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On the distribution of a random variable occurring in 1D disordered systems

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Abstract. We consider the random variable: $z = 1 + x_1 + x_1x_2 + x_1x_2x_3 + \ldots$, where the x_i are independent, identically distributed variables. We derive some asymptotic properties of the distribution of z, which are related e.g. to the low-temperature behaviour of the random field Ising chain. For a special class of distributions of the x_i, exact solutions are presented. We also study the cases where the distribution function of z exhibits a power-law fall-off modulated by a 'periodic critical amplitude'.

1. Introduction

Since the pioneering work of Dyson (1953), one-dimensional random systems have been of considerable interest. Besides the well known localisation problem (Schrödinger equation with a random potential or, equivalently, eigenmode equation of a harmonic chain with random masses and/or spring constants: see Lieb and Mattis (1966) for a review), other examples of disordered systems have been investigated recently. In particular we mention the hopping models of diffusion in a random medium (Alexander *et al* 1981, Derrida and Pomeau 1982, Bernasconi and Schneider 1983, Derrida 1983, Nieuwenhuizen and Ernst 1984) and the Ising chain in a random magnetic field (Derrida *et al* 1978, Brandt and Gross 1978, Bruinsma and Aeppli 1983, Györgyi and Rujan 1983, Normand *et al* 1984).

One of us (Nieuwenhuizen 1983, 1984a, b) recently introduced a method which allows us to find exactly soluble classes of the above mentioned problems. These very different situations share an interesting feature: they lead to linear recursion relations with random coefficients, just as in the study of infinite products of random matrices, where the quantity of particular interest is the (largest) Liapunov exponent of the infinite product. This problem has also been of much interest in probability theory (Kesten 1973, Kesten *et al* 1975, Solomon 1975, Ruelle 1979, Sinaï 1982, Key 1984). The close analogy between certain products of 2×2 matrices and physical systems has been pointed out by Matsuda and Ishii (1970), and Derrida and Hilhorst (1983).

Randomness has two kinds of interesting effect on some of these systems: different physical quantities develop singularities at different points; some quantities are singular in a whole range of parameters, and exhibit continuously varying critical exponents. Some particular distributions of random interactions may lead to a more spectacular phenomenon, namely the modulation of the usual power-law critical singularities by oscillatory functions (Derrida and Hilhorst 1983, Bernasconi and Schneider 1983, Derrida 1983). On the other hand, the support of some quantities in the 1D random field Ising model is a fractal or Cantor set (Bruinsma and Aeppli 1983, Györgyi and Rujan 1983, Normand *et al* 1984).

In this paper we present a prototype of a 1D disordered system, which is simple enough to allow for a deep analytical and numerical study. It exhibits most characteristic features of more realistic models.

Consider an infinite sequence of independent real random variables $x_1, x_2, x_3, \ldots$, each x_i being distributed according to the same probability measure $\mathrm{d}\mu(x)$, the support of which is contained in an interval $[a, b]$ with $0 \leq a < b < \infty$. Define the random variable z through

$$z = \sum_{n \geq 0} \prod_{1 \leq i \leq n} x_i = 1 + x_1 + x_1 x_2 + x_1 x_2 x_3 + \ldots . \tag{1.1}$$

The solution of this model consists of the determination of the measure $\mathrm{d}\rho(z)$ according to which z is distributed. Particular attention is paid to the large-z behaviour of the density $R(z)$ and the integrated density $N(z)$ defined by

$$\begin{aligned} N(z) &= \int_z^\infty \mathrm{d}\rho(z') \\ R(z)\,\mathrm{d}z &= -N'(z)\,\mathrm{d}z = \mathrm{d}\rho(z). \end{aligned} \tag{1.2}$$

These quantities generally have the following large-z behaviour (see § 2)

$$\begin{aligned} N(z) &\sim (C_0/\alpha^*) z^{-\alpha^*} \\ R(z) &\sim C_0 z^{-(1+\alpha^*)} \end{aligned} \tag{1.3}$$

where α^* depends continuously upon the measure $\mathrm{d}\mu(x)$. In some particular cases, the constant C_0 is replaced by a periodic function of $\ln z$ (see § 5).

Let us discuss briefly to what extent the present model is a prototype of 1D random systems. Although as far as we know the measure density $R(z)$ has no direct physical interpretation, its behaviour for large z is closely related to critical singularities of physical quantities, at least in the following two circumstances. The first one is the random field Ising chain at low temperature (Derrida and Hilhorst 1983). These authors express the quenched averaged free energy as

$$F(\varepsilon) = \lim_{N \to \infty} \frac{1}{N} \ln \operatorname{Tr} \prod_{1 \leq i \leq N} \begin{pmatrix} 1 & \varepsilon \\ z_i \varepsilon & z_i \end{pmatrix} \tag{1.4}$$

where $\varepsilon = \mathrm{e}^{-2J/T}$ (J is the nearest-neighbour coupling) and $z_i = \mathrm{e}^{-2h_i/T}$ (h_i is the field acting on site i). The critical point of this system is $T = 0$, i.e. $\varepsilon = 0$. The small-ε behaviour of F is in some cases

$$F(\varepsilon) \sim C_1 \varepsilon^{2\alpha^*} \tag{1.5}$$

where $\alpha^* < 1$ depends continuously upon the distribution of the h_i. For some discrete distributions, the constant C_1 is replaced by a periodic function of $|\ln \varepsilon| = 2J/T$. The mathematical analogy between $F(\varepsilon)$ and our $R(z)$, discussed by Derrida and Hilhorst, will not be reproduced here. Disorder-dependent exponents and periodic amplitudes also occur in the random hopping problem (diffusion on a random chain), where α^* characterises the long-time behaviour of the mean position

$$\overline{x(t)} \sim C_2 t^{\alpha^*} \qquad (0<\alpha^*<1) \tag{1.6}$$

in the range of parameters where the physical velocity vanishes (Derrida and Pomeau 1982, Bernasconi and Schneider 1983). The special case $\alpha^*=0$ has been studied in more detail by Sinaï (1982) and Golosov (1984).

The plan of this paper is as follows. In § 2 we determine the asymptotic expansions of $R(z)$ and $N(z)$ for large z, as well as for z close to its lower bound. Our method, based upon the Mellin transformation, gives systematically subleading terms (for $z\to\infty$, the leading term is known from probability theory). Section 3 presents closed formulae and algorithms to determine the amplitudes occurring in the large-z expansion of $N(z)$. The cases where the exponent α^* is an integer happen to be particularly simple. In § 4, we use the above mentioned method introduced by one of us (Nieuwenhuizen 1983, 1984a, b) to solve the problem exactly for a three-parameter family of measures $d\mu(x)$. We obtain an infinite product representation for the Mellin transform of a shifted version of $d\rho(z)$. Section 5 is devoted to the more exciting case of a class of discrete measures $d\mu(x)$ which give rise to oscillatory critical amplitudes. We present some pictures of these functions, and some conjectures concerning the support properties of $d\rho(z)$, which are reminiscent of the Cantor sets found in the above cited papers on the random field Ising model. In § 6, we discuss the implications of our results for realistic models. As far as the periodic amplitudes are concerned, we show that, whenever the amplitude ψ of $R(z)$ exhibits oscillations, the amplitude of the Ising free energy reflects this periodic structure in a much smoother way.

2. Basic properties and asymptotic behaviour

2.1. *Behaviour at $z\to\infty$*

The random variable defined in (1.1) may also be written as

$$z = 1 + x_1 y \tag{2.1}$$

where

$$y = 1 + x_2 + x_2x_3 + x_2x_3x_4 + \dots. \tag{2.2}$$

Let us call $d\rho$ the probability measure for z, and R its density (if it exists): $d\rho(z) = R(z)\,dz$. From this definition the y and x variables are independent, and y is distributed according to the same measure, $d\rho$, as z is. If $d\rho$ has a density R, it satisfies the following Dyson–Schmidt-type integral equation (Dyson 1953, Schmidt 1957)

$$R(z) = \int d\mu(x) x^{-1} R\left(\frac{z-1}{x}\right). \tag{2.3}$$

More generally, the mean value of any function $F(z)$ must satisfy

$$\int d\rho(z) F(z) = \int d\mu(x)\, d\rho(y) F(1+xy). \tag{2.4}$$

Now let us introduce the moments of both measures $d\mu$ and $d\rho$ extended to complex values α and β of the exponents

$$f(\alpha)=\int d\mu(x)x^{\alpha} \tag{2.5}$$

$$g(\beta)=\int d\rho(z)z^{\beta}. \tag{2.6}$$

f and g are just the Mellin transforms of the densities of the measures $d\mu$ and $d\rho$. They are of interest because their analytic properties are related to the asymptotic behaviour of the measures. If g is meromorphic for Re $\beta<r$

$$g(\beta)=\sum_i \frac{g_i}{\beta-\beta_i}+g_1(\beta) \tag{2.7}$$

where $g_1(\beta)$ is analytic for Re $\beta\leqslant r$, then we have

$$d\rho(z)=\left(\sum_i g_i z^{-\beta_i-1}\right)dz+d\rho_1(z) \tag{2.8}$$

where $d\rho_1$ is integrable with any smooth function growing not faster than z^r at infinity. If $d\rho$ has a density R, then R is given by the inverse Mellin transform

$$R(z)=\int_{-\infty}^{\infty}\frac{d\,\mathrm{Im}\,\beta}{2\pi}z^{-\beta-1}g(\beta). \tag{2.9}$$

When $g(\beta)$ decreases fast enough for large values of Im β, the integral is absolutely convergent. By moving the integration path, using the assumed meromorphy (2.7) and the Cauchy theorem, we find

$$R(z)=\sum_i g_i z^{-\beta_i-1}+R_1(z) \tag{2.10}$$

where

$$|R_1(z)|<Cz^{-r-1}.$$

In the more general situation, the determination of the asymptotic expansion (2.10) amounts to the desingularisation of $g(\beta)$, which will now be performed.

Taking $F(z)=z^{\beta}$ in (2.4), we use the Mellin representation

$$\Gamma(-\beta)(1+xy)^{\beta}=\int_{-\infty}^{\infty}\frac{d\,\mathrm{Im}\,u}{2\pi}\Gamma(-u)\Gamma(u-\beta)(xy)^{u} \tag{2.11}$$

with

$$\mathrm{Re}\,\beta<\mathrm{Re}\,u<0. \tag{2.12}$$

We restrict ourselves to the physically interesting cases where $d\rho$ exists, is normed to 1 and has $[1, +\infty[$ as support. In these cases, the integral in (2.6) is absolutely convergent for any negative value of Re β. On the other hand, the Euler function $\Gamma(x)$ decreases exponentially for large values of Im x. Thus we can exchange the x, y and u integrations, and we obtain for $g(\beta)$ the following integral equation

$$g(\beta)=\frac{1}{\Gamma(-\beta)}\int_{\mathrm{Re}\,\beta<\mathrm{Re}\,u<0}\frac{d\,\mathrm{Im}\,u}{2\pi}\Gamma(-u)\Gamma(u-\beta)f(u)g(u). \tag{2.13}$$

In order to find the meromorphy properties of g for higher values of Re β, we shift the integration path by crossing the simple pole of $\Gamma(u-\beta)$ at $u=\beta$

$$g(\beta)=f(\beta)g(\beta)+\frac{1}{\Gamma(-\beta)}\int_{\operatorname{Re}\beta-1<\operatorname{Re}u<\operatorname{Re}\beta<0}\frac{\mathrm{d}\operatorname{Im}u}{2\pi}\Gamma(-u)\Gamma(u-\beta)f(u)g(u) \tag{2.14}$$

or

$$g(\beta)=\frac{1}{\Gamma(-\beta)[1-f(\beta)]}\times\int\frac{\mathrm{d}\operatorname{Im}u}{2\pi}\Gamma(-u)\Gamma(u-\beta)f(u)g(u),\qquad \begin{array}{l}\operatorname{Re}\beta-1<\operatorname{Re}u<\operatorname{Re}\beta\\ \operatorname{Re}u<0.\end{array} \tag{2.15}$$

Equation (2.15) is the starting point for our analysis. Before using it, let us give our notation for the zeros of $1-f(\beta)$. For real values of β, it is easy to verify that $f(\beta)$ is a convex function. Of course

$$f(0)=\int \mathrm{d}\mu(x)=1, \tag{2.16}$$

but the zero of $1-f$ at $\beta=0$ is cancelled by the pole of $\Gamma(-\beta)$ and does not give a singularity for $g(\beta)$ at $\beta=0$, as should be, since

$$g(0)=\int \mathrm{d}\rho(z)=1. \tag{2.17}$$

The variable z defined in (2.1) only takes finite values under the restriction

$$\langle\log x\rangle=\int \mathrm{d}\mu(x)\log x=f'(0)<0. \tag{2.18}$$

Therefore we are left with two possibilities

(i) $f(\beta)$ remains smaller than 1 for any real positive value of β. Since

$$|f(\beta)|\leq\int \mathrm{d}\mu(x)x^{\operatorname{Re}\beta}=f(\operatorname{Re}\beta), \tag{2.19}$$

$1-f$ never vanishes. From equation (2.15), $g(\beta)$ is also analytic in the half-plane $\operatorname{Re}\beta>0$ and for large z, $R(z)$ decreases faster than any power.

(ii) $1-f$ vanishes for one real positive value $\beta=\alpha^*$ (and only one, since f is convex). From (2.19), $1-f$ does not vanish in the strip $0<\operatorname{Re}\beta<\alpha^*$. But $1-f$ may vanish at other complex values, say $\beta=\alpha_i$, with $\operatorname{Re}\alpha_i\geq\alpha^*$. For a very peculiar class of measures $\mathrm{d}\mu$, already exhibited by Derrida and Hilhorst (1983), there is an infinite set of zeros of $1-f$, all with $\operatorname{Re}\alpha_i=\alpha^*$. This phenomenon appears only when

$$\mathrm{d}\mu/\mathrm{d}x=\sum_{l\in\mathbf{Z}}p_l\delta(x-a^l) \tag{2.20}$$

and will be studied in more detail in § 5. For all other measures, all the complex zeros of $1-f$ satisfy $\operatorname{Re}\alpha_i>\alpha^*$.

We restrict ourselves to case (ii) in the rest of this section and in § 3. In § 4 case (i) will also be considered. Let us come back to equation (2.15). It shows that for $\operatorname{Re}\beta<1$, the only singularities of g are the possible poles of $(1-f)^{-1}$. We can shift the integration path by crossing the poles of $g(u)$ and $\Gamma(-u)$, without relaxing the inequalities $\mathrm{Re}\,\beta-1<\mathrm{Re}\,u<\mathrm{Re}\,\beta$. Iterating this procedure up to $\mathrm{Re}\,u=r$ for a given r we obtain a representation of g, valid for $r<\mathrm{Re}\,\beta<r+1$

$$g(\beta)=\sum_{0\leq n<r}\frac{\Gamma(n-\beta)}{\Gamma(-\beta)}\frac{f(n)g(n)}{1-f(\beta)}+\sum_{i,\mathrm{Re}\,\alpha_i<r}\frac{\Gamma(-\alpha_i)\Gamma(\alpha_i-\beta)}{\Gamma(-\beta)[1-f(\beta)]}f(\alpha_i)g_i$$
$$+\frac{1}{\Gamma(-\beta)[1-f(\beta)]}\int_{\mathrm{Re}\,u=r}\frac{\mathrm{d}\,\mathrm{Im}\,u}{2\pi}\Gamma(-u)\Gamma(u-\beta)f(u)g(u) \tag{2.21}$$

which shows that for $\mathrm{Re}\,\beta<r+1$, g has the poles at $\beta=\alpha_i$ we have mentioned already and new poles at $\beta=\alpha_i+l$, with $l=1,2,\ldots$. Shifting the integration to arbitrary high values of $\mathrm{Re}\,u$, we obtain a proof of the meromorphy of g. If we call C_iD_{il}, with $D_{i0}=1$, the residues of the poles of g at $\beta=\alpha_i+l$, we have the corresponding asymptotic expansion for $R(z)$, when it exists, or more generally for the integrated probability $N(z)$ (see equation (1.2))

$$N(z)=\int_z^\infty \mathrm{d}\rho(z')=\sum_i\frac{C_i}{\alpha_i}z^{-\alpha_i}\left(1+\sum_{l=1}^{\infty}D_{il}z^{-l}\right). \tag{2.22}$$

We show in the next section how to compute the coefficients of the asymptotic expansion. Let us note that, apart from the first real pole at $\beta=\alpha^*$, the other (complex) poles of g contribute in the asymptotic behaviour by oscillating terms: if α_i is a complex pole, then $\overline{\alpha_i}$ is also and we have terms like

$$z^{-\mathrm{Re}\,\alpha_i}\{C_i\exp[-\mathrm{i}(\mathrm{Im}\,\alpha_i)\ln z]+\overline{C_i}\exp[\mathrm{i}(\mathrm{Im}\,\alpha_i)\ln z]\} \tag{2.23}$$

which lead to sine and cosine functions of $\ln z$.

