Topics on random walks

Reversibility and its consequences
Harmonic properties, isoperimetry, heat kernels

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Notes downloadable from URL
http://perso.univ-rennes1.fr/dimitri.petritis/enseignement/markov/blq-topics-rw2.pdf

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Motivation

Consider s.r.w. on $\mathbb{X} = \mathbb{Z}$. Define, for $A \subset \mathbb{X}$

$$\begin{aligned} \tau_B^\flat &:= \inf\{n \geq \flat : X_n \in A\}, \text{ for } \flat = 0, 1, \\ h(x) &:= \mathbb{P}_x(\tau_0^0 < \infty). \end{aligned}$$

Obviously $h(0) = 1$ and for $x \neq 0$:

$$\begin{aligned} h(x) &= \mathbb{P}_x(\cup_{n\geq 0}\{\tau_0^0 = n\}) = \sum_{n\geq 0} \mathbb{P}_x(\cup_{n\geq 0}\{\tau_0^0 = n\}) \\ &= \sum_{n\geq 1} [\mathbb{P}_x(X_1 = x-1)\mathbb{P}_x(\tau_0^0 = n | X_1 = x-1) + (x-1 \to x+1)] \\ &= \frac{1}{2}\sum_{n\geq 1}[\mathbb{P}_{x-1}(\tau_0^0 = n-1) + \mathbb{P}_{x+1}(\tau_0^0 = n-1)] \\ &= \frac{1}{2}[h(x-1) + h(x+1)]. \end{aligned}$$

i.e. $\Delta h(x) = 0$.

Solution of the discrete equation $\Delta h = 0$ in $d = 1$

Blackboard 2

Harmonic functions

$$\mathrm{Dom}(P) := \{f : \mathbb{X} \to \mathbb{R} | \forall x, \sum_y P(x,y)|f(y)| < \infty\}.$$

Definition

Let $f \in \mathrm{Dom}(P)$ and define $Pf(x) := \sum_y P(x,y)f(y)$. The function f is called

- harmonic if $Pf = f$,
- superharmonic if $Pf \leq f$, and
- subharmonic if $Pf \geq f$.

Remark

Any constant function is harmonic.

Elementary properties of harmonic functions

Lemma

For an irreducible matrix P, any bounded harmonic function that reaches its maximum at some point z is constant.

Proof: Let $M := \sup_{x\in\mathbb{X}} f(x) = f(z)$. Then
$$Pf(z) = \sum_y P(z,y)f(y) = f(z) = M.$$

Hence
$$\sum_y P(z,y)(M - f(y)) = 0 \Rightarrow f(y) = M, \forall y : P(z,y) > 0. \quad \square$$

Lemma

f superharmonic implies $(f(X_n))$ supermartingale.

Proof:
$$\mathbb{E}(f(X_{n+1})|\mathcal{F}_n) = \sum_y P(X_n,y)f(y) \le f(X_n). \quad \square$$

Probabilistic vs. electrical estimates

Blackboard 5

Lessons

- Probabilistic (combinatorial and Fourier transform) estimates greatly simplified by use of harmonic functions.
- Harmonic functions appear in many other disciplines (especially in electric networks). Probabiliistic quantities governed by same equations as physical quantities. $\Rightarrow$ probabilistic estimates obtained by electrical intuition.
- Transforming (X_n) by an harmonic function produces a martingale. $\Rightarrow$ one expects semi-martingale techniques to be instrumental (Lyapunov functions).

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Past vs. future

Remark (Past and future are independent conditionally to present)

Let $(X_n) \in MC(\mathbb{X}, P, \mu)$ *and*

$$\mathcal{F}_n = \sigma(X_1, \ldots, X_n); \quad \mathcal{T}_n = \sigma(X_n, X_{n+1}, \ldots).$$

For all $A \in \mathcal{F}_n$ *and all* $B \in \mathcal{T}_n$,

$$\mathbb{P}_\mu(A \cap B | \sigma(X_n)) = \mathbb{P}_\mu(A|\sigma(X_n))\mathbb{P}_\mu(B|\sigma(X_n)).$$

Past and future play symmetric role w.r.t. conditioning.

