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Critical behaviour of the Edwards random walk in two dimensions: a case where the fractal and Hausdorff dimensions are not equal†

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Abstract. We study by Monte Carlo simulations, using two different algorithms, the Edwards walk in two dimensions and extract its critical exponents. We establish that the fractal dimension computed in terms of critical exponents is different from the Hausdorff dimension computed by measure-theoretic methods.

1. Introduction

Random walks arise in many different areas of applied mathematics, physics and chemistry (see des Cloizeaux and Jannink (1987), Le Gall (1987), McKenzie (1976) and references therein). The ordinary random walks (ORW) are well understood because they arise as discretisation of Brownian motion. This is not the case, however, for random walks with constraints like self-avoiding walks (SAW), excluded-volume walks, restricted walks and so on.

Random walks have non-trivial dimensional properties. For instance, two-dimensional ordinary (Brownian) walks, although topologically one-dimensional objects, are known to fill the plane. The Hausdorff dimension was introduced by Hausdorff (1919) and Besicovitch (1927) to describe this property. The Hausdorff dimension of random walks is usually identified with the inverse of their critical exponent ν. The critical exponent ν governs the asymptotic behaviour of the squared end-to-end Euclidean distance $\langle r^2\rangle_L$ of a random walk after L steps, i.e. $\langle r^2\rangle_L \sim L^{2\nu}$. The identification $d_H = 1/\nu$ can be rigorously proved to be exact in the case of Brownian motion (i.e. for an ordinary random walk) where $d_H = 2$ and $\nu = \frac{1}{2}$. In the case of constrained random walks it is only believed to be valid (heuristic arguments in favour of this conjecture are given in des Cloizeaux and Jannink (1987)). This belief is, however, so strong that, very often, values of $1/\nu$ are reported in the literature as the Hausdorff dimension, not even mentioning that $1/\nu$ is expected to be equal to the Hausdorff dimension! In this paper we shall concentrate on a special kind of constrained random walk, known as the Edwards random walk (Edwards 1965). Very often the Edwards random walk is quoted as the weakly self-avoiding walk (WSAW).

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Roughly speaking, the Edwards walk arises by assigning a weight $\exp(-\lambda N(\omega))$ to each ordinary random walk ω, $N(\omega)$ being the 'number of self-intersections' of the walk ω. The quotation marks are here to remind us that in the continuum this 'number' is not well defined since $N(\omega)$ is an infinite quantity (in any dimension $d \geqslant 2$) which can be given a rigorous meaning by renormalisation only.

In a more physical way, the Edwards walk provides a quite realistic model for polymers because the weight factor with positive λ implements somehow the excluded-volume effect. (There exist many constrained walks like the 'true' self-avoiding walk (Amit *et al* 1983), the polygonally restricted walk, the k-tolerant walk (Domb 1969) and so on; these objects are, however, distinct from the Edwards walk and must not be confused with it.) Moreover, the Edwards walk arises as a representation of quantum field theory, i.e. the Schwinger functions of lattice $(\boldsymbol{\phi} \cdot \boldsymbol{\phi})^2$ theory can be expanded (in the limit $N \to 0$) in terms of sums over the Edwards random walks (Aragão de Carvalho *et al* 1983, de Gennes 1972).

Intuitively, the Edwards walk is an interpolation between ordinary ($\lambda = 0$) and self-avoiding ($\lambda \to \infty$) walks. However, this intuition is misleading for various reasons.

(i) The limiting behaviour depends crucially on the dimension, d, of the lattice. It is believed—and this belief is supported by renormalisation group arguments (see Brydges and Spencer (1985), Derrida (1981), Duplantier (1986), Le Guillou and Zinn-Justin (1980), Hilhorst (1977) and references therein)—that for $d = 2$ or 3 and $\lambda > 0$, the critical behaviour of the Edwards walk is governed by the critical exponents of the self-avoiding walk. In other words, if $\lambda > 0$, no matter how small λ may be, it is driven by the renormalisation group flow to $\lambda = \infty$. In particular, we expect the exponent ν, associated with the end-to-end Euclidean distance, to be the same as for the SAW, i.e. $\nu = \frac{3}{4}$ in $d = 2$.

