Type and type transition for random walks on finitely presented semi-groupoids

Classical probabilities with some pertinence to quantum evolution

Dimitri Petritis

Institut de recherche mathématique
Université de Rennes 1 and CNRS (UMR 6625)
France

QP33, CIRM, 4 Octobre 2012

What is the type problem for random walks?

- How often does a random walker on a denumerably infinite graph $\mathbb{X}$ returns to its starting point?
- For $\mathbb{X} = \mathbb{Z}^d$ with symmetric jumps on n.n. problem solved [1] by direct combinatorial and Fourier estimates.
- **Distinctive property of the simple random walk on $\mathbb{Z}^d$:** Abelian group of finite type generated by the support of the law of the simple random walk, the graph on which r.w. evolves is the Cayley($\mathbb{Z}^d$, supp μ).

[1] Georg Pólya, Über eine Aufgabe der Wahrscheinlichkeitsrechnung betreffend die Irrfahrt im Straßennetz, Ann. Math. (1921).

What is the type problem for random walks?

- How often does a random walker on a denumerably infinite graph $\mathbb{X}$ returns to its starting point?
- For $\mathbb{X} = \mathbb{Z}^d$ with symmetric jumps on n.n. problem solved [1] by direct combinatorial and Fourier estimates.
- **Distinctive property of the simple random walk on $\mathbb{Z}^d$:** Abelian group of finite type generated by the support of the law of the simple random walk, the graph on which r.w. evolves is the Cayley($\mathbb{Z}^d$, supp μ).

[1] Georg Pólya, Über eine Aufgabe der Wahrscheinlichkeitsrechnung betreffend die Irrfahrt im Straßennetz, Ann. Math. (1921).

What is the type problem for random walks?

- How often does a random walker on a denumerably infinite graph $\mathbb{X}$ returns to its starting point?
- For $\mathbb{X} = \mathbb{Z}^d$ with symmetric jumps on n.n. problem solved [1] by direct combinatorial and Fourier estimates.
- **Distinctive property of the simple random walk on $\mathbb{Z}^d$:** Abelian group of finite type generated by the support of the law of the simple random walk, the graph on which r.w. evolves is the Cayley$(\mathbb{Z}^d, \operatorname{supp} \mu)$.

[1] Georg Pólya, Über eine Aufgabe der Wahrscheinlichkeitsrechnung betreffend die Irrfahrt im Straßennetz, Ann. Math. (1921).

Why random walk are studied?

- Mathematical interest: simple models with three interwoven structures:
  - low-level algebraic structure conveying combinatorial information,
  - high-level algebraic structure conveying geometric information,
  - stochastic structure adapted to the two previous structures.
- Modelling transport phenomena
  - in crystals (metals, semiconductors, ionic conductors, etc.)
  - or on networks.
- Intervening in all models described by differential equations involving a Laplacian (cassical electrodynamics, statistical mechanics, quantum mechanics, quantum field theory, etc.)
- Discretised (in time/space) versions of stochastic processes.

Why random walk are studied?

- Mathematical interest: simple models with three interwoven structures:
  - low-level algebraic structure conveying combinatorial information,
  - high-level algebraic structure conveying geometric information,
  - stochastic structure adapted to the two previous structures.
- Modelling transport phenomena
  - in crystals (metals, semiconductors, ionic conductors, etc.)
  - or on networks.
- Intervening in all models described by differential equations involving a Laplacian (cassical electrodynamics, statistical mechanics, quantum mechanics, quantum field theory, etc.)
- Discretised (in time/space) versions of stochastic processes.

UNIVERSITÉ DE RENNES 1

Why random walk are studied?

- Mathematical interest: simple models with three interwoven structures:
  - low-level algebraic structure conveying combinatorial information,
  - high-level algebraic structure conveying geometric information,
  - stochastic structure adapted to the two previous structures.
- Modelling transport phenomena
  - in crystals (metals, semiconductors, ionic conductors, etc.)
  - or on networks.
- Intervening in all models described by differential equations involving a Laplacian (cassical electrodynamics, statistical mechanics, quantum mechanics, quantum field theory, etc.)
- Discretised (in time/space) versions of stochastic processes.

Why random walk are studied?

- Mathematical interest: simple models with three interwoven structures:
  - low-level algebraic structure conveying combinatorial information,
  - high-level algebraic structure conveying geometric information,
  - stochastic structure adapted to the two previous structures.
- Modelling transport phenomena
  - in crystals (metals, semiconductors, ionic conductors, etc.)
  - or on networks.
- Intervening in all models described by differential equations involving a Laplacian (cassical electrodynamics, statistical mechanics, quantum mechanics, quantum field theory, etc.)
- Discretised (in time/space) versions of stochastic processes.

Why random walk are studied?

- Mathematical interest: simple models with three interwoven structures:
  - low-level algebraic structure conveying combinatorial information,
  - high-level algebraic structure conveying geometric information,
  - stochastic structure adapted to the two previous structures.
- Modelling transport phenomena
  - in crystals (metals, semiconductors, ionic conductors, etc.)
  - or on networks.
- Intervening in all models described by differential equations involving a Laplacian (cassical electrodynamics, statistical mechanics, quantum mechanics, quantum field theory, etc.)
- Discretised (in time/space) versions of stochastic processes.

Why random walk are studied?

- Mathematical interest: simple models with three interwoven structures:
  - low-level algebraic structure conveying combinatorial information,
  - high-level algebraic structure conveying geometric information,
  - stochastic structure adapted to the two previous structures.
- Modelling transport phenomena
  - in crystals (metals, semiconductors, ionic conductors, etc.)
  - or on networks.
- Intervening in all models described by differential equations involving a Laplacian (cassical electrodynamics, statistical mechanics, quantum mechanics, quantum field theory, etc.)
- Discretised (in time/space) versions of stochastic processes.

Why random walk are studied?

- Mathematical interest: simple models with three interwoven structures:
  - low-level algebraic structure conveying combinatorial information,
  - high-level algebraic structure conveying geometric information,
  - stochastic structure adapted to the two previous structures.
- Modelling transport phenomena
  - in crystals (metals, semiconductors, ionic conductors, etc.)
  - or on networks.
- Intervening in all models described by differential equations involving a Laplacian (cassical electrodynamics, statistical mechanics, quantum mechanics, quantum field theory, etc.)
- Discretised (in time/space) versions of stochastic processes.

How can we generalise?

- Generalisation to non-commutative groups:
  - The three interwoven structures and harmonic analysis survive.
  - Very active domain (e.g. products of **fixed size** random matrices, random dynamical systems, etc.) (only marginally touched in this lecture).
- Weakening of the group structure to groupoid.
- Further weakening to semi-groupoid.

Very instructive to see why standard methods fail to treat random walks on these new structures.

How can we generalise?

- Generalisation to non-commutative groups:
  - The three interwoven structures and harmonic analysis survive.
  - Very active domain (e.g. products of **fixed size** random matrices, random dynamical systems, etc.) (only marginally touched in this lecture).
- Weakening of the group structure to groupoid.
- Further weakening to semi-groupoid.

Very instructive to see why standard methods fail to treat random walks on these new structures.

How can we generalise?

- Generalisation to non-commutative groups:
  - The three interwoven structures and harmonic analysis survive.
  - Very active domain (e.g. products of **fixed size** random matrices, random dynamical systems, etc.) (only marginally touched in this lecture).
- Weakening of the group structure to groupoid.
- Further weakening to semi-groupoid.

Very instructive to see why standard methods fail to treat random walks on these new structures.

How can we generalise?

- Generalisation to non-commutative groups:
  - The three interwoven structures and harmonic analysis survive.
  - Very active domain (e.g. products of **fixed size** random matrices, random dynamical systems, etc.) (only marginally touched in this lecture).
- Weakening of the group structure to groupoid.
- Further weakening to semi-groupoid.

Very instructive to see why standard methods fail to treat random walks on these new structures.

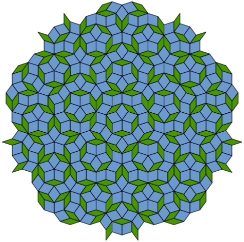
How can we generalise?

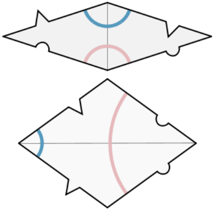
- Generalisation to non-commutative groups:
  - The three interwoven structures and harmonic analysis survive.
  - Very active domain (e.g. products of **fixed size** random matrices, random dynamical systems, etc.) (only marginally touched in this lecture).
- Weakening of the group structure to groupoid.
- Further weakening to semi-groupoid.

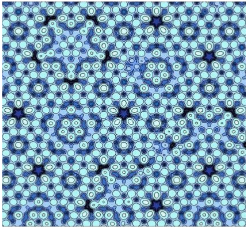
Very instructive to see why standard methods fail to treat random walks on these new structures.

And when the graph is not a group? (cont'd)

R.w. on quasi-periodic tilings of $\mathbb{R}^d$ of Penrose type: the groupoid case





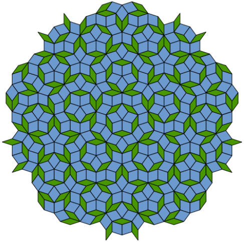


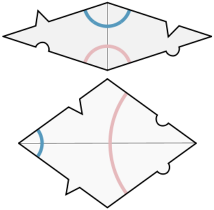
- Transport properties on quasi-periodic structures[2].
- Spectral properties of Schrödinger operators on quasi-periodic structures.
- Random walks on groupoids, non-random inhomogeneity.

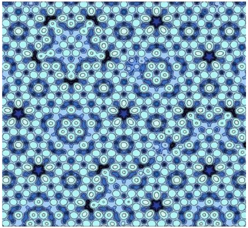
[2] Introduced as mathematical curiosities by Sir Roger Penrose (1974–1976), observed in nature as crystalline structures of Al-Mn alloys by Shechtman (1982), Nobel Prize in Chemistry 2011, obtained by an algorithmically much more efficient way by Duneau-Katz (1985).

And when the graph is not a group? (cont'd)

R.w. on quasi-periodic tilings of $\mathbb{R}^d$ of Penrose type: the groupoid case





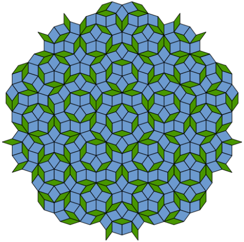


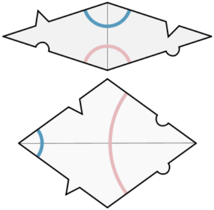
- Transport properties on quasi-periodic structures[2].
- Spectral properties of Schrödinger operators on quasi-periodic structures.
- Random walks on groupoids, non-random inhomogeneity.

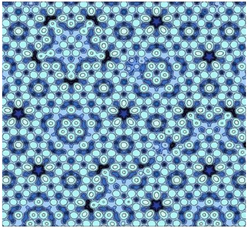
[2] Introduced as mathematical curiosities by Sir Roger Penrose (1974–1976), observed in nature as crystalline structures of Al-Mn alloys by Shechtman (1982) - Nobel Prize in Chemistry 2011, obtained by an algorithmically much more efficient way by Duneau-Katz (1985).

And when the graph is not a group? (cont'd)

R.w. on quasi-periodic tilings of $\mathbb{R}^d$ of Penrose type: the groupoid case





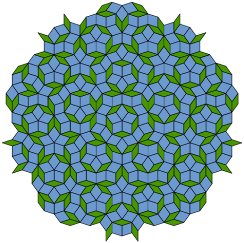
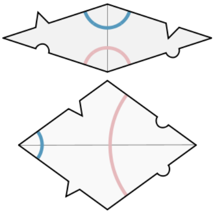
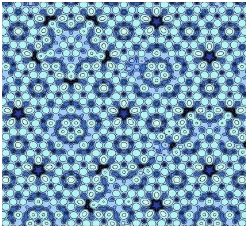


- Transport properties on quasi-periodic structures[2].
- Spectral properties of Schrödinger operators on quasi-periodic structures.
- Random walks on groupoids, non-random inhomogeneity.

[2] Introduced as mathematical curiosities by Sir Roger Penrose (1974–1976), observed in nature as crystalline structures of Al-Mn alloys by Shechtman (1982) - Nobel Prize in Chemistry 2011, obtained by an algorithmically much more efficient way by Duneau-Katz (1985).

And when the graph is not a group? (cont'd)

R.w. on quasi-periodic tilings of $\mathbb{R}^d$ of Penrose type: the groupoid case

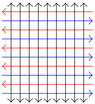
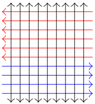
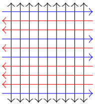




- Transport properties on quasi-periodic structures[2].
- Spectral properties of Schrödinger operators on quasi-periodic structures.
- Random walks on groupoids, non-random inhomogeneity.

[2] Introduced as mathematical curiosities by Sir Roger Penrose (1974–1976), observed in nature as crystalline structures of Al-Mn alloys by Shechtman (1982) - Nobel Prize in Chemistry 2011, obtained by an algorithmically much more efficient way by Duneau-Katz (1985).

And when the graph is not a group?

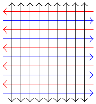
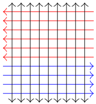
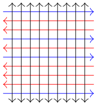
R.w. on directed graphs: the semi-groupoid case

Alternate lattice	Half-plane one-way	Random horizontal
		

- Hydrodynamic dispersion in porous rocks Matheron and Marsily (1980), numerical simulations Redner (1997).
- Propagation of information on directed networks (pathway signalling networks in genomics, neural system, world wide web, etc.)
- Differential geometry, causal structures in quantum gravity.
- Random walks on semi-groupoids (and their C^*-algebras), failure of the reversibility condition.

And when the graph is not a group?

R.w. on directed graphs: the semi-groupoid case

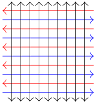
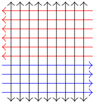
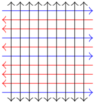
Alternate lattice	Half-plane one-way	Random horizontal
		

- Hydrodynamic dispersion in porous rocks Matheron and Marsily (1980), numerical simulations Redner (1997).
- Propagation of information on directed networks (pathway signalling networks in genomics, neural system, world wide web, etc.)
- Differential geometry, causal structures in quantum gravity.
- Random walks on semi-groupoids (and their C^*-algebras), failure of the reversibility condition.

UNIVERSITÉ DE RENNES 1

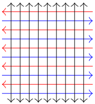
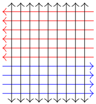
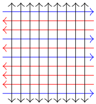
And when the graph is not a group?

R.w. on directed graphs: the semi-groupoid case

Alternate lattice	Half-plane one-way	Random horizontal
		

- Hydrodynamic dispersion in porous rocks Matheron and Marsily (1980), numerical simulations Redner (1997).
- Propagation of information on directed networks (pathway signalling networks in genomics, neural system, world wide web, etc.)
- Differential geometry, causal structures in quantum gravity.
- Random walks on semi-groupoids (and their C^*-algebras), failure of the reversibility condition.

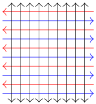
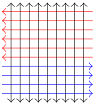
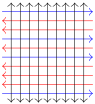
And when the graph is not a group?
R.w. on directed graphs: the semi-groupoid case

Alternate lattice	Half-plane one-way	Random horizontal
		

- Hydrodynamic dispersion in porous rocks Matheron and Marsily (1980), numerical simulations Redner (1997).
- Propagation of information on directed networks (pathway signalling networks in genomics, neural system, world wide web, etc.)
- Differential geometry, causal structures in quantum gravity.
- Random walks on semi-groupoids (and their C^*-algebras), failure of the reversibility condition.

And when the graph is not a group?

R.w. on directed graphs: the semi-groupoid case

Alternate lattice	Half-plane one-way	Random horizontal
		

- Hydrodynamic dispersion in porous rocks Matheron and Marsily (1980), numerical simulations Redner (1997).
- Propagation of information on directed networks (pathway signalling networks in genomics, neural system, world wide web, etc.)
- Differential geometry, causal structures in quantum gravity.
- Random walks on semi-groupoids (and their C^*-algebras), failure of the reversibility condition.

Results
For groupoids

Theorem (de Loynes, thm 3.1.2 in PhD thesis (2012)[a])

[a] Available at http://tel.archives-ouvertes.fr/tel-00726483.

The simple random walk on (adjacent edges of) a generic Penrose tiling of the d-dimensional space is

- *recurrent, if $d \leq 2$, and*
- *transient, if $d \geq 3$.*

Results
For semi-groupoids

Theorem (Campanino and P. (2003))

The simple random walk

- *on the alternate 2-dimensional lattice is recurrent,*
- *on the half-plane one-way 2-dimensional lattice is transient,*
- *on the randomly horizontally directed 2-dimensional lattice, where $(\theta_{x_2})_{x_2 \in \mathbb{Z}}$ is an i.i.d. $\{0, 1\}$-distributed sequence of average 1/2, is transient for almost all realisations of the sequence.*

Various subsequent developments in relation with this model: Guillotin and Schott (2006), Guillotin and Le Ny (2007), Pete (2008), Pène (2009), Devulder and Pène (2011), de Loynes (2012).

Results (cont'd)

For semi-groupoids

Theorem (Campanino and P. (2012), ArXiV:1204.5297)

- *$f : \mathbb{Z} \to \{0,1\}$ a Q-periodic function ($Q \geq 2$): $\sum_{y=1}^{Q} f(y) = 1/2$.*
- *$(\rho_y)_{y\in\mathbb{Z}}$ i.i.d. Rademacher sequence.*
- *$(\lambda_y)_{y\in\mathbb{Z}}$ i.i.d. $\{0,1\}$-valued sequence such that $\mathbb{P}(\lambda_y = 1) = \frac{c}{|y|^\beta}$ for large $|y|$.*
- *$\theta_y = (1-\lambda_y)f(y) + \lambda_y \frac{1+\rho_y}{2}$.*

1. *If $\beta < 1$ then the simple random walk is almost surely transient.*
2. *If $\beta > 1$ then the simple random walk is almost surely recurrent.*

Groupoids and semigroupoids

Definition

Let $\Gamma \neq \emptyset$. $(\Gamma, \cdot)$ is a

semigroup monoid group	semigroupoid groupoid
if $\cdot : \Gamma \times \Gamma \to \Gamma$ and $\forall a, b, c \in \Gamma$	if $\exists \Gamma^2 \subseteq \Gamma \times \Gamma$ and $\cdot : \Gamma^2 \to \Gamma$
$(cb)a = c(ba)$	$(c, b), (b, a) \in \Gamma^2 \Rightarrow$ $(cb, a), (c, ba) \in \Gamma^2$ and $(cb)a = c(ba)$
$\exists! e \in \Gamma : ea = ae = a$	units not necessarily unique,
$\exists a^{-1} \in \Gamma : aa^{-1} = a^{-1}a = e$	$\exists a^{-1} : (a^{-1})^{-1} = a$, $(a, a^{-1}), (a^{-1}, a) \in \Gamma^2$ and $(a, b) \in \Gamma^2 \Rightarrow a^{-1}(ab) = b$; $(b, a) \in \Gamma^2 \Rightarrow (ba)a^{-1} = b$.

Groupoids and semigroupoids

Definition

Let $\Gamma \neq \emptyset$. $(\Gamma, \cdot)$ is a

semigroup **monoid** group	semigroupoid groupoid
if $\cdot : \Gamma \times \Gamma \to \Gamma$ and $\forall a, b, c \in \Gamma$	if $\exists \Gamma^2 \subseteq \Gamma \times \Gamma$ and $\cdot : \Gamma^2 \to \Gamma$
$(cb)a = c(ba)$	$(c, b), (b, a) \in \Gamma^2 \Rightarrow$ $(cb, a), (c, ba) \in \Gamma^2$ and $(cb)a = c(ba)$
$\exists! e \in \Gamma : ea = ae = a$	units not necessarily unique,
$\exists a^{-1} \in \Gamma : aa^{-1} = a^{-1}a = e$	$\exists a^{-1} : (a^{-1})^{-1} = a$, $(a, a^{-1}), (a^{-1}, a) \in \Gamma^2$ and $(a, b) \in \Gamma^2 \Rightarrow a^{-1}(ab) = b$; $(b, a) \in \Gamma^2 \Rightarrow (ba)a^{-1} = b$.

Groupoids and semigroupoids

Definition

Let $\Gamma \neq \emptyset$. $(\Gamma, \cdot)$ is a

semigroup monoid group	semigroupoid groupoid
if $\cdot : \Gamma \times \Gamma \to \Gamma$ and $\forall a, b, c \in \Gamma$	if $\exists \Gamma^2 \subseteq \Gamma \times \Gamma$ and $\cdot : \Gamma^2 \to \Gamma$
$(cb)a = c(ba)$	$(c, b), (b, a) \in \Gamma^2 \Rightarrow$ $(cb, a), (c, ba) \in \Gamma^2$ and $(cb)a = c(ba)$
$\exists ! e \in \Gamma : ea = ae = a$	units not necessarily unique,
$\exists a^{-1} \in \Gamma : aa^{-1} = a^{-1}a = e$	$\exists a^{-1} : (a^{-1})^{-1} = a$, $(a, a^{-1}), (a^{-1}, a) \in \Gamma^2$ and $(a, b) \in \Gamma^2 \Rightarrow a^{-1}(ab) = b$; $(b, a) \in \Gamma^2 \Rightarrow (ba)a^{-1} = b$.

