Topics on random walks

Introduction and main results

Combinatorial, geometric, and probabilistic structures for random walks

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Notes downloadable from URL
http://perso.univ-rennes1.fr/dimitri.petritis/enseignement/markov/blq-topics-rw1.pdf

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Type problem

- How often does a random walker on a denumerably infinite set $\mathbb{X}$ returns to its starting point?
- It depends on $\mathbb{X}$ and on the law of jumps.
- Typically a dichotