Type and type transition for random walks on randomly directed lattices

To Iain MacPhee, in memoriam

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Aspects of random walks
1 April 2014

What is the type problem for random walks?

- How often does a random walker on a denumerably infinite graph $\mathbb{X}$ returns to its starting point?
- It depends on $\mathbb{X}$ and on the law of jumps.
- Typically a dichotomy
  - either almost surely infinitely often (recurrence),
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- $\mathbb{X} = \mathbb{Z}^d$ is an Abelian group with generating set, e.g. the minimal generating set

$$\mathbb{A} = \{\mathbf{e}_1, -\mathbf{e}_1, \ldots, \mathbf{e}_d, -\mathbf{e}_d\}; \quad \mathrm{card}\mathbb{A} = 2d.$$

- μ probability on $\mathbb{A} \Rightarrow$ probability on $\mathbb{X}$ with $\mathrm{supp}\,\mu = \mathbb{A}$.
  - Uniform: $\forall x \in \mathbb{A} : \mu(x) \equiv \frac{1}{\mathrm{card}\mathbb{A}} = \frac{1}{2d}$.
  - Symmetric: $\forall x \in \mathbb{A} : \mu(x) = \mu(-x)$.
  - Zero mean: $\sum_{x\in\mathbb{A}} x\mu(x) = 0$.
- $\xi = (\xi_n)_{n\in\mathbb{N}}$ i.i.d. sequence with $\xi_1 \sim \mu$.
- Define $X_0 = x \in \mathbb{X}$ and $X_{n+1} = X_n + \xi_{n+1}$. Then

$$P(x,y) = \mathbb{P}(X_{n+1} = y | X_n = x) = \mathbb{P}(\xi_{n+1} = y - x) = \mu(y-x).$$

- Simple (=uniform on the minimal generating set) random walk on $\mathbb{X}$: the $\mathbb{X}$-valued Markov chain $(X_n)_{n\in\mathbb{N}}$ of $\mathrm{MC}(\mathbb{X}, P, \epsilon_x)$

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Theorem (Georg Pólya[a])

[a] Über eine Aufgabe der Wahrscheinlichkeitsrechnung betreffend die Irrfahrt im Straßennetz, Ann. Math. (1921)

For $\mathbb{X} = \mathbb{Z}^d$ with uniform jumps on n.n.

$$d \geq 3: \text{ transcience,}$$
$$d = 1, 2: \text{ recurrence.}$$

Proof by direct combinatorial and Fourier estimates.

- $P^n(x, y) := \sum_{x_1, \ldots x_{n-1}} \mathbb{P}(X_0 = x, X_1 = x_1, \ldots, X_n = y) = \mu^{*n}(y - x)$.
- For $\xi \sim \mu$ and μ uniform, $\chi(t) = \mathbb{E}\exp(i\langle t \,|\, \xi \rangle) = \sum_x \exp(i\langle t \,|\, x \rangle)\mu(x) = \frac{1}{d}\sum_{k=1}^d \cos(t_k)$.
- $P^{2n}(0, 0) \sim \frac{1}{(2\pi)^d} \int_{[-\pi,\pi]^d} \left(\frac{1}{d}\sum_{k=1}^d \cos(t_k)\right)^{2n} d^d t \sim \frac{c_d}{n^{d/2}}$ as $n \to \infty$.
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Why simple random walk are studied?

- Mathematical interest: simple models with three interwoven structures:
  - low-level algebraic structure conveying combinatorial information,
  - high-level algebraic structure conveying geometric information,
  - stochastic structure adapted to the two previous structures.
- Discretised (in time/space) versions of stochastic processes, numerous interesting mathematical problems still open.
- Modelling transport (of energy, information, charge, etc.) phenomena
  - in crystals (metals, semiconductors, ionic conductors, etc.)
  - or on networks.
- Intervening in models described by PDE's involving a Laplacian hence in harmonic analysis
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Short algebraic reminder
Groups, groupoids and semigroupoids

Definition

Let $\Gamma \neq \emptyset$. $(\Gamma, \cdot)$ is a

semigroup monoid group	semigroupoid groupoid
if $\cdot : \Gamma \times \Gamma \to \Gamma$ and $\forall a, b, c \in \Gamma$	if $\exists \Gamma^2 \subseteq \Gamma \times \Gamma$ and $\cdot : \Gamma^2 \to \Gamma$
$(cb)a = c(ba)$	$(c, b), (b, a) \in \Gamma^2 \Rightarrow$ $(cb, a), (c, ba) \in \Gamma^2$ and $(cb)a = c(ba)$
$\exists! e \in \Gamma : ea = ae = a$	units not necessarily unique,
$\exists a^{-1} \in \Gamma : aa^{-1} = a^{-1}a = e$	$\exists a^{-1} : (a^{-1})^{-1} = a$, $(a, a^{-1}), (a^{-1}, a) \in \Gamma^2$ and $(a, b) \in \Gamma^2 \Rightarrow a^{-1}(ab) = b$; $(b, a) \in \Gamma^2 \Rightarrow (ba)a^{-1} = b$.