Remark (Asymptotic behaviour is asymmetrical in time)

The following are equivalent (Blackboard 7):

- *The only bounded functions that are harmonic for the space-time chain are constant.*
- *For all* $\mu, \nu \in \mathcal{M}_1(\mathbb{X}) : \lim_n \|\mu P^n - \nu P^n\| = 0$.

If π *the invariant probability (i.e.* $\pi P = \pi$*),* $\forall \mu \in \mathcal{M}_1(\mathbb{X}), \mu P^n \to \pi$.
Hence to restore symmetry, must initialise chain with π.

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Strong and weak reversibility

Theorem

Let $X = (X_n)_{n\in\mathbb{N}} \in MC(\mathbb{X}, P, \mu)$ *irreducible and* $\mu P = \mu$ *with* $\mu \in \mathcal{M}_+(\mathbb{X})$. *For any large integer* N *define* $Y_n = X_{N-n}$ *forn* $= 0, \ldots, N$. *Then* $(Y_n)_{0\leq n\leq N} \in MC(\mathbb{X}, Q, \mu)$ *and* $\pi Q = \pi$ *where*

$$Q(x, y) := Q^{(P,\mu)}(x, y) = \mu(y)\frac{P(y, x)}{\mu(x)}.$$

Definition

The Markov chain as above (or equivalently (P, μ)) is

- in detailed balance if $\forall x, y \in \mathbb{X} : \mu(x)P(x, y) = \mu(y)P(y, x)$,
- weakly reversible if $Q^{(P,\mu)} = P$, with $\mu \in \mathcal{M}_+(\mathbb{X})$,
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Lemma

If (P, μ) with $\mu \in \mathcal{M}_+(\mathbb{X})$ are in detailed balance, then μ is invariant ($\mu P = \mu$).

Lemma

If P irreducible and $\mu \in \mathcal{M}_1(\mathbb{X})$, then equivalence between:

- *(P, μ) strongly reversible, and*
- *(P, μ) in detailed balance.*

Remark

A simple random walk on a directed graph can never be reversible (Blackboard 9).

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Graphs revisited

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- $s, t : \mathbb{G}^1 \to \mathbb{G}^0$ source and terminal functions. If $\alpha = (x, y)$ then $s(\alpha) = x$ and $t(\alpha) = y$.
- $\forall n, \mathbb{G}^n := \{\alpha = (\alpha_1, \ldots, \alpha_n) : \alpha_i \in \mathbb{G}^1 \;\&\; s(\alpha_{i+1}) = t(\alpha_i), \forall i\}$.
- $\mathbb{G}^* = \cup_{n \in \mathbb{N}} \mathbb{G}^n$.
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Directed graphs stemming from Markov chains

Let P stochastic matrix on $\mathbb{X}$. Then there is a directed graph $\mathbb{G} := \mathbb{G}(P)$ defined by

- $\mathbb{G}^0 = \mathbb{X}$,
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- Irreducibility of $P \Leftrightarrow$ transitivity of $\mathbb{G}$, i.e. $\forall x, y \in \mathbb{G}^0, \mathbb{G}^*(x, y) \neq \emptyset$.
- Any $\mathbb{D} \subseteq \mathbb{G}^0$ inherits edges $\mathbb{D}^1$ from $\mathbb{G}^1$ and in turn becomes a subgraph $\mathbb{D} = (\mathbb{D}^0, \mathbb{D}^1)$ with all derived objects (like $\mathbb{D}^*$, $\mathbb{D}^*(x)$, $\mathbb{D}^*(x, y)$, etc.).

Remark

Directed graph stemming from irreducible P is transitive but fails to be a metric space. Connectedness does not hold in general.

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Remark

Directed graph stemming from irreducible P is transitive but fails to be a metric space. Connectedness does not hold in general.

(Blackboard 11)

Directed graphs stemming from Markov chains

Let P stochastic matrix on $\mathbb{X}$. Then there is a directed graph $\mathbb{G} := \mathbb{G}(P)$ defined by

- $\mathbb{G}^0 = \mathbb{X}$,
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Undirected graphs from weakly reversible Markov chains

Let P stochastic matrix on $\mathbb{X}$ and $\mu \in \mathcal{M}_+(\mathbb{X})$ s.t. (P, μ) weakly reversible. Let $\mathbb{G} := \mathbb{G}(P)$ be the graph of P.