Groupoids and semigroupoids

Definition

Let $\Gamma \neq \emptyset$. $(\Gamma, \cdot)$ is a

semigroup monoid group	semigroupoid groupoid
if $\cdot : \Gamma \times \Gamma \to \Gamma$ and $\forall a, b, c \in \Gamma$	if $\exists \Gamma^2 \subseteq \Gamma \times \Gamma$ and $\cdot : \Gamma^2 \to \Gamma$
$(cb)a = c(ba)$	$(c, b), (b, a) \in \Gamma^2 \Rightarrow$ $(cb, a), (c, ba) \in \Gamma^2$ and $(cb)a = c(ba)$
$\exists! e \in \Gamma : ea = ae = a$	units not necessarily unique,
$\exists a^{-1} \in \Gamma : aa^{-1} = a^{-1}a = e$	$\exists a^{-1} : (a^{-1})^{-1} = a$, $(a, a^{-1}), (a^{-1}, a) \in \Gamma^2$ and $(a, b) \in \Gamma^2 \Rightarrow a^{-1}(ab) = b$; $(b, a) \in \Gamma^2 \Rightarrow (ba)a^{-1} = b$.

Groupoids and semigroupoids

Definition

Let $\Gamma \neq \emptyset$. $(\Gamma, \cdot)$ is a

semigroup monoid group

if $\cdot : \Gamma \times \Gamma \to \Gamma$ and $\forall a, b, c \in \Gamma$

$(cb)a = c(ba)$

$\exists! e \in \Gamma : ea = ae = a$

$\exists a^{-1} \in \Gamma : aa^{-1} = a^{-1}a = e$

semigroupoid groupoid

if $\exists \Gamma^2 \subseteq \Gamma \times \Gamma$ and $\cdot : \Gamma^2 \to \Gamma$

$(c, b), (b, a) \in \Gamma^2 \Rightarrow$
$(cb, a), (c, ba) \in \Gamma^2$ and $(cb)a = c(ba)$

units not necessarily unique,

$\exists a^{-1} : (a^{-1})^{-1} = a$,
$(a, a^{-1}), (a^{-1}, a) \in \Gamma^2$ and
$(a, b) \in \Gamma^2 \Rightarrow a^{-1}(ab) = b$;
$(b, a) \in \Gamma^2 \Rightarrow (ba)a^{-1} = b$.

Two archetypal examples of (semi)groupoids

Directed graphs

Example

- Directed graph: $\mathbb{G} = (\mathbb{G}^0, \mathbb{G}^1, s, t)$ with $\mathbb{G}^0$ and $\mathbb{G}^1$ denumerable (finite or infinite) sets of vertices (paths of length 0) and edges (paths of length 1) and $s, t : \mathbb{G}^1 \to \mathbb{G}^0$ the source and terminal maps.
- For $n \geq 2$ define

$$\mathbb{G}^n = \{\alpha = \alpha_n \dots \alpha_1, \alpha_i \in \mathbb{G}^1, s(\alpha_{i+1}) = t(\alpha_i)\} \subseteq (\mathbb{G}^1)^n,$$

  and $\mathrm{PS}(\mathbb{G}) = \cup_{n\geq 0}\mathbb{G}^n$ the path space of $\mathbb{G}$. Maps s, t extend trivially to $\mathrm{PS}(\mathbb{G})$.
- On defining $\Gamma = \mathrm{PS}(\mathbb{G})$, $\Gamma^2 = \{(\beta, \alpha) \in \Gamma \times \Gamma : s(\beta) = t(\alpha)\}$ and $\cdot : \Gamma^2 \to \mathbb{G}$ the left admissible concatenation, $(\Gamma, \Gamma^2, \cdot)$ is a semigroupoid with space of units $\mathbb{G}^0$.

Two archetypal examples of (semi)groupoids

Admissible words on an alphabet

Example

$\mathbb{A}$ alphabet, $A = (A_{b,a})_{a,b\in\mathbb{A}}$ with $A_{a,b} \in \{0,1\}$, $\mathbb{A}^0 = \{()\}$,
$\mathbb{A}^n = \{\alpha = (\alpha_n \cdots \alpha_1), \alpha_i \in \mathbb{A}\}$,

- set of words of arbitrary length $\mathbb{A}^* = \cup_{n\in\mathbb{N}}\mathbb{A}^n$ equipped with left concatenation is a monoid,
- $W_A(\mathbb{A}) = \{\alpha \in \mathbb{A}^* : A(\alpha_{i+1}, \alpha_i) = 1, i = 1, \ldots, |\alpha|\}$ (set of A-admissible words) is a semigroupoid with (β, α) composable pair if $A(\beta_1, \alpha_{|\alpha|}) = 1$.

Remark

A semigroupoid is not always a category. Consider, for example,
$\mathbb{A} = \{a, b\}$ *and* $A = \begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix}$.

How these two models related?

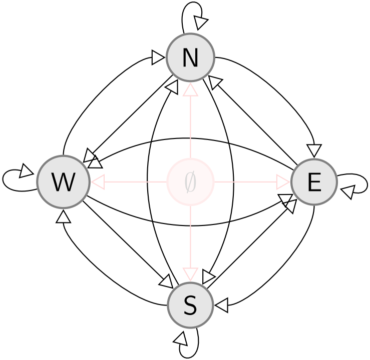
Lemma

- $\mathbb{A}$ *a finite alphabet.*
- $\mathbb{K} := \mathbb{K}_{\mathbb{A}}$ *the complete graph on* $\mathbb{A}$ *i.e.* $\mathbb{K} = (\mathbb{K}^0, \mathbb{K}^1)$*, where* $\mathbb{K}^0 = \mathbb{A}$ *the vertex set of* $\mathbb{K}$ *and* $\mathbb{K}^1 = \mathbb{A} \times \mathbb{A}$ *the edge set of* $\mathbb{K}$*.*
- $\hat{\mathbb{K}} := \hat{\mathbb{K}}_{\mathbb{A}}$ *the complete graph with source i.e.* $\hat{\mathbb{K}} = (\hat{\mathbb{K}}^0, \hat{\mathbb{K}}^1)$*, where* $\hat{\mathbb{K}}^0 = \mathbb{A} \cup \{\emptyset\}$ *the vertex set of* $\hat{\mathbb{K}}$ *and* $\hat{\mathbb{K}}^1 = \mathbb{A} \times \mathbb{A} \cup \{\emptyset\} \times \mathbb{A}$ *the edge set of* $\hat{\mathbb{K}}$*. The special vertex* $\emptyset$ *is the source of the graph.*
- *The path space* $PS(\hat{\mathbb{K}}_{\mathbb{A}})$ *is isomorphic to* $\mathbb{A}^*$*.*

The complete graphs $\mathbb{K}_{\mathbb{A}}$ and $\hat{\mathbb{K}}_{\mathbb{A}}$

$$\mathbb{A} = \{E, N, W, S\}.$$

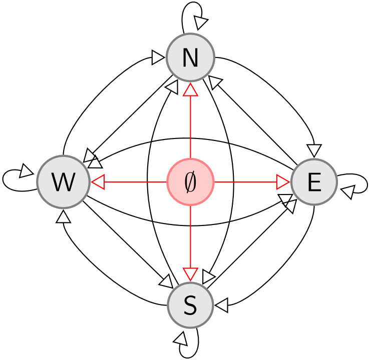
$\mathbb{K}_{\mathbb{A}}$
$\hat{\mathbb{K}}_{\mathbb{A}}$ =



The complete graphs $\mathbb{K}_{\mathbb{A}}$ and $\hat{\mathbb{K}}_{\mathbb{A}}$