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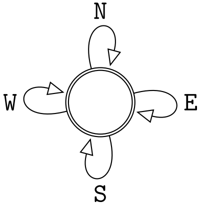
Monoidal closure of $\mathbb{A}$

$$\mathbb{A} = \{E, N, W, S\}; \ \mathbb{A}^* = \cup_{n=0}^{\infty} \mathbb{A}^n,$$

$$\mathbb{A}^0 = \{\varepsilon\}, \ \mathbb{A}^n = \{\boldsymbol{\alpha} = (\alpha_1, \ldots, \alpha_n), \alpha_i \in \mathbb{A}\}$$

$\mathrm{FA}_1 =$ 

Proposition

$(\mathbb{A}^*, \circ)$ *is a monoid, the monoidal closure of* $\mathbb{A}$.

$\alpha \circ \varepsilon = \varepsilon \circ \alpha = \alpha$. If $\alpha = EENNESW; \beta = WSN$ then
$\alpha \circ \beta = EENNESWWSN \neq WSNEENNESW = \beta \circ \alpha$.

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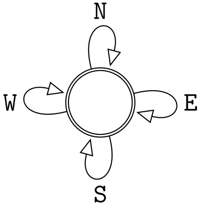

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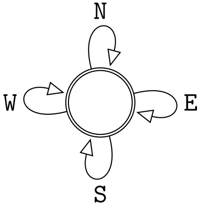

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Combinatorial information $\neq$ geometric information

- $\mathbb{A}^* \simeq$ path space. Combinatorial information encoded into the finite automaton FA. Paths define a regular language recognised by FA_1.
- Road map needed to translate into geometric information $E = a, W = a^{-1}; N = b, S = b^{-1}$ and relations on reduced words.

Example

$\mathbb{Z}^2 = \langle \mathbb{A} | \mathcal{R}_1 \rangle$: $\mathcal{R}_1 = \{aba^{-1}b^{-1} = e\}$ (Abelian).

$\mathbb{F}_2 = \langle \mathbb{A} | \mathcal{R}_2 \rangle$: $\mathcal{R}_2 = \emptyset$ (free).

$\mathbb{Z}^2$ and $\mathbb{F}_2$ have same combinatorial description but are very different groups.

Geometric information encoded into the group structure $\Gamma = \langle \mathbb{A} | \mathcal{R} \rangle$. Natural surjection $g : \mathbb{A}^* \twoheadrightarrow \Gamma$.

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$\mathbb{Z}^2 = \langle \mathbb{A} | \mathcal{R}_1 \rangle$: $\mathcal{R}_1 = \{aba^{-1}b^{-1} = e\}$ (Abelian).

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The Cayley graph of finitely generated groups

Definition

Let $\Gamma = \langle \mathbb{A} \,|\, \mathcal{R} \rangle$. The **Cayley graph** Cayley$(\Gamma, \mathbb{A})$ is the graph

- vertex set Γ and
- edge set the pairs $(x, y) \in \Gamma^2$ such that $y = ax$ for some $a \in \mathbb{A}$.

Remark

Since $\mathbb{A}$ symmetric, graph undirected.

Example

For $\mathbb{A} = \{a, b, a^{-1}, b^{-1}\}$,

- Cayley$(\mathbb{F}_2, \mathbb{A})$ is the homogeneous tree of degree 4,
- Cayley$(\mathbb{Z}^2, \mathbb{A})$ is the standard $\mathbb{Z}^2$ lattice with edges over n.n.