(ii) The fractal properties (given by d_{H}) of the Edwards walk are the same as for the ordinary random walk when $d = 2$ and $\lambda > 0$. This assertion is strongly supported by a rigorous result of Varadhan (1969) concerning the weakly self-avoiding renormalised random bridge (WSARB). This means (see §5) that the Hausdorff dimension, d_{H}, of the Edwards random walk is expected to be 2.

The previous remarks show why the intuition is misleading: in fact, the Edwards random walk behaves as an ORW with respect to its fractal properties and as a SAW with respect to its critical properties. This gives rise to an explicit contradiction because obviously $2 = d_{\mathrm{H}} \neq 1/\nu = \frac{4}{3}$.

It is firmly believed that the critical exponent $\nu = \frac{3}{4}$ for the SAW in two dimensions. A conjecture asserts it (Nienhuis 1982) and many subsequent numerical studies confirm it, such as exact enumeration results (Guttmann 1984), renormalisation group (Derrida 1981) and Monte Carlo simulations (Berretti and Sokal 1985, de Forcrand *et al* 1986). Although the critical exponents of the two-dimensional SAW are well known and calculated by many different methods, there is no explicit simulation of the Edwards walk in two dimensions. Some preliminary (poor) results for the Edwards walk in four dimensions are known (Guha and Roberts 1986) but they do not cover the two-dimensional case where the theoretical predictions are contradictory. The purpose of this paper is therefore twofold:

(i) to compute numerically the critical exponents for the Edwards walk in $d = 2$ in order to numerically test the validity of the renormalisation group arguments, and

(ii) to establish in a clearcut way whether the value $\nu = \frac{1}{2}$ (dictated by Hausdorff dimension equality) or $\nu = \frac{3}{4}$ (dictated by renormalisation group arguments) actually occurs.

The paper is organised as follows. In § 2 we define the model and recall briefly the definitions of the critical exponents. In § 3 we present two different Monte Carlo algorithms we used to simulate the Edwards walks both in grand canonical and canonical ensembles. In § 4 we describe the results obtained with both methods. Finally in § 5 we discuss the results and present some further developments.

2. Definition of the model and the critical exponents

Let us first introduce some notation and definitions. We consider an ordinary random walk, ω, on the $\mathbb{Z}^2$ lattice, i.e. a map $\omega: \mathbb{N}_0 \to \mathbb{Z}^2$ such that $\omega(0)=0$ and $|\omega(i)-\omega(i+1)|=1$, where $|\cdot|$ denotes the Euclidean distance on $\mathbb{Z}^2$ (by abuse of language we identify the walk with the map describing it). $|\omega|$ denotes the length of the walk ω, i.e. the number of bonds $(\omega(i), \omega(i+1))$ in the walk. The number of self-intersections, $N(\omega)$, of a walk ω is defined as $N(\omega)=\mathrm{card}\{i \in \mathbb{N}_0 \mid \exists j > i \text{ such that } \omega(j)=\omega(i)\}$.

The purpose of the model is to give a weight of the form $\exp(-\lambda N(\omega))$ to each ordinary random walk, thus suppressing walks with many self-intersections (λ being a positive real constant).

The probability on the space of discrete random walks is hence given by

$$\pi_{\beta,\lambda}(\omega)=Z(\beta,\lambda)^{-1}\beta^{|\omega|}\exp(-\lambda N(\omega)) \tag{1}$$

$Z(\beta,\lambda)$ being the statistical sum $Z(\beta,\lambda)=\Sigma_\omega\, \beta^{|\omega|}\exp(-\lambda N(\omega))$ at monomer activity β. The probability (1) completely determines the model in the sense that all the thermodynamic quantities can be computed as averages with respect to $\pi_{\beta,\lambda}$. For example, the mean length $\langle L\rangle_{\beta,\lambda}$ of the walk is given by

$$\langle L\rangle_{\beta,\lambda}=\sum_\omega \pi_{\beta,\lambda}(\omega)|\omega|.$$

For every λ, there is a critical $\beta_{\mathrm{cr}}(\lambda)$ such that $\lim_{\beta\to\beta_{\mathrm{cr}}(\lambda)}\langle L\rangle_{\beta,\lambda}=\infty$.