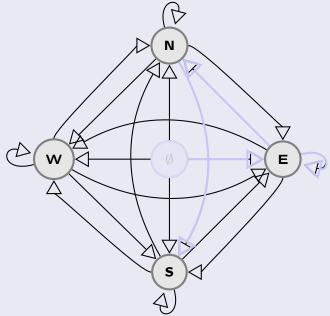
$$\mathbb{A} = \{E, N, W, S\}.$$

$\mathbb{K}_{\mathbb{A}}$
$\hat{\mathbb{K}}_{\mathbb{A}}$ $=$

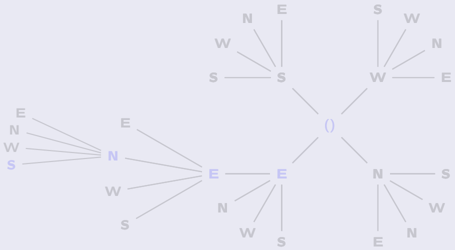


Path space tree

A path on $\hat{\mathbb{K}}_{\mathbb{A}}$



The path on $PS(\hat{\mathbb{K}}_{\mathbb{A}})$

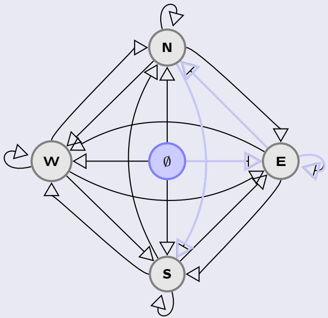


The path on $\mathbb{A}^*$

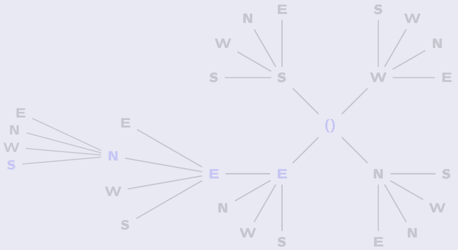
()EENS

Path space tree

A path on $\hat{\mathbb{K}}_{\mathbb{A}}$



The path on $\mathrm{PS}(\hat{\mathbb{K}}_{\mathbb{A}})$

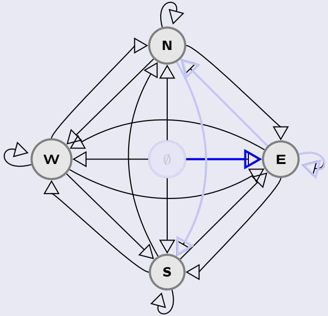


The path on $\mathbb{A}^*$

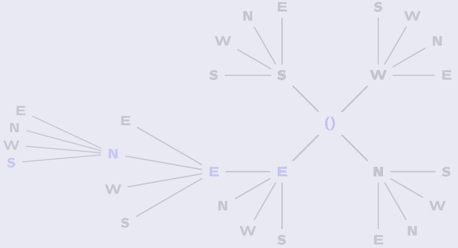
()EENS

Path space tree

A path on $\hat{\mathbb{K}}_{\mathbb{A}}$



The path on $\mathrm{PS}(\hat{\mathbb{K}}_{\mathbb{A}})$

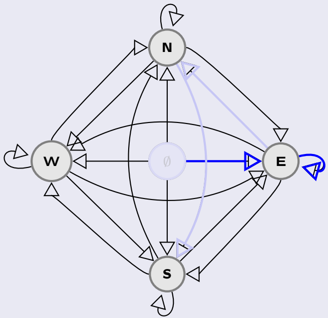


The path on $\mathbb{A}^*$

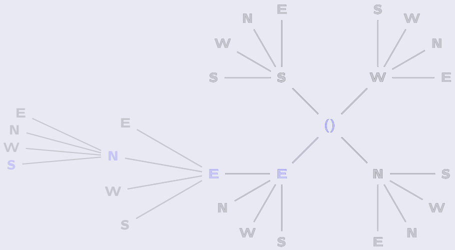
()EENS

Path space tree

A path on $\hat{\mathbb{K}}_{\mathbb{A}}$



The path on $PS(\hat{\mathbb{K}}_{\mathbb{A}})$

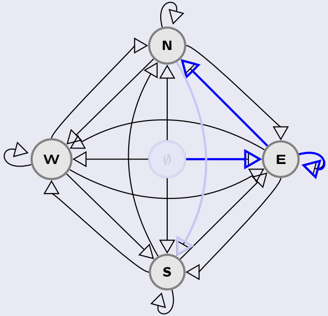


The path on $\mathbb{A}^*$

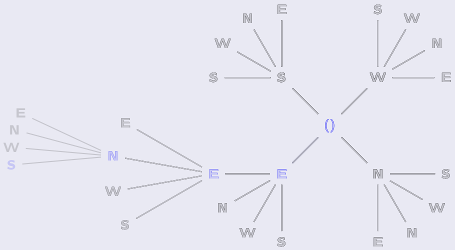
()EENS

Path space tree

A path on $\hat{\mathbb{K}}_{\mathbb{A}}$



The path on $\mathrm{PS}(\hat{\mathbb{K}}_{\mathbb{A}})$

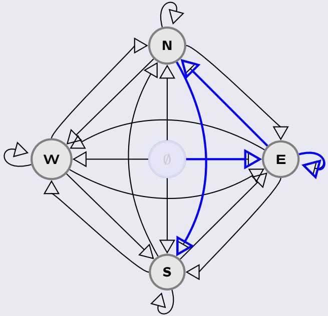


The path on $\mathbb{A}^*$

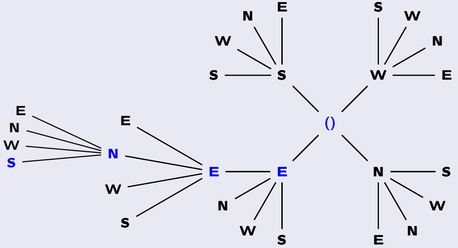
()EENS

Path space tree

A path on $\hat{\mathbb{K}}_{\mathbb{A}}$



The path on $\mathrm{PS}(\hat{\mathbb{K}}_{\mathbb{A}})$



The path on $\mathbb{A}^*$

()EENS

Constrained Cayley graphs

$$EW = WE = e, \; NS = SN = e,$$
$$E = a \Rightarrow W = a^{-1} \text{ and } N = b \Rightarrow S = b^{-1}.$$
$$\mathbb{A} = \{a, a^{-1}, b, b^{-1}\}.$$

Definition

Let $\mathbb{A}$ finite be given (generating) and $\Gamma = \langle \mathbb{A} \,|\, \mathcal{R} \rangle$. Let $c : \Gamma \times \mathbb{A} \to \{0, 1\}$ be a choice function. Define the constrained Cayley graph $\mathbb{G} = (\mathbb{G}^0, \mathbb{G}^1) = \mathrm{Cayley}_c(\Gamma, \mathbb{A}, \mathcal{R})$ by

- $\mathbb{G}^0 = \Gamma$,
- $\mathbb{G}^1 = \{(x, xz) \in \Gamma \times \Gamma : z \in \mathbb{A}; c(x, z) = 1\}$.
- $d_x^- = \mathrm{card}\{y \in \Gamma : (x, y) \in \mathbb{G}^1\}$.

Properties of constrained Cayley graphs

- $0 \leq d_x^- \leq \mathrm{card}\mathbb{A}$.
- If $d_x^- = 0$ for some x, then x is a sink. All graphs considered here have $d_x^- > 0$.
- If $c \equiv 1$ then $(\mathbb{G}^1)^{-1} = \mathbb{G}^1$ (the graph is undirected).
- The graph can fail to be transitive. All graphs considered here are transitive i.e. for all $x, y \in \mathbb{G}^0$, there exists a finite sequence $(x_0 = x, x_1, \ldots, x_n = y)$ with $(x_{i-1}, x_i) \in \mathbb{G}^1$ for all $i = 1, \ldots, n$.
- Algebraic structure of $\mathrm{Cayley}_c(\Gamma, \mathbb{A}, \mathcal{R})$: a groupoid or a semi-groupoid.

Properties of constrained Cayley graphs

- $0 \leq d_x^- \leq \mathrm{card}\mathbb{A}$.
- If $d_x^- = 0$ for some x, then x is a sink. All graphs considered here have $d_x^- > 0$.
- If $c \equiv 1$ then $(\mathbb{G}^1)^{-1} = \mathbb{G}^1$ (the graph is undirected).
- The graph can fail to be transitive. All graphs considered here are transitive i.e. for all $x, y \in \mathbb{G}^0$, there exists a finite sequence $(x_0 = x, x_1, \ldots, x_n = y)$ with $(x_{i-1}, x_i) \in \mathbb{G}^1$ for all $i = 1, \ldots, n$.
- Algebraic structure of $\mathrm{Cayley}_c(\Gamma, \mathbb{A}, \mathcal{R})$: a groupoid or a semi-groupoid.

Properties of constrained Cayley graphs

- $0 \leq d_x^- \leq \text{card}\mathbb{A}$.
- If $d_x^- = 0$ for some x, then x is a sink. All graphs considered here have $d_x^- > 0$.
- If $c \equiv 1$ then $(\mathbb{G}^1)^{-1} = \mathbb{G}^1$ (the graph is undirected).
- The graph can fail to be transitive. All graphs considered here are transitive i.e. for all $x, y \in \mathbb{G}^0$, there exists a finite sequence $(x_0 = x, x_1, \ldots, x_n = y)$ with $(x_{i-1}, x_i) \in \mathbb{G}^1$ for all $i = 1, \ldots, n$.
- Algebraic structure of $\text{Cayley}_c(\Gamma, \mathbb{A}, \mathcal{R})$: a groupoid or a semi-groupoid.

Properties of constrained Cayley graphs

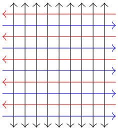
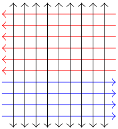
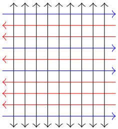
- $0 \leq d_x^- \leq \operatorname{card}\mathbb{A}$.
- If $d_x^- = 0$ for some x, then x is a sink. All graphs considered here have $d_x^- > 0$.
- If $c \equiv 1$ then $(\mathbb{G}^1)^{-1} = \mathbb{G}^1$ (the graph is undirected).
- The graph can fail to be transitive. All graphs considered here are transitive i.e. for all $x, y \in \mathbb{G}^0$, there exists a finite sequence $(x_0 = x, x_1, \ldots, x_n = y)$ with $(x_{i-1}, x_i) \in \mathbb{G}^1$ for all $i = 1, \ldots, n$.
- Algebraic structure of $\text{Cayley}_c(\Gamma, \mathbb{A}, \mathcal{R})$: a groupoid or a semi-groupoid.

Properties of constrained Cayley graphs

- $0 \leq d_x^- \leq \mathrm{card}\mathbb{A}$.
- If $d_x^- = 0$ for some x, then x is a sink. All graphs considered here have $d_x^- > 0$.
- If $c \equiv 1$ then $(\mathbb{G}^1)^{-1} = \mathbb{G}^1$ (the graph is undirected).
- The graph can fail to be transitive. All graphs considered here are transitive i.e. for all $x, y \in \mathbb{G}^0$, there exists a finite sequence $(x_0 = x, x_1, \ldots, x_n = y)$ with $(x_{i-1}, x_i) \in \mathbb{G}^1$ for all $i = 1, \ldots, n$.
- Algebraic structure of $\mathrm{Cayley}_c(\Gamma, \mathbb{A}, \mathcal{R})$: a groupoid or a semi-groupoid.

Examples of semi-groupoids

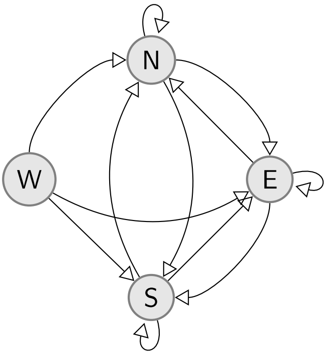
Vertex set $\mathbb{X} = \mathbb{Z}^2$, i.e. for all $x \in \mathbb{X}$, we write $x = (x_1, x_2)$; generating set $\mathbb{A} = \{\mathbf{e}_1, -\mathbf{e}_1, \mathbf{e}_2, -\mathbf{e}_2\}$.

Alternate lattice	Half-plane one-way	Random horizontal
		
$c(\mathbf{x}, \mathbf{e_2}) = c(\mathbf{x}, -\mathbf{e_2}) = 1$ $c(\mathbf{x}, \mathbf{e_1}) = 1, x_2 \in 2\mathbb{Z}$ $c(\mathbf{x}, -\mathbf{e_1}) = 1, x_2 + 1 \in 2\mathbb{Z}$	$c(\mathbf{x}, \mathbf{e_2}) = c(\mathbf{x}, -\mathbf{e_2}) = 1$ $c(\mathbf{x}, \mathbf{e_1}) = 1, x_2 < 0$ $c(\mathbf{x}, -\mathbf{e_1}) = 1, x_2 \geq 0$	$c(\mathbf{x}, \mathbf{e_2}) = c(\mathbf{x}, -\mathbf{e_2}) = 1$ $c(\mathbf{x}, \mathbf{e_1}) = \theta_{x_2}$ $c(\mathbf{x}, -\mathbf{e_1}) = 1 - \theta_{x_2}$

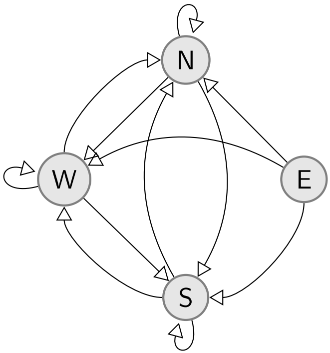
For all three lattices: $\forall x \in \mathbb{Z}^2, d^-_{\mathbf{x}} = 3$.

Here $\mathbb{G}^1 \subset \mathbb{G}^0 \times \mathbb{G}^0$. Hence maps s, t superfluous.

Path space generated by context-dependent grammar



Eastwardly



Westwardly

Random walk on the horizontally directed graph
Probability of jumps

- μ probability measure on $\mathbb{X} = \mathbb{Z}^2$ with $\operatorname{supp} \mu = \mathbb{A}$.
- Markov chain (Z_n) on augmented space $\mathbb{X} \times \mathbb{A}$.

$$Z_{n+1} = (X_{n+1}, \Xi_{n+1}) = (X_n + \Xi_{n+1}, \Xi_{n+1}).$$

$$\begin{aligned}\mathbb{P}(Z_{n+1} = (y, \xi) | Z_n = (x, \zeta)) &= \frac{\mu(\xi)}{\sum_{\eta: c(x,\eta)=1} \mu(\eta)} \delta_{1,c(x,\xi)} \delta_{y,x+\xi} \\ &= \begin{cases} \frac{1}{3} & y_2 = x_2 \pm 1 \\ \frac{1}{3} & y_1 = x_1 + \epsilon_{x_2} \\ 0 & \text{otherwise.} \end{cases}\end{aligned}$$

- $(X_n)_{n \in \mathbb{N}}$ is a hidden Markov chain on $\mathbb{X}$.
- Graph of the stochastic matrix of X = the above constrained Cayley graph.
- The combinatorial, geometric, and stochastic structures are mutually adapted.

Random walk on the horizontally directed graph

Probability of jumps

- μ probability measure on $\mathbb{X} = \mathbb{Z}^2$ with $\operatorname{supp}\mu = \mathbb{A}$.
- Markov chain (Z_n) on augmented space $\mathbb{X} \times \mathbb{A}$.

$$Z_{n+1} = (X_{n+1}, \Xi_{n+1}) = (X_n + \Xi_{n+1}, \Xi_{n+1}).$$

$$\begin{aligned}\mathbb{P}(Z_{n+1} = (y, \xi) | Z_n = (x, \zeta)) &= \frac{\mu(\xi)}{\sum_{\eta : c(x,\eta)=1} \mu(\eta)} \delta_{1,c(x,\xi)} \delta_{y,x+\xi} \\ &= \begin{cases} \frac{1}{3} & y_2 = x_2 \pm 1 \\ \frac{1}{3} & y_1 = x_1 + \epsilon_{x_2} \\ 0 & \text{otherwise.} \end{cases}\end{aligned}$$

- $(X_n)_{n\in\mathbb{N}}$ is a hidden Markov chain on $\mathbb{X}$.
- Graph of the stochastic matrix of X = the above constrained Cayley graph.
- The combinatorial, geometric, and stochastic structures are mutually adapted.

Random walk on the horizontally directed graph

Probability of jumps

- μ probability measure on $\mathbb{X} = \mathbb{Z}^2$ with $\operatorname{supp} \mu = \mathbb{A}$.
- Markov chain (Z_n) on augmented space $\mathbb{X} \times \mathbb{A}$.

$$Z_{n+1} = (X_{n+1}, \Xi_{n+1}) = (X_n + \Xi_{n+1}, \Xi_{n+1}).$$

$$\begin{aligned} \mathbb{P}(Z_{n+1} = (y, \xi) | Z_n = (x, \zeta)) &= \frac{\mu(\xi)}{\sum_{\eta : c(x,\eta)=1} \mu(\eta)} \delta_{1,c(x,\xi)} \delta_{y,x+\xi} \\ &= \begin{cases} \frac{1}{3} & y_2 = x_2 \pm 1 \\ \frac{1}{3} & y_1 = x_1 + \epsilon_{x_2} \\ 0 & \text{otherwise.} \end{cases} \end{aligned}$$

- $(X_n)_{n \in \mathbb{N}}$ is a hidden Markov chain on $\mathbb{X}$.
- Graph of the stochastic matrix of X = the above constrained Cayley graph.
- The combinatorial, geometric, and stochastic structures are mutually adapted.

Random walk on the horizontally directed graph
Probability of jumps

- μ probability measure on $\mathbb{X} = \mathbb{Z}^2$ with $\operatorname{supp} \mu = \mathbb{A}$.
- Markov chain (Z_n) on augmented space $\mathbb{X} \times \mathbb{A}$.

$$Z_{n+1} = (X_{n+1}, \Xi_{n+1}) = (X_n + \Xi_{n+1}, \Xi_{n+1}).$$

$$\begin{aligned}\mathbb{P}(Z_{n+1} = (y, \xi) | Z_n = (x, \zeta)) &= \frac{\mu(\xi)}{\sum_{\eta : c(x,\eta)=1} \mu(\eta)} \delta_{1,c(x,\xi)} \delta_{y,x+\xi} \\ &= \begin{cases} \frac{1}{3} & y_2 = x_2 \pm 1 \\ \frac{1}{3} & y_1 = x_1 + \epsilon_{x_2} \\ 0 & \text{otherwise.} \end{cases}\end{aligned}$$

- $(X_n)_{n \in \mathbb{N}}$ is a hidden Markov chain on $\mathbb{X}$.
- Graph of the stochastic matrix of X = the above constrained Cayley graph.
- The combinatorial, geometric, and stochastic structures are mutually adapted.

Random walk on the horizontally directed graph

Probability of jumps

- μ probability measure on $\mathbb{X} = \mathbb{Z}^2$ with $\operatorname{supp} \mu = \mathbb{A}$.
- Markov chain (Z_n) on augmented space $\mathbb{X} \times \mathbb{A}$.

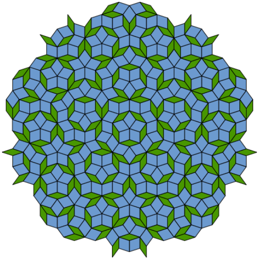
$$Z_{n+1} = (X_{n+1}, \Xi_{n+1}) = (X_n + \Xi_{n+1}, \Xi_{n+1}).$$

$$\begin{aligned} \mathbb{P}(Z_{n+1} = (y,\xi) | Z_n = (x,\zeta)) &= \frac{\mu(\xi)}{\sum_{\eta : c(x,\eta)=1} \mu(\eta)} \delta_{1,c(x,\xi)} \delta_{y,x+\xi} \\ &= \begin{cases} \frac{1}{3} & y_2 = x_2 \pm 1 \\ \frac{1}{3} & y_1 = x_1 + \epsilon_{x_2} \\ 0 & \text{otherwise.} \end{cases} \end{aligned}$$

- $(X_n)_{n\in\mathbb{N}}$ is a hidden Markov chain on $\mathbb{X}$.
- Graph of the stochastic matrix of X = the above constrained Cayley graph.
- The combinatorial, geometric, and stochastic structures are mutually adapted.

Example of groupoid

- Choose integer $N \geq 2$; decompose $\mathbb{R}^N = E \oplus E^\perp$ with $\dim E = d$ and $\dim E^\perp = N - d$, $1 \leq d < N$.
- K the unit hypercube in $\mathbb{R}^N$.
- $\pi : \mathbb{R}^N \to E$ and $\pi^\perp : \mathbb{R}^N \to E^\perp$ projections.
- For generic orientation of E and $t \in E_\perp$ let $\mathcal{K}_t := \{x \in \mathbb{Z}^N : \pi^\perp(E + t) \in \pi^\perp(K)\}$.
- $\pi(\mathcal{K}_t)$ is a quasi-periodic tiling of $E \cong \mathbb{R}^d$ (of Penrose type).
- For generic orientations of E, points in $\mathcal{K}_t$ are in bijection with points of the tiling.
- $\mathbb{A} = \{\pm \mathbf{e}_1, \ldots, \pm \mathbf{e}_N\}$.
- $c(x, z) = \mathbb{1}_{\mathcal{K}_t \times \mathcal{K}_t}(x, x + z)$, $z \in \mathbb{A}$.



$\text{Cayley}_c(\mathbb{Z}^N, \mathbb{A})$

- $\text{Cayley}_c(\mathbb{Z}^N, \mathbb{A})$ is undirected (groupoid).
- d_x^- can be made arbitrarily large.

Representation of PS($\mathbb{G}$)

- Locally-finite directed graph: $\mathbb{G} = (\mathbb{G}^0, \mathbb{G}^1, s, t)$ (hence PS($\mathbb{G}$)),
- Fock space: $\mathcal{H}_\mathbb{G} = \ell^2(\mathrm{PS}(\mathbb{G}))$ with $(\psi_\alpha, \alpha \in \mathrm{PS}(\mathbb{G}))$ an orthonormal basis [Exel (2008)].
- For $\beta \in \mathrm{PS}(\mathbb{G})$, $a \in \mathbb{G}^1$, and $x \in \mathbb{G}^0$, define creation operators

$$L_a|\psi_\beta\rangle = \begin{cases} |\psi_{a\beta}\rangle & s(a) = t(\beta), \\ 0 & \text{otherwise}, \end{cases}$$

$$L_x|\psi_\beta\rangle = \begin{cases} |\psi_{x\beta}\rangle = |\psi_\beta\rangle & x = t(\beta), \\ 0 & \text{otherwise}. \end{cases}$$

- $(L_a)_{a\in\mathbb{G}^1}$ partial isometries; $(L_x)_{x\in\mathbb{G}^0}$ projections verifying

$$L_a^* L_a = L_{s(a)}, \quad \sum_{a\in t^{-1}(x)} L_a L_a^* = L_x.$$

- Left free semigroupoid algebra:

$$\mathfrak{L}_\mathbb{G} = \overline{\mathrm{Alg}}^{\mathrm{wot}}\{L_x, L_a, x \in \mathbb{G}^0, a \in \mathbb{G}^1\}.$$

Representation of PS($\mathbb{G}$)

- Locally-finite directed graph: $\mathbb{G} = (\mathbb{G}^0, \mathbb{G}^1, s, t)$ (hence PS($\mathbb{G}$)),
- Fock space: $\mathcal{H}_\mathbb{G} = \ell^2(\text{PS}(\mathbb{G}))$ with $(\psi_\alpha, \alpha \in \text{PS}(\mathbb{G}))$ an orthonormal basis [Exel (2008)].
- For $\beta \in \text{PS}(\mathbb{G})$, $a \in \mathbb{G}^1$, and $x \in \mathbb{G}^0$, define creation operators

$$\begin{aligned} L_a|\psi_\beta\rangle &= \begin{cases} |\psi_{a\beta}\rangle & s(a) = t(\beta), \\ 0 & \text{otherwise}, \end{cases} \\ L_x|\psi_\beta\rangle &= \begin{cases} |\psi_{x\beta}\rangle = |\psi_\beta\rangle & x = t(\beta), \\ 0 & \text{otherwise}. \end{cases} \end{aligned}$$

- $(L_a)_{a\in\mathbb{G}^1}$ partial isometries; $(L_x)_{x\in\mathbb{G}^0}$ projections verifying

$$L_a^* L_a = L_{s(a)}, \quad \sum_{a\in t^{-1}(x)} L_a L_a^* = L_x.$$

- Left free semigroupoid algebra:

$$\mathfrak{L}_\mathbb{G} = \overline{\text{Alg}}^{\text{wot}}\{L_x, L_a, x \in \mathbb{G}^0, a \in \mathbb{G}^1\}.$$

Representation of PS($\mathbb{G}$)

- Locally-finite directed graph: $\mathbb{G} = (\mathbb{G}^0, \mathbb{G}^1, s, t)$ (hence PS($\mathbb{G}$)),
- Fock space: $\mathcal{H}_{\mathbb{G}} = \ell^2(\mathrm{PS}(\mathbb{G}))$ with $(\psi_\alpha, \alpha \in \mathrm{PS}(\mathbb{G}))$ an orthonormal basis [Exel (2008)].
- For $\beta \in \mathrm{PS}(\mathbb{G})$, $a \in \mathbb{G}^1$, and $x \in \mathbb{G}^0$, define creation operators

$$L_a|\,\psi_\beta\,\rangle = \begin{cases} |\,\psi_{a\beta}\,\rangle & s(a) = t(\beta), \\ 0 & \text{otherwise}, \end{cases}$$

$$L_x|\,\psi_\beta\,\rangle = \begin{cases} |\,\psi_{x\beta}\,\rangle = |\,\psi_\beta\,\rangle & x = t(\beta), \\ 0 & \text{otherwise}. \end{cases}$$