The probabilistic structure

- $\mu := (p_1, \ldots, p_{\mathrm{card}\mathbb{A}}) \in \mathcal{M}_1(\mathbb{A})$ transforms FA into PFA.
- Path space $\mathbb{A}^*$ acquires natural probability $\mathbb{P}^\mu(\{\alpha\}) = \prod_{i=1}^{|\alpha|} p_{\alpha_i}$.
- Due to the surjection g, PFA induces natural Markov chain (X_n):

$$\mathbb{P}(X_{n+1} = y | X_n = x) = \mu(\{x^{-1}y\}) = p_{x^{-1}y}, x, y \in \Gamma.$$

- Probabilistic structure adapted to combinatorial/geometric structure if $\operatorname{supp} \mu = \mathbb{A}$.
- When μ replaced by family $(\mu_x)_{x\in\Gamma}$ not necessarily $\operatorname{supp} \mu_x = \mathbb{A}, \forall x \in \Gamma$ (i.e. ellipticity can fail).
- Suppose there exist $a \in \mathbb{A}$ and $x, y \in \Gamma$, with $x \neq y$, such that

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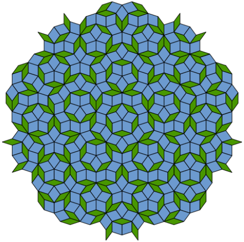
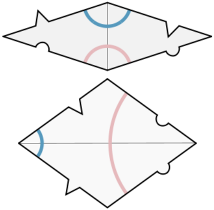
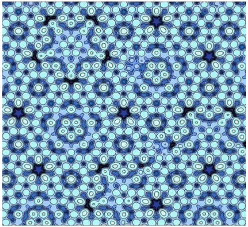
And when the graph is not a group?

R.w. on quasi-periodic tilings of $\mathbb{R}^d$ of Penrose type: the groupoid case







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- Spectral properties of Schrödinger operators on quasi-periodic structures.
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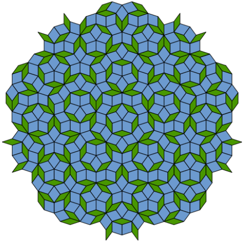
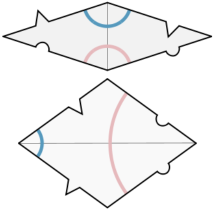
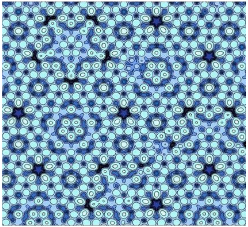

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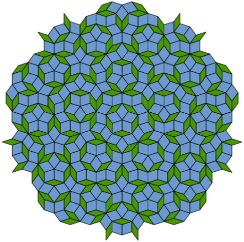
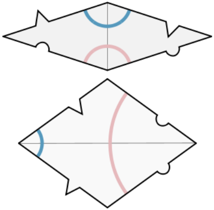
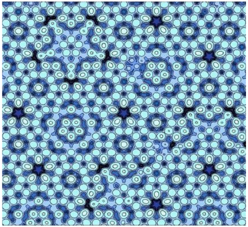

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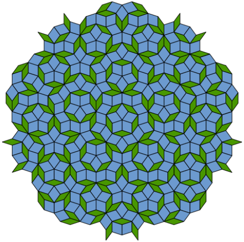
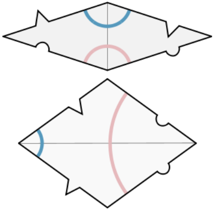
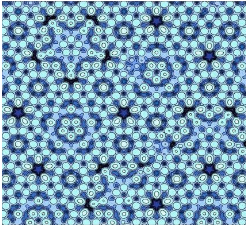

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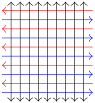	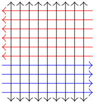	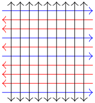

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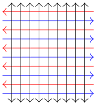	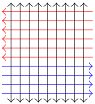	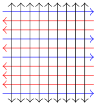

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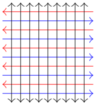	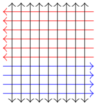	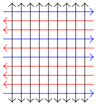

- Hydrodynamic dispersion in porous rocks Matheron and Marsily (1980), numerical simulations Redner (1997).
- Propagation of information on directed networks (pathway signalling networks in genomics, neural system, world wide web, etc.)
- Differential geometry, causal structures in quantum gravity.
- Random walks on semi-groupoids (and their C^*-algebras), failure of the reversibility condition.

And when the graph is not a group?

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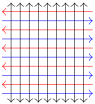	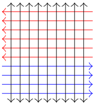	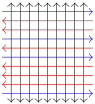

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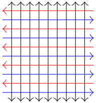	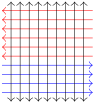	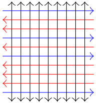

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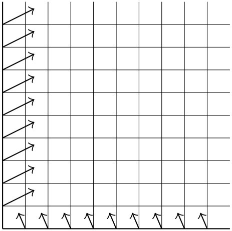
R.w. on quadrants with reflecting boundaries



In the interior of the quadrant: zero drift, non-diagonal covariance matrix.