- Define weight $\kappa \in \mathcal{M}_+(\mathbb{G}^1)$ by
$$\begin{aligned} \mathbb{G}^1 \ni \alpha = (x, y) \quad &\mapsto \quad \kappa(x, y) := \mu(x)P(x, y) \\ &= \quad \mu(s(\alpha))P(\alpha) = \kappa(\alpha) \in \mathbb{R}^+, \end{aligned}$$
- (P, μ) reversible $\Rightarrow \forall \alpha \in \mathbb{G}^1, \exists! \overline{\alpha} : s(\overline{\alpha}) = t(\alpha)$ & $t(\overline{\alpha}) = s(\alpha)$ (undirectedness of $\mathbb{G}$).
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$$\forall x \in \mathbb{G}^0, \exists \alpha \in \mathbb{G}^1 : s(\alpha) = x \quad \text{(no sink condition)}.$$
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Weakly reversible Markov chains from weighted undirected graphs

Let $(\mathbb{G}, \kappa)$ be a transitive, undirected, graph with symmetric weight $\kappa \in \mathcal{M}_+(\mathbb{G}^1)$.

- Transitivity of $\mathbb{G} \Rightarrow \forall x \in \mathbb{G}^0, \mu(x) = \sum_{\alpha \in s^{-1}(\{x\})} \kappa(\alpha) > 0$.
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Bijection between irreducible weakly reversible Markov chains (P, μ) and transitive undirected weighted graphs $(\mathbb{G}, \kappa)$ with symmetric weight $\kappa \in \mathcal{M}_+(\mathbb{G}^1)$.

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- $(\mathbb{G}, \kappa)$ undirected $\Rightarrow$ graph distance defined by
$$\mathrm{dist}(x, y) := \inf\{|\alpha| : \alpha \in \mathbb{G}^*(x, y)\} = \mathrm{dist}(y, x).$$
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- Let $\mathbb{D}^0$ connected subset of $\mathbb{G}^0$. Vertex and edge boundaries:
$$\partial_0\mathbb{D}^0 := \{y \in (\mathbb{D}^0)^c : (x, y) \in \mathbb{G}^1, \text{ for some } x \in \mathbb{D}^0\}$$
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- $\overline{\mathbb{D}^0} = \mathbb{D}^0 \cup \partial_0\mathbb{D}^0$.

Elementary topological properties of undirected graphs

- $(\mathbb{G}, \kappa)$ undirected $\Rightarrow$ graph distance defined by
$$\mathrm{dist}(x, y) := \inf\{|\alpha| : \alpha \in \mathbb{G}^*(x, y)\} = \mathrm{dist}(y, x).$$
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Maximum principle

Bijection between $(\mathbb{G}, \kappa)$ and (P, μ) used repeatedly without notice. All graphs considered row finite.

Theorem

Let $\mathbb{D}$ connected subgraph of $\mathbb{G}$ and $f : \mathbb{D}^0 \to \mathbb{R}$. Suppose

- $Pf = f$ *on* $\mathbb{D}^0$, *and*
- f *reaches its maximum at some* $z \in \mathbb{D}^0$.

Then f *is constant on* $\overline{\mathbb{D}^0}$.

Proof: (Blackboard 15) □

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Proof: (Blackboard 15) □

Unicity

Theorem

Let $\mathbb{D}$ finite proper subgraph of $\mathbb{G}$ and $f, g : \mathbb{G}^0 \to \mathbb{R}$. Suppose

- *f and g harmonic on $\mathbb{D}^0$, and*
- *$f = g$ on $(\mathbb{D}^0)^c$.*

Then $f = g$ on $\mathbb{G}^0$.

Proof: (Blackboard 16) □

Consequences of unicity

- Harmonicity of f on a finite set $\mathbb{D}$ and boundary conditions (i.e. the values of f on the set where f is not guaranteed to be harmonic) characterises f.
- If f, f_1, f_2 harmonic on some proper subset $\mathbb{D}^0$ and $f = a_1 f_1 + a_2 f_2$ outside $\mathbb{D}^0$ for some real a_1, a_2, then $f = a_1 f_1 + a_2 f_2$ everywhere (superposition principle).

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Existence

Theorem

$\mathbb{D}^0 \subseteq \mathbb{G}^0$ and $g : (\mathbb{D}^0)^c \to \mathbb{R}$ bounded. Then $\exists f : \mathbb{G}^0 \to \mathbb{R}$ s.t.

