A note on matrix multiplicative cascades and bindweeds[1]

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Abstract: We study the problem of Mandelbrot's multiplicative cascades, but with random matrices instead of random variables. Then, we introduce a new model (which we call the bindweed model), which can be viewed as a random string in a random environment on a tree, and show that the classification of this model from the point of view of positive recurrence can be obtained from the corresponding classification of the matrix-valued multiplicative cascades.

1 Introduction

1.1 Motivation

Random walks in random environments on various types of graphs are known to display a behaviour dramatically differing from the one for ordinary random walks on the same graph. Among the different types of graphs, infinite trees provide a rich variety of mathematical problems, particularly connected to their non-amenability; beyond their mathematical interest they arise as more or less realistic models in several applied fields like random search algorithms in large data structures, Internet traffic, random grammars and probabilistic Turing machines, DNA coding, interacting random strings and automated languages, etc.

As a simple example, consider a rooted tree with constant branching b, and let us construct a random walk in random environment on it by sampling the transition probabilities (or transition rates, for continuous time) in each vertex from a given distribution, independently. When studying the question of (positive) recurrence of such a walk, one naturally arrives (see [16, 19]) to the following model. On each edge a of the tree, independently of the others, we place a

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positive random variable $\hat{\xi}_a$. Then, for each vertex v denote $\hat{\xi}[v] = \hat{\xi}_{a_1} \dots \hat{\xi}_{a_n}$, where $a_1, \dots, a_n$ is the (unique) path connecting the root to v. Models of this type are called *multiplicative cascades* (see e.g. [13, 15]), and, to study the positive recurrence of the corresponding random walk in random environment, one has to answer the question whether the sum of $\hat{\xi}[v]$ is finite. It turns out (see e.g. [16]) that the classification parameter for this problem is

$$\hat{\lambda} = \inf_{s \in [0,1]} \mathbb{E} \hat{\xi}_a^s, \tag{1}$$

which is then compared to $1/b$ (in fact, in [16] the case of general tree was considered).

The main goal of this paper is to answer the same type of questions for the above multiplicative cascades model with random matrices in place of random variables. The main classification parameter will be defined in (2) and it provides an appropriate generalisation of the formula (1), and the main results are summarised in the theorems 2.2 and 2.3 (a lot of preliminary work is required however, before formulating these results.)

This model is also closely related to a model of random walk in a random environment on a multiplexed tree, a process we term *bindweed* in the sequel for which we determine the classifying parameter that delimits recurrence from transience regimes. Again for this problem, the main classification parameter will be defined by the formula (2) and the main result summarised in Theorem 2.5. This model is a generalisation of the class of models known as one dimensional random string models, first introduced in [7, 6] in view of some queueing applications. String model was subsequently generalised in a random environment in [3] and there it appeared that the classifying parameter is defined only implicitly through the Lyapunov exponent for products of random matrices. In the model we study here, a random walk evolves on a tree but its transition probabilities are determined by random strings on the tree. The classifying parameter (2) — as was already the case in [3] — is implicitly determined by the asymptotic behaviour of the product of random matrices.

It was already remarked in [19] that asymptotic properties like recurrence/transience of random walk on trees with constant branching b are intimately connected to the existence of non-trivial solutions for the so-called multiplicative chaos equation of order b, first introduced as a simple turbulence model in [17]. The simplest variant of the multiplicative chaos equation is the following: let $(\xi_i)_{i=1,\dots,b}$, with $b \in \mathbb{N}$, be a finite family of non-negative random variables having known joint distribution and $(Y_i')_{i=1,\dots,b}$ and Y be a family of $b+1$ independent non-negative random variables distributed according to the same unknown law and verifying

$$Y \stackrel{\text{law}}{=} \sum_{i=1}^{b} Y_i' \xi_i.$$

The multiplicative chaos problem consists in determining under which conditions on the joint distribution of the ξ's the above equation has a non-trivial solution. This scalar problem is thoroughly studied in the literature, see e.g. [2, 4, 14]. As we remark later in this paper, the matrix multiplicative chaos equation may be an interesting problem to study as well.

1.2 Notation

In this section we give the formal definitions concerning trees, in particular, we define the notions of the growth rate and the branching number.

We denote $\mathbb{R}_+ = [0,\infty[$, $\mathbb{N} = \{0,1,2,\ldots\}$, $\mathbb{N}_+ = \{1,2,3,\ldots\}$, and for every $n \in \mathbb{N}_+$, $\mathbb{N}_n = \{1,2,\ldots,n\}$, while $\mathbb{N}_0 = \emptyset$. Let $\mathcal{A} \equiv \mathcal{A}^1$ be a finite or infinite denumerable set, called the *alphabet*. Define $\mathcal{A}^0 = \{\emptyset\}$ and for every $n \in \mathbb{N}_+$ denote

$$\mathcal{A}^n = \{\boldsymbol{\alpha} = \alpha_1 \cdots \alpha_n : \alpha_i \in \mathcal{A} \text{ for } i \in \mathbb{N}_n\}$$

the set of words of length n (*i.e.* having n letters),

$$\mathcal{A}^* = \cup_{n\in\mathbb{N}} \mathcal{A}^n$$

the set of words of arbitrary (finite) length, and

$$\partial\mathcal{A}^* \equiv \mathcal{A}^\infty = \{\boldsymbol{\alpha} = \alpha_1\alpha_2 \cdots : \alpha_i \in \mathcal{A} \text{ for } i \in \mathbb{N}_+\}$$

the set of infinite words. Finally, denote $\overline{\mathcal{A}^*} = \mathcal{A}^* \cup \partial\mathcal{A}^*$ and $\overset{\circ}{\mathcal{A}}{}^* = \mathcal{A}^* \setminus \mathcal{A}^0$.

For every $\boldsymbol{\alpha} \in \mathcal{A}^*$, there exists $n \in \mathbb{N}$ such that $\boldsymbol{\alpha} \in \mathcal{A}^n$; in this situation $|\boldsymbol{\alpha}| := n$ denotes the *length* of the word $\boldsymbol{\alpha}$ with the convention $|\emptyset| = 0$. Consistently, for every $\boldsymbol{\alpha} \in \partial\mathcal{A}^*$, we have $|\boldsymbol{\alpha}| = \infty$. For $\boldsymbol{\alpha} \in \overline{\mathcal{A}^*}$ with $|\boldsymbol{\alpha}| \geq n$ we denote by $\boldsymbol{\alpha}\restriction_n = \alpha_1 \cdots \alpha_n \in \mathcal{A}^n$ the *restriction* of $\boldsymbol{\alpha}$ to its n first letters with the convention $\boldsymbol{\alpha}\restriction_0 = \emptyset$. For every $\boldsymbol{\alpha} \in \overset{\circ}{\mathcal{A}}{}^*$, the *ancestor* $\hat{\boldsymbol{\alpha}}$ of $\boldsymbol{\alpha}$ is defined by $\hat{\boldsymbol{\alpha}} = \boldsymbol{\alpha}\restriction_{|\boldsymbol{\alpha}|-1}$. For $\boldsymbol{\alpha} \in \mathcal{A}^*$ and $\boldsymbol{\beta} \in \overline{\mathcal{A}^*}$, the *concatenation* of $\boldsymbol{\alpha}$ followed by $\boldsymbol{\beta}$ is the word $\boldsymbol{\alpha}\boldsymbol{\beta} = \alpha_1 \cdots \alpha_{|\boldsymbol{\alpha}|}\beta_1\beta_2 \cdots$ and for $\boldsymbol{\alpha}, \boldsymbol{\beta} \in \mathcal{A}^*$, their *common radix* $\boldsymbol{\alpha} \wedge \boldsymbol{\beta}$ is the longest word $\boldsymbol{\gamma} \in \mathcal{A}^*$ such that $\boldsymbol{\alpha} = \boldsymbol{\gamma}\boldsymbol{\alpha'}$ and $\boldsymbol{\beta} = \boldsymbol{\gamma}\boldsymbol{\beta'}$ for some words $\boldsymbol{\alpha'}, \boldsymbol{\beta'} \in \mathcal{A}^*$. We write $\boldsymbol{\alpha} \leq \boldsymbol{\beta}$ if $\boldsymbol{\alpha} = \boldsymbol{\alpha} \wedge \boldsymbol{\beta}$.