The critical exponents are defined as usually. Let c_L denote the weighted number of walks starting at the origin of the lattice, ending anywhere on the lattice, and having length L, defined as

$$c_L=\sum_{\omega:|\omega|=L}\exp(-\lambda N(\omega)).$$

The asymptotic behaviour of c_L for large L is governed by a critical exponent γ as

$$c_L \sim \mu^L L^{\gamma-1}.$$

The mean end-to-end Euclidean distance $\langle r^2\rangle_L$ for walks of length L behaves, asymptotically for large L, as

$$\langle r^2\rangle_L \sim L^{2\nu}$$

and the mean number of self-intersections $\langle N\rangle_L$ for walks of length L as

$$\langle N\rangle_L \sim L^{\rho}.$$

Remark. The symbol $\langle\cdot\rangle_L=\Sigma_{\omega:|\omega|=L}(\cdot)/c_L$ denotes the canonical average at fixed length L, not to be confused with $\langle\cdot\rangle_{\beta,\lambda}=\Sigma_\omega(\cdot)\pi_{\beta,\lambda}$ that denotes the grand canonical average.

A priori, one expects the effective coordination number μ and the critical exponents γ, ν and ρ to be functions of λ. But the renormalisation group arguments attest that γ and ν do not depend on λ, for $\lambda > 0$.

Finally, the fractal properties of the walk are supposed to be governed by the critical exponent ν, in the sense that the Hausdorff dimension, d_H, is equal to $1/\nu$ as explained in the introduction.

Remark. The model described above is discrete ($\mathbb{N}_0$ and $\mathbb{Z}^2$ are discrete spaces), but it has of course a continuous version. It is well known that an ordinary random walk has a continuous limit called the Brownian path. Similarly, a two-dimensional Edwards walk has a continuous version which could be called the Edwards path. In particular, if λ is positive the probability (1) has (in a certain sense) a continuous limit which we shall call the Edwards measure by analogy with the Wiener measure which describes Brownian paths. It is this continuous limit which must be used to compute analytically the Hausdorff dimension (see Koukiou *et al* (1988) for more details).

3. The algorithms

We use two different algorithms to simulate the Edwards walk on the two-dimensional lattice with coordination number q. In the grand canonical simulation neither the length L nor the number of self-intersections N is kept fixed. In the canonical simulation, L is kept fixed and N is allowed to vary.

3.1. The grand canonical simulation

We use a dynamical local algorithm to generate walks having (1) as their unique equilibrium distribution. The algorithm allows only two elementary moves: either add a link of the $\mathbb{Z}^2$ lattice at the end of the walk or delete the last link of the walk. Thus, the elementary moves are the same as for the algorithm of Berretti and Sokal (1985) for generating SAW; the transition probabilities are radically different, however. The algorithm can be described by the following flow chart.

Begin from the empty walk anchored at the origin. Choose a monomer activity $\beta \leq 1$ and a self-repulsion constant λ.
Repeat many times:
{Choose a random number r uniformly distributed in $[0, 1]$.
If $r > \frac{1}{2}$
  then try to append a link l in one of the $q = 2d$ possible equivalent directions at the end of the walk ω;
    Compute $\delta = N(\omega \cup l) - N(\omega)$;
    Choose a random number $s \in [0, 1]$;
    If $s < A(\beta, \lambda, \delta)$
      then change ω to $\omega' = \omega \cup l$;
      else keep ω once more, i.e. $\omega' = \omega$;
  else try to delete last bond;
    Choose a random number $t \in [0, 1]$;
    If $t < S(\beta, \lambda, \delta)$ and $\omega \neq \varnothing$
      then delete the last bond;
      else keep ω once more.}

The functional forms of $A(\beta, \lambda, \delta)$ and $S(\beta, \lambda, \delta)$ are not uniquely determined. A convenient choice is

$$A(\beta, \lambda, \delta) = \begin{cases} \beta \exp(-\lambda\delta) & \text{if } \lambda > 0 \\ \beta & \text{otherwise} \end{cases}$$

and

$$S(\beta, \lambda, \delta) = \begin{cases} q^{-1} & \text{if } \lambda > 0 \\ q^{-1} \exp(\lambda\delta) & \text{otherwise.} \end{cases}$$

Remark. The algorithm described above allows spikes to occur, i.e. after a step we can have an immediate return thus forming a loop. An additional test can be included in the 'insert' part of the algorithm to prevent spikes. We verified that the suppression of spikes does not modify the critical exponents. The results quoted in § 4 are for the algorithm without spikes.