- *$f = g$ outside $\mathbb{D}^0$ and*
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Proof: Exercise.
Hint: For any $x \in \mathbb{D}^0$ start P-random walk (X_n) at x. Define

$$f(x) = \mathbb{E}_x(\mathbb{1}_{\{\tau^0_{(\mathbb{D}^0)^c} < \infty\}} g(X_{\tau^0_{(\mathbb{D}^0)^c}})).$$

Check (exercise!) that $Pf = f$ on $\mathbb{D}^0$. □

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Vector spaces associated with graphs

- Any transitive, undirected, weighted graph $(\mathbb{G}, \kappa)$ — already in bijection with irreducible weakly reversible Markov chain (P, μ) — is in bijection with electrical circuit composed solely from nodes and resistances. Resistance of edge $\alpha : \rho(\alpha) = \frac{1}{\kappa(\alpha)}$.
- Probabilistic quantities involving harmonic functions can be estimated by electrical analogs.
- Vector spaces: $\mathbb{V}_0 := \{f : \mathbb{G}^0 \to \mathbb{R}\}$ and $\mathbb{V}_1 := \{f : \mathbb{G}^1 \to \mathbb{R}\}$.
- Co-boundary and boundary operators d and d^*:

$$\mathbb{V}_0 \ni f \quad \mapsto \quad df \in \mathbb{V}_1; df(\alpha) := f(t(\alpha)) - f(s(\alpha))$$

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Memories of your young days

on your lessons in electricity

Definition

- A voltage $v : \mathbb{G}^0 \to \mathbb{R}$ is harmonic at every vertex $x \in \mathbb{G}^0$ not directly connected to an electric source.
- A current $i : \mathbb{G}^1 \to \mathbb{R}$ is associated with the voltage by Ohm's law:

$$i(\alpha) = \pm \kappa(\alpha) dv(\alpha) \Leftrightarrow dv(\alpha) = \pm \rho(\alpha) i(\alpha).$$

Remark

- $\forall \alpha \in \mathbb{G}^1 : i(\overline{\alpha}) = -i(\alpha)$ *(antisymmetry of current).*
- *v harmonic at $x \in \mathbb{G}^0 \Rightarrow Pv(x) - v(x) = 0$, hence*

$$0 = \sum_{y \sim x} \kappa(x, y)[v(y) - v(x)] = \sum_{\alpha \in s^{-1}(x)} \kappa(\alpha) dv(\alpha) = \sum_{\alpha \in s^{-1}(x)} i(\alpha).$$

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Flows

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Let $\mathbb{L}$ and $\mathbb{M}$ be subgraphs of $(\mathbb{G}, \kappa)$. A function $\phi : \mathbb{G}^1 \to \mathbb{R}$ is a flow between $\mathbb{L}$ and $\mathbb{M}$ if

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Remark

Current is a flow.

Exercise

Current satisfies Kirchoff's cycle law: if $\alpha \in \mathbb{G}^*$ and $\alpha = (\alpha_1, \ldots, \alpha_n)$ with $s(\alpha) = t(\alpha)$ (i.e. α is a cycle) not containing any source node, then $\sum_{i=1}^{n} \rho(\alpha_i) i(\alpha_i) = 0$.

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Equivalent circuits and their probabilistic interpretation

- Let $\mathbb{L} = \{z\}$ and $\mathbb{M} \subseteq \mathbb{G}^0$.
- Interpret $\mathbb{P}(z \rhd \mathbb{M}) = \mathbb{P}_z(\tau^0_{\mathbb{M}} < \tau^1_z) = \mathbb{P}(\exists \text{ flow from } z \text{ to } \mathbb{M})$.
- Apply voltage $v(z)$ at z and 0 at $\mathbb{M}$ (and harmonic elsewhere).
- By superposition principle: $\mathbb{P}_x(\tau^0_z < \tau^0_{\mathbb{M}}) = \frac{v(x)}{v(z)}$. Establish then

(Blackboard 22)

$$\begin{aligned}
\mathbb{P}(z \rhd \mathbb{M}) &= \sum_y P(z,y)[1 - \mathbb{P}_y(\tau^0_z < \tau^0_{\mathbb{M}})] \\
&= \sum_y \frac{\kappa(z,y)}{\mu(z)}[1 - \frac{v(y)}{v(z)}] \\
&= \frac{1}{v(z)\mu(z)} \sum_y \kappa(z,y)[v(z) - v(y)] \\
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- $v(z) = \frac{\text{incoming current at } z}{\mu(z)\mathbb{P}(z \rhd \mathbb{M})}$ or $C_{\text{eff}}(z \rhd \mathbb{M}) = \mu(z)\mathbb{P}(z \rhd \mathbb{M})$.

