exp(-gN_r(\omega)) \in L^1(\mathrm{d}P)$ for every finite g, and Fubini's theorem, it is easy to verify that for every $\varepsilon > 0$, $I_t(\nu)$ is finite $\forall t < 2 - \varepsilon$, for μ-almost all ω. Now, any Borel set is a Souslin set. Since both $[0, T]$ and $\mathbb{R}^2$ are complete metric spaces, the almost surely continuity of the mapping $\omega: [0, T] \to \mathbb{R}^2$ proved in the previous lemma is enough to guarantee that Edwards' paths are Souslin subsets of $\mathbb{R}^2$. We can thus conclude by the previous proposition. □

We introduced in § 2 the notion of spherical dimension. It is easy to compute it as a corollary of the previous theorem. By taking the infimum only over δ covers consisting of balls, one obtains a measure $\mathcal{S}^s$ not smaller than the Hausdorff measure $\mathcal{H}^s$. Therefore, $d_H \leq S_H$. By virtue of the theorem, we have $2 \leq S_H \leq 2$ (the second inequality is trivial because $\omega([0, T])$ is a subset of $\mathbb{R}^2$). We may thus conclude that $S_H = 2$. So, the Hausdorff and spherical dimensions of the Edwards' random walk coincide (the same is obviously true for the ORW).

Remark. A notion related to the Hausdorff dimension is that of 'capacity.' For every compact set $K \subset \mathbb{R}^n$, denote by $N(\varepsilon, K)$ the minimum number of balls of same radius ε needed to cover K. The capacity $C(K)$ of K is defined as

$$C(K) = \limsup_{\varepsilon \to 0} \frac{\log N(\varepsilon, K)}{\log(1/\varepsilon)}.$$

It can be seen that for every compact set K, $d_H(K) \leq C(K)$. Obviously, the capacity of almost every Edwards' path is two, thus it coincides with the Hausdorff dimension for the two-dimensional Edwards' paths.

5. Discussion

It is not difficult to understand *a posteriori* why the Hausdorff dimension of the Edwards' path is two; if one looks carefully at the proof of the theorem, it is clear that d_H is essentially a metric property of the path. The explicit Edwards' probability measure enters only through Fubini's theorem to ensure the almost sure finiteness of the energy integral. Thus, the details of this probability measure—contained in the factor $\exp(-gN_r(\omega))$—do not enter into consideration. Indeed, we verified that if a stochastic process has a Radon–Nikodým derivative with respect to the two-dimensional Wiener measure which is L^p integrable for every finite p ($p \geq 1$), then the Hausdorff dimension of its trajectories is almost surely two. It is however possible to imagine a fractal dimension where the characteristics of the probability measure are taken into account. It would be interesting to establish the mathematical properties of such a dimension; work in this direction is in progress.

In view of the conjecture (which claims that $d_H = 1/\nu$), the theorem on d_H proved in last section and the results on ν quoted in § 1 seem to be in contradiction. Of course this work does not provide a proof that the conjecture is false: we computed the Hausdorff dimension only for (arbitrarily large but) finite T whereas the exponent ν is defined in the limit $T \to \infty$; note, however, that d_H does not depend on T. Moreover we were not able to compute ν analytically.

In spite of the above remark some comments can be made against the conjecture. First, we proved that in the continuum the Edwards' random walk has different short scale (Hausdorff dimension) and large scale ($1/\nu$) behaviours. This phenomenon occurs as well in the study of the ϕ^4 field theory in two dimensions [21], and it is well known that Edwards' random walks provide a representation of that same theory [22]. Thus, the field-theory results completely support ours.

How does d_H behave in the limit $T \to \infty$? This limit corresponds to the infrared-scaling limit of the ϕ^4 field theory. Such a 'limit' has never been proven. Roughly speaking, there are two possibilities.

(i) The limit measure is still an Edwards' measure in the sense that it has the same form as the measure constructed in § 3. In that case, one may expect the Hausdorff dimension to be still two because, as explained above, the dimension does not depend on all the details of the measure. This would explicitly contradict the conjecture.

(ii) In the opposite case, the limit measure does not describe Edwards' walks (as constructed in § 3) any longer, but probably some kind of self-avoiding random walks. So the Hausdorff dimension computed with this limit measure is no longer the Hausdorff dimension of the Edwards' random walk. This means that in this case the conjecture cannot be applied. Furthermore, since the unnormalised Edwards' measure becomes singular with respect to the Wiener measure in the limit $T \to \infty$, it is expected that there is not a unique limit measure.

The conclusion of the above reasoning is that Edwards' random walks provide an example where either the conjecture is false (case (i)) or is not applicable (case (ii)).

Our last comment is the following. Even if $d_H = 1/\nu$ in some very special cases (e.g. Brownian motion), we claim that $1/\nu$ cannot correspond to a Hausdorff dimension, or even to any fractal dimension in general. It is true that $1/\nu$ gives information about the large-scale behaviour of the considered random geometrical object (walk, surface, percolation cluster, etc). It must be stressed, however, that it cannot be a dimension with all the mathematical physical or intuitive connotations the word 'dimension' implies. In fact, a quantity D can be reasonably called a dimension if

(i) it coincides with the topological dimension on regular objects,

(ii) it defines a transition point for some outer measure in the sense that some outer measure m_s, parameterised by s, is infinite if $s < D$ and zero if $s > D$.

A corollary of the latter property is that D is not decreasing, i.e. $A \subset B \Rightarrow D(A) \leq D(B)$. Now, these conditions are violated by $1/\nu$. For example, the Edwards' walk with $g < 0$ in two dimensions has $\nu = 0$ [11]. If $1/\nu$ was a dimension then one would expect $D = \infty$. Now, the dimension must be bounded above by 2 by virtue of (i) and (ii) which leads to the explicit contradiction $2 > \infty$! In conclusion, it is vain to try to identify $1/\nu$ with the Hausdorff dimension (or any other dimension). From a general point of view, there is a lot of confusion about Hausdorff dimension. In the physical literature one can find several more or less physical quantities, related to the capacity but with the limit $\varepsilon \to 0$ replaced by $\varepsilon \to \infty$, which are wrongfully used as 'definitions' of the Hausdorff dimension. Note, however, that the limit $\varepsilon \to \infty$ equals $1/\nu$; it does not define some independent quantity that is later shown to be equal to $1/\nu$. Although the limits $\varepsilon \to 0$ and $\varepsilon \to \infty$ coincide in the very special case of Brownian motion ($g = 0$), there is no reason for these limits to coincide in general. Most of the confusion stems from the fact that the trust in the conjecture is so strong that nowadays a lot of physicists, when speaking about Hausdorff dimension, mean in fact $1/\nu$. We hope that this paper will help to clarify the situation.

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