The previous algorithm defines a Markov process on the space of all possible walks with transition probability $p(\omega \to \omega')$ given by

$$p(\omega \to \omega') = \begin{cases} \dfrac{1}{2q} P_{\beta\lambda}(\omega) & \text{if } \omega' = \omega \cup l \\ \frac{1}{2} Q_{\beta\lambda}(\omega)(1 - \delta_{|\omega|,0}) & \text{if } \omega' = \omega \backslash m \\ \dfrac{1}{2q} \sum_l (1 - P_{\beta\lambda}(\omega)) + \frac{1}{2}(1 - Q_{\beta\lambda}(\omega))(1 - \delta_{|\omega|,0}) + \frac{1}{2}\delta_{|\omega|,0} & \text{if } \omega' = \omega \\ 0 & \text{otherwise.} \end{cases}$$

In this formula l denotes one of the links adjacent to the last point of the walk; m is the last link (with respect to the natural ordering obtained by scanning the walk starting from the origin of the lattice) belonging to the walk. $P_{\beta\lambda}(\omega) = A(\beta, \lambda, N(\omega \cup l) - N(\omega))$ and $Q_{\beta\lambda}(\omega) = S(\beta, \lambda, N(\omega \cup l) - N(\omega))$. The matrix $p(\omega \to \omega')$ is a stochastic matrix because it has non-negative matrix elements and

$$\sum_{\omega'} p(\omega \to \omega') = 1.$$

It accepts $\pi_{\beta,\lambda}(\omega)$ as left eigenvector with eigenvalue 1 (weak form of detailed balance), i.e.

$$\sum_{\omega} \pi_{\beta,\lambda}(\omega) p(\omega \to \omega') = \pi_{\beta,\lambda}(\omega').$$

Moreover, the process is aperiodic and ergodic. Therefore $\pi_{\beta,\lambda}(\omega)$ is the unique equilibrium distribution for the Markov process defined in terms of the stochastic matrix $p(\omega \to \omega')$. Henceforth, the empirical distribution obtained from the statistical sample converges to $\pi_{\beta,\lambda}(\omega)$ for every initial condition. This remark guarantees that we can use this algorithm for a Monte Carlo simulation.

Successive Monte Carlo steps are highly correlated; it is pointless to store on disk the data after every step. We can estimate, however, the Monte Carlo steps we have to skip for every datum stored. In fact, all memory is lost each time we reach the

empty walk. For a process with $\langle L\rangle$ as mean length this happens roughly every $\langle L\rangle^2$ MC steps, i.e. the autocorrelation time $\tau \sim K\langle L\rangle^2$. As pointed out in Berretti and Sokal (1985) this heuristic argument gives only a first estimate; the proportionality constant K may be rather large. We decided to skip 10 000 MC steps for every stored walk. The resulting sample is still correlated but we analyse its statistical properties experimentally to determine the autocorrelation time as explained in § 4.

As data structure we used a matrix monitoring the site occupation with the corresponding multiplicity. Hence, the operations needed at each MC step are independent of the length of the walk.

The advantage of this algorithm is that its overall complexity (de Forcrand *et al* 1988) is proportional to $\langle L\rangle^2$; therefore it allows us to approach the asymptotic regime within a reasonable computational time.

3.2. The canonical simulation

We use a dynamical algorithm related to the pivot algorithm of (Lal 1969, Madras and Sokal 1987). In contrast to the grand canonical algorithm this one is non-local and it generates walks with a fixed length L. The equilibrium probability is given by

$$\pi_L(\omega) = Z_L(\lambda)^{-1} \exp(-\lambda N(\omega)) \tag{2}$$

where

$$Z_L(\lambda) = \sum_{\omega:|\omega|=L} \exp(-\lambda N(\omega)).$$

The algorithm goes as follows.