Remark: Notice that, consistently with the above notation, the symbol $\mathbb{N}^*$ denotes the set of finite words on the alphabet $\mathbb{N}$, contrary to some tradition (especially the French one) where this symbol is used to denote what we call here $\mathbb{N}_+$.

Definition 1.1 A mapping $B : \mathbb{N}^* \to \mathbb{N}$ is called a *branching function.*

To each branching function corresponds a uniquely determined rooted tree $\mathbb{T} = (\mathbb{V}, \mathbb{A})$ with vertex set $\mathbb{V} \equiv \mathbb{V}^*(B) \subseteq \mathbb{N}^*$ and edge set $\mathbb{A} = \overset{\circ}{\mathbb{V}}$ defined as follows: $\mathbb{V}^*(B) = \cup_{n\in\mathbb{N}}\mathbb{V}^n(B)$ where $\mathbb{V}^0(B) = \{\emptyset\} = \{\text{root}\} \equiv \{\mathbf{0}\}$ and for $n \in \mathbb{N}_+$,

$$\mathbb{V}^n(B) = \{\mathbf{v} = v_1 \cdots v_n : v_l \in \mathbb{N}_{B(\mathbf{v}\restriction_{l-1})}, \text{ for } l = 1, \ldots, n\}.$$

The branching function is said to be *without extinction* if the corresponding tree has non-trivial boundary $\partial\mathbb{V}$. The edge set is the subset of unordered pairs of vertices $[\mathbf{u}, \mathbf{v}] = [\mathbf{v}, \mathbf{u}]$ such that either $\mathbf{v} = \hat{\mathbf{u}}$ or $\mathbf{u} = \hat{\mathbf{v}}$. Since every vertex has a unique ancestor, every edge is indexed by its out-most vertex, *i.e.* for every $\mathbf{v} \in \overset{\circ}{\mathbb{V}}$, the corresponding edge is $a(\mathbf{v}) = [\hat{\mathbf{v}}, \mathbf{v}]$, showing thus that $\mathbb{A} \simeq \overset{\circ}{\mathbb{V}}$.

If $\mathbf{u}, \mathbf{v} \in \mathbb{V}$ and $\mathbf{u} \leq \mathbf{v}$ we define the *path* $[\mathbf{u}, \mathbf{v}]$ as the collection of the $|\mathbf{v}| - |\mathbf{u}|$ edges $[\mathbf{u}, \mathbf{v}\restriction_{|\mathbf{u}|+1}], \ldots, [\hat{\mathbf{v}}, \mathbf{v}]$, and if $\mathbf{u} = \emptyset$ then we simply denote by $[\mathbf{v}]$ the path $[\emptyset, \mathbf{v}]$ for every $\mathbf{v} \in \overset{\circ}{\mathbb{V}}$.

Example 1.2 If B is the constant function defined by $B(\mathbf{v}) = 2$ for all $\mathbf{v} \in \mathbb{V}$, then the corresponding tree is the *regular rooted binary tree.*

Example 1.3 Let $(\Psi, \mathcal{Y}, \mathfrak{w})$ be some abstract probability space and $B : \Psi \times \mathbb{N}^* \to \mathbb{N}$ be a family of independent random variables distributed according to the same law ν, *i.e.* $\mathfrak{w}(\{\psi \in \Psi : B(\psi, \mathbf{v}) = k\}) = \nu(k)$ for all $\mathbf{v} \in \mathbb{N}^*$ and all $k \in \mathbb{N}_+$. The corresponding tree is the *Galton-Watson* tree with offspring distribution ν.

In the sequel we shall consider only *deterministic branching functions without extinction.*

Definition 1.4 Let $\kappa_n = \mathsf{card}\mathbb{V}^n(B)$ denote the cardinality of the n^{th} generation of the tree defined by the branching function without extinction B. We call *lower growth rate* of the tree

$$\underline{\mathsf{gr}}(\mathbb{V}) = \liminf_n \kappa_n^{1/n},$$

upper growth rate of the tree

$$\overline{\mathsf{gr}}(\mathbb{V}) = \limsup_n \kappa_n^{1/n},$$

and, if $\underline{\mathsf{gr}}(\mathbb{V}) = \overline{\mathsf{gr}}(\mathbb{V})$, we call the common value *growth rate*

$$\mathsf{gr}(\mathbb{V}) = \lim_n \kappa_n^{1/n}.$$

For $\mathbf{u}, \mathbf{v} \in \partial\mathbb{V}$, define $\delta(\mathbf{u}, \mathbf{v}) = \exp(-|\mathbf{u} \wedge \mathbf{v}|)$. It can be shown that δ is a distance on $\partial\mathbb{V}$. Moreover if $\|B\|_\infty = \sup_{\mathbf{v}\in\mathbb{V}} B(\mathbf{v}) < \infty$ then the space $(\partial\mathbb{V}, \delta)$ is compact and we can define its Hausdorff dimension $\dim_H \partial\mathbb{V}$ as usual (see [5] for instance).

Definition 1.5 For a tree $\mathbb{V}$ generated by a branching function B with $\|B\|_\infty < \infty$, we define its *branching rate*

$$\mathsf{br}(\mathbb{V}) = \exp(\dim_H \partial\mathbb{V}).$$

It is shown in [18, 16] that $\mathsf{br}(\mathbb{V}) = \sup\{\lambda : \inf \sum_{v\in C} \lambda^{-|\mathbf{v}|} > 0\}$ where the infimum is evaluated over all cut-sets C of $\mathbb{V}$. We have in general that $\mathsf{br}(\mathbb{V}) \leq \underline{\mathsf{gr}}(\mathbb{V})$.