- $(L_a)_{a\in\mathbb{G}^1}$ partial isometries; $(L_x)_{x\in\mathbb{G}^0}$ projections verifying

$$L_a^* L_a = L_{s(a)}, \quad \sum_{a\in t^{-1}(x)} L_a L_a^* = L_x.$$

- Left free semigroupoid algebra:

$$\mathfrak{L}_{\mathbb{G}} = \overline{\mathrm{Alg}}^{\mathrm{wot}}\{L_x, L_a, x \in \mathbb{G}^0, a \in \mathbb{G}^1\}.$$

Representation of PS($\mathbb{G}$)

- Locally-finite directed graph: $\mathbb{G} = (\mathbb{G}^0, \mathbb{G}^1, s, t)$ (hence PS($\mathbb{G}$)),
- Fock space: $\mathcal{H}_{\mathbb{G}} = \ell^2(\mathrm{PS}(\mathbb{G}))$ with $(\psi_\alpha, \alpha \in \mathrm{PS}(\mathbb{G}))$ an orthonormal basis [Exel (2008)].
- For $\beta \in \mathrm{PS}(\mathbb{G})$, $a \in \mathbb{G}^1$, and $x \in \mathbb{G}^0$, define creation operators

$$L_a | \psi_\beta \rangle = \begin{cases} | \psi_{a\beta} \rangle & s(a) = t(\beta), \\ 0 & \text{otherwise,} \end{cases}$$

$$L_x | \psi_\beta \rangle = \begin{cases} | \psi_{x\beta} \rangle = | \psi_\beta \rangle & x = t(\beta), \\ 0 & \text{otherwise.} \end{cases}$$

- $(L_a)_{a\in\mathbb{G}^1}$ partial isometries; $(L_x)_{x\in\mathbb{G}^0}$ projections verifying

$$L_a^* L_a = L_{s(a)}, \quad \sum_{a \in t^{-1}(x)} L_a L_a^* = L_x.$$

- Left free semigroupoid algebra:

$$\mathfrak{L}_{\mathbb{G}} = \overline{\mathrm{Alg}}^{\mathrm{wot}} \{L_x, L_a, x \in \mathbb{G}^0, a \in \mathbb{G}^1\}.$$

Representation of $\mathrm{PS}(\mathbb{G})$

- Locally-finite directed graph: $\mathbb{G} = (\mathbb{G}^0, \mathbb{G}^1, s, t)$ (hence $\mathrm{PS}(\mathbb{G})$),
- Fock space: $\mathcal{H}_{\mathbb{G}} = \ell^2(\mathrm{PS}(\mathbb{G}))$ with $(\psi_\alpha, \alpha \in \mathrm{PS}(\mathbb{G}))$ an orthonormal basis [Exel (2008)].
- For $\beta \in \mathrm{PS}(\mathbb{G})$, $a \in \mathbb{G}^1$, and $x \in \mathbb{G}^0$, define creation operators

$$L_a | \psi_\beta \rangle = \begin{cases} | \psi_{a\beta} \rangle & s(a) = t(\beta), \\ 0 & \text{otherwise}, \end{cases}$$

$$L_x | \psi_\beta \rangle = \begin{cases} | \psi_{x\beta} \rangle = | \psi_\beta \rangle & x = t(\beta), \\ 0 & \text{otherwise}. \end{cases}$$

- $(L_a)_{a \in \mathbb{G}^1}$ partial isometries; $(L_x)_{x \in \mathbb{G}^0}$ projections verifying

$$L_a^* L_a = L_{s(a)}, \quad \sum_{a \in t^{-1}(x)} L_a L_a^* = L_x.$$

- Left free semigroupoid algebra:

$$\mathfrak{L}_{\mathbb{G}} = \overline{\mathrm{Alg}}^{\mathrm{wot}} \{ L_x, L_a, x \in \mathbb{G}^0, a \in \mathbb{G}^1 \}.$$

Cuntz-Krieger algebra of $\mathbb{G}$

- Let $\{S_a, a \in \mathbb{G}^1\}$ be partial isometries and $\{P_x, x \in \mathbb{G}^0\}$ projections s.t.:

$$\begin{aligned}
x, y \in \mathbb{G}^0, x \neq y &\quad\Rightarrow\quad P_x P_y = 0,\\
a, b \in \mathbb{G}^1, a \neq b &\quad\Rightarrow\quad S_a^* S_b = 0,\\
a \in \mathbb{G}^1 &\quad\Rightarrow\quad S_a^* S_a = P_{s(a)},\\
a \in \mathbb{G}^1 &\quad\Rightarrow\quad S_a S_a^* \leq P_{t(a)},\\
x \in \mathbb{G}^0, |t^{-1}(x)| \neq 0, \infty &\quad\Rightarrow\quad \sum_{a \in t^{-1}(x)} S_a S_a^* = P_x.
\end{aligned}$$

- There exists universal C^*-algebra of $\mathbb{G}$, $C^*(\mathbb{G})$, the Cuntz-Krieger algebra. [Cuntz-Krieger (1980), Raeburn-Szymański (2004)].
- For $\alpha \in \mathrm{PS}(\mathbb{G})$, $|\alpha| \geq 1$, $S_\alpha = S_{\alpha_{|\alpha|}} \cdots S_{\alpha_1}$.

Quantum channels and general quantum operations

$\mathcal{K}$ separable Hilbert space. $\Phi : \mathfrak{B}(\mathcal{K}) \to \mathfrak{B}(\mathcal{K})$ completely positive map.

- There exists [Kraus (1971)] a "row vector" of operators $(A_1, A_2, \ldots, A_N)$ on $\mathfrak{B}(\mathcal{K})$ s.t.

$$\mathfrak{B}(\mathcal{K}) \ni X \mapsto \Phi(X) = \sum_i A_i X A_i^*.$$

- Quantum operations: unital Φ's (hence $\sum_i A_i A_i^* = \mathbb{1}_{\mathcal{K}}$).
- Quantum channels: unital and trace-preserving Φ's (hence also $\sum_i A_i^* A_i = \mathbb{1}_{\mathcal{K}}$).
- $\mathcal{A}' = \{X \in \mathfrak{B}(\mathcal{K}) : XA_i = A_i X \text{ and } XA_i^* = A_i^* X, \forall i\}$.
- $\mathrm{Fix}(\Phi) = \{X \in \mathfrak{B}(\mathcal{K}) : \Phi(X) = X\}$ the Poisson boundary of Φ.
-

$$\begin{aligned} \Phi^{\circ n}(X) &= \sum_{i_1, \ldots, i_n} A_{i_n} \cdots A_{i_1} X A_{i_1}^* \cdots A_{i_n}^* \\ &= \sum_{\alpha \in \mathrm{PS}(\mathbb{G}) : |\alpha| = n} A_\alpha X A_\alpha^*. \end{aligned}$$

Quantum operations

Finite case Kribs (2003).

Theorem

If Φ unital, then $\mathcal{A}'$ is a von Neumann algebra contained in $\mathrm{Fix}(\Phi)$.

Question: When $\mathcal{A}' = \mathrm{Fix}(\Phi)$?

Theorem

Let Φ unital and p projection. Then

- $$\exists\lambda \geq 0 : \Phi(p) \leq \lambda p \Leftrightarrow \forall i, A_i p = pA_i p \Leftrightarrow \Phi(p) \leq p.$$
- *If additionally Φ trace-preserving, then:* $\Phi(p) = p \Leftrightarrow p \in \mathcal{A}'$.

Corollary

- *The largest C^*-algebra contained in* $\mathrm{Fix}(\Phi)$ *is* $\mathcal{A}'$.
- *If* $\mathrm{Fix}(\Phi)$ *is a sub-algebra of* $\mathfrak{B}(\mathcal{K})$, *then* $\mathrm{Fix}(\Phi) = \mathcal{A}'$.

Quantum operations
Repeated filtered POVM

Finite case Maassen-Kümmerer (2006), infinite case Lim (2010).

- $(A_j)_{j \in \mathbb{J}}$ Kraus operators corresponding to unital, trace-preserving channel Φ.
$$\Phi(\rho) = \sum_j A_j \rho A_j^*.$$
- $\pi_j(\rho) = \mathrm{tr}(A_j \rho A_j^*)$, $\phi_j(\rho) = A_j \rho A_j^* / \pi_j(\rho)$. Hence,
$$\Phi(\rho) = \sum_j \frac{A_j \rho A_j^*}{\pi_j(\rho)} \pi_j(\rho) = \sum_j \phi_j(\rho) \pi_j(\rho).$$
- Classical Markov chain $(Z_n)_n$ induced on augmented space $\mathfrak{D} \times \mathbb{J}$
$$\mathbb{P}(Z_{n+1} \in B \times J | Z_n = (\rho, k)) = \sum_{j \in \mathbb{J}} \pi_j(\rho) \delta_{\phi_j(\rho)}(B) \delta_j(J)$$
for $k \in \mathbb{J}$, $J \subseteq \mathbb{J}$ and $B \subseteq \mathfrak{D}$.
- Classical hidden Markov chain (X_n) on $\mathfrak{D}$, with transition matrix
$$P(\rho, B) := \mathbb{P}(X_{n+1} \in B | X_n = \rho) = \sum \pi_j(\rho) \delta_{\phi_j(\rho)}(B).$$

Quantum operations
Repeated filtered POVM

Finite case Maassen-Kümmerer (2006), infinite case Lim (2010).

- $(A_j)_{j\in\mathbb{J}}$ Kraus operators corresponding to unital, trace-preserving channel Φ.

$$\Phi(\rho) = \sum_j A_j \rho A_j^*.$$

- $\pi_j(\rho) = \operatorname{tr}(A_j\rho A_j^*)$, $\phi_j(\rho) = A_j\rho A_j^*/\pi_j(\rho)$. Hence,

$$\Phi(\rho) = \sum_j \frac{A_j\rho A_j^*}{\pi_j(\rho)}\pi_j(\rho) = \sum_j \phi_j(\rho)\pi_j(\rho).$$

- Classical Markov chain $(Z_n)_n$ induced on augmented space $\mathfrak{D}\times\mathbb{J}$

$$\mathbb{P}(Z_{n+1}\in B\times J | Z_n = (\rho,k)) = \sum_{j\in\mathbb{J}} \pi_j(\rho)\delta_{\phi_j(\rho)}(B)\delta_j(J)$$

for $k\in\mathbb{J}$, $J\subseteq\mathbb{J}$ and $B\subseteq\mathfrak{D}$.