Begin from an arbitrary walk, ω, of length L, with a given number, $N(\omega)$, of self-intersections. (Usually a straight rod with $N(\omega)=0$.) Fix its zeroth point at the origin of the lattice and compute the number of its self-intersections $N(\omega)$.
Repeat many times: {
1. Choose at random an integer k in the set $\{1,\ldots,L-1\}$ and use the kth vertex of the walk as a pivot. This vertex divides the walk in two parts. Let S be the shortest part, i.e. the part of shortest length.
2. Choose with equal probability one of the symmetry operations, g, of the symmetry group, G, of the $\mathbb{Z}^2$ lattice.
3. Apply g to the shortest part S of the walk ω to obtain a candidate walk ω_{trial} with number of self-intersections $N(\omega_{\text{trial}})$. Compute the difference, δ, of the numbers of self-intersections, $\delta = N(\omega_{\text{trial}}) - N(\omega)$.
4. Draw a random number r uniformly distributed in $[0, 1]$.
If $r \leqslant \min\{1, \exp(-\lambda\delta)\}$
  then accept the trial, i.e. $\omega' = \omega_{\text{trial}}$ and parallel translate ω' (if it becomes necessary to recentre it);
  else reject the trial, i.e. $\omega' = \omega$.}

Remark. The algorithm can be simplified in two respects.

(i) G can be any subset of the full symmetry group of the lattice, provided that ergodicity is ensured. Here we assign equal weight to rotations of $\pi/2$, π and $3\pi/2$. A single rotation of $\pm\pi/2$ would be sufficient, however.

(ii) The algorithm works equally for $\lambda < 0$ or $\lambda > 0$. However, for $\lambda > 0$ it is much more efficient to take step 4 before step 3 in the algorithm. Then one can compute the maximum allowable number of self-intersections N_{max}. As soon as the current count of $N(\omega_{trial})$ exceeds N_{max}, counting stops and ω_{trial} is rejected.

In a similar way as for the grand canonical algorithm the reader can convince him/herself that this algorithm is Markovian, ergodic and fulfils the weak form of detailed balance. It admits (2) as the unique equilibrium probability.

The merit of this algorithm comes from its global moves, which decorrelate very quickly global observables such as the end-to-end distance. In fact, with the second modification mentioned in the remark above (for $\lambda > 0$), the computer time per independent datum grows only slightly faster than L. Here we wanted to compare the two approaches and be able to extract results conveniently.

4. Results

We performed various simulations using both algorithms at different values of the parameters β and λ. Since both the analysis and the addressed quantities differ in the two cases, we present the results separately.

4.1. Grand canonical simulation

For each λ, we chose a β near $\beta_{cr}(\lambda)$ such that $\langle L \rangle \sim 150$ and we did approximately 5×10^9 MC steps, recording every 10^4 steps. When analysing the results we always skipped the first 10 000 records, corresponding to 10^8 MC steps, to allow the system to thermalise. We use periodic boundary conditions for the monitoring matrix. For most simulations we used a monitoring matrix of 1024×1024; for the simulations very close to the critical point this matrix was 4096×4096. Thus this matrix is so huge that we can safely say that the walk never touches the borders during the simulation and in fact we simulate in an infinite box from the point of view of the walk. This simulation needs 72 μs per MC step on a VAX-8550 computer.

The method we used to extract μ, γ, ν from the data is described in Berretti and Sokal (1985) and de Forcrand *et al* (1986, 1987). We simply mention that μ and γ are determined simultaneously by maximum likelihood fit and ν by least squares fit as for the SAW. To extract ρ we used the same method as for ν. The fit program allows us to select intervals of walk lengths of the form $[L_{min}, L_{max}]$ and fit uniquely inside them to obtain an effective value of ν. If the fitted values do not change significantly when this interval slides through all the obtained values of lengths, we say that the asymptotic regime is obtained. The asymptotic regime is attained for bigger and bigger L as λ diminishes and we must go to L values as large as 2000 to attain the asymptotic regime for $\lambda \simeq 0.1$.

The systematic errors are computed in the standard way (de Forcrand *et al* 1986, 1987).

To take into account the statistical correlation of the data we determined explicitly the autocorrelation time τ. This autocorrelation time is $\tau \sim K\langle L \rangle^2$ with $K \simeq 5$. It seems that the proportionality constant is somewhat smaller than the corresponding quantity for the SAW using the algorithm of Berretti and Sokal (1985). This may be due to the fact that the Edwards walk is less constrained than SAW and hence it decorrelates more

rapidly. The determination of this parameter is not so accurate to decide if there is a systematic trend with the variation of λ.