2 The model and main results

2.1 Matrix multiplicative cascades and the corresponding results

Let $(\Omega, \mathcal{F}, \mathbb{P})$ be some abstract probability space which carries all the random variables that will be needed in the model. Let $(\mathbb{V}, \mathbb{A})$ be the rooted tree associated with a given branching function B. Let G be the multiplicative semi-group $\mathcal{M}_d(\mathbb{R}_+)$ of $d \times d$ matrices with non-negative coefficients that are allowable, i.e. every row and every column contains a strictly positive element. We denote by G_+ the subset of G consisting of matrices with strictly positive elements. The semi-group G can also be identified with the set $\mathsf{End}(\mathbb{R}^d)$ leaving invariant the first octant $\overline{C} = \{\mathbf{x} \in \mathbb{R}^d : \forall i = 1, \ldots, d, x_i \geq 0\}$. When the vector space $\mathbb{R}^d$ is equipped with a vector norm $\|\cdot\|$, a matrix norm, also denoted by $\|\cdot\|$, is automatically induced on G, by $\|g\| = \sup\{\|g\mathbf{x}\|, \mathbf{x} \in \mathbb{R}^d, \|\mathbf{x}\| = 1\}$. In the sequel, we always assume that the vector norm is $\|\mathbf{x}\| = |x_1| + \ldots + |x_d|$, for all $\mathbf{x} \in \mathbb{R}^d$.

Equipped with the norm $\|\cdot\|$, the semigroup is naturally endowed with a Borel σ-algebra $\mathcal{G}$. Let $(\xi_a)_{a\in\mathbb{A}}$ be an edge-indexed family of independent and identically distributed G-valued random variables defined on the probability space $(\Omega,\mathcal{F},\mathbb{P})$ and denote by μ their common distribution on $(G,\mathcal{G})$, i.e.

$$\mathbb{P}(\xi_a \in dg) = \mu(dg), \text{ for all } a \in \mathbb{A}.$$

Denote by $\sigma_\mu = \mathsf{supp}\,\mu \subset G$ the support of the measure μ and by Σ_μ the semi-group generated by σ_μ.

For $\mathbf{u},\mathbf{v} \in \mathbb{V}$ with $\mathbf{u} \le \mathbf{v}$ define

$$\xi[\mathbf{u},\mathbf{v}] \equiv \overleftarrow{\prod_{a\in[\mathbf{u},\mathbf{v}]}} \xi_a,$$

where $\overleftarrow{\prod}$ denotes the product in reverse order, *i.e.* if $[\mathbf{u},\mathbf{v}] = a_1\cdots a_k$ then $\xi[\mathbf{u},\mathbf{v}] = \xi_{a_k}\cdots\xi_{a_1}$ with the convention $\xi[\mathbf{v},\mathbf{v}] = e$ where e is the neutral element of G. We introduce the following G-valued random processes: the *matrix-multiplicative cascade process*

$$\psi_n = \sum_{\mathbf{v}\in\mathbb{V}^n} \xi[\mathbf{v}], \quad n \in \mathbb{N}$$

and the *integrated matrix-multiplicative cascade process*

$$\zeta_n = \sum_{k=1}^{n} \psi_k, \quad n \in \mathbb{N}_+.$$

As we mentioned in Section 1.1, the main goal of this paper is to investigate the asymptotics of ζ_n. However, to formulate our main results (Theorems 2.2 and 2.3 below), we have to introduce more notation and conditions. We begin by defining, for a fixed $\mathbf{v} \in \partial\mathbb{V}$ and all $n \in \mathbb{N}$

$$X_n \equiv X_n(\mathbf{v}) = \xi[\mathbf{v}\restriction_n].$$

It is immediate to see (cf. [20]) that $(X_n)_{n\in\mathbb{N}_+}$ is a G-valued multiplicative Markov chain with stochastic kernel

$$P(g, \mathrm{d}\, g') \equiv \mathbb{P}(X_{n+1} \in \mathrm{d}\, g' | X_n = g) = \mu \star \delta_g(\mathrm{d}\, g'), \quad n \in \mathbb{N}_+,$$

where for two measures μ,μ' on $(G,\mathcal{G})$ their convolution $\mu\star\mu'$ is defined by its dual action on the set $C_K(G)$ of continuous functions on G with comact support, via

$$\langle\, \mu\star\mu', f\,\rangle \equiv \int_G f(g)\mu\star\mu'(\mathrm{d}\, g) = \int_G\int_G f(gg')\mu(\mathrm{d}\, g)\mu'(\mathrm{d}\, g'),$$

for all $f \in C_K(G)$. The stochastic kernel P defines an operator, denoted also $P : C_K(G) \to C_K(G)$ by

$$Pf(g) = \int_G P(g,\mathrm{d}\, g')f(g') = \int_G \mu(\mathrm{d}\, g')f(g'g),$$

for all $g \in G$. Its adjoint can be seen as a left action on positive measures, m, on $(G,\mathcal{G})$:

$$mP(A) = \int_G m(\mathrm{d}\, g)P(g,A) = \mu\star m(A),$$

for all $A \in \mathcal{G}$. The stochastic kernel P induces a probability measure $\mathbb{P}_\mu$ on the space of trajectories $G^{\otimes \mathbb{N}}$ of the Markov chain $(X_n)_{n\in\mathbb{N}_+}$.

We require the following conditions on μ.

Condition 1 (Integrability): Let

$$I_\mu = \{s \in \mathbb{R} : \int_G \|g\|^s \mu(\mathrm{d}\, g) < \infty\}.$$

It is assumed that $[0,1] \subseteq I_\mu$.

Observe that

$$\mathbb{E}_\mu(\|X_n \mathbf{x}\|^s) = \int_G \|g\mathbf{x}\|^s \mu^{\star n}(\mathrm{d}\, g).$$

Condition 2 (Strict positivity): We assume that $\sigma_\mu \subseteq G$ is such that $\mu(G \setminus G_+) = 0$

Remark: Notice that the condition of strict positivity can be slightly if only large deviation bounds for products of random matrices are seeked. It is necessary however to obtain *almost sure* divergence in theorem 2.3.

Lemma 2.1 *Let*

$$k(s) = \lim_n \left(\int_G \|g\|^s \mu^{\star n}(\mathrm{d}\, g) \right)^{1/n}.$$

The limit always exists in $[0, \infty]$ *and defines a log-convex* $[0,\infty]$*-valued function on* $\mathbb{R}$. *If* $s \in I_\mu$ *then* $k(s) < \infty$.

We define in the sequel the quantity λ, that turns out to be the main classification parameter for the matrix multiplicative cascades model, by

$$\lambda = \inf_{s\in[0,1]} k(s) \tag{2}$$

(compare (2) with (1)).

We are now in the position to state our main results.

Theorem 2.2 *Let* $(\mathbb{V}, \mathbb{A})$ *be some tree defined in terms of a given branching function* B *and* $\overline{\mathsf{gr}}(\mathbb{V})$ *and* λ *defined as in definition 1.4 and equation (2) respectively. Under the conditions 1 and 2,*

$$\lambda \overline{\mathsf{gr}}(\mathbb{V}) < 1 \Rightarrow \zeta_{\infty,ij} < \infty \text{ almost surely, for all } i,j = 1,\dots,d.$$

Theorem 2.3 *Let* $\mathsf{br}(\mathbb{V})$ *and* λ *be the quantities introduced in definition 1.5 and equation 2 respectively and let* $\chi \in \mathbb{R}^d$ *be the vector having all its components equal to 1:* $\chi_i = 1$, *for all* $i = 1,\dots,d$. *Let* $(\mathbb{V},\mathbb{A})$ *be some tree defined in terms of a given branching function* B *without extinction. Under the conditions 1 and 2,*

$$\lambda \mathsf{br}(\mathbb{V}) > 1 \Rightarrow Z_\infty := (\chi, \zeta_\infty \chi) = \infty \text{ almost surely.}$$

Remark: Similarly to [16], there is a gap between Theorems 2.2 and 2.3, since in general the branching number need not be equal to the growth rate. However, this is not very important, because in most of the practical examples these quantities do coincide.