- Many models in queuing theory.
- No algebraic structure encoding the geometry survives.
- Studied by Markov chain methods.
- Thoroughly studied with Lyapunov functions: Fayolle, Malyshev, Menshikov (1994), Asymont, Fayolle Menshikov (1995), Aspiandiarov, Iasnogorodsli, Menshikov (1996), Menshikov, P. (2002).

Results
For groupoids

Theorem (de Loynes, thm 3.1.2 in PhD thesis (2012)[a])

[a] Available at http://tel.archives-ouvertes.fr/tel-00726483.

The simple random walk on (adjacent edges of) a generic Penrose tiling of the d-dimensional space is

- *recurrent, if $d \leq 2$, and*
- *transient, if $d \geq 3$.*

Theorem (de Loynes (2014))

- *The asymptotic entropy of the simple random walk on generic Penrose tiling vanishes,*
- *hence, the tail and invariant σ-algebras are trivial.*

Results

For semi-groupoids

Theorem (Campanino and P., MPRF 2003)

The simple random walk

- *on the alternate 2-dimensional lattice is recurrent,*
- *on the half-plane one-way 2-dimensional lattice is transient,*
- *on the randomly horizontally directed 2-dimensional lattice, where $(\varepsilon_{x_2})_{x_2 \in \mathbb{Z}}$ is an i.i.d. $\{0, 1\}$-distributed sequence of average 1/2, is transient for almost all realisations of the sequence.*

Various subsequent developments in relation with this model: Guillotin and Schott (2006), Guillotin and Le Ny (2007), Pete (2008), Pène (2009), Devulder and Pène (2011), de Loynes (2012).

Results (cont'd)

For semi-groupoids

Theorem (Campanino and P., JAP 2014, in press)

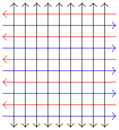
- $f : \mathbb{Z} \to \{-1, 1\}$ *a Q-periodic function (*$Q \geq 2$*):* $\sum_{y=1}^{Q} f(y) = 0$.
- $(\rho_y)_{y\in\mathbb{Z}}$ *i.i.d. Rademacher sequence.*
- $(\lambda_y)_{y\in\mathbb{Z}}$ *i.i.d.* $\{0, 1\}$*-valued sequence such that* $\mathbb{P}(\lambda_y = 1) = \frac{c}{|y|^\beta}$ *for large* $|y|$.
- $\varepsilon_y = (1 - \lambda_y)f(y) + \lambda_y \rho_y$.

1. *If* $\beta < 1$ *then the simple random walk is almost surely transient.*
2. *If* $\beta > 1$ *then the simple random walk is almost surely recurrent.*

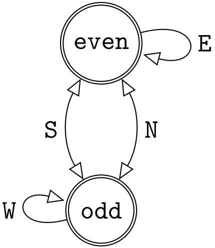
Remark

$\boldsymbol{\lambda}$ *deterministic sequence with* $\|\boldsymbol{\lambda}\|_1 < \infty \Rightarrow$ *recurrence. Nevertheless, there exist deterministic sequences with* $\|\boldsymbol{\lambda}\|_1 = \infty$ *leading to recurrence.*

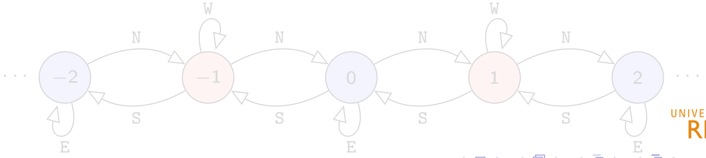
And when it is not a group?



$\mathsf{FA}_2 =$



- For alternate lattice, again a finite automaton, FA_2, governs combinatorics. E.g. starting at even, NSWWNW $\notin$ language.
- Vertical projection of walk = Markov chain on $\mathbb{Z}$ with transitions



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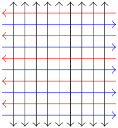

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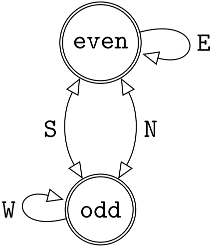