2.2. *Behaviour at the lower bound*

The same kind of method can be used to determine the behaviour of $\mathrm{d}\rho$ at the lower bound of its support. Let $0\leq a<1$ [resp. $b=1/(1-a)$] be the lower bound of the support of $\mathrm{d}\mu$ [resp. $\mathrm{d}\rho$]. Then in a similar way as before the expansion of $\mathrm{d}\rho$ in the neighbourhood of $z=b$ is related to the meromorphy properties of the Mellin transform

$$k(\beta)=\int\mathrm{d}\rho(z)(z-b)^{-\beta} \tag{2.24}$$

for $\mathrm{Re}\,\beta\geq 0$. Again using (2.4), we write

$$k(\beta)=\int\mathrm{d}\mu(x)\,\mathrm{d}\rho(y)(1+xy-b)^{-\beta} \tag{2.25}$$

and we introduce the Mellin representation, valid for $\mathrm{Re}\,\beta>0$

$$\Gamma(\beta)(1+xy-b)^{-\beta}=\int_{0<\mathrm{Re}\,u<\mathrm{Re}\,\beta}\frac{\mathrm{d}\,\mathrm{Im}\,u}{}\Gamma(u)\Gamma(\beta-u)x^{-u}(y-\beta)^{-u}b^{u-\beta}(x-a)^{u-\beta}. \tag{2.26}$$

Then by substituting (2.26) into (2.25) we find

$$k(\beta)=\int_{0<\mathrm{Re}\,u<\mathrm{Re}\,\beta}\frac{\mathrm{d}\,\mathrm{Im}\,u}{2\pi}\frac{\Gamma(u)\Gamma(\beta-u)}{\Gamma(\beta)}k(u)b^{u-\beta}h(-u,u-\beta) \tag{2.27}$$

where

$$h(-u, u-\beta) = \int d\mu(x) x^{-u}(x-a)^{u-\beta}. \quad (2.28)$$

(i) Let us assume that $d\mu$ is integrable with a small negative power of $x-a$. Then h is analytic for

$$\mathrm{Re}\,\beta < \mathrm{Re}\,u + \varepsilon \quad (2.29)$$

and arbitrary high values of Re u if a is not vanishing. By increasing simultaneously the values of Re u and Re β, in such a way that (2.29) remains satisfied, equation (2.27) shows that $k(\beta)$ is analytic for arbitrary high values of Re β. This implies that $R(z)$ or $N(z)$ decreases faster than any power of $z-b$ when $z \to b$.

(ii) On the other hand, if $a=0$, $b=1$, (2.25) gives directly

$$k(\beta) = g(-\beta) f(-\beta). \quad (2.30)$$

Since $g(-\beta)$ is analytic for Re $\beta > 0$, $k(\beta)$ has exactly the same singularities as $f(-\beta)$. The remarkable difference between cases (i) and (ii) can be traced easily if we remember that from equation (1.1) $z=b$ if and only if each x_i equals a, for $a \neq 0$. Conversely for $a=0$, $z=1$ if and only if $x_1 = 0$.

(iii) Finally if $d\mu$ is not integrable with any negative power of $x-a$, the preceding argument does not apply and the exact singularity of $d\mu$ at its lower bound has to be given explicitly. As an example, let us consider the following case

$$d\mu(x) = p\delta(x-a)\,dx + (1-p)\,d\nu(x) \quad (2.31)$$

where the lower bound of the support of $d\nu$ is $C > a$. In the same way as before, we get

$$k(\beta) = pa^{-\beta}k(\beta) + (1-p)\int_{0<\mathrm{Re}\,u<\mathrm{Re}\,\beta} \frac{d\,\mathrm{Im}\,u}{2\pi} \frac{\Gamma(u)\Gamma(\beta-u)}{\Gamma(\beta)} \times k(u) b^{u-\beta} l(-u, u-\beta) \quad (2.32)$$

where

$$l(-u, u-\beta) = \int d\nu(x) x^{-u}(x-a)^{u-\beta}. \quad (2.33)$$

Now l is analytic, but $k(\beta)$ has poles for $pa^{-\beta} = 1$, or

$$\beta = (\log p + 2in\pi)/\log a. \quad (2.34)$$

We shall consider in § 4 (case 3) an explicit example of such behaviour.

3. Determination of the asymptotic expansion at $z \to \infty$

A first equation giving the asymptotic expansion can be obtained from equation (2.15). By shifting the integration path to lower values of Re u, and crossing the poles of $\Gamma(u-\beta)$ at $u=\beta, \beta-1, \beta-2, \ldots$ we find

$$g(\beta) = \frac{1}{1-f(\beta)} \sum_{n=1}^{\infty} \frac{(-1)^n}{n!} g(\beta-n) f(\beta-n) \frac{\Gamma(n-\beta)}{\Gamma(\beta)}. \quad (3.1)$$

Actually we shall use a simpler equation coming directly from equation (2.1). We write

$$(x_1 y)^\beta = (z-1)^\beta = z^\beta \sum_{n=0}^{\infty} \binom{\beta}{n} (-z)^{-n}. \tag{3.2}$$

By taking the mean values of both sides, we find

$$g(\beta) = \frac{-1}{1-f(\beta)} \sum_{n=1}^{\infty} (-1)^n \binom{\beta}{n} g(\beta-n). \tag{3.3}$$

For real integer values of β, the RHS of equation (3.3) reduces to a polynomial, with a finite number of terms. Taking successively $\beta = 1, 2, \ldots$ we find

$$g(1) = \frac{1}{1-f(1)}, \qquad g(2) = \frac{2g(1)-1}{1-f(2)}, \qquad g(3) = \frac{3g(2)-3g(1)+1}{1-f(3)} \text{ etc.} \ldots \tag{3.4}$$

which provides a direct determination of the moments of $\mathrm{d}\rho$, given the moments of $\mathrm{d}\mu$.

For non-integer values of β, the RHS of (3.3) is an infinite series. We may approximate $g(\beta)$ by the following approximation scheme

$$g(\beta - N - p) = 0 \qquad \forall p > 0 \tag{3.5}$$

$$g(\beta - N) = Kb^{\beta - N}. \tag{3.6}$$

Then $g(\beta - N + l)$ is given by the set of equations (3.3), taken for $l = 0, 1, 2, \ldots, N$, and truncated by (3.5) and we find

$$g(\beta) \simeq -\frac{1}{\Gamma(\beta)} \frac{1}{1-f(\beta)} K S_N(\beta) \tag{3.7}$$

where S_N is explicitly determined by the above computation. We want to determine the residues C_i of $g(\beta)$ at the different poles $\beta = \alpha_i$. Applying first equation (3.7) to the point $\beta = 0$ gives

$$1 = g(0) = [1/f'(0)] K S_N(0). \tag{3.8}$$

For $\beta = \alpha^*$ we find similarly

$$C_0 = -[1/\Gamma(\alpha^*) f'(\alpha^*)] K S_N(\alpha^*) \tag{3.9}$$

giving the first residue

$$C_0 = -\frac{f'(0)}{f'(\alpha^*)} \frac{1}{\Gamma(\alpha^*)} \frac{S_N(\alpha^*)}{S_N(0)}. \tag{3.10}$$

In principle all the other coefficients in the asymptotic expansion (2.22) can be evaluated in the same way. But what is questionable is the convergence of such an algorithm when the order N of the approximation increases. By a computer calculation we find that the algorithm is convergent for the residue of the real pole, that is, for the dominant term in the asymptotic expansion. In the particular case of an integer value of α^*, we even obtain an exact determination of C_0 by equation (3.4) in a closed form. For the complex poles, the convergence is more dubious, as can be seen from the following argument: from the above equations, we must have for $\mathrm{d}\mu$ as in (2.20) and $\omega = 2\pi/\ln a$

$$\frac{g(r+\mathrm{i}\omega)}{g(r)} = \lim_{N\to\infty} \frac{\Gamma(N-r-\mathrm{i}\omega)\Gamma(-r)g(r+\mathrm{i}\omega-N)}{\Gamma(N-r)\Gamma(-r-\mathrm{i}\omega)g(r-N)}. \tag{3.11}$$

Now if $g(\beta - N) \sim b^{-N}$, we find that the RHS behaves like $\exp(-i\omega \ln N)$ which has no limit.

We shall study this case in § 5, where the complex poles contribute to the dominant behaviour, and see how singular the distribution may appear.

Remark. Consider the case of a real dominant pole at $\beta = \alpha^*$. If we vary the parameters of the measure $d\mu$ in such a way that we tend to the limiting case $\alpha^* = 0$, we see from (3.10) that $C_0/\alpha^* \to 1$. This is caused by the fact that the variable z becomes infinite with probability 1

$$N(z) \xrightarrow[\alpha^* \to 0]{} 1 \qquad \text{for all } z. \tag{3.12}$$

The following logarithmic behaviour studied by Sinaï (1982) and Golosov (1984) is not present in our simple model.

