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Absence of Localization in a Class of Schrödinger Operators with Quasiperiodic Potential

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Abstract. We prove that a class of discrete Schrödinger operators with a quasi-periodic potential taking only a finite number of values, exhibits purely continuous spectrum; in particular they cannot have localized eigenvectors.

Introduction

We consider discrete Schrödinger operators acting on $\ell^2(\mathbb{Z})$:

$$(H\Psi)_n = \Psi_{n+1} + \Psi_{n-1} + \lambda V_n \Psi_n, \tag{1}$$

where V_n is a quasiperiodic sequence. In [1], Kohmoto, Kadanoff and Tang proposed a model where V is given by:

$$V_n = \chi_A(n\omega + \theta), \tag{2}$$

where χ_A is the characteristic function of an interval A on the circle and the argument of χ_A has to be understood mod 1. In their case ω was the golden mean, A was the interval $]-\omega^3, \omega^2]$ and θ was 0. In a subsequent paper by Kohmoto and Oono [2], it is shown by renormalization arguments that for some values of λ the spectrum of H is a Cantor set of zero Lebesgue measure. Furthermore these authors conjectured that the spectral measure is singular continuous. In this paper A is an arbitrary interval, ω an irrational number and we use the notation $H(\theta, \lambda)$ to take into account the dependence of H on λ and θ. We prove below the following theorem:

Theorem 1. *For Lebesgue almost every ω and for any A, then for Lebesgue almost every θ, the spectral measure of $H(\theta, \lambda)$ is continuous for any value of λ.*

Remark. The set of ω for which our proof is valid is described in Theorem 2; unlike most of the works on the subject, this set does not contain the golden mean.

This theorem is obtained by showing that any solution Ψ of the eigenvalue problem $H\Psi = E\Psi$ cannot decay at infinity. More precisely it is shown that Ψ is "recurrent" on a set of sites associated with the decomposition of ω in a continued fraction. In some cases [2, 3] the spectrum is expected to have no absolutely

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continuous component and the recurrent behaviour of Ψ on these sites may be characteristic of the hierarchical structure of the generalized eigenfunctions of H [4].

Our approach is easily extended to the case where the potential is generated in (2) by a piecewise constant function instead of the characteristic function of an interval A.

Proof of Theorem 1

Our proof relies on an idea due to Gordon [5] and already used by Avron and Simon [6] to prove the absence of point spectrum when V is generated by a continuous function and ω is a Liouville number. A straightforward adaptation of the proof of Lemma 7.6 of [7] to matrices of determinant 1 provides the following lemma:

Lemma 1. *Let Ψ be a solution of the eigenvalue equation $H\Psi = E\Psi$ and suppose that there exists an integer r such that $V_{n+ir} = V_n$ for $i = -1, 1$ and $0 < n \leqq r$ (Hypothesis $H1$), then we have for all E:*

$$\mathrm{Max}(\|u_{-r}\|, \|u_r\|, \|u_{2r}\|) \geqq \|u_0\|/2, \tag{3}$$

where u_n is the vector (Ψ_n, Ψ_{n+1}).

Proof of Lemma 1. As in [7], let us call B the 2×2 transfer matrix which maps u_0 on u_r; then B satisfies:

$$B^2 + cB + I = 0,$$

since the determinant of B is 1 (c is the trace of B). The result follows easily by applying this equality to u_{-r} (respectively u_0) if $|c|$ is smaller (respectively larger) than 1.

Hence if there exists, for a.e. ω and for a.e. θ, an infinite increasing sequence r_n satisfying the hypothesis $H1$ of Lemma 1, then the theorem is proven. Indeed, since in this case $\mathrm{Max}(\|u_{-r_n}\|, \|u_{r_n}\|, \|u_{2r_n}\|)$ does not go to 0 as n goes to $+\infty$, $H(\theta, \lambda)$ cannot have eigenvectors in ℓ^2 for a.e. θ and all λ. Thus the Theorem 1 is a mere consequence of the following Lemma 2:

Lemma 2. For almost every ω there exists for almost every θ a sequence $r_n(\theta)$ for which the hypothesis $H1$ of Lemma 1 is fulfilled.

Proof. Let p_n/q_n be the n^{th} principal convergent of the irrational number ω so that we have:

$$q_{n+1} = a_n q_n + q_{n-1},$$

$$p_{n+1} = a_n p_n + p_{n-1},$$

where a_n is the n^{th} partial quotient of the continued fraction of ω. The rate of convergence of the p_n/q_n to ω is given by:

$$|\omega - p_n/q_n| \leqq |p_{n+1}/q_{n+1} - p_n/q_n| = 1/q_n q_{n+1} \leqq 1/(a_n q_n^2). \tag{4}$$

Now the set $E(n)$ of values of θ such that hypothesis $H1$ is fulfilled with $r = q_n$ is given by:

$$E(n) = E_1(n) \cap E_2(n),$$
$$E_i(n) = \{\theta;\ \underset{0<m\leqq q_n}{\mathrm{Inf}}\ (|m\omega + \theta - x_i|) > q_n|\omega - p_n/q_n|\}, \tag{5}$$

where x_i ($i = 1, 2$) are the two endpoints of the interval A. Indeed, (5) expresses the fact that the argument $m\omega + \theta$ of $\chi_A(.)$ for m in $[1, q_n]$ must be distant from x_i by at least the phase shift corresponding to a translation of $\pm q_n$ in order to ensure the exact repetition of the potential considered in Lemma 1. Clearly we have:

$$\mu(E_i(n)) \geqq 1 - 2q_n^2|\omega - p_n/q_n|, \tag{6}$$

where μ is the Lebesgue measure on $[0, 1]$; thus using (4):

$$\mu(E(n)) \geqq 1 - 4q_n^2|\omega - p_n/q_n| \geqq 1 - 4/a_n. \tag{7}$$

Now, it is known [8] that for a.e. ω we have:

$$\limsup_{n\to\infty} a_n = +\infty, \tag{8}$$

which yields that:

$$\limsup_{n\to\infty} \mu(E_n) = 1, \tag{9}$$

whence:

$$\mu(\limsup_{n\to\infty} E_n) = 1. \tag{10}$$

Consequently, for a.e. ω, there exists for μ a.e. θ an infinite sequence $r_k(\theta) = q_{n_k}(\theta)$ such that $H1$ is fulfilled. This ends the proof of Lemma 2 which, using Lemma 1, provides Theorem 1.

Extensions

Extension to a larger class of irrational numbers. By the definition (5) of $E_i(n)$, one can easily check that these sets are asymptotically independent in the sense:

$$\mu(E_i(n) \cap E_i(m)) \to \mu(E_i(n))\mu(E_i(m))$$

as the ratio q_n/q_m goes to 0 (or to $+\infty$). Thus using Borel–Cantelli lemma, $\limsup_{n\to+\infty} a_n(\omega) > 4$ is sufficient to ensure that:

$$\mu(\limsup_{n\to+\infty} E(n)) = 1,$$

which proves the following Theorem:

Theorem 2. *For every irrational number ω such that $\limsup_{n\to+\infty} a_n(\omega) > 4$ (a_n is the n^{th} partial quotient of ω) and for any A, then for Lebesgue almost every θ, the spectral measure of $H(\theta, \lambda)$ is continuous for any value of λ.*

The case of piecewise constant functions. In this case, the proof is similar except the fact that factor 4 in (7) and in Theorem 2 has to be replaced by twice the number of discontinuities of the function generating the sequences V_n.

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