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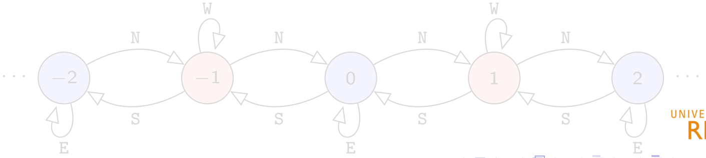

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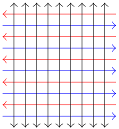

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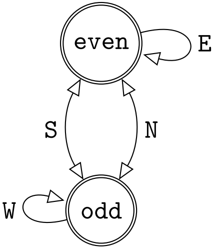

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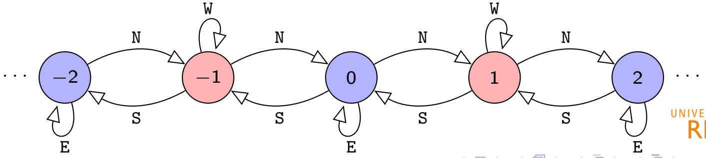

And when it is not a group? (cont'd)

- For alternate lattice $\Rightarrow$ path space generated by finite automaton $\Rightarrow$ admissible paths form regular language.
- For half-plane lattice $\Rightarrow$ path space generated by push down automaton $\Rightarrow$ admissible paths form context-free language.
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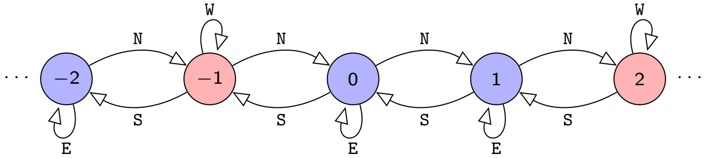

Two archetypal examples of (semi)groupoids

Directed graphs

Example

- Directed graph: $\mathbb{G} = (\mathbb{G}^0, \mathbb{G}^1, s, t)$ with $\mathbb{G}^0$ and $\mathbb{G}^1$ denumerable (finite or infinite) sets of vertices (paths of length 0) and edges (paths of length 1) and $s, t : \mathbb{G}^1 \to \mathbb{G}^0$ the source and terminal maps.
- For $n \geq 2$ define

$$\mathbb{G}^n = \{\alpha = \alpha_n \ldots \alpha_1, \alpha_i \in \mathbb{G}^1, s(\alpha_{i+1}) = t(\alpha_i)\} \subseteq (\mathbb{G}^1)^n,$$

  and $\mathrm{PS}(\mathbb{G}) = \cup_{n \geq 0} \mathbb{G}^n$ the path space of $\mathbb{G}$. Maps s, t extend trivially to $\mathrm{PS}(\mathbb{G})$.
- On defining $\Gamma = \mathrm{PS}(\mathbb{G})$, $\Gamma^2 = \{(\beta, \alpha) \in \Gamma \times \Gamma : s(\beta) = t(\alpha)\}$ and $\cdot : \Gamma^2 \to \mathbb{G}$ the left admissible concatenation, $(\Gamma, \Gamma^2, \cdot)$ is a semigroupoid with space of units $\mathbb{G}^0$.

Two archetypal examples of (semi)groupoids

Admissible words on an alphabet

Example

$\mathbb{A}$ alphabet, $A = (A_{b,a})_{a,b \in \mathbb{A}}$ with $A_{a,b} \in \{0,1\}$, $\mathbb{A}^0 = \{()\}$,
$\mathbb{A}^n = \{\alpha = (\alpha_n \cdots \alpha_1), \alpha_i \in \mathbb{A}\}$,

- set of words of arbitrary length $\mathbb{A}^* = \cup_{n \in \mathbb{N}} \mathbb{A}^n$ equipped with left concatenation is a monoid,
- $W_A(\mathbb{A}) = \{\alpha \in \mathbb{A}^* : A(\alpha_{i+1}, \alpha_i) = 1, i = 1, \ldots, |\alpha|\}$ (set of A-admissible words) is a semigroupoid with (β, α) composable pair if $A(\beta_1, \alpha_{|\alpha|}) = 1$.

Remark

A semigroupoid is not always a category. Consider, for example, $\mathbb{A} = \{a, b\}$ *and* $A = \begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix}$.