4. Exact solutions

In this section, we consider a family of measures $d\mu(x)$ for which the Mellin transform of $d\rho(z)$ can be obtained exactly through an infinite product representation. The method was introduced by one of us (Nieuwenhuizen 1983, 1984a, b) for Kronig-Penney, harmonic, X-Y, tight-binding and hopping models and for diluted random systems. The basic idea is to search for distributions $d\mu(x)$ such that the Dyson-Schmidt equation for the density $R(z)$ can be transformed into a differential-difference equation. This is done as follows: we assume $d\mu(x) = r(x)\, dx$ where r is differentiable, and we employ the identity

$$\frac{\partial}{\partial z}(z-1)\int_a^b r(x)\frac{dx}{x}R\left(\frac{z-1}{x}\right) = -r(x)R\left(\frac{z-1}{x}\right)\bigg|_a^b + \int_a^b [xr(x)]'\frac{dx}{x}R\left(\frac{z-1}{x}\right). \tag{4.1}$$

First we require that $[xr(x)]'$ is proportional to $r(x)$, implying $r(x) \sim x^{\sigma-1}$ for some σ. Moreover, we demand that one of the boundary terms in the RHS of equation (4.1) vanishes, implying either $a = 0$ or $b = \infty$. Following Nieuwenhuizen (1984b), we can also allow for an atom in $d\mu(x)$ at the other boundary, and consider the more general distribution

$$d\mu(x) = [p\delta(x-a) + r(x)]\, dx$$

with (4.2)

$$r(x) = (1-p)(|\sigma|/a)(x/a)^{\sigma-1} \qquad \text{for } x \in J.$$

According to the values of the three independent parameters (a, p, σ), four cases can be considered:

Case 1 ($a > 1$, $\sigma > 0$, $J = [0; a]$) is an example of the general case (finite index α^*) discussed in the previous section.

Case 2 ($a = 1$, $\sigma > 0$, $J = [0; 1]$) corresponds formally to an infinite order singularity ($\alpha^* = \infty$). It is of a particular interest, because we do not have asymptotic estimates of $R(z)$ for arbitrary measures in this case.

Case 3 ($a < 1$, $\sigma > 0$, $J = [0; a]$) does not exhibit any critical singularity, since the support of ρ is bounded, but shows an interesting behaviour in the vicinity of the upper bound.

Case 4 ($a<1$, $\sigma<0$, $J=[a;\infty[$) is an example of the general case with a support of μ extending up to ∞; the oscillations of case 3 are now present at the lower bound of the support of ρ.

In the four cases, the moment function $f(t)$ reads

$$f(t)=a^t(\sigma+pt)/(\sigma+t). \tag{4.3}$$

The moments of $\mathrm{d}\mu$ given by (4.3) exist only for Re $t>-\sigma$ in cases 1, 2, 3, and only for Re $t<-\sigma$ in case 4. The derivative of $f(t)$ is given by

$$f'(t)=a^t\frac{(\sigma+t)(\sigma+pt)\ln a-(1-p)\sigma}{(\sigma+t)^2}. \tag{4.4}$$

Case 1

$$a>1, \qquad \sigma>0, \qquad 0\leq p\leq 1$$

$$\mathrm{Supp}\ \mu=[0,a]$$

$$\mathrm{Supp}\ \rho=[1,\infty[.$$

Due to equation (4.4), the condition $f'(0)<0$ (see equation (2.18)) reads

$$\sigma\ln a+p-1<0. \tag{4.5}$$

With our choice of measure $\mathrm{d}\mu(x)$, equation (2.3) reads

$$R(z)=\frac{p}{a}R\left(\frac{z-1}{a}\right)+\int_0^a\frac{\mathrm{d}x}{x}r(x)R\left(\frac{z-1}{x}\right). \tag{4.6}$$

From this we obtain, assuming that R has a derivative,

$$R'(z)=\frac{p}{a^2}R'\left(\frac{z-1}{a}\right)+\int_0^a\frac{\mathrm{d}x}{x^2}r(x)R'\left(\frac{z-1}{x}\right). \tag{4.7}$$

We perform a partial integration, use the fact that

$$xr'(x)=(\sigma-1)r(x)$$

and delete the integral by again using (4.7).

This leads to the following differential-difference equation

$$(z-1)R'(z)-(\sigma-1)R(z)=(z-1)\frac{p}{a^2}R'\left(\frac{z-1}{a}\right)+\frac{p-\sigma}{a}R\left(\frac{z-1}{a}\right). \tag{4.8}$$

Finally, one gets rid of the inhomogeneous term in the argument of R by putting

$$R(z)=S(y) \qquad \text{with } y=z+1/(a-1) \tag{4.9}$$

$$\left(y-\frac{a}{a-1}\right)S'(y)-(\sigma-1)S(y)=\left(y-\frac{a}{a-1}\right)\frac{p}{a^2}S'\left(\frac{y}{a}\right)+\frac{p-\sigma}{a}S\left(\frac{y}{a}\right). \tag{4.10}$$

Define the shifted moment function of $S(y)$ as

$$M(t)=\int_0^\infty S(y)y^t\,\mathrm{d}y. \tag{4.11}$$

The lower bound is harmless for any t, since $S(y)$ is identically zero for $y<a/(a-1)$. Under the same assumptions as in § 2, $M(t)$ is defined through (4.11) for Re $t<0$.

Equation (4.10) implies therefore

$$M(t)=M(t-1)\frac{a}{a-1}\frac{1-pa^{t-1}}{1-pa^{t}+\sigma(1-a^{t})/t} \tag{4.12}$$

for Re $t<0$.

$M(t)$ is the unique solution of equation (4.12) analytic for Re $t<0$ and such that $M(0)=1$. It can be found by iteration to be the following infinite product

$$M(t)=\left(\frac{a}{a-1}\right)^{t}\frac{1-p-\sigma\ln a}{1-pa^{t}+(\sigma/t)(1-a^{t})}$$
$$\times\prod_{n>1}\left(\frac{1-pa^{t-n}}{1-pa^{-n}}\frac{1-pa^{-n}-(\sigma/n)(1-a^{-n})}{1-pa^{t-n}+[\sigma/(t-n)](1-a^{t-n})}\right). \tag{4.13}$$

Indeed, it can be checked that $M(t)$ given by (4.13) decreases at fixed Re t and large $|\text{Im } t|$ as

$$M(t)\underset{|\text{Im } t|\to\infty}{\sim}|\text{Im } t|^{-\sigma} \tag{4.14}$$

and hence any other solution of equation (4.12) would grow exponentially for $|\text{Im } t|\to\infty$, which is not acceptable.

This argument is also valid for cases 2, 3 and 4 below.

Equation (4.13) defines $M(t)$ as a meromorphic function in the whole plane, with poles located at

$t=u+n$

with $n=0,1,2,\ldots$

and u such that $f(u)=1$.

In § 2, we have emphasised that only those solutions of $f(u)=1$ with positive real part are actually poles of $M(t)$, while equation (4.13) suggests that possible solutions with negative real part are also poles. The contradiction is easily solved by checking that the equation $f(u)=1$ has no solution with Re $u<0$ for $\mathrm{d}\mu$ given by (4.2). Indeed, writing $u=X+iY$, the condition $|f(u)|=1$ reads

$$Y^{2}=[a^{2X}(\sigma+pX)^{2}-(\sigma+X)^{2}]/(1-p^{2}a^{2X}). \tag{4.15}$$

This curve in the X-Y plane has one 'physical branch' for $\alpha^{*}\leq X\leq -\ln p/\ln a$ bearing an infinity of poles of $M(t)$, and a 'spurious branch' for $X_{L}\leq X\leq 0$ with $-\sigma/p\leq X_{L}\leq -\sigma$. The latter curve is closed, and is the pre-image of the unit circle by the conformal map φ: $u\to 1/f(u)$. $f(u)$ therefore takes the value 1 only once. Since $f(0)=1$ by definition, $f(u)=1$ has no solution for Re $u<0$.

The explicit form (4.13) of $M(t)$ allows first for a cross-check of some results derived in the preceding sections. When α^{*} is a positive integer, all terms in the infinite product (4.13) cancel except the first α^{*} ones, and we easily reobtain the general results (3.4). When α^{*} goes to zero, it is easy to check that $C_{0}/\alpha^{*}\to 1$.

The exact solution we have derived here also furnishes some information which was not available in the general case. One interesting example is the behaviour of the leading amplitude C_{0} when α^{*} becomes large. This limit can be reached for fixed p and σ by letting a go to unity. In this limit, the infinite product goes to the exponential of a definite integral, and we easily obtain the following asymptotic formula

$$C_0 \sim \frac{1-p}{p^{3/2}} \left(\frac{\alpha^*}{-\ln p} \right)^{1+\alpha^*} e^{\sigma\varphi(p)}$$

with

$$\varphi(p) = \frac{1-p}{p \ln p} + \frac{1}{p} \int_0^1 \frac{ds}{s} \frac{(1-p)^2 s \ln s - (1-s)^2 p \ln p}{(1-s)(1-ps) \ln s \ln(s/p)} \tag{4.16}$$

for fixed σ and $p \neq 0$.