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Equivalent circuits and their probabilistic interpretation

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- Interpret $\mathbb{P}(z \triangleright \mathbb{M}) = \mathbb{P}_z(\tau^0_{\mathbb{M}} < \tau^1_z) = \mathbb{P}(\exists \text{ flow from } z \text{ to } \mathbb{M})$.
- Apply voltage $v(z)$ at z and 0 at $\mathbb{M}$ (and harmonic elsewhere).
- By superposition principle: $\mathbb{P}_x(\tau^0_z < \tau^0_{\mathbb{M}}) = \frac{v(x)}{v(z)}$. Establish then

$$\begin{aligned}
\mathbb{P}(z \triangleright \mathbb{M}) &\overset{\text{(Blackboard 22)}}{=} \sum_y P(z,y)[1 - \mathbb{P}_y(\tau^0_z < \tau^0_{\mathbb{M}})] \\
&= \sum_y \frac{\kappa(z,y)}{\mu(z)}[1 - \frac{v(y)}{v(z)}] \\
&= \frac{1}{v(z)\mu(z)} \sum_y \kappa(z,y)[v(z) - v(y)] \\
&= -\frac{\sum_y i(z,y)}{v(z)\mu(z)}.
\end{aligned}$$

- $v(z) = \frac{\text{incoming current at } z}{\mu(z)\mathbb{P}(z \triangleright \mathbb{M})}$ or $C_{\text{eff}}(z \triangleright \mathbb{M}) = \mu(z)\mathbb{P}(z \triangleright \mathbb{M})$.

Other probabilistic quantities and their electrical counterpart

Exercise

- *Let* $\mathbb{Y}$ *be the two-element set* $\mathbb{Y} := \{z, \mathbb{M}\}$. *Consider the* $\mathbb{Y}$*-valued Markov chain* $(Y_n)_{n\in\mathbb{N}}$ *with transition matrix*
$$\Pi = \begin{pmatrix} 1-p & p \\ 0 & 1 \end{pmatrix}, 0 \leq p \leq 1.$$
For $y \in \mathbb{Y}$, *let* $\eta(y) = \sum_{k=0}^{\infty} \mathbb{1}_{\{y\}}(Y_k)$. *Compute* $\mathbb{E}_z \eta(z)$.
- *Consider* $(X_n)_{n\in\mathbb{N}}$, *the random walk stemming from the graph* $(\mathbb{G}, \kappa)$ *that gets absorbed when attains the set* $\mathbb{M} \subseteq \mathbb{G}^0$ *and denote* ${}_{\mathbb{M}}G(z,z)$ *its Green function. Use the previous question to establish that*
$${}_{\mathbb{M}}G(z,z) = \frac{1}{\mathbb{P}(z \triangleright \mathbb{M})} = \mu(z)\rho_{eff}(z \triangleright \mathbb{M}),$$
where $\rho_{eff}(z \triangleright \mathbb{M}) = \frac{1}{\mu(z)\mathbb{P}(z \triangleright \mathbb{M})}$.

(Blackboard 23)

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Applications of electric networks

Theorem

The s.r.w. on $\mathbb{Z}^1$ is recurrent.

Proof: Exercise. Hint: compute $\rho_{\text{eff}}(z \triangleright \infty)$. □

Theorem

The s.r.w. on $\mathbb{T}_3$ (the homogeneous tree of constant degree 3) is transient.

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Exercise (Side question)

We have seen that $\mathbb{F}_2$ (the free group with 2 generators and their inverses) is isomorphic to $\mathbb{T}_4$ (the homogeneous tree of constant degree 4). Hence $\mathbb{T}_4$ is a group. What can you say about $\mathbb{T}_3$?