Constrained Cayley graphs

$$EW = WE = e, \; NS = SN = e,$$
$$E = a \Rightarrow W = a^{-1} \text{ and } N = b \Rightarrow S = b^{-1}.$$
$$\mathbb{A} = \{a, a^{-1}, b, b^{-1}\}.$$

Definition

Let $\mathbb{A}$ finite be given (generating) and $\Gamma = \langle \mathbb{A} \,|\, \mathcal{R} \rangle$. Let $c : \Gamma \times \mathbb{A} \to \{0, 1\}$ be a choice function. Define the constrained Cayley graph $\mathbb{G} = (\mathbb{G}^0, \mathbb{G}^1) = \mathrm{Cayley}_c(\Gamma, \mathbb{A}, \mathcal{R})$ by

- $\mathbb{G}^0 = \Gamma$,
- $\mathbb{G}^1 = \{(x, xz) \in \Gamma \times \Gamma : z \in \mathbb{A}; c(x, z) = 1\}$.
- $d_x^- = \mathrm{card}\{y \in \Gamma : (x, y) \in \mathbb{G}^1\}$.

Properties of constrained Cayley graphs

- $0 \leq d_x^- \leq \text{card}\mathbb{A}$.
- If $d_x^- = 0$ for some x, then x is a sink. All graphs considered here have $d_x^- > 0$.
- If $c \equiv 1$ then $(\mathbb{G}^1)^{-1} = \mathbb{G}^1$ (the graph is undirected).
- The graph can fail to be transitive. All graphs considered here are transitive i.e. for all $x, y \in \mathbb{G}^0$, there exists a finite sequence $(x_0 = x, x_1, \ldots, x_n = y)$ with $(x_{i-1}, x_i) \in \mathbb{G}^1$ for all $i = 1, \ldots, n$.
- Algebraic structure of $\text{Cayley}_c(\Gamma, \mathbb{A}, \mathcal{R})$: a groupoid or a semi-groupoid.

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Examples of semi-groupoids

Vertex set $\mathbb{X} = \mathbb{Z}^2$, i.e. for all $x \in \mathbb{X}$, we write $x = (x_1, x_2)$; generating set $\mathbb{A} = \{\mathbf{e}_1, -\mathbf{e}_1, \mathbf{e}_2, -\mathbf{e}_2\}$.

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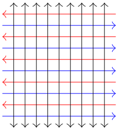
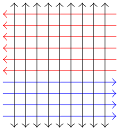
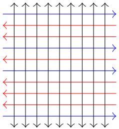
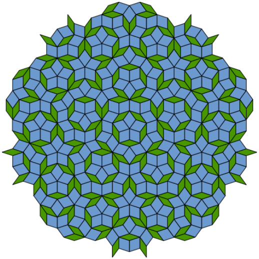
		
$c(\mathbf{x}, \mathbf{e_2}) = c(\mathbf{x}, -\mathbf{e_2}) = 1$ $c(\mathbf{x}, \mathbf{e_1}) = 1, x_2 \in 2\mathbb{Z}$ $c(\mathbf{x}, -\mathbf{e_1}) = 1, x_2 + 1 \in 2\mathbb{Z}$	$c(\mathbf{x}, \mathbf{e_2}) = c(\mathbf{x}, -\mathbf{e_2}) = 1$ $c(\mathbf{x}, \mathbf{e_1}) = 1, x_2 < 0$ $c(\mathbf{x}, -\mathbf{e_1}) = 1, x_2 \geq 0$	$c(\mathbf{x}, \mathbf{e_2}) = c(\mathbf{x}, -\mathbf{e_2}) = 1$ $c(\mathbf{x}, \mathbf{e_1}) = \theta_{\mathbf{x_2}}$ $c(\mathbf{x}, -\mathbf{e_1}) = 1 - \theta_{\mathbf{x_2}}$

For all three lattices: $\forall x \in \mathbb{Z}^{\mathbf{2}}, d_{\mathbf{x}}^{-} = 3$.

Here $\mathbb{G}^1 \subset \mathbb{G}^0 \times \mathbb{G}^0$. Hence maps s, t superfluous.

Example of groupoid

- Choose integer $N \geq 2$; decompose $\mathbb{R}^N = E \oplus E^\perp$ with $\dim E = d$ and $\dim E^\perp = N - d$, $1 \leq d < N$.
- K the unit hypercube in $\mathbb{R}^N$.
- $\pi : \mathbb{R}^N \to E$ and $\pi^\perp : \mathbb{R}^N \to E^\perp$ projections.
- For generic orientation of E and $t \in E_\perp$ let $\mathcal{K}_t := \{x \in \mathbb{Z}^N : \pi^\perp(E + t) \in \ \pi^\perp(K)\}$.
- $\pi(\mathcal{K}_t)$ is a quasi-periodic tiling of $E \cong \mathbb{R}^d$ (of Penrose type).
- For generic orientations of E, points in $\mathcal{K}_t$ are in bijection with points of the tiling.
- $\mathbb{A} = \{\pm \mathbf{e}_1, \ldots, \pm \mathbf{e}_N\}$.
- $c(x, z) = \mathbb{1}_{\mathcal{K}_t \times \mathcal{K}_t}(x, x + z)$, $z \in \mathbb{A}$.