When $p = 0$, we get a slightly different behaviour

$$C_0 \sim (\sigma\alpha^*)^{1/2} \left(\frac{\alpha^*}{\ln(\alpha^*/\sigma)} \right)^{1+\alpha^*} \exp[\alpha^*/\ln(\alpha^*/\sigma)]. \tag{4.17}$$

In both cases, C_0 grows very rapidly (roughly speaking like $\Gamma(\alpha^*)$).

Case 2

$$a = 1, \qquad \sigma > 0, \qquad 0 \leqslant p < 1$$

$$\text{Supp}\ \mu = [0, 1]$$

$$\text{Supp}\ \rho = [1, \infty[.$$

In this case, we have to go back to equation (4.8) which now reads

$$(z-1)R'(z) = (z-1)pR'(z-1) + (\sigma-1)R(z) + (p-\sigma)R(z-1). \tag{4.18}$$

The adequate transform to solve this equation is now the Laplace transform

$$F(s) = \int_1^\infty e^{-sz} R(z)\, dz \tag{4.19}$$

which exists at least for Re $s > 0$, under the same assumptions as in § 2. Equation (4.18) is equivalent to the following differential equation for $F(s)$

$$\frac{F'(s)}{F(s)} = \frac{\sigma(e^{-s}-1)-s}{s(1-p\,e^{-s})} = \psi(s) \tag{4.20}$$

with the normalisation condition $F(0) = 1$.

The function $\psi(s)$ has simple poles at $s = s_k = \ln p + 2\pi i k$ ($k \in \mathbb{Z}$), with residues

$$\rho_k = [\sigma(1-p)/ps_k] - 1. \tag{4.21}$$

Therefore $F(s)$ is analytic for Re $s > \ln p$, where we have

$$F(s) = \exp \int_0^s \psi(t)\, dt. \tag{4.22}$$

It has branch points at $s = s_k$ of the form

$$F(s) \sim F_k (s - s_k)^{\rho_k}. \tag{4.23}$$

The density $R(z)$ is given by the inverse Laplace formula

$$R(z) = \int_{\text{Re}\, t > \ln p} \frac{d\,\text{Im}\, t}{2\pi} e^{zt} F(t). \tag{4.24}$$

The branch points of $F(s)$ at $s = s_k$ are responsible for the following asymptotic expansion of $R(z)$ at large z

$$R(z) \sim e^{-z|\ln p|} \sum_k G_k z^{-(1+\rho_k)} e^{2\pi i k z}. \tag{4.25}$$

For $z \to \infty$, the sum is dominated by the ρ_k with smallest real part, which is $k = 0$ (for $\sigma > 0$)

$$R(z) \sim G_0 \, e^{-z|\ln p|} z^{\sigma(1-p)/p|\ln p|}. \tag{4.26}$$

Unfortunately, we have not been able to find a generalisation of equation (4.26) for an arbitrary measure $d\mu(x)$ with a support ending exactly at unity. The leading exponential $e^{-z|\ln p|}$ is very likely to be universal, but the subleading power of z is not easy to predict in general. We expect that the behaviour (4.26) remains valid for the class of measures which have an atom at $x = 1$ on top of a continuous density $\sim (1-x)^{\sigma-1}$.

In the limit where σ vanishes, equation (4.25) indicates that the exponential $e^{-z|\ln p|}$ is modulated by a periodic function of z, since all ρ_k equal -1. The next section is devoted to a more general study of periodic amplitudes.

Case 3

$$a < 1, \qquad \sigma > 0, \qquad 0 \leqslant p \leqslant 1$$
$$\mathrm{Supp}\, \mu = [0, a]$$
$$\mathrm{Supp}\, \rho = [1, 1/(1-a)].$$

In this case, equation (4.8) is still valid, and the convenient way to solve it is to now define

$$R(z) = S(y) \qquad \text{with } y = [1/(1-a)] - z. \tag{4.27}$$

Equation (4.8) then becomes

$$\left(y - \frac{a}{1-a}\right) S'(y) - (\sigma - 1) S(y) = \left(y - \frac{a}{1-a}\right) \frac{p}{a^2} S'\left(\frac{y}{a}\right) + \frac{p-\sigma}{a} S\left(\frac{y}{a}\right). \tag{4.28}$$

Since $S(y)$ is non-zero only for $0 \leqslant y \leqslant a/(1-a)$, its moment function, defined as

$$M(t) = \int_0^\infty S(y) y^t \, dy,$$

is analytic for $\mathrm{Re}\, t > 0$, and equation (4.28) implies that

$$M(t) = M(t-1) \frac{a}{1-a} \frac{1 - pa^{t-1}}{1 - pa^t + (\sigma/t)(1-a^t)} \tag{4.29}$$

for $\mathrm{Re}\, t > 0$.

In analogy with case 1, we deduce that $M(t)$ is given by the infinite product

$$M(t) = \left(\frac{a}{1-a}\right)^t \frac{1-p}{1-pa^t} \prod_{n \geqslant 1} \left(\frac{1-pa^n}{1-pa^{t+n}} \frac{1 - pa^{t+n} + [\sigma/(t+n)](1-a^{t+n})}{1 - pa^n + (\sigma/n)(1-a^n)} \right). \tag{4.30}$$

Consider first the case where p is non-zero. Then $M(t)$ has poles at the following values of t

$$t = -\beta + \mathrm{i}k\omega - l$$
$$\text{with } l = 0, 1, 2, \ldots; \; k = 0, \pm 1, \pm 2, \ldots \tag{4.31}$$
$$\text{and } \beta = \ln p/\ln a > 0, \qquad \omega = -2\pi/\ln a > 0.$$

Of special interest is the behaviour of $R(z)$ for z close to its upper bound $1/(1-a)$. This behaviour is dominated by the leading string of poles corresponding to $l = 0$ in (4.31). The associated residues $\tilde{P}_k$ are given by

$$\tilde{P}_k = \left(\frac{a}{1-a}\right)^{-\beta+\mathrm{i}k\omega} \frac{1-p}{-\ln a} \prod_{n\geq 1} \frac{1-pa^n}{1-a^n} \frac{1-a^n+[\sigma/(n-\beta+\mathrm{i}k\omega)](1-a^n/p)}{1-pa^n+(\sigma/n)(1-a^n)}. \tag{4.32}$$

It is easy to check that the infinite product in (4.32) decreases as $C(\mathrm{i}k\omega)^{-\sigma}$ for large $|k|$.

This implies that we have

$$R(z) \underset{z\to 1/(1-a)}{\sim} \left(\frac{1}{1-a} - z\right)^{\beta-1} P\left(\frac{\ln(1/(1-a)-z)}{\ln a}\right)\left[1 + O\left(\frac{1}{1-a} - z\right)\right] \tag{4.33}$$

where P is a periodic function of period unity, and its Fourier coefficients are precisely $\tilde{P}_k$:

$$P(\xi) = \sum_{k\in \mathbb{Z}} \tilde{P}_k \, \mathrm{e}^{2\pi\mathrm{i}k\xi}, \tag{4.34}$$

since $|\tilde{P}_k| \sim |k|^{-\sigma}$, the derivative of $P(\xi)$ of order $[\sigma]$ is singular (at one point per period). In particular, if $1 < \sigma < 2$, $R(z)$ is continuous but not differentiable; if $\sigma < 1$, the integrated density $N(z)$ is continuous but has no derivative: $R(z)$ is infinite somewhere. More generally, only the Hölder-derivatives of N and R of order σ and $\sigma - 1$ respectively exist.

In order to illustrate this behaviour by a numerical computation of $P(\xi)$, we first transform equation (4.34) into an exponentially convergent series representation for $P(\xi)$.

Using the expression (4.32) for $\tilde{P}_k$, we get

$$P(\xi) = \sigma(1-p)\left(\frac{a}{1-a}\right)^{-\beta} Q\left(\xi + \frac{\ln(1-a)}{\ln a}\right) \tag{4.35}$$

where $Q(\xi)$ is another periodic function, with Fourier coefficients $\tilde{Q}_k$ given by

$$\tilde{Q}_k = S(\mathrm{i}k\omega); \qquad S(z) = \prod_{n\geq 1} \frac{1-pa^n}{1-a^n} \frac{1-a^n+[\sigma/(n-\beta+z)](1-a^n/p)}{1-pa^n+(\sigma/n)(1-a^n)}. \tag{4.36}$$

$S(z)$ has poles at $z = \beta - m$ $(m \geq 1)$. (The origin is never a pole.)