Remark: As mentioned above, the classification parameter for this problem is $\lambda = \inf_{s\in[0,1]} k(s)$. This parameter is not explicitly computable in general since it involves the infinite product of matrices. However for some particular cases this quantity can be computed explicitly as shown in the following proposition.

Proposition 2.4 *Suppose that the measure μ is such that $g_{ij} < 1/d$ almost surely for all $i, j = 1, \dots, d$. Then λ is the largest eigenvalue of the matrix $\mathbb{E}g$.*

Remark: It is interesting to consider the *chaos equation* for the case of matrix-valued random variables and constant branching b:

$$Y \stackrel{\text{law}}{=} \sum_{j=1}^{b} Y'_j \xi_j, \tag{3}$$

where Y, Y'_j, ξ_j are G-valued random variables, and ξ_j (which are not necessarily independent) are distributed according to μ; Y'_j, $j = 1, \dots, b$, are i.i.d. and have the same (unknown) law as Y. Analogously to [19] we can get that (at least in the case when ξ_1 satisfies conditions 1 and 2) $\lambda d = 1$ is a necessary condition for the existence of solution of (3). It is an open problem whether this condition is sufficient.

Remark: The condition of independence of the random variables ξ_a can be relaxed; what is important is that if ξ_a and ξ_b are not adjacent to the same vertex then they must be independent, and that ξ-s that belong to any path emanating from the root must be independent.

2.2 The bindweed model

In this section we introduce a model describing an evolution of a random string in random environment on a tree (which is somewhat similar to the model studied in [3]) which we call the bindweed model. Then, we show that its classification from the point of view of positive recurrence can be obtained by using theorems 2.2 and 2.3.

Let $\mathcal{S} = \{1, \dots, d\}$ be a finite alphabet and denote, in accordance with the notations introduced in Section 1.2, $\mathcal{S}^{n+1} = \{\boldsymbol{\sigma} = \sigma_0 \cdots \sigma_n : \sigma_i \in \mathcal{S}\}$ the set of words of length $n+1$ composed from the symbols of the alphabet $\mathcal{S}$, $\mathcal{S}^0$ the set containing only the empty word and $\mathcal{S}^*$ the set of words of arbitrary length. Suppose that a branching function B is given on $\mathbb{N}^*$ and denote $\mathbb{V}^n \equiv \mathbb{V}^n(B)$ the corresponding generations of the tree determined by B. Therefore, the rooted tree $\mathbb{T} = (\mathbb{V}, \mathbb{A})$ is uniquely defined.

Now we are going to construct a continuous-time Markov chain with state space $\mathfrak{S}$, defined by

$$\mathfrak{S} = \{\hat{\emptyset}\} \cup \bigcup_{n=1} (\mathbb{V}^n \times \mathcal{S}^{n+1}),$$

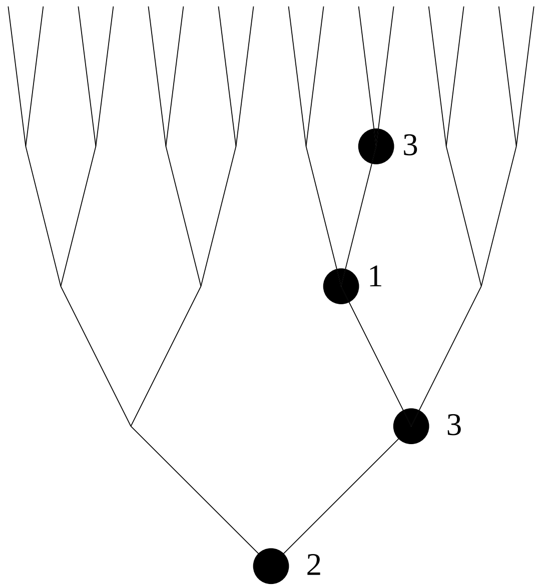

Figure 1: A typical state of the bindweed model for $\mathcal{S} = \{1,2,3\}$, $\boldsymbol{\sigma} = 2313$, and $\mathbb{T}$ the binary tree.

where $\hat{\emptyset}$ is a special state to be defined later. In fact, what happens is the following: we place a word $\boldsymbol{\sigma} = \sigma_0 \dots \sigma_n$ on the tree $\mathbb{T}$ in such a way that the 0^{th} symbol of the word is placed on the root $\mathbf{0}$, for any $i = 1, \dots, n$ the i^{th} symbol of the word is placed somewhere in $\mathbb{V}^i$, and, if the i^{th} symbol σ_i is placed on vertex $\mathbf{u}$, and σ_{i+1} on $\mathbf{v}$, then $\mathbf{u} < \mathbf{v}$ and $[\mathbf{u}, \mathbf{v}] \in \mathbb{A}$ (see figure 1). The state $\hat{\emptyset}$ means that nothing is placed on the tree.

Now, let us define the dynamics of the bindweed model. Suppose that for any $a \in \mathbb{A}$ two collections of positive numbers $(\nu_{yz}(a), y, z \in \mathcal{S})$, $(\mu_y(a), y \in \mathcal{S})$ are given. If the bindweed model is in the state $(\mathbf{u}, \boldsymbol{\sigma})$, where $\boldsymbol{\sigma} = \sigma_0 \dots \sigma_{n-1} y$, $\mathbf{u} \in \mathbb{V}^n$, then

- for $n \geq 0$ it jumps to the state $(\mathbf{v}, \sigma_0 \dots \sigma_{n-1} yz)$ with rate $\nu_{yz}(a(\mathbf{v}))$, for all $\mathbf{v} \in \mathbb{V} : \mathbf{u} = \hat{\mathbf{v}}$;
- for $n \geq 1$ it jumps to the state $(\hat{\mathbf{u}}, \sigma_0 \dots \sigma_{n-1})$ with rate $\mu_y(a(\mathbf{u}))$.

For any $\sigma_0 \in \mathcal{S}$ the transitions $\hat{\emptyset} \to (\mathbf{0}, \sigma_0)$ and $(\mathbf{0}, \sigma_0) \to \hat{\emptyset}$ occur with rate 1. Thus, we have defined a continuous-time Markov chain with state space $\mathfrak{S}$.

Let us describe now how to choose the transition rates. Let ρ be any probability measure on $\mathbb{R}_+^{d^2+d}$. Suppose that for any $a \in \mathbb{A}$ the vector $\Xi(a) = (\nu_{yz}(a), y, z \in \mathcal{S}, \mu_y, y \in \mathcal{S})$ is random, having distribution ρ, and $(\Xi(a), a \in \mathbb{A})$ are independent and identically distributed. Fix a realisation of that collection of random vectors and consider the bindweed model with the transition rates ruled by that realisation. So, the model that we constructed is a continuous-time Markov chain in a quenched random environment.