$\mathrm{Cayley}_c(\mathbb{Z}^N, \mathbb{A})$

- $\mathrm{Cayley}_c(\mathbb{Z}^N, \mathbb{A})$ is undirected (groupoid).
- d_x^- can be made arbitrarily large.

Decomposition

into vertical skeleton and horizontally embedded process

- Condition the Markov chain $(\mathbf{M}_n)$ on the directed version of $\mathbb{Z}^2$ to perform vertical moves.
- The so conditionned process is a simple random walk (Y_n) on the vertical $\mathbb{Z}$. Denote $\eta_n(y)$ its occupation measure.
- Let $(\xi_n^{(y)})_{n\in\mathbb{N},y\in\mathbb{Z}}$ be a doubly infinite sequence of geometric r.v. of parameter $p = 1/3$.
- $X_n = \sum_{y\in\mathbb{Z}} \varepsilon_y \sum_{i=1}^{\eta_{n-1}(y)} \xi_i^{(y)}$ is the horizontally embedded walk, where ε_y direction of level y.

Lemma

Let $T_n = n + \sum_{y\in\mathbb{Z}} \sum_{i=1}^{\eta_{n-1}(y)} \xi_i^{(y)}$ the instant after n^{th} vertical move. Then

$$\mathbf{M}_{T_n} = (X_n, Y_n).$$

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- Let $(\xi_n^{(y)})_{n\in\mathbb{N},y\in\mathbb{Z}}$ be a doubly infinite sequence of geometric r.v. of parameter $p = 1/3$.
- $X_n = \sum_{y\in\mathbb{Z}} \varepsilon_y \sum_{i=1}^{\eta_{n-1}(y)} \xi_i^{(y)}$ is the horizontally embedded walk, where ε_y direction of level y.

Lemma

Let $T_n = n + \sum_{y\in\mathbb{Z}} \sum_{i=1}^{\eta_{n-1}(y)} \xi_i^{(y)}$ the instant after n^{th} vertical move. Then

$$\mathbf{M}_{T_n} = (X_n, Y_n).$$

Decomposition

into vertical skeleton and horizontally embedded process

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Comparison

Lemma

Let (σ_n) sequence of successive returns to 0 for (Y_n).

- *If (X_{σ_n}) is transient then (M_n) is transient.*
- *If $\sum_{n=0}^{\infty} \mathbb{P}_0(X_{\sigma_n} = 0|\mathcal{F} \vee \mathcal{G}) = \infty$ then $\sum_{l=0}^{\infty} \mathbb{P}(\mathbf{M}_l = (0,0)|\mathcal{F} \vee \mathcal{G}) = \infty$.*

$$\chi(\theta) = \mathbb{E}\exp(i\theta\xi) = \frac{q}{1 - p\exp(i\theta)} = r(\theta)\exp(i\alpha(\theta)), \quad \theta \in [-\pi, \pi],$$

where

$$r(\theta) = |\chi(\theta)| = \frac{q}{\sqrt{q^2 + 2p(1-\cos\theta)}} = r(-\theta);$$

$$\alpha(\theta) = \arctan\frac{p\sin\theta}{1 - p\cos\theta} = -\alpha(-\theta).$$

Notice that $r(\theta) < 1$ for $\theta \in [-\pi, \pi] \setminus \{0\}$.

Lemma

$$\mathbb{E}\exp(i\theta X_{\sigma_n}) = \mathbb{E}\left(\prod_{y\in\mathbb{Z}} \chi(\theta\varepsilon_y)^{\eta_{\sigma_n - 1}(y)}\right)$$
$$= \mathbb{E}\left[r(\theta)^{\sigma_n}\exp\left(\alpha(\theta)(\sum_{y\in\mathbb{Z}}\varepsilon_y\eta_{\sigma_n-1}(y))\right)\right].$$

Alternate and half-plane lattices

- For alternate lattice:
  $\sum_{n\in\mathbb{N}} \mathbb{P}(X_{\sigma_n} = 0) = \lim_{\epsilon\to 0} 2\int_\epsilon^\pi \frac{1}{\sqrt{1-r(\theta)^2}} d\theta = \infty.$
- For half-plane lattice:
  $\sum_{n\in\mathbb{N}} \mathbb{P}(\mathbf{M}_{\sigma_n} = (0,0)) = \lim_{\epsilon\to 0} \int_\epsilon^\pi [2\,\mathrm{Re}\,\chi(\theta)\frac{1}{1-g(\theta)}]d\theta = C < \infty.$
  Notice that $(X_{\sigma_n})_n$ are heavy-tailed symmetric $\mathbb{R}$-valued variables.
- Quid for randomly horizontally directed lattice? Very technical.