The function $Q(\xi)$ is given by

$$Q(\xi) = -\oint \frac{\mathrm{d}z}{2\pi\mathrm{i}} \frac{\pi/\omega}{\sinh(\pi/\omega)z} \exp\left[\frac{2\pi}{\omega} z\left(\xi - \frac{1}{2}\right)\right] S(z) \tag{4.37}$$

where the contour encloses the imaginary axis, but does not contain poles of $S(z)$.

Shifting the contour, we get finally the following representation for $0<\xi<1$

$$Q(\xi)=\sum_{m\geq 1}\frac{a^{\xi(m-\beta)}(1-pa^m)}{(1-a^m)[1-pa^m+(\sigma/m)(1-a^m)]}\times\prod_{\substack{n\geq 1\\ n\neq m}}\frac{1-pa^n}{1-a^n}\,\frac{1-a^n+\sigma/(n-m)(1-a^n/p)}{1-pa^n+(\sigma/n)(1-a^n)}. \tag{4.38}$$

Figures 1 to 3 show plots of the function $Q(\xi)$ in three typical cases. Figure 1 corresponds to $p=0.6$, $a=0.5$ and $\sigma=0.5$: $R(z)$, and therefore $Q(\xi)$, becomes infinite.

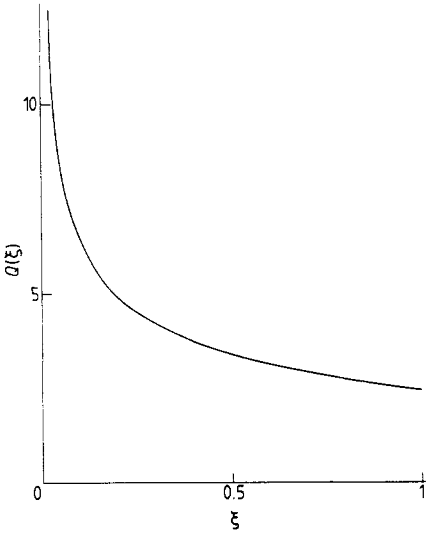

Figure 1. Plot of the period function $Q(\xi)$ over one period. $Q(\xi)$ modulates the leading behaviour of the probability density $R(z)$ in case 3 for z close to its upper bound $(1-a)^{-1}$. The parameters read: $p=0.6$, $a=0.5$ and $\sigma=0.5$.

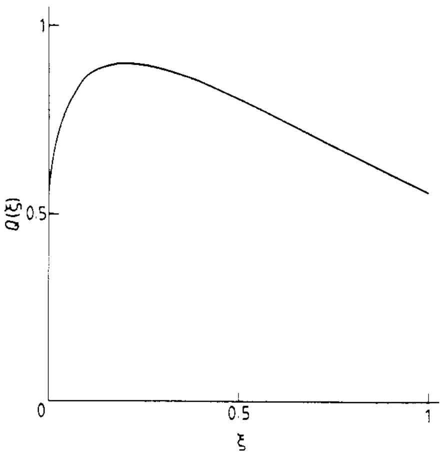

Figure 2. The same as figure 1, with $p=0.6$, $a=0.25$ and $\sigma=1.5$.

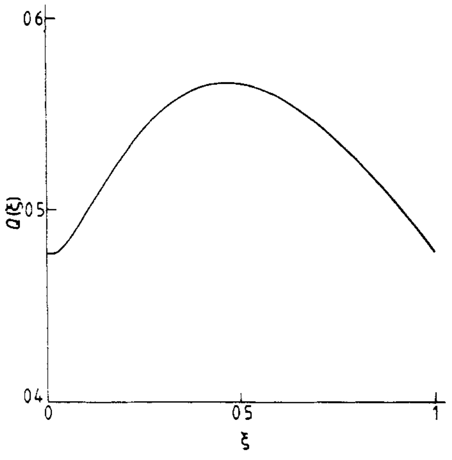

Figure 3. The same as figure 1, with $p = 0.6$, $a = 0.25$ and $\sigma = 2.5$.

The function Q is defined such that the divergence occurs at integer values of ξ. Figure 2 corresponds to $p = 0.6$, $a = 0.25$ and $\sigma = 1.5$: $Q(\xi)$ is not differentiable at integer values of ξ. The singular part of $Q(\xi)$ is proportional to $\xi^{1/2}$ when $\xi \to 0^+$. Figure 3 corresponds to $p = 0.6$, $a = 0.25$ and $\sigma = 2.5$: the singular part of $Q(\xi)$ at the origin is proportional to $\xi^{3/2}$.

In § 5, we shall consider another type of periodic amplitudes where even $N(z)$ is not differentiable. It is therefore instructive to have an explicit example of a non-infinitely differentiable function.

When p is zero, $M(t)$ is analytic in the entire plane, and $S(y)$ decreases more rapidly than every power of y as $y \to 0$, as expected from § 2. In the present case, we can go further and determine the small-y behaviour of $S(y)$ more precisely. From equation (4.29) we deduce that $\ln M(t)$ behaves for Re $t \to -\infty$ as

$$\ln M(t) = -\tfrac{1}{2}t^2 \ln a + t\ln(-t) + (K-1)t + \mathrm{O}[\ln(-t)] \tag{4.39}$$

with

$$K = \ln[a^{3/2}/(1-a)\sigma].$$

We can now use (4.39) to evaluate the inverse Mellin integral defining S

$$S(y) = \int \frac{\mathrm{d}\,\mathrm{Im}\,t}{2\pi} M(t) y^{-t-1}\,\mathrm{d}t \tag{4.40}$$

using the saddle point method. It yields easily

$$S(y) \sim \exp\left\{\frac{\ln^2 y}{2\ln a}\left[1 + \mathrm{O}\left(\frac{\ln(-\ln y)}{\ln y}\right)\right]\right\} \tag{4.41}$$

Case 4

$$a < 1, \qquad \sigma < 0, \qquad 0 \leqslant p \leqslant 1$$

$$\mathrm{Supp}\,\mu = [a, \infty[$$

$$\mathrm{Supp}\,\rho = [1/(1-a), \infty[.$$

In this last case, the condition $f'(0)<0$ reads

$$\sigma \ln a + p - 1 > 0. \tag{4.42}$$

Since $f(t)$ diverges as $t \to -\sigma^-$, the exponent α^* is necessarily less than $(-\sigma)$. As in case 1, we define the shifted variable y through equation (4.9). Then the density $R(z)=S(y)$ still obeys equation (4.10). Consider the Mellin transform of $S(y)$ defined as

$$M(t)=\int_0^\infty S(y)y^t\,\mathrm{d}y. \tag{4.43}$$

Since the support of $S(y)$ now ranges from 0 to ∞, we do not have any *a priori* analyticity requirement for $M(t)$. But the results of § 2 ensure that $M(t)$ is analytic for $0 \leqslant \operatorname{Re} t \leqslant \alpha^*$, and therefore we have as in case 1

$$\frac{M(t)}{M(t-1)}=\frac{a}{a-1}\,\frac{1-pa^{t-1}}{1-pa^t+(\sigma/t)(1-a^t)}=\varphi(t) \tag{4.44}$$

provided $\alpha^*>1$ and $1 \leqslant \operatorname{Re} t < \alpha^*$.

Before solving equation (4.44), let us remark that the equation $f(t)=1$ has only *one* solution for $\operatorname{Re} t>0$, namely the real solution α^*. The proof of this fact is a straightforward extension of the analogous result for $\operatorname{Re} t \leqslant 0$ in case 1. According to § 2, $M(t)$ is expected to have poles for $\operatorname{Re} t>0$ only at α^*, $\alpha^*+1, \ldots$. This suggests that to set

$$M(t)=[\sin \pi\alpha^*/\sin \pi(\alpha^*-t)]J(t), \tag{4.45}$$

$J(t)$ has to be analytic for $\operatorname{Re} t>0$ and has to obey

$$J(t)/J(t-1)=-\varphi(t); \qquad J(0)=1 \tag{4.46}$$

which is more satisfactory than (4.44) since $-\varphi(t)$ goes to the positive constant $a/(1-a)$ as $\operatorname{Re} t \to \infty$. $J(t)$ is therefore easily obtained as an infinite product, and $M(t)$ finally reads

$$M(t)=\left(\frac{a}{1-a}\right)^t \frac{\sin \pi\alpha^*}{\sin \pi(\alpha^*-t)}\,\frac{1-p}{1-pa^t} \times \prod_{n\geqslant 1}\left(\frac{1-pa^n}{1-pa^{t+n}}\,\frac{1-pa^{t+n}+\sigma/(t+n)(1-a^{t+n})}{1-pa^n+(\sigma/n)(1-a^n)}\right). \tag{4.47}$$

The assumption $\alpha^*>1$ can be removed, since the solution (4.47) can be analytically continued (for instance in a) from $\alpha^*>1$ to $\alpha^*\leqslant 1$. Moreover, it is easy to convince oneself that our solution has all desired analyticity properties.