(Blackboard 24)

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A refinement on electric circuits

Definition

Let $\mathbb{G}$ transitive graph and $\mathbb{L}, \mathbb{M}$ subsets of $\mathbb{G}^0$. A subset C of $\mathbb{G}^1$ is called a cutset if every path from $\mathbb{L}$ to $\mathbb{M}$ contains an edge of C.

Theorem (Nash-Williams criterion)

Let $(\mathbb{G}, \kappa)$ be a transitive, undirected, locally finite, weighted graph and $(C_n)_{n\in\mathbb{N}}$ a sequence of finite disjoint cutsets, each C_n separating a given reference vertex $o \in \mathbb{G}^0$ from ∞. Then

$$\rho_{\text{eff}}(o \rhd \infty) \geq \sum_{n\in\mathbb{N}} \frac{1}{\sum_{\alpha\in C_n} \kappa(\alpha)}.$$

Corollary

The s.r.w. on $\mathbb{Z}^2$ is recurrent.

Proof: Exercise. Hint: minorate $\rho_{\text{eff}}(o \rhd \infty)$.
(Blackboard 25)

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Isoperimetric inequalities

Definition

Let $\psi : [a, \infty[\to \mathbb{R}_+$ be defined for some $a \geq 0$. $(\mathbb{G}, \kappa)$ satisfies ψ-isoperimetric inequality (is IP_ψ) if

$$\forall F \subset \mathbb{G}^0, F \neq \emptyset, F \text{ finite} : \kappa(\partial_1 F) \geq c_\psi \psi(\mu(F)).$$

The maximal c_ψ for which IP_ψ holds is called isoperimetric constant.

Remark

Always assume ψ well-defined for $a \geq \inf_{x \in \mathbb{G}^0} \mu(x)$.

Example

$\mathbb{Z}^d$ for $\psi(t) = t^{1-1/d}, t > 0$, is $\mathrm{IP}_\psi \equiv \mathrm{IP}_d$.

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Graph Hilbert spaces

Definition

- Vertex Hilbert space $\mathcal{H}_0 = \ell^2(\mathbb{G}^0, \mu)$.
- Edge Hilbert space $\mathcal{H}_1 = \ell^2(\mathbb{G}^1, \kappa)$.
- Co-boundary operator $d : \mathcal{H}_0 \to \mathcal{H}_1$ defined by $df(\alpha) = f(t(\alpha)) - f(s(\alpha))$.
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- $\Delta f(x) = d^* df(x) = \frac{1}{\mu(x)} \sum_{\alpha \in s^{-1}(x)} \kappa(\alpha) df(\alpha) = (P - I)f(x)$.
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For directed graph, Markov evolution expressed in terms of d^* (reminiscent of Dirac operator), not $\mathcal{L}$!

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Spectral estimates

Definition

- For $F \subset \mathbb{G}^0$, finite, define $C_F = \{f : F \to \mathbb{R}\}$, extended to $\mathbb{G}^0$ by $f(x) = 0$ on F^c.
- For $f \in C_F$ define $\mathcal{L}_F f(x) = f(x) - \sum_{y \sim x} P(x,y) f(y)$.

Lemma

For all $f, g \in C_F$:

$$(\mathcal{L}_F f, g) \overset{\text{(Blackboard 28)}}{=} \frac{1}{2} \sum_{x,y \in \overline{F}} \kappa(x,y) df(x,y) dg(x,y).$$

Remark

Since $\mathcal{L}_F$ self-adjoint, $\operatorname{spec} \mathcal{L}_F = \{\lambda_1 \leq \cdots \leq \lambda_{|F|}\}$ *and*

$$\lambda_1(F) = \inf_{f \in C_F \setminus \{0\}} \frac{\sum_{x,y \in \overline{F}} \kappa(x,y)(df(x,y))^2}{2 \sum_{x \in F} \mu(x) f^2(x)}.$$

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Cheeger's inequality

Definition

For $F \subset \mathbb{G}^0$, finite. Define the Cheeger's constant

$$h(F) := \inf_{\emptyset \subset S \subseteq F} \frac{\kappa(\partial_1 S)}{\mu(S)},$$

i.e. the largest constant $h(F)$ s.t. $\kappa(\partial_1 S) \geq h(F)\mu(S)$.