- Classical hidden Markov chain (X_n) on $\mathfrak{D}$, with transition matrix

$$P(\rho,B) := \mathbb{P}(X_{n+1}\in B|X_n=\rho) = \sum \pi_j(\rho)\delta_{\phi_j(\rho)}(B).$$

Quantum operations
Repeated filtered POVM

Finite case Maassen-Kümmerer (2006), infinite case Lim (2010).

- $(A_j)_{j\in\mathbb{J}}$ Kraus operators corresponding to unital, trace-preserving channel Φ.

$$\Phi(\rho) = \sum_j A_j \rho A_j^*.$$

- $\pi_j(\rho) = \mathrm{tr}(A_j\rho A_j^*)$, $\phi_j(\rho) = A_j\rho A_j^*/\pi_j(\rho)$. Hence,

$$\Phi(\rho) = \sum_j \frac{A_j\rho A_j^*}{\pi_j(\rho)}\pi_j(\rho) = \sum_j \phi_j(\rho)\pi_j(\rho).$$

- Classical Markov chain $(Z_n)_n$ induced on augmented space $\mathfrak{D}\times\mathbb{J}$

$$\mathbb{P}(Z_{n+1}\in B\times J | Z_n = (\rho,k)) = \sum_{j\in\mathbb{J}} \pi_j(\rho)\delta_{\phi_j(\rho)}(B)\delta_j(J)$$

for $k\in\mathbb{J}$, $J\subseteq\mathbb{J}$ and $B\subseteq\mathfrak{D}$.

- Classical hidden Markov chain (X_n) on $\mathfrak{D}$, with transition matrix

$$P(\rho, B) := \mathbb{P}(X_{n+1}\in B|X_n=\rho) = \sum \pi_j(\rho)\delta_{\phi_j(\rho)}(B).$$

Quantum operations

Repeated filtered POVM

Finite case Maassen-Kümmerer (2006), infinite case Lim (2010).

- $(A_j)_{j\in\mathbb{J}}$ Kraus operators corresponding to unital, trace-preserving channel Φ.

$$\Phi(\rho) = \sum_j A_j \rho A_j^*.$$

- $\pi_j(\rho) = \mathrm{tr}(A_j\rho A_j^*)$, $\phi_j(\rho) = A_j\rho A_j^*/\pi_j(\rho)$. Hence,

$$\Phi(\rho) = \sum_j \frac{A_j\rho A_j^*}{\pi_j(\rho)}\pi_j(\rho) = \sum_j \phi_j(\rho)\pi_j(\rho).$$

- Classical Markov chain $(Z_n)_n$ induced on augmented space $\mathfrak{D}\times\mathbb{J}$

$$\mathbb{P}(Z_{n+1}\in B\times J | Z_n = (\rho,k)) = \sum_{j\in\mathbb{J}} \pi_j(\rho)\delta_{\phi_j(\rho)}(B)\delta_j(J)$$

for $k\in\mathbb{J}$, $J\subseteq\mathbb{J}$ and $B\subseteq\mathfrak{D}$.

- Classical hidden Markov chain (X_n) on $\mathfrak{D}$, with transition matrix

$$P(\rho, B) := \mathbb{P}(X_{n+1}\in B|X_n=\rho) = \sum \pi_j(\rho)\delta_{\phi_j(\rho)}(B).$$

Quantum operations

Repeated filtered POVM

Finite case Maassen-Kümmerer (2006), infinite case Lim (2010).

- $(A_j)_{j\in\mathbb{J}}$ Kraus operators corresponding to unital, trace-preserving channel Φ.

$$\Phi(\rho) = \sum_j A_j \rho A_j^*.$$

- $\pi_j(\rho) = \mathrm{tr}(A_j\rho A_j^*)$, $\phi_j(\rho) = A_j\rho A_j^*/\pi_j(\rho)$. Hence,

$$\Phi(\rho) = \sum_j \frac{A_j\rho A_j^*}{\pi_j(\rho)}\pi_j(\rho) = \sum_j \phi_j(\rho)\pi_j(\rho).$$

- Classical Markov chain $(Z_n)_n$ induced on augmented space $\mathfrak{D}\times\mathbb{J}$

$$\mathbb{P}(Z_{n+1}\in B\times J | Z_n = (\rho,k)) = \sum_{j\in\mathbb{J}} \pi_j(\rho)\delta_{\phi_j(\rho)}(B)\delta_j(J)$$

for $k\in\mathbb{J}, J\subseteq\mathbb{J}$ and $B\subseteq\mathfrak{D}$.

- Classical hidden Markov chain (X_n) on $\mathfrak{D}$, with transition matrix

$$P(\rho,B) := \mathbb{P}(X_{n+1}\in B|X_n=\rho) = \sum \pi_j(\rho)\delta_{\phi_j(\rho)}(B).$$

How to merge general relativity and quantum theory?

- At Planck scale[3], quantum effects of gravity become strong. At those scales, the description of space-time as a 1+3-dimensional continuum breaks down.
- Causality remains valid at these scales.
- Quantum theory remains valid at these scales.
- In a fundamental theory of the universe, any finite region should be described by a finite number of degrees of freedom.
- The theory should be background independent.

The problem still remains widely open. Various alternative approaches proposed: string theory, M-theory, loop gravity, causal set theory.

[3] 1.22×10^{19} GeV $= 1.952$ J; 5.39121×10^{-44} s; 2.17645×10^{-8} kg; 1.616252×10^{-35} m.

How to merge general relativity and quantum theory?

- At Planck scale[3], quantum effects of gravity become strong. At those scales, the description of space-time as a 1+3-dimensional continuum breaks down.
- Causality remains valid at these scales.
- Quantum theory remains valid at these scales.
- In a fundamental theory of the universe, any finite region should be described by a finite number of degrees of freedom.
- The theory should be background independent.

The problem still remains widely open. Various alternative approaches proposed: string theory, *M*-theory, loop gravity, causal set theory.

[3] 1.22×10^{19} GeV $= 1.952$ J; 5.39121×10^{-44} s; 2.17645×10^{-8} kg; 1.616252×10^{-35} m.

How to merge general relativity and quantum theory?

- At Planck scale[3], quantum effects of gravity become strong. At those scales, the description of space-time as a 1+3-dimensional continuum breaks down.
- Causality remains valid at these scales.
- Quantum theory remains valid at these scales.
- In a fundamental theory of the universe, any finite region should be described by a finite number of degrees of freedom.
- The theory should be background independent.

The problem still remains widely open. Various alternative approaches proposed: string theory, M-theory, loop gravity, causal set theory.

[3] 1.22×10^{19} GeV $= 1.952$ J; 5.39121×10^{-44} s; 2.17645×10^{-8} kg; 1.616252×10^{-35} m.

How to merge general relativity and quantum theory?

- At Planck scale[3], quantum effects of gravity become strong. At those scales, the description of space-time as a 1+3-dimensional continuum breaks down.
- Causality remains valid at these scales.
- Quantum theory remains valid at these scales.
- In a fundamental theory of the universe, any finite region should be described by a finite number of degrees of freedom.
- The theory should be background independent.

The problem still remains widely open. Various alternative approaches proposed: string theory, M-theory, loop gravity, causal set theory.

[3] 1.22×10^{19} GeV $= 1.952$ J; 5.39121×10^{-44} s; 2.17645×10^{-8} kg; 1.616252×10^{-35} m.

How to merge general relativity and quantum theory?

- At Planck scale[3], quantum effects of gravity become strong. At those scales, the description of space-time as a 1+3-dimensional continuum breaks down.
- Causality remains valid at these scales.
- Quantum theory remains valid at these scales.
- In a fundamental theory of the universe, any finite region should be described by a finite number of degrees of freedom.
- The theory should be background independent.

The problem still remains widely open. Various alternative approaches proposed: string theory, M-theory, loop gravity, causal set theory.

[3] 1.22×10^{19} GeV $= 1.952$ J; 5.39121×10^{-44} s; 2.17645×10^{-8} kg; 1.616252×10^{-35} m.

How to merge general relativity and quantum theory?

- At Planck scale[3], quantum effects of gravity become strong. At those scales, the description of space-time as a 1+3-dimensional continuum breaks down.
- Causality remains valid at these scales.
- Quantum theory remains valid at these scales.
- In a fundamental theory of the universe, any finite region should be described by a finite number of degrees of freedom.
- The theory should be background independent.

The problem still remains widely open. Various alternative approaches proposed: string theory, M-theory, loop gravity, causal set theory.

[3] 1.22×10^{19} GeV $= 1.952$ J; 5.39121×10^{-44} s; 2.17645×10^{-8} kg; 1.616252×10^{-35} m.

Quantum causal histories

Discrtete causal structure induced from graph ordering

Markopoulou-Smolin (1997), Malyshev (2001), Hawkins-Markopoulou-Sahlmann (2003), Kribs-Markopoulou (2005). Let $\mathbb{G} = (\mathbb{G}^0, \mathbb{G}^1, s, t)$ **without cycles** and $x, y \in \mathbb{G}^0$.

- Induced partial ordering (discretised causality):

$$[x \prec y] \Leftrightarrow [\exists \alpha \in \mathrm{PS}(\mathbb{G}) : s(\alpha) = x, t(\alpha) = y],$$

- Events x and y (causally) related: if $x \prec y$ or $y \prec x$. Otherwise space-like separated ($x \sim y$).
- Acausal set: $\xi \subset \mathbb{G}^0$ s.t. $x, y \in \xi \Rightarrow x \sim y$.
- An acausal set ξ is a complete future of x, $x \overline{\prec} \xi$, if any inextendible future path from x intersects ξ. Define similarly $\zeta \preceq y$ the complete past of y.
- If ξ, ζ acausal sets s.t. ξ complete past of ζ and ζ complete future of ξ then write $\xi \overline{\preceq} \zeta$: complete pair.

Quantum causal histories

Discrtete causal structure induced from graph ordering

Markopoulou-Smolin (1997), Malyshev (2001), Hawkins-Markopoulou-Sahlmann (2003), Kribs-Markopoulou (2005). Let $\mathbb{G} = (\mathbb{G}^0, \mathbb{G}^1, s, t)$ **without cycles** and $x, y \in \mathbb{G}^0$.

- Induced partial ordering (discretised causality):

$$[x \prec y] \Leftrightarrow [\exists \alpha \in \mathrm{PS}(\mathbb{G}) : s(\alpha) = x, t(\alpha) = y],$$

- Events x and y (causally) related: if $x \prec y$ or $y \prec x$. Otherwise space-like separated ($x \sim y$).
- Acausal set: $\xi \subset \mathbb{G}^0$ s.t. $x, y \in \xi \Rightarrow x \sim y$.
- An acausal set ξ is a complete future of x, $x \overline{\prec} \xi$, if any inextendible future path from x intersects ξ. Define similarly $\zeta \preceq y$ the complete past of y.
- If ξ, ζ acausal sets s.t. ξ complete past of ζ and ζ complete future of ξ then write $\xi \overline{\preceq} \zeta$: complete pair.