A particular feature of this last case is that one single expression for $M(t+1)$ allows us to explore the behaviour of $R(z)$ both for large z and for z close to $1/(1-a)$. The large-z behaviour is dominated by the poles at α^*, $\alpha^*+1, \ldots$ in agreement with § 2. The lower bound limit exhibits a periodic amplitude which is very similar to the upper bound limit of case 3, the only difference between equations (4.30) and (4.47) being the sine ratio and the sign of τ.

5. Periodic amplitudes

We have pointed out in § 2 that a particular class of discrete measures $d\mu$ gives rise to a periodic 'critical amplitude' for $N(z)$. These measures read:

$$d\mu/dx = \sum_{-\infty<l<+\infty} p_l \delta(x-a^l) \tag{5.1}$$

where $a>1$, and p_l are positive weights such that

$$\begin{aligned} & p_l \neq 0 \text{ at least for one } l>0 \\ & \sum_l p_l = 1 \\ & \sum_l l p_l = f'(0)/\ln a < 0. \end{aligned} \tag{5.2}$$

In these cases, the leading behaviour of $N(z)$ for large z is

$$N(z) \sim z^{-\alpha^*} \psi(\xi) \tag{5.3}$$

where ψ is a periodic function of $\xi = \ln z/\ln a$, with period unity, and α^* is the real positive solution of $f(\beta)=1$.

This section is devoted to the periodic amplitude ψ. Its Fourier coefficients $\tilde{\psi}_n$ are the residues of $g(\beta)$ at the points

$$\alpha_n = \alpha^* + 2\pi i n/\ln a. \tag{5.4}$$

The algorithms presented in § 3 allow us therefore to compute only the average $\tilde{\psi}_0$.

Let us consider first one very simple limiting case of the family (5.1), given by

$$d\mu/dx = (1-p)\delta(x) + p\delta(x-a) \tag{5.5}$$

with $a>1$ and $0<p<1$. This measure is formally given by (5.1) with only p_1 and $p_{-\infty}$ different from zero. This case is exactly soluble: it is soon realised that z assumes only the values

$$z_n = 1 + a + \ldots a^n = (a^{n+1}-1)/(a-1) \qquad (n \geq 0) \tag{5.6}$$

with probabilities $q_n = (1-p)p^n$.

We have therefore asymptotically $N(z)$ given by (5.3), with $\alpha^* = -\ln p/\ln a$, and with

$$\psi(\xi) = (1-p) \sum_{-\infty<n<+\infty} \exp[\alpha \ln a(\xi-n)]\theta\left(n+1-\frac{\ln(a-1)}{\ln a}-\xi\right). \tag{5.7}$$

Note that this is nothing other than the $\sigma \to 0$ limit of case 1 in the previous section.

The amplitude ψ has one discontinuity per period

$$\begin{aligned} \lim_{\varepsilon\to 0^-} \psi\left(m-\frac{\ln(a-1)}{\ln a}+\varepsilon\right) &= a^{\alpha^*}(a-1)^{-\alpha^*} \\ \lim_{\varepsilon\to 0^+} \psi\left(m-\frac{\ln(a-1)}{\ln a}+\varepsilon\right) &= (a-1)^{-\alpha^*} \end{aligned} \tag{5.8}$$

for every integer m; the discontinuity strength therefore reads

$$\Delta\psi = (a^{\alpha^*}-1)(a-1)^{-\alpha^*}. \tag{5.9}$$

At fixed a, $\Delta\psi$ vanishes linearly in α^* for $\alpha^* \to 0$, as expected from § 2.

Consider now the general oscillating case (5.1). In order to explore the properties of ψ numerically, we take measures for which only p_{-1} and p_{+1} are non-vanishing

$$\mathrm{d}\mu/\mathrm{d}x = p\delta(x-a^{-1})+(1-p)\delta(x-a) \tag{5.10}$$

with $a>1$ and $\frac{1}{2}<p<1$. The corresponding index α^* reads

$$\alpha^* = \frac{\ln[p/(1-p)]}{\ln a}. \tag{5.11}$$

The amplitude ψ is easily computed numerically by the following procedure: consider the sequence of random variables $z_{(n)}$ defined through

$$z_{(0)} = 1, \qquad z_{(n+1)} = 1 + z_{(n)}x_n \tag{5.12}$$

where each x_n is distributed with $\mathrm{d}\mu$. The $z_{(n)}$ are asymptotically distributed with $\mathrm{d}\rho$ when $n\to\infty$. Since $z_{(n)}$ assumes only 2^n values, one can extract approximants $\psi_{(n)}$ of ψ. The stability of the approximants with respect to the order n provides a check of the convergence of the method. The values $n=18$ and $z\in[100; 100a]$ give ψ with a very good accuracy, except for small values of α^*.

Figures 4 to 7 show plots of the function $\psi(\xi)$ over one period for the distribution (5.10) with $\alpha=2.5$ and $a=2$, 2.5, 4 and 5 respectively. The most remarkable fact is that ψ *seems* to be discontinuous at a lot of points. Nevertheless we can show that ψ *is continuous.* Assume that $N(z)$ has its largest discontinuity Δ at z_0. Since $N(z)$ satisfies the equation

$$N(z) = pN[a(z-1)] + (1-p)N[a^{-1}(z-1)] \tag{5.13}$$

it follows that $N(z)$ also has discontinuities of strength Δ at the points $f_1(z_0)$ and $f_{-1}(z_0)$ with

$$f_l(z) = a^l(z-1) \tag{5.14}$$

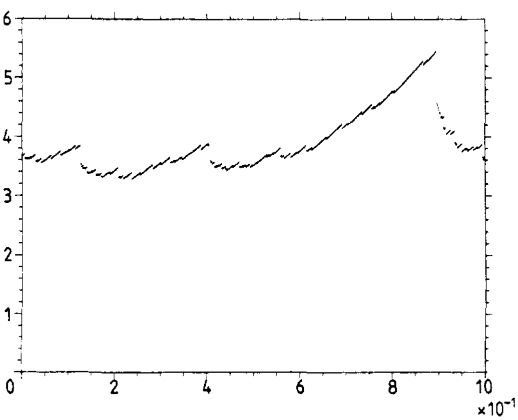

Figure 4. Plot of the periodic function $\psi(\xi)$ over one period. $\psi(\xi)$ modulates the large-z behaviour of the integrated density $N(z)$ for the binary distribution (5.10). The parameters read: $a=2.5$ and p such that $\alpha^*=2$.

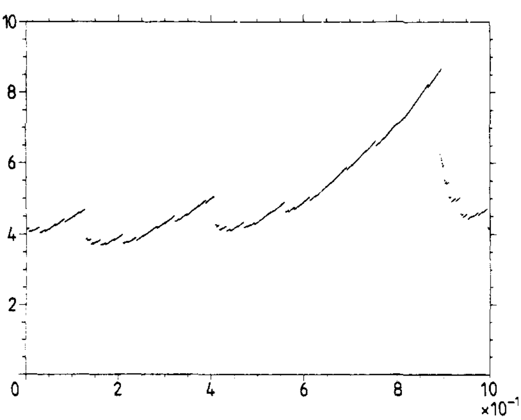

Figure 5. The same as figure 4, with $\alpha^*=2.5$.

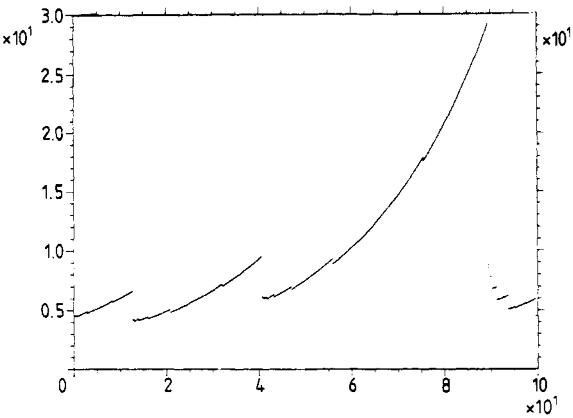

Figure 6. The same as figure 4, with $\alpha^* = 4$.