The naive errors of the fitting procedure are multiplied by $\sqrt{2\tau}$ to give the statistical errors.

Figure 1 plots $\frac{1}{2}\ln\langle r^2\rangle_L$ against $\ln L$ to give an idea of the quality of the simulation and table 1 summarises the results obtained for different λ.

As expected from the renormalisation group, the values of the critical exponents γ and ν do not show any systematic dependence on λ. The values obtained lie within less than a standard deviation to the corresponding critical exponents for the self-avoiding walk. The effective coordination number μ shows however a systematic variation with λ and figure 2 exhibits this trend. The critical monomer activity, β_{cr}, is equal to $1/\mu$ and figure 2 allows us to read the critical activity as a function of λ. A similar figure plotting the critical activity as a function of λ for the Edwards walk in four dimensions is presented as the 'phase diagram' in Guha and Roberts (1986).

4.2. Canonical simulation

For each value of λ we did five simulations with L fixed at 200, 400, 800, 1600 and 3200. Each simulation consists of approximately 15×10^6 MC steps. For all practical

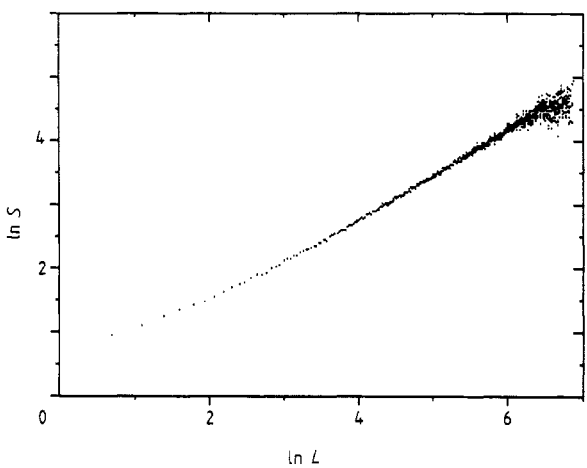

Figure 1. Plot of the function $\ln S = \ln\sqrt{\langle r^2\rangle_L}$ against $\ln L$ for $\beta = 0.365$ and $\lambda = 0.5$.

Table 1. The values of the effective coordination number μ and the critical exponents γ, ν and ρ for different values of λ as computed by grand canonical simulation. The format is central value ± statistical error ± systematic error. Numbers in parentheses in the first column give the value of $\exp(-\lambda)$.

λ	μ	γ	ν	ρ
0.105 (0.9)	$2.8993\pm0.0001\pm0.0010$	$1.329\pm0.002\pm0.076$	$0.762\pm0.037\pm0.012$	$1.062\pm0.049\pm0.016$
0.357 (0.7)	$2.7847\pm0.0001\pm0.0010$	$1.386\pm0.002\pm0.080$	$0.750\pm0.010\pm0.012$	$0.987\pm0.005\pm0.009$
0.693 (0.5)	$2.7160\pm0.0001\pm0.0010$	$1.431\pm0.003\pm0.106$	$0.734\pm0.013\pm0.013$	$0.999\pm0.010\pm0.011$
1.203 (0.3)	$2.6760\pm0.0001\pm0.0006$	$1.249\pm0.025\pm0.052$	$0.743\pm0.011\pm0.017$	
2.303 (0.1)	$2.6485\pm0.0001\pm0.0011$	$1.321\pm0.025\pm0.027$	$0.772\pm0.012\pm0.030$	$0.999\pm0.020\pm0.056$