Theorem (Cheeger's inequality)

$$\lambda_1(F) \geq \frac{h^2(F)}{2}.$$

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Assume $(\mathbb{G}, \kappa)$ be IP_ψ with ψ s.t. $\psi(s)/s$ decreasing. Then $\forall F \subset \mathbb{G}^0$, finite, $\neq \emptyset$,

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If $(\mathbb{G}, \kappa)$ *satisfies:*

- $\forall \alpha \in \mathbb{G}^1 : 1 \leq \kappa(\alpha) \leq K$,
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$$IP_\psi, \int^\infty \frac{1}{\psi^2(s)} ds < \infty \quad \Rightarrow \quad \textit{transience},$$

$$FK_\Lambda, \int^\infty \frac{1}{s^2\Lambda^2(s)} ds < \infty \quad \Rightarrow \quad \textit{transience}.$$

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The s.r.w. on $\mathbb{Z}^d$ is transient for $d \geq 3$.
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$$IP_\psi, \int^\infty \frac{1}{\psi^2(s)} ds < \infty \quad \Rightarrow \quad \textit{transience},$$

$$FK_\Lambda, \int^\infty \frac{1}{s^2 \Lambda^2(s)} ds < \infty \quad \Rightarrow \quad \textit{transience}.$$

Example

The s.r.w. on $\mathbb{Z}^d$ is transient for $d \geq 3$.
(Blackboard 30)

Criteria for transience

Theorem

If $(\mathbb{G}, \kappa)$ satisfies:

- $\forall \alpha \in \mathbb{G}^1 : 1 \leq \kappa(\alpha) \leq K$,
- $\forall x \in \mathbb{G}^0 : 1 \leq d(x) \leq D$, *then*

$$IP_\psi, \int^\infty \frac{1}{\psi^2(s)} ds < \infty \quad \Rightarrow \quad \textit{transience},$$

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Example

The s.r.w. on $\mathbb{Z}^d$ is transient for $d \geq 3$.
(Blackboard 30)

Heat kernel estimates

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Theorem (Varopoulos 1985)

Let $\beta \geq 2$ and $r = 2\beta/(\beta - 2)$. If for any $f \in c_0(\mathbb{G}^0)$,

$$\|f\|_r \leq C\|f\|_{Dir},$$

(where $\|f\|_{Dir}^2 = \frac{1}{2}\sum_{x,y\in\mathbb{G}^0} \kappa(x,y)|f(x) - f(y)|^2$) then

$$\sup_{x,y\in\mathbb{G}^0} \frac{P^n(x,y)}{\mu(y)} = \mathcal{O}(n^{-\beta/2}).$$

Theorem (de Loynes, thm 3.1.2 in PhD thesis (2012)[a])

[a] Available at http://tel.archives-ouvertes.fr/tel-00726483.

The simple random walk on (adjacent edges of) a generic Penrose tiling of the d-dimensional space is

- *recurrent, if $d \leq 2$, and*
- *transient, if $d \geq 3$.*

Idea of the proof: For technical reasons, not possible to show IP_d for Penrose graph but only for $\mathrm{Fuzz}_k(\mathbb{G})$ (the graph with same $\mathbb{G}^0$ and $\mathbb{G}^1$ all $x, y \in \mathbb{G}^0$ such that $1 \leq d_{\mathbb{G}}(x, y) \leq k$) and k-fuzz leaves type invariant.

- $\mathrm{IP}_d \Rightarrow [\|f\|_{d/(d-1)} \leq C\|f\|_{\mathrm{Sob}}$ (Sobolev inequality), where $\|f\|_{\mathrm{Sob}} = \frac{1}{2}\sum_{x,y\in\mathbb{G}^0} \kappa(x, y)|f(x) - f(y)|$.
- But
$$[\|f\|_{d/(d-1)} \leq C\|f\|_{\mathrm{Sob}}] \Rightarrow [\|f\|_{2d/(d-2)} \leq C'\|f\|_{\mathrm{Dir}}].$$
- Varopoulos theorem allow then to conclude.

Why heat kernel methods apply to groupoids?

The s.r.w. on Penrose is reversible!