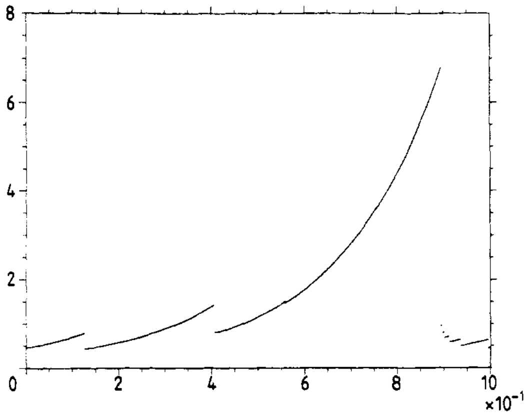

Figure 7. The same as figure 4, with $\alpha^* = 5$.

and, by induction, that $N(z)$ has a discontinuity Δ at every point of the form

$$f_{l_1} \circ f_{l_2} \ldots \quad \ldots \circ f_{l_N}(z_0) \tag{5.15}$$

with $l_i = \pm 1$ and N arbitrary.

A subsequence of this set, namely

$$z_k = (f_1 \circ f_{-1})^k (z_0) = z_0 - k(1+a) \tag{5.16}$$

lies for k great enough at the left of $a/(a-1)$. But in $]-\infty, a/(a-1)]$ the function $N(z)$ equals unity and Δ must be zero, implying that $N(z)$ is continuous. The discontinuity (5.9) is therefore a peculiarity of the limiting case (5.5).

Let us end this section with a conjecture on the location of the *seemingly* discontinuities of $N(z)$. In the limiting case (5.5), they are real discontinuities located at $(a^{n+1}-1)/(a-1) = (f_{-1})^{-n}(1)$, where 1 is the lower bound of the support of $\mathrm{d}\rho$. We conjecture that, in the general oscillating case, $\mathrm{d}\rho$ looks like a pure point measure, with atoms at the points

$$f_{l_1}^{-1} \circ \ldots \circ f_{l_N}^{-1}(z_{\mathrm{inf}}) \tag{5.17}$$

where z_{inf} is the lower bound of the support of $\mathrm{d}\rho$ and l_i such that $p_{l_i} \neq 0$. This conjecture is strongly supported by the fact that the figures show one outstanding largest seemingly discontinuity (per period), at a location which always corresponds to $(f_{-1})^{-n}(z_{\mathrm{inf}})$ with $z_{\mathrm{inf}} = a/(a-1)$. Nearby seemingly discontinuities, corresponding to longer and longer 'words' in (5.17), are expected to have smaller and smaller 'strengths'. This behaviour is very reminiscent of the spectral density of a chain of harmonic oscillators with equal spring contants and random masses which take either a certain value or are infinite. Domb *et al* (1959) showed that the spectral density consists of a dense denumerable set of delta functions.

6. Discussion—relation to more realistic models

The physical implications of the dependence of critical exponents upon disorder, and of the existence of several critical points in some 1D random systems, have already been discussed elsewhere (Derrida and Hilhorst 1983, Derrida 1983). We shall therefore restrict the present discussion to another striking property of such disordered systems, namely the periodic amplitudes.

We have shown in § 5 that our model exhibits critical amplitudes ψ which are periodic in $\ln z$, look *discontinuous* but are continuous. A question which naturally arises is to know which physical quantity may have such a support in realistic models. In the random field Ising model, the distributions of local field and local magnetisation are given by equations which are very similar to our Dyson–Schmidt equation (2.3). And these quantities are known to have a fractal or Cantor-like support in some cases (Groeneveld 1980, Bruinsma and Aeppli 1983, Györgyi and Rujan 1983, Normand *et al* 1984).

Let us now consider thermodynamical quantities, and in particular the free energy $F(\varepsilon)$ (1.4). It was argued by Derrida and Hilhorst that it exhibits a periodic amplitude

$$F(\varepsilon) \sim \varepsilon^{2\alpha^*} \varphi(\ln \varepsilon^2/\ln a) \tag{6.1}$$

whenever $N(z)$ does. We shall now compare briefly the functions ψ and φ, and begin with the soluble case (5.5), already considered by Derrida and Hilhorst. They found that the singular part of $F(\varepsilon)$ reads

$$F_{sg}(\varepsilon) = (1-p)^2 \sum_{n \geq 1} p^n \ln(1 + b\lambda^{n+1}) \tag{6.2}$$

with

$$\begin{aligned} b &= 1 + \frac{(a-1)^2}{2a\varepsilon^2}\left[1 - \left(1 + \frac{4a\varepsilon^2}{(a-1)^2}\right)^{1/2}\right] \\ \lambda &= (a-b)/(1-ab). \end{aligned} \tag{6.3}$$

The Fourier coefficients $\tilde{\varphi}_k$ of the amplitude φ can be extracted from equation (6.2) by taking the Mellin transform of $F_{sg}(\varepsilon)$ with respect to b

$$M(s) = \int_0^\infty b^{s-1} F_{sg}(\varepsilon)\, \mathrm{d}b. \tag{6.4}$$

This integral can be done exactly

$$M(s) = \frac{(1-p)^2}{p} \frac{\lambda^s}{\lambda^s - p} \frac{\pi}{s \sin(\pi s)} \tag{6.5}$$

and therefore the coefficients $\tilde{\varphi}_k$ read

$$\tilde{\varphi}_k = \frac{p(1-p)^2}{\ln a}\left(\frac{a}{(a-1)^2}\right)^{\alpha^* + ik\omega} \frac{\pi}{(\alpha^* + ik\omega)\sin[\pi(\alpha^* + ik\omega)]} \tag{6.6}$$

with $\alpha^* = -\ln p/\ln a$ and $\omega = 2\pi/\ln a$. The essential property of the $\tilde{\varphi}_k$ is their behaviour at $k \to \infty$

$$|\tilde{\varphi}_k| \sim \exp(-2\pi^2|k|/\ln a). \tag{6.7}$$

This exponential decay corresponds to the fact that $F(\varepsilon)$ is analytic in the half-plane: $\mathrm{Re}\ \varepsilon > 0$. Indeed, the series

$$F(\varepsilon) = \sum_k \tilde{\varphi}_k \exp(2\pi i k \ln \varepsilon^2/\ln a)$$

converges only for $|\arg \varepsilon| < \frac{1}{2}\pi$.

In the general oscillating case, the very same behaviour (6.7) of $\tilde{\varphi}_k$ at large k is expected. The easiest way to realise it is to follow Derrida and Hilhorst's derivation of the analogy between $F(\varepsilon)$ and $N(z)$: it is clear from their calculations that the partition function of a finite chain can only vanish for real negative ε^2, at least in the small-ε limit, therefore $F(\varepsilon)$ is always analytic for $|\arg \varepsilon| < \frac{1}{2}\pi$.

We have performed a numerical determination of the periodic amplitude φ for the binary distribution (5.10), just as we did in § 5 for ψ, by enumeration of the 2^n (up to $n = 16$) values of $F(\varepsilon)$. Figure 8 shows a plot of φ for $a = 1000$, $\alpha = 0.4$. Notice that the magnitude of the oscillations, given roughly by

$$|\tilde{\varphi}_1/\tilde{\varphi}_0| \sim \exp(-2\pi^2/\ln a) \tag{6.8}$$

is usually extremely small (4.10^{-13} for $a = 2$). That explains why we chose a very large value of a to draw figure 8.

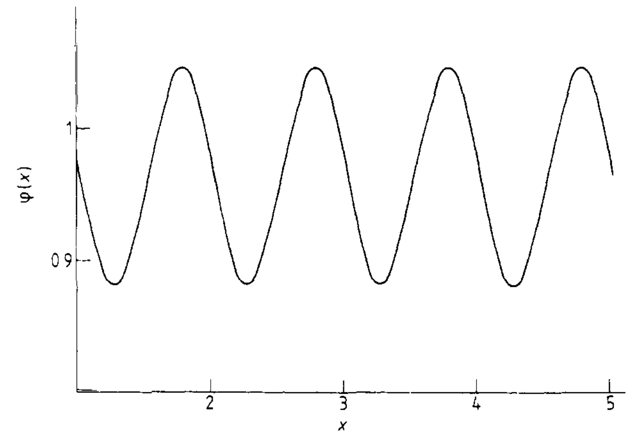

Figure 8. Plot of the periodic function $\varphi(\xi)$, which modulates the power-law behaviour of the free energy in the random field Ising model. The parameters read: $\alpha^* = 0.4$ and $a = 1000$.

Another example of smooth periodic critical amplitudes has been recently studied (Derrida *et al* 1984) on hierarchical lattices. In their case, the oscillations are due to the discrete character of the exact renormalisation transform they use to solve the model. In the present situation, the oscillations reflect 'resonances', the physical interpretation of which remains somehow puzzling. The presence of such periodic functions implies that a critical behaviour is not necessarily characterised by pure power laws. It would be interesting to investigate whether Cantor-like supports and oscillatory amplitudes also occur in dimensions higher than one.

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