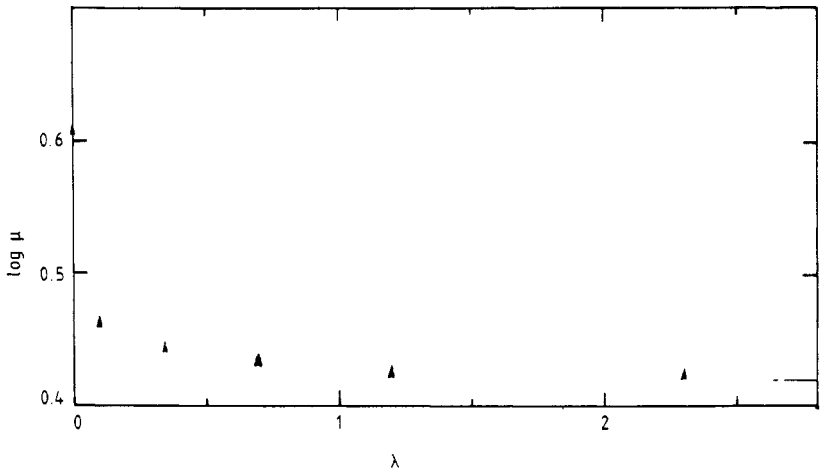

Figure 2. The effective coordination number μ as a function of λ. The value at $\lambda = 0$ is exactly known, $\mu = 4$, and the asymptotic value at $\lambda \to \infty$ is known from independent works (Berretti and Sokal 1985). The small horizontal line at the right-hand bottom corner of the figure corresponds to $\mu(\lambda = \infty) = 2.6382$.

purposes the autocorrelation time estimated out of our data can be taken equal to 40 000 MC steps (independent of L). This autocorrelation time proves unexpectedly large for this method. The first 2×10^6 MC steps are discarded to allow the thermalisation of the system. Measurements are taken every 100 MC steps.

For each L we compute $\langle r^2 \rangle_L$ and $\langle N \rangle_L$. Then, assuming asymptotic formulae to be valid we extract effective values for ν_i and ρ_i for the intervals $[200 \times 2^{i-1}, 200 \times 2^i]$, $i = 1, \ldots, 4$. We observe that ν_i and ρ_i tend to asymptotic values as i increases. The smaller λ is the slower is the rate of convergence of the sequences of effective critical exponents. Figure 3 shows this sequence of values of ν_i for the worst case we simulated, corresponding to $\lambda = 0.1$.

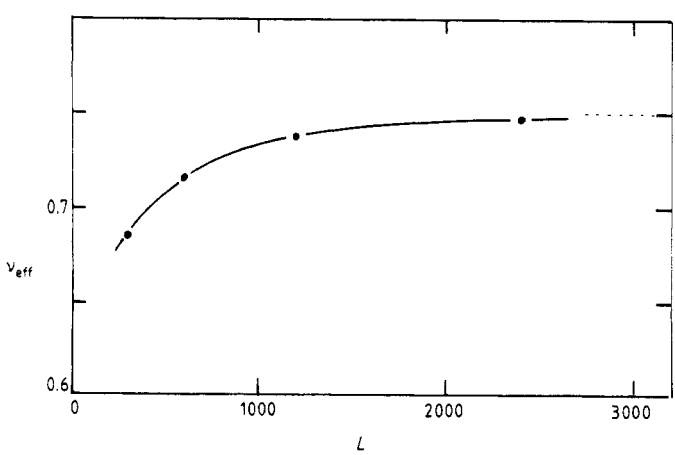

Figure 3. The sequence of effective values of ν (computed in the intervals $[100 \times 2^{i-1}, 100 \times 2^i]$) plotted against the median values L of the corresponding intervals. The broken line at the end corresponds to the asymptotic value $\nu = \frac{3}{4}$.

Table 2. The critical exponents ν and ρ as computed by canonical simulation.

λ	ν	ρ
0.1	$0.746 \pm 0.050 \pm 0.010$	$1.004 \pm 0.050 \pm 0.010$
−0.1	$0.05 \pm 0.05 \pm 0.01$	$1.066 \pm 0.050 \pm 0.010$

Table 3. The acceptance rate for different values of λ.

		L				
λ	g	200	400	800	1600	3200
−0.1	$\pi/2$	0.7531	0.6085	0.4460	0.3149	0.2183
−0.1	π	0.7030	0.5386	0.3652	0.2396	0.1582
+0.1	$\pi/2$		0.7215	0.6352	0.5534	0.4806
+0.1	π		0.6424	0.5265	0.4150	0.3167

Table 2 summarises the results (obtained by canonical simulation) for the critical exponents ν and ρ and for two different values of λ. The quoted values are those obtained for the effective exponents in the region [1600, 3200].