Remark

Here crucial ingredients:

- *reversibility holds but not space homogeneity ($\Rightarrow$ not Fourier transform),*
- *quasi-isometric embedding of k-fuzz of Penrose lattice into $\mathbb{Z}^2$.*

Foster's criteria

Here Markov chain X is $\text{MC}(\mathbb{X}, P, \mu)$ on denumerable set $\mathbb{X}$ without further condition on P beyond irreducibility.

Theorem (Transience)

If $\text{card}\mathbb{X} = \aleph_0$, *equivalence between:*

- X *transient,*
- $\exists f \in \text{Dom}_+(P)$ *and* $\exists A \subset \mathbb{X}$ *s.t.*
  - $\mathbb{E}(f(X_{n+1}) - f(X_n) | X_n = y) \leq 0$ *for all* $y \in A^c$,
  - $\exists y \in A^c : f(y) < \inf_{z \in A} f(z)$.

Theorem (Recurrence)

If $\text{card}\mathbb{X} = \aleph_0$, *equivalence between:*

- X *recurrent,*
- $\exists f \in \text{Dom}_+(P)$, $f \to \infty$ *and* $\exists F \subset \mathbb{X}$ *finite s.t.*

$$x \in F^c \Rightarrow \mathbb{E}(f(X_{n+1}) - f(X_n) | X_n = x) \leq 0.$$

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- $\exists f \in \mathrm{Dom}_+(P)$, $f \to \infty$ *and* $\exists F \subset \mathbb{X}$ *finite s.t.*
$$x \in F^c \Rightarrow \mathbb{E}(f(X_{n+1}) - f(X_n) | X_n = x) \leq 0.$$

Foster's criteria (cont'd)

Theorem (Positive recurrence)

Equivalence between:

- X *positive recurrent,*
- $\exists f \in \mathrm{Dom}_+(P)$, $\exists F \subset \mathbb{X}$ *finite, and* $\exists \epsilon > 0$ *s.t.*
$$x \in F^c \Rightarrow \mathbb{E}(f(X_{n+1} - f(X_n)|X_n = x) \leq -\epsilon.$$

Definition

The function $f \in \mathrm{Dom}_+(P)$, entering into the 3 above theorems, is called a Lyapunov function for the Markov chain.

Example

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The s.r.w. on $\mathbb{Z}^d$ is recurrent for $d = 1, 2$, transient for $d \geq 3$.

Exercise

For each d, determine Lyapunov functions f_d allowing to establish the above result.

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Exercise

For each d, determine Lyapunov functions f_d allowing to establish the above result.

Exercises

Exercise (Easy problem, easily obtained result)

Let $\mathbb{X} = \mathbb{N}$ *and* $(\xi_n)_{n\geq 1}$ *i.i.d. sequence of* $\{-1, 1\}$*-valued variables s.t.*

$$\mathbb{P}(\xi_1 = -1) = 1 - \mathbb{P}(\xi_1 = 1) = p \in]0, 1[.$$

Define $X_{n+1} = (X_n + \xi_{n+1})^+$, *and* $\lambda = \ln \frac{p}{1-p}$. *Show*

- $\lambda > 0 \Rightarrow$ *positive recurrence,*
- $\lambda = 0 \Rightarrow$ *(null) recurrence,*
- $\lambda < 0 \Rightarrow$ *transience.*

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Exercises (cont'd)

Exercise (Hard (Salomon (1975), Sinai (1982)), easily obtained result)

Let $\mathbb{X} = \mathbb{N}$ *and* $(p_x)_{x\in\mathbb{X}})$ *i.i.d. sequence of* $[0,1]$*-valued r.v. Let* $(\xi_n)_{n\geq 1}$ *i.i.d. sequence of* $\{-1,1\}$*-valued r.v. and define* $X_{n+1} = (X_n + \xi_{n+1})^+$, $cF_n = \sigma(X_0,\ldots,X_n)$, *and*

$$\mathbb{P}(\xi_{n+1} = -1|\mathcal{F}_n) = 1 - \mathbb{P}(\xi_{n+1} = 1|\mathcal{F}_n) = p_{X_n}.$$

Let $\lambda = \mathbb{E}(\ln \frac{p_1}{1-p_1})$. *Show*

$\lambda > 0 \Rightarrow$ *positive recurrence,*

$\lambda = 0 \Rightarrow$ *(null) recurrence,*

$\lambda < 0 \Rightarrow$ *transience.*

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