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Randomly horizontally directed lattices

Proof of transience ($\beta < 1$)

- Introduce $A_n = A_{n,1} \cap A_{n_2}$ and B_n with

$$A_{n,1} = \left\{ \omega \in \Omega : \max_{0 \le k \le 2n} |Y_k| < n^{\frac{1}{2}+\delta_1} \right\}$$

$$A_{n,2} = \left\{ \omega \in \Omega : \max_{y \in \mathbb{Z}} \eta_{2n-1}(y) < n^{\frac{1}{2}+\delta_2} \right\},$$

$$B_n = \left\{ \omega \in A_n : \left| \sum_{y \in \mathbb{Z}} \varepsilon_y \eta_{2n-1}(y) \right| > n^{\frac{1}{2}+\delta_3} \right\}.$$

- Estimate separately

$$p_{n,1} = \mathbb{P}(X_{2n} = 0, Y_{2n} = 0; B_n)$$
$$p_{n,2} = \mathbb{P}(X_{2n} = 0, Y_{2n} = 0; A_n \setminus B_n)$$
$$p_{n,3} = \mathbb{P}(X_{2n} = 0, Y_{2n} = 0; A_n^c).$$

- Establish that $\sum_n p_{n,1} < \infty$; $\sum_n p_{n,3} < \infty$ and for $\beta < 1$ also $\sum_n p_{n,2} < \infty$.

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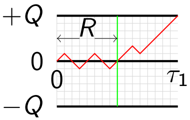
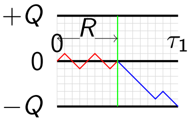
Randomly horizontally directed lattices

Proof of recurrence ($\beta > 1$)

- $\tau_0 \equiv 0$ and $\tau_{n+1} = \inf\{k : k > \tau_n, |Y_k - Y_{\tau_n}| = Q\}$ for $n \geq 0$.





- Periodise the lattice $\mathbb{Z}_Q = \mathbb{Z}/Q\mathbb{Z} = \{0, 1, \ldots, Q-1\}$ and define $N_n(\overline{y}) := \overline{\eta}_{\tau_{n-1}, \tau_n - 1}(\overline{y}) = \sum_{k=\tau_{n-1}}^{\tau_n - 1} \mathbb{1}_{\overline{y}}(\overline{Y}_k)$.
- $\mathbb{E}_0 N_1(\overline{y}) = \mathbb{E}_0\left(N_1(\overline{y}) \mid Y_{\tau_1} = Q\right) = \mathbb{E}_0\left(N_1(\overline{y}) \mid Y_{\tau_1} = -Q\right) = \frac{\mathbb{E}_0 \tau_1}{Q}$.

Randomly horizontally directed lattices

Proof of recurrence ($\beta > 1$) cont'd

- If $\beta > 1$ then $\sum_y \mathbb{P}(\lambda_y = 1) < \infty$.
- Hence $\exists L := L(\omega) < \infty$ s.t. $\lambda_y = 0$ for $|y| \geq L$.

$$F_{L,2n}(\omega) = \Big\{k : 0 \leq k \leq 2n-1; |Y_{\tau_k(\omega)}(\omega)| \leq L(\omega)Q; |Y_{\tau_{k+1}(\omega)}(\omega)| \leq L(\omega$$

$$G_{L,2n}(\omega) = \Big\{k : 0 \leq k \leq 2n-1; |Y_{\tau_k(\omega)}(\omega)| \geq L(\omega)Q; |Y_{\tau_{k+1}(\omega)}(\omega)| \geq L(\omega$$

- Write $\theta_k = X_{\tau_{k+1}} - X_{\tau_k}$ and observe that

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- Finally prove $\sum_{k\in\mathbb{N}} \mathbb{P}_0\left(X_{\sigma_k} = 0, Y_{\sigma_k} = 0 \,\middle|\, \mathcal{G}\right) = \infty$ a.s.

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