We also measured the acceptance fraction $f_L(g)$ of the proposed transformation g (g from step 2 of the algorithm above) as a function of g and L. We confirm the disturbing fact, observed in Madras and Sokal (1987) for the SAW, that the asymptotic L behaviour of these acceptance rates depends on g. Specifically, we observed that $f_L(\pi/2)/f_L(\pi)$ keeps increasing with L, pointing to an unsuspected persistency in the orientation of the walk. The acceptance rates we measured are reported in table 3 for further investigation. The acceptance rate is expected to behave as $f_L \sim L^{-p}$ for large L. It turns out that the critical exponent p depends not only on g as pointed out in Madras and Sokal (1987) but also on λ, i.e. $p = p(g, \lambda)$. We find $p(\pi, -0.1) = 0.6$ and $p(\pi/2, 0.1) = 0.2$; the value obtained for $p(\pi, 0.1)$ is in agreement with the value of $\gamma - 1$ computed by grand canonical simulation.

5. Discussion and further developments

Results from the two different simulations are compatible and mutually supported. Comparing ν and ρ for the two different simulations we conclude that, for $\lambda > 0$, $\nu = \frac{3}{4}$ and $\rho = 1$.

The value $\nu = \frac{3}{4}$ is the one expected by renormalisation group arguments (Le Guillou and Zinn-Justin 1980) and even Flory-type arguments (Flory 1967). Our results completely support the renormalisation group arguments.

Now, we can rigorously prove that the Hausdorff dimension for the Edwards walk is $d_H = 2$. The rigorous proof of this fact is outside the scope of this numerically oriented paper (see Koukiou *et al* (1988) for mathematical details). Intuitively we can say that the Hausdorff dimension is a metric property of sets. That means that we can consider the trajectory of the Edwards walk as a subset of the real space without wondering about its statistical weight. Then we can define the Hausdorff dimension of this set. Now the Edwards measure being absolutely continuous with respect to the

Wiener measure, the previous set surely occurs when considering Brownian motion instead of the Edwards walk. The Brownian path is known to have Hausdorff dimension equal to 2. We conclude that the Edwards path has also a Hausdorff dimension equal to 2.

The obtained numerical results, strengthened by renormalisation group arguments, establish in a clearcut way that $d_H \neq 1/\nu$ for the Edwards walk in two dimensions.

Some cases, mainly deterministic, are known in the literature where the various fractal dimensions do not coincide (Mandelbrot 1986). The Edwards walk provides a non-deterministic example where this discrepancy between the different fractal dimensions occurs. Although it is clear that $1/\nu$ has a physical meaning, since it describes the large-scale behaviour of the walk, it is not obvious whether this quantity has any sensible mathematical interpretation as a fractal dimension.

The case $\lambda < 0$ is also very interesting to study; it corresponds to a self-attracting polymer. In this case one expects that the system collapses to a compact conglomerate. Intuitively one expects that the critical exponent $\nu = 0$, i.e. the polymer does not expand to infinity but remains in a bounded domain. Actually $\nu = 0$ does not exclude an infinite extent of the polymer with logarithmic growth. We conjecture, and this conjecture is supported by some intuitive probabilistic arguments, that, for λ slightly negative, the polymer extends to infinity logarithmically and for λ negative enough, it remains with probability 1 inside a bounded domain. However, one has to remember that there is no theorem proving the existence of the 'Edwards' measure in the case $\lambda < 0$.

As for the exponent ρ, we found that $\rho = 1$ for all $\lambda > 0$. Following Varadhan (1969) one easily proves that, for Brownian paths on a finite time interval T, $N(\omega) \sim T$ (in two dimensions). Hence, the critical exponent ρ is driven by the renormalisation group flow to its Gaussian value.

Finally, we remark that the effective coordination number, μ, varies in a systematic way with λ. One need not be impressed, however, by this dependence. The value of μ may be fixed arbitrarily to any positive value just by a finite renormalisation of the number of self-intersections. In the continuum the number of self-intersections is an infinite quantity and must be renormalised. Therefore the finite part of the renormalised number of self-intersections is defined only up to an additive constant. This finite renormalisation does not change the physical model but only the renormalisation condition. This arbitrariness is a well known phenomenon in quantum field theory (see de Calan (1982) for a pedagogical explanation).

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