Now, we are interested in obtaining a classification of this Markov chain with respect to positive recurrence. For $(\mathbf{v}, \boldsymbol{\sigma}) \in \mathfrak{S}$ denote by $\pi(\mathbf{v}, \boldsymbol{\sigma})$ the stationary measure. For any $a \in \mathbb{A}$ let ξ_a be a $d \times d$ matrix whose matrix elements are defined in the following way: $\xi_{a,xy} = \nu_{xy}(a)/\mu_y(a)$, $x, y \in \mathcal{S}$. It is not difficult to see that we have a reversible Markov chain, so it is clear that

$\pi(\hat{\emptyset}) = \pi(\mathbf{0}, x)$, for all $x \in \mathcal{S}$, and, for any $\mathbf{v} \in \mathbb{V}^n$, $n \geq 1$, and $x, y, \sigma_0, \ldots, \sigma_{n-2} \in \mathcal{S}$, we can formally write

$$\begin{aligned}\pi(\mathbf{v}, \sigma_0 \ldots \sigma_{n-2} xy) &= \frac{\nu_{xy}(a(\mathbf{v}))}{\mu_y(a(\mathbf{v}))} \pi(\hat{\mathbf{v}}, \sigma_0 \ldots \sigma_{n-2} x) \\ &= \xi_{a(\mathbf{v}),xy} \pi(\hat{\mathbf{v}}, \sigma_0 \ldots \sigma_{n-2} x). \end{aligned} \tag{4}$$

For $\mathbf{v} \in \mathbb{V}^n$ denote by $m(\mathbf{v})$ the matrix with elements

$$m_{zy}(\mathbf{v}) = \sum_{\sigma_1, \ldots, \sigma_{n-1} \in \mathcal{S}} \pi(\mathbf{v}, z\sigma_1 \ldots \sigma_{n-1} y).$$

Using (4), we have

$$\begin{aligned} m_{zy}(\mathbf{v}) &= \sum_x \sum_{\sigma_1, \ldots, \sigma_{n-2} \in \mathcal{S}} \pi(\mathbf{v}, z\sigma_1 \ldots \sigma_{n-2} xy) \\ &= \sum_x \sum_{\sigma_1, \ldots, \sigma_{n-2} \in \mathcal{S}} \xi_{a(\mathbf{v}),xy} \pi(\hat{\mathbf{v}}, z\sigma_1 \ldots \sigma_{n-2} x) \\ &= \sum_x \xi_{a(\mathbf{v}),xy} m_{zx}(\hat{\mathbf{v}}), \end{aligned}$$

so

$$m(\mathbf{v}) = m(\hat{\mathbf{v}}) \xi_{a(\mathbf{v})}. \tag{5}$$

Let χ be a vector of order d with all its coordinates equal to 1. As $\pi(\hat{\emptyset}) = \pi(\mathbf{0}, x)$, for all $x \in \mathcal{S}$, we have

$$\sum_{\substack{\mathbf{v} \in \mathbb{V}^1 \\ \sigma_0 \sigma_1 \in \mathcal{S}^2}} \pi(\mathbf{v}, \sigma_0 \sigma_1) = \sum_{\substack{x,y \in \mathcal{S} \\ \mathbf{v} \in \mathbb{V}^1}} m_{xy}(\mathbf{v}) = \pi(\hat{\emptyset}) \left(\chi, \sum_{\mathbf{v} \in \mathbb{V}^1} \xi_{a(\mathbf{v})} \chi \right).$$

This fact together with equation (5) shows that

$$\sum_{\substack{\mathbf{v} \in \mathbb{V}^n \\ \boldsymbol{\sigma} \in \mathcal{S}^n}} \pi(\mathbf{v}, \boldsymbol{\sigma}) = \pi(\hat{\emptyset}) \left(\chi, \sum_{\mathbf{v} \in \mathbb{V}^n} \xi[\mathbf{v}] \chi \right)$$

and so $\sum_{(\mathbf{v}, \boldsymbol{\sigma}) \in \mathfrak{S}} \pi(\mathbf{v}, \boldsymbol{\sigma})$ is finite if and only if Z_∞ is finite. Thus, theorems 2.2 and 2.3 allow us to obtain the classification of the bindweed model in random environment from the point of view of positive recurrence, in the following way:

Theorem 2.5 *Suppose that the distribution of the random matrix ξ_a is such that the Conditions 1 and 2 are satisfied. Let λ be the quantity defined as in Section 2.1. Then*

- *if $\lambda \overline{\mathsf{gr}}(\mathbb{V}) < 1$, then the bindweed model is positive recurrent;*
- *if $\lambda \mathsf{br}(\mathbb{V}) > 1$, then the bindweed model is not positive recurrent.*

3 Proof of positive recurrence

Since the theorem 2.2 is easier to prove than theorem 2.3, let us start from its proof.

Proof of theorem 2.2: Write

$$\zeta_n = \sum_{k=1}^{n} \sum_{\mathbf{v}\in\mathbb{V}^n} \xi[\mathbf{v}] \ \text{ and } \ z_n(s) = \sum_{k=1}^{n} \sum_{\mathbf{v}\in\mathbb{V}^n} \|\xi[\mathbf{v}]\|^s$$

and compute

$$\mathbb{E}z_n(s) = \sum_{k=1}^{n} \mathsf{card}\mathbb{V}^n \int_G \|g\|^s \mu^{\star n}(\mathrm{d}\, g) = \sum_{k=1}^{n} \kappa_n \int_G \|g\|^s \mu^{\star n}(\mathrm{d}\, g).$$

Now if $\lambda = \inf_{s\in[0,1]} k(s) = k(s_0)$ and $\overline{\mathsf{gr}}(\mathbb{V})\lambda < 1$ then the series defining $z_\infty(s_0)$ converges, hence $z_\infty(s_0) < \infty$ and consequently $\sum_{\mathbf{v}\in\mathbb{V}} \|\xi[\mathbf{v}]\|^{s_0} < \infty$ almost surely. Therefore, the set $V = \{\mathbf{v} \in \mathbb{V} : \|\xi[\mathbf{v}]\| > 1\}$ is finite. Hence

$$\begin{aligned} \sum_{\mathbf{v}\in\mathbb{V}} \|\xi[\mathbf{v}]\| &= \sum_{\mathbf{v}\in V} \|\xi[\mathbf{v}]\| + \sum_{\mathbf{v}\in\mathbb{V}\setminus V} \|\xi[\mathbf{v}]\| \\ &\le \sum_{\mathbf{v}\in V} \|\xi[\mathbf{v}]\| + \sum_{\mathbf{v}\in\mathbb{V}\setminus V} \|\xi[\mathbf{v}]\|^{s_0} \\ &\le \sum_{\mathbf{v}\in V} \|\xi[\mathbf{v}]\| + z_\infty(s_0) < \infty \text{ almost surely.} \end{aligned}$$

A fortiori $\|\zeta_\infty\| < \infty$ almost surely which proves also the statement concerning the matrix elements $\zeta_{\infty,ij}$ for $i, j = 1, \ldots, d$ since $\Sigma_\mu \subseteq G$. □

4 Large deviation estimates and proof of theorem 2.3

In order to prove theorem 2.3, we need some preparation.