Quantum causal histories

Discrtete causal structure induced from graph ordering

Markopoulou-Smolin (1997), Malyshev (2001), Hawkins-Markopoulou-Sahlmann (2003), Kribs-Markopoulou (2005). Let $\mathbb{G} = (\mathbb{G}^0, \mathbb{G}^1, s, t)$ **without cycles** and $x, y \in \mathbb{G}^0$.

- Induced partial ordering (discretised causality):

$$[x \prec y] \Leftrightarrow [\exists \alpha \in \mathrm{PS}(\mathbb{G}) : s(\alpha) = x, t(\alpha) = y],$$

- Events x and y (causally) related: if $x \prec y$ or $y \prec x$. Otherwise space-like separated ($x \sim y$).
- Acausal set: $\xi \subset \mathbb{G}^0$ s.t. $x, y \in \xi \Rightarrow x \sim y$.
- An acausal set ξ is a complete future of x, $x \overline{\prec} \xi$, if any inextendible future path from x intersects ξ. Define similarly $\zeta \preceq y$ the complete past of y.
- If ξ, ζ acausal sets s.t. ξ complete past of ζ and ζ complete future of ξ then write $\xi \overline{\preceq} \zeta$: complete pair.

Quantum causal histories

Discrtete causal structure induced from graph ordering

Markopoulou-Smolin (1997), Malyshev (2001), Hawkins-Markopoulou-Sahlmann (2003), Kribs-Markopoulou (2005). Let $\mathbb{G} = (\mathbb{G}^0, \mathbb{G}^1, s, t)$ **without cycles** and $x, y \in \mathbb{G}^0$.

- Induced partial ordering (discretised causality):
$$[x \prec y] \Leftrightarrow [\exists \alpha \in \mathrm{PS}(\mathbb{G}) : s(\alpha) = x, t(\alpha) = y],$$
- Events x and y (causally) related: if $x \prec y$ or $y \prec x$. Otherwise space-like separated ($x \sim y$).
- Acausal set: $\xi \subset \mathbb{G}^0$ s.t. $x, y \in \xi \Rightarrow x \sim y$.
- An acausal set ξ is a complete future of x, $x \overline{\prec} \xi$, if any inextendible future path from x intersects ξ. Define similarly $\zeta \preceq y$ the complete past of y.
- If ξ, ζ acausal sets s.t. ξ complete past of ζ and ζ complete future of ξ then write $\xi \overline{\preceq} \zeta$: complete pair.

Quantum causal histories

Discrtete causal structure induced from graph ordering

Markopoulou-Smolin (1997), Malyshev (2001), Hawkins-Markopoulou-Sahlmann (2003), Kribs-Markopoulou (2005). Let $\mathbb{G} = (\mathbb{G}^0, \mathbb{G}^1, s, t)$ **without cycles** and $x, y \in \mathbb{G}^0$.

- Induced partial ordering (discretised causality):

$$[x \prec y] \Leftrightarrow [\exists \alpha \in \mathrm{PS}(\mathbb{G}) : s(\alpha) = x, t(\alpha) = y],$$

- Events x and y (causally) related: if $x \prec y$ or $y \prec x$. Otherwise space-like separated ($x \sim y$).
- Acausal set: $\xi \subset \mathbb{G}^0$ s.t. $x, y \in \xi \Rightarrow x \sim y$.
- An acausal set ξ is a complete future of x, $x \overline{\prec} \xi$, if any inextendible future path from x intersects ξ. Define similarly $\zeta \preceq y$ the complete past of y.
- If ξ, ζ acausal sets s.t. ξ complete past of ζ and ζ complete future of ξ then write $\xi \overline{\preceq} \zeta$: complete pair.

Quantum causal histories

Discrtete causal structure induced from graph ordering

Markopoulou-Smolin (1997), Malyshev (2001), Hawkins-Markopoulou-Sahlmann (2003), Kribs-Markopoulou (2005).
Let $\mathbb{G} = (\mathbb{G}^0, \mathbb{G}^1, s, t)$ **without cycles** and $x, y \in \mathbb{G}^0$.

- Induced partial ordering (discretised causality):

$$[x \prec y] \Leftrightarrow [\exists \alpha \in \mathrm{PS}(\mathbb{G}) : s(\alpha) = x, t(\alpha) = y],$$

- Events x and y (causally) related: if $x \prec y$ or $y \prec x$. Otherwise space-like separated ($x \sim y$).
- Acausal set: $\xi \subset \mathbb{G}^0$ s.t. $x, y \in \xi \Rightarrow x \sim y$.
- An acausal set ξ is a complete future of x, $x \overline{\prec} \xi$, if any inextendible future path from x intersects ξ. Define similarly $\zeta \preceq y$ the complete past of y.
- If ξ, ζ acausal sets s.t. ξ complete past of ζ and ζ complete future of ξ then write $\xi \overline{\preceq} \zeta$: complete pair.

Quantum causal histories

Discrtete causal structure induced from graph ordering

Markopoulou-Smolin (1997), Malyshev (2001),
Hawkins-Markopoulou-Sahlmann (2003), Kribs-Markopoulou (2005).
Let $\mathbb{G} = (\mathbb{G}^0, \mathbb{G}^1, s, t)$ **without cycles** and $x, y \in \mathbb{G}^0$.

- Induced partial ordering (discretised causality):

$$[x \prec y] \Leftrightarrow [\exists \alpha \in \mathrm{PS}(\mathbb{G}) : s(\alpha) = x, t(\alpha) = y],$$

- Events x and y (causally) related: if $x \prec y$ or $y \prec x$. Otherwise space-like separated ($x \sim y$).
- Acausal set: $\xi \subset \mathbb{G}^0$ s.t. $x, y \in \xi \Rightarrow x \sim y$.
- An acausal set ξ is a complete future of x, $x \overline{\prec} \xi$, if any inextendible future path from x intersects ξ. Define similarly $\zeta \preceq y$ the complete past of y.
- If ξ, ζ acausal sets s.t. ξ complete past of ζ and ζ complete future of ξ then write $\xi \overline{\preceq} \zeta$: complete pair.

Quantum causal histories

Quantum evolution between complete acausal pairs

- With every $x \in \mathbb{G}^0$ associate finite-dimensional $\mathcal{H}_x$.
- $\mathfrak{A}_x = \mathfrak{B}(\mathcal{H}_x) =$ full matrix algebra acting on $\mathcal{H}_x$.
- If $x \sim y$ then joint Hilbert space $\mathcal{H}_x \otimes \mathcal{H}_y$.
- If $\xi \subset \mathbb{G}^0$ acausal set, then $\mathcal{H}_\xi = \otimes_{x \in \xi} \mathcal{H}_x$.
- $\forall a \in \mathbb{G}^1$, there exists a quantum channel $\phi_a : \mathfrak{A}_{t(a)} \to \mathfrak{A}_{s(a)}$.

Quantum causal histories

Quantum evolution between complete acausal pairs

- With every $x \in \mathbb{G}^0$ associate finite-dimensional $\mathcal{H}_x$.
- $\mathfrak{A}_x = \mathfrak{B}(\mathcal{H}_x)$ = full matrix algebra acting on $\mathcal{H}_x$.
- If $x \sim y$ then joint Hilbert space $\mathcal{H}_x \otimes \mathcal{H}_y$.
- If $\xi \subset \mathbb{G}^0$ acausal set, then $\mathcal{H}_\xi = \otimes_{x \in \xi} \mathcal{H}_x$.
- $\forall a \in \mathbb{G}^1$, there exists a quantum channel $\phi_a : \mathfrak{A}_{t(a)} \to \mathfrak{A}_{s(a)}$.

UNIVERSITÉ DE RENNES 1

Quantum causal histories

Quantum evolution between complete acausal pairs

- With every $x \in \mathbb{G}^0$ associate finite-dimensional $\mathcal{H}_x$.
- $\mathfrak{A}_x = \mathfrak{B}(\mathcal{H}_x) =$ full matrix algebra acting on $\mathcal{H}_x$.
- If $x \sim y$ then joint Hilbert space $\mathcal{H}_x \otimes \mathcal{H}_y$.
- If $\xi \subset \mathbb{G}^0$ acausal set, then $\mathcal{H}_\xi = \otimes_{x \in \xi} \mathcal{H}_x$.
- $\forall a \in \mathbb{G}^1$, there exists a quantum channel $\phi_a : \mathfrak{A}_{t(a)} \to \mathfrak{A}_{s(a)}$.

Quantum causal histories

Quantum evolution between complete acausal pairs

- With every $x \in \mathbb{G}^0$ associate finite-dimensional $\mathcal{H}_x$.
- $\mathfrak{A}_x = \mathfrak{B}(\mathcal{H}_x) =$ full matrix algebra acting on $\mathcal{H}_x$.
- If $x \sim y$ then joint Hilbert space $\mathcal{H}_x \otimes \mathcal{H}_y$.
- If $\xi \subset \mathbb{G}^0$ acausal set, then $\mathcal{H}_\xi = \otimes_{x \in \xi} \mathcal{H}_x$.
- $\forall a \in \mathbb{G}^1$, there exists a quantum channel $\phi_a : \mathfrak{A}_{t(a)} \to \mathfrak{A}_{s(a)}$.

Quantum causal histories

Quantum evolution between complete acausal pairs

- With every $x \in \mathbb{G}^0$ associate finite-dimensional $\mathcal{H}_x$.
- $\mathfrak{A}_x = \mathfrak{B}(\mathcal{H}_x)$ = full matrix algebra acting on $\mathcal{H}_x$.
- If $x \sim y$ then joint Hilbert space $\mathcal{H}_x \otimes \mathcal{H}_y$.
- If $\xi \subset \mathbb{G}^0$ acausal set, then $\mathcal{H}_\xi = \otimes_{x\in\xi}\mathcal{H}_x$.
- $\forall a \in \mathbb{G}^1$, there exists a quantum channel $\phi_a : \mathfrak{A}_{t(a)} \to \mathfrak{A}_{s(a)}$.

Quantum causal histories

Unitary maps from cp maps

Theorem (Hawkins-Markopoulou-Sahlmann (2003))

If acausal sets ξ and ζ form a complete pair $\xi \overline{\preceq} \zeta$, then there exists an isomorphism $\phi_{\xi\zeta} : \mathfrak{A}_\zeta \to \mathfrak{A}_\xi$ s.t.

$$\forall x \in \xi, \forall z \in \zeta : red_{xz}(\phi_{\xi\zeta}) = \phi_{xz}.$$

...and for those so inclined to philosophical meditation

"I think of my lifetime in physics as divided into three periods. In the first period [...] I was in the grip of the idea that Everything is Particles. [...] I call my second period Everything is Fields. [...] Now I am in the grip of a new vision, that Everything is Information."

– John Archibald Wheeler[4]
Geons, Black Holes, and Quantum Foam
W. W. Norton & Co., NY (1998)

[4] 9 July 1911 – 13 April 2008. Among the last collaborators of A. Einstein, tried to finish their joint work on "Unified theory" by introducing the concept of geometrodynamics (programme abandoned in 1970). Niels Bohr medal 1939, Franklin medal 1969. PhD thesis supervisor of R. Feynman.