4.1 Results on products of random matrices

Products of random matrices are extensively studied in the literature. The objective of this subsection is to use results and ideas of [11, 12, 9, 10] in order to obtain large deviation estimates for the sequence $(\|X_n\|^s)_{n\in\mathbb{N}_+}$, for $s \ge 0$. [8].

Recall that the first Lyapunov's characteristic exponent is defined by

$$\gamma_1 = \lim_{n\to\infty} \frac{1}{n} \log \|g_n \cdots g_1\|. \tag{6}$$

First we prove lemma 2.1 and proposition 2.4.

Proof of lemma 2.1: The sequence $a_n(s) = \int \|g\|^s \mu^{\star n}(\mathrm{d}\, g)$ is submultiplicative, i.e. $a_{m+n}(s) \leq a_m(s) a_n(s)$, due to the independence of the matrices; this guarantees the existence of the limit. Log-convexity follows from Jensen's inequality. □

Proof of proposition 2.4: Under the condition $g_{ij} < 1/d$ for all $i, j = 1, \ldots, d$, it follows that $\|g\| < 1$ and the infimum is attained at $s = 1$, *i.e.* $\inf_{s\in[0,1]} k(s) = k(1)$. We know moreover from lemma 2.1 that for $s \in I_\mu$ we have $k(s) < \infty$. For an arbitrary $\mathbf{x}$ with $\|\mathbf{x}\| = 1$ and $x_i > 0$, let $\alpha = \min_i x_i > 0$. We have $\|g\mathbf{x}\| \leq \|g\|$ and

$$\|g\mathbf{x}\| = \sum_i \sum_j g_{ij} x_j \geq \alpha \max_i \sum_j g_{ij} = \alpha \|g\|.$$

Therefore, for any $\mathbf{x} \in \mathbb{R}^d$ such that $\|\mathbf{x}\| = 1$ and $\alpha = \min_i x_i > 0$, we have

$$\left(\alpha \int_G \|g\|^s \mu^{\star n}(\mathrm{d}\, g)\right)^{1/n} \leq \left(\int_G \|g\mathbf{x}\|^s \mu^{\star n}(\mathrm{d}\, g)\right)^{1/n} \leq \left(\int_G \|g\|^s \mu^{\star n}(\mathrm{d}\, g)\right)^{1/n}.$$

Choose $\mathbf{x}$ to be the normalised eignevector corresponding to the largest eigenvalue ρ of $\mathbb{E}g$. Then

$$\int_G \|g\mathbf{x}\|^s \mu^{\star n}(\mathrm{d}\, g) = \mathbb{E}\|g_n \cdots g_1 \mathbf{x}\| = \mathbb{E}\left(\sum_{i=1}^d \sum_{j=1}^d X_{n,ij} x_j\right) = \rho^n \sum_{j=1}^d x_j = \rho^n$$

showing that $\lambda = \rho$. □

Now we pass to the most important large deviation result in our case, namely the establishment of the Chernoff-Cramér bound.

4.2 The Chernoff-Cramér bound

Denote by

$$L(s) = \begin{cases} \log k(s) & \text{if } s \in I_\mu \\ +\infty & \text{if } s \notin I_\mu, \end{cases}$$

$\lambda = \inf_{s\in[0,1]} k(s)$ and $\gamma_1(s) = \frac{\mathrm{d}}{\mathrm{d}\, s} \log k(s), s \in I$. The value $\gamma_1(0)$ is the largest Lyapunov's characteristic exponent.

Lemma 4.1 *For every $\rho \in]0, \exp\gamma_1(0)[$ and every $\delta > 0$, there exists $N = N(\delta, \rho)$ such that for all $n \geq N$ and for all $\mathbf{v} \in \partial\mathbb{V}$,*

$$\mathbb{P}(\|\xi[\mathbf{v}\restriction_n]\| > \rho^n) > 1 - \delta.$$

Proof: Denote $\Xi_n = \frac{\log \|\xi[\mathbf{v}\restriction_n]\|}{n}$. It is a well known result (see [1] for instance) that $\lim \Xi_n = \gamma_1(0)$ almost surely, *i.e.*

for any $\varepsilon, \delta > 0$, there exists $N_0 = N_0(\varepsilon, \delta)$ such that
for all $n \geq N_0$ it holds that $\mathbb{P} \cap_{k\geq n} \{|\Xi_k - \gamma_1(0)| \leq \varepsilon\}) \geq 1 - \delta$.

Hence $\mathbb{P}(|\Xi_n - \gamma_1(0)| \leq \varepsilon) \geq 1 - \delta$ and a fortiori $\mathbb{P}(\Xi_n \geq \gamma_1(0) - \varepsilon) \geq 1 - \delta$. We conclude by choosing $\varepsilon = \gamma_1(0) - \log\rho > 0$. □

Theorem 4.2 *Assume that* $\lambda \mathsf{br}(\mathbb{V}) > 1$ *and let* $\rho \in]0,1]$ *be such that* $\lambda = \inf_{s\in[0,1]} \rho^{1-s} k(s)$. *Then there exists* n_0 *such that for all* $n \geq n_0$,

$$\mathbb{P}(\|\xi[\mathbf{v}\lceil_n]\| > \rho^n) > \frac{1}{(\rho \mathsf{br}(\mathbb{V}))^n}.$$

Proof: If $\rho \leq \exp \gamma_1(0)$, there is nothing to prove since the previous lemma provides an even better bound. Assume thus that $\rho > \exp \gamma_1(0)$ and write for the Legendre transform

$$\Lambda(s^*) = \sup_s (s^* s - L(s)) = \begin{cases} 0, & \text{if } s^* \leq \gamma_1(0), \\ s^* s_0(s^*) - L(s_0(s^*)), & \text{if } s^* > \gamma_1(0), \end{cases}$$

where $s_0(s^*)$ is solution of $s^* = L'(s^*) = \gamma_1(s)$. Since $k(s)$ is log-convex, we have that $\gamma(s) \geq \gamma_1(0)$. For $s \geq 0$, the limit $k(s) = \lim_n (\int_G \|g\|^s \mu^{\star n}(\mathrm{d}\, g))^{1/n}$ exists and the random matrix $X_n = \xi[\mathbf{v}\lceil_n]$ is distributed according to $\mu^{\star n}$. Denote $S_n = \log \|X_n\|$ (for fixed $\mathbf{v} \in \partial \mathbb{V}$). The sequence S_n verifies (cf. [12]) a large deviation principle with rate function Λ, *i.e.*

$$\lim_n \frac{1}{n} \log \mathbb{P}(S_n > n\theta) = \inf_{s^* > \theta} \Lambda(s^*) = \Lambda(\theta).$$

Therefore, for any $\varepsilon > 0$ there exists $N = N(\varepsilon)$ such that for all $n \geq N$ we have $\frac{1}{n} \log \mathbb{P}(S_n > n\theta) \geq -\Lambda(\theta) - \varepsilon$, or $\exp(n\varepsilon)\mathbb{P}(\|X_n\| > \exp(n\theta)) \geq \exp(-n\Lambda(\theta))$. Now,

$$\lambda = \inf_{s\in[0,1]} k(s) = \sup_{r\in]0,1]} \inf_{s\geq 0} r^{1-s} k(s).$$

Let $\rho \in]0,1]$ be such that $\lambda = \inf_{s\geq 0} \rho^{1-s} k(s)$. For this fixed ρ,

$$\begin{aligned} \inf_{s\geq 0} \rho^{1-s} k(s) &= \rho \exp\left(\inf_{s \geq 0} (-s \log \rho + L(s)) \right) \\ &= \rho \exp\left(- \sup_{s\geq 0} (s \log \rho - L(s)) \right) \\ &= \rho \exp(-\Lambda(\log \rho)) \\ &= \lambda \\ &> \frac{1}{\mathsf{br}(\mathbb{V})}. \end{aligned}$$

Two cases can in fact appear: either $\gamma_1(0) \geq 0$ or $\gamma_1(0) < 0$.

- If $\gamma_1(0) \geq 0$, then $\lambda = 1$ and $\log \rho \leq \gamma_1(0)$. Therefore $\Lambda(\log \rho) = 0$. So, either $\log \rho < \gamma_1(0)$ and in that case the previous lemma provides an even better bound, or else $\log \rho = \gamma_1(0)$ and then we obtain $\exp \gamma_1(0) = \lambda = 1 > \frac{1}{\mathsf{br}(\mathbb{V})}$ which can happen only if $\gamma_1(0) = 0$.
- If $\gamma_1(0) < 0$, then $\lambda < 1$ because $k(s)$ is decreasing, at least for a small interval $[0, \eta]$, $\eta > 0$.
  - If $\log \rho = \gamma_1(0)$ then $\Lambda(\log \rho) = 0$ and we have $\frac{1}{\mathsf{br}(\mathbb{V})} < \exp \gamma_1(0) = \lambda < 1$.
  - If $\log \rho > \gamma_1(0)$ then $\Lambda(\log \rho) > 0$ and we have $\frac{1}{\mathsf{br}(\mathbb{V})} < \lambda = \rho \exp(-\Lambda(\log \rho)) \leq 1$ and in this case there exists a $\rho < 1$ verifying the previous inequality. Choosing $\theta = \log \rho$, we have

    $$\exp(n\varepsilon)\mathbb{P}(\|X_n\| > \rho^n) \geq (\exp(-\Lambda(\log \rho)))^n > \left(\frac{1}{\rho \mathsf{br}(\mathbb{V})} \right)^n,$$

    because $\exp(-\Lambda(\log \rho)) = \lambda / \rho > (\rho \mathsf{br}(\mathbb{V}))^{-1}$.

Hence there exists $\delta > 0$ such that $\exp(-\Lambda(\log\rho)) = \exp(\delta)\frac{1}{\rho\mathsf{br}(\mathbb{V})}$. Therefore $\mathbb{P}(\|X_n\| > \rho^n) \geq \frac{\exp((\delta-\varepsilon)n)}{(\rho\mathsf{br}(\mathbb{V}))^n}$. Choosing $0 < \varepsilon < \delta$, we have for every $n \geq N$ that

$$\mathbb{P}(\|\xi[\mathbf{v}\restriction_n]\| > \rho^n) > \frac{1}{(\rho\mathsf{br}(\mathbb{V}))^n}.$$

□

Corollary 4.3 *There exist $\varepsilon_1, \varepsilon_2 > 0$ such that for all $\rho_1 \in [\rho(1-\varepsilon_1), \rho]$ and for all $\rho_2 \in [\rho, \rho(1+\varepsilon_2)]$,*

$$\mathbb{P}(\|\xi[\mathbf{v}\restriction_n]\| > \rho_2^n) > \frac{1}{(\rho_1\mathsf{br}(\mathbb{V}))^n}.$$

4.3 Matrix estimates

Definition 4.4 Let $g \in \mathcal{M}_d(\mathbb{R}_+)$ and $\mathbf{x} \in \mathbb{R}_+^d$ and suppose that there exists $K \geq 1$ such that

1. for all $i, j = 1, \ldots, d$ the matrix elements verify: $\frac{1}{K} \leq g_{ij} \leq K$. Then the matrix g is called *K-equilibrated.*
2. for all $i, j = 1, \ldots, d$ the vector coordinates verify: $\frac{1}{K} \leq \frac{x_i}{x_j} \leq K$. Then the vector $\mathbf{x}$ is called *K-equilibrated.*

Lemma 4.5 *Let $g \in \mathcal{M}_d(\mathbb{R}_+)$ and $\mathbf{x} \in \mathbb{R}_+^d$ be an arbitrary vector with $\|\mathbf{x}\| \neq 0$. Let $\mathbf{y} = g\mathbf{x} \in \mathbb{R}_+^d$. If the matrix g is K-equilibrated, then the vector $\mathbf{y}$ is K^2-equilibrated.*

Proof: Write $y_i = \sum_{j=1}^d g_{ij}x_j$ and observe that

$$y_i \leq \max_l y_l = \max_l \sum_j g_{lj}x_j \leq K(x_1 + \ldots + x_d) = K\|\mathbf{x}\|,$$

and

$$y_i \geq \min_l y_l = \min_l \sum_j g_{lj}x_j \geq \frac{1}{K}(x_1 + \ldots + x_d) = \frac{1}{K}\|\mathbf{x}\|,$$

so, since $\|\mathbf{x}\| \neq 0$, the lemma follows. □

Lemma 4.6 *Let $g, g', h \in \mathcal{M}_d(\mathbb{R}_+)$ and suppose that h is K-equilibrated. As before, denote by $\chi \in \mathbb{R}^d$ the vector having all its coordinates equal to 1. Then*

$$(\chi, ghg'\chi) \geq \frac{(\chi, g\chi)(\chi, g'\chi)}{K^5}.$$

Proof: Write

$$\begin{aligned}(\chi, ghg'\chi) &= (g^*\chi, hg'\chi) \\ &\geq \min\{(hg'\chi)_i, i = 1, \ldots, d\}(g^*\chi, \chi).\end{aligned}$$

For every K^2-equilibrated vector $\mathbf{y}$, we have

$$\|\mathbf{y}\| = y_1 + \ldots + y_d \leq d \max_i y_i \leq K^4 d \min_i y_i,$$

hence $\min_i y_i \geq \frac{\|\mathbf{y}\|}{dK^4}$. Now if the matrix h is K-equilibrated, then the vector $hg'\chi$ is K^2-equilibrated, hence $\min\{(hg'\chi)_i, i = 1, \ldots, d\} \geq \frac{\|hg'\chi\|}{K^4 d} = \frac{(\chi, hg'\chi)}{K^4 d}$. Similarly, $(\chi, hg'\chi) = \sum_{i,j} h_{ij}(g'\chi)_j \geq \frac{d}{K}(\chi, g'\chi)$. □

Lemma 4.7 *Let l and n be some strictly positive integers and ρ and K some strictly positive reals. Let $\mathbf{v} \in \mathbb{V}_{l(n+1)}$ and for $k = 1, \ldots, l$ denote $A_k = \xi[\mathbf{v}\restriction_{(k-1)(n+1)}, \mathbf{v}\restriction_{k(n+1)-1}]$ and $g_k = \xi[\mathbf{v}\restriction_{k(n+1)-1}, \mathbf{v}\restriction_{k(n+1)}]$. If for all $k = 1, \ldots, l$, $\|A_k\| > \rho^n$ and g_k is K-equilibrated, then*

$$(\chi, \xi[\mathbf{v}]\chi) \geq \rho^{ln}\left(\frac{1}{K^5}\right)^l.$$

Proof: Writing $C_k = g_k A_k$, we have that $\xi[\mathbf{v}] = C_l \cdots C_1$. Hence using iteratively l times the previous lemma and the fact that $(\chi, A\chi) = \|A\chi\| \geq \|A\|$ for any matrix A, we obtain that

$$(\chi, \xi[\mathbf{v}]\chi) \geq \prod_{k=1}^{l} \frac{(\chi, A_k\chi)}{K^5} \geq \frac{\prod_{k=1}^{l} \|A_k\|}{(K^5)^l}.$$

□

4.4 Probabilistic estimates

Lemma 4.8 *Let g be a matrix on G distributed according to the probability measure μ satisfying conditions 1 and 2. Then, for every $\delta > 0$, there exists $K = K(\delta)$ such that*

$$\mathbb{P}(\frac{1}{K} \leq g_{ij} \leq K, \textit{ for all } i, j = 1, \ldots, d) \geq 1 - \delta.$$

Proof: Since μ verifies condition 1, for all $\delta_1 > 0$, there exists $K_1 = K_1(\delta)$ such that $\mathbb{P}(g_{ij} > K_1$, for all $i, j = 1, \ldots, d) \leq \mathbb{P}(\|g\| > K_1) \leq \frac{\mathbb{E}\|g\|}{K_1} < \delta_1$. On the other hand, because μ satisfies condition 2, it does not have an atom at 0, so for all $\delta_2 > 0$ there exists $K_2 = K_2(\delta_2)$ such that $\mathbb{P}(g_{ij} < 1/K_2$ for all $i, j = 1, \ldots, d) \leq \delta_2$. Choose $K = \max(K_1, K_2)$ and $\delta = \delta_1 + \delta_2$ to conclude. □

Now we are able to complete the proof of theorem 2.3, i.e., to prove that if $\lambda\mathsf{br}(\mathbb{V}) > 1$ then $Z_\infty = \|\zeta_\infty\chi\| = (\chi, \zeta_\infty\chi) = +\infty$ almost surely.

Proof of theorem 2.3: Let $\beta = \mathsf{br}(\mathbb{V}) = \sup\{b : \inf_C \sum_{\mathbf{v}\in C} b^{-|\mathbf{v}|} > 0\}$, where C is any cutset. Let $\delta_1 = \log(\lambda\beta) > 0$. From theorem 4.2, there exists $N_1 = N_1(\delta_1)$, $\rho \in]0,1]$, and ρ_1', ρ_2' with $\rho_1' < \rho < \rho_2'$ such that for all $n \geq N_1$ we have

$$\mathbb{P}(\|X_n\| > (\rho_2')^n) > \frac{1}{(\rho_1'\beta)^n}.$$

Fix any $n \geq N_1$; combining this bound with the result of lemma 4.8, we can choose ρ_1, ρ_2 with $\rho_1' < \rho_1 < \rho < \rho_2 < \rho_2'$ and K sufficiently large so that for all $\mathbf{u}, \mathbf{v} \in \mathbb{V}$ with $\mathbf{u} < \mathbf{v}$ and $|\mathbf{v}| - |\mathbf{u}| = n+1$ we have

$$q := \mathbb{P}(\|\xi[\mathbf{u}, \hat{\mathbf{v}}]\| > \rho_2^n; \frac{1}{K} \leq (\xi[\hat{\mathbf{v}}, \mathbf{v}])_{ij} < K, \text{ for all } i,j = 1,\dots,d) > \frac{1}{(\rho_1\beta)^n}$$

(it is important to note that the choice of K does not depend on n).

Now, we use some ideas from the proof of theorem 1 (i) in [16]. As $q\beta^{n+1} > \rho_1^{-n} > 1$, proceeding as in [16], we get that there exists a set $\mathbb{U} \subset \mathbb{V}$ with the following properties ($\mathbb{U}$ is, in fact, one of the infinite clusters in the corresponding quasi-Bernoulli percolation process). The set $\mathbb{U}$ is an infinite subset of $\mathbb{V}$ whose subsequent levels are separated by multiples of $n+1$, *i.e.* for all $\mathbf{u}, \mathbf{v} \in \mathbb{U}$ there exists $k \in \mathbb{Z}$ such that $|\mathbf{v}| - |\mathbf{u}| = k(n+1)$. Moreover, if $\mathbf{u}, \mathbf{v} \in \mathbb{U}$ and $|\mathbf{v}| - |\mathbf{u}| = (n+1)$, then $\|\xi[\mathbf{u}, \hat{\mathbf{v}}]\| > \rho_2^n$ while $\frac{1}{K} \leq (\xi[\hat{\mathbf{v}}, \mathbf{v}])_{ij} \leq K$ for all $i,j = 1,\dots,d$. Hence,

$$\begin{aligned} Z_\infty &= \left(\chi, \sum_{\mathbf{v}\in\mathbb{V}} \xi[v]\chi\right) \\ &\geq \left(\chi, \sum_{\mathbf{v}\in\mathbb{U}} \xi[v]\chi\right) \\ &= \sum_{k=1}^{\infty} \left(\chi, \sum_{\mathbf{v}\in\mathbb{U}\cap\mathbb{V}^{k(n+1)}} \xi[v]\chi\right). \end{aligned}$$

For $\mathbf{u} \in \mathbb{U} \cap \mathbb{V}^{k(n+1)}$, let $A_l = \xi[\mathbf{v}\restriction_{(l-1)(n+1)}, \mathbf{v}\restriction_{l(n+1)-1}]$ and $g_l = \xi[\mathbf{v}\restriction_{l(n+1)-1}, \mathbf{v}\restriction_{l(n+1)}]$, for $l = 1,\dots,k$. These matrices verify the conditions of applicability of lemma 4.7. Therefore,

$$Z_\infty \geq \sum_{k=1}^{\infty} \sum_{\mathbf{v}\in\mathbb{U}\cap\mathbb{V}^{k(n+1)}} \rho_2^{kn} \left(\frac{1}{K^5}\right)^k.$$

Let us fix n so large that we can chose ρ_2'' with $\rho_2 > \rho_2'' > \rho$ such that $\frac{\rho_2^n}{K^5} > (\rho_2'')^{n+1}$. Let $\mathbb{U}'$ be the infinite subtree of $\mathbb{V}$ induced by $\mathbb{U}$ (*i.e.*, $v \in \mathbb{U}'$ iff $v' \leq v \leq v''$ for some $v', v'' \in \mathbb{U}$). As, by construction, $\mathsf{br}(\mathbb{U}') > (\rho_2'')^{-1}$, we have

$$Z_\infty \geq \sum_{k=1}^{\infty} \sum_{\mathbf{v}\in\mathbb{U}\cap\mathbb{V}^{k(n+1)}} (\rho_2'')^{k(n+1)} = \infty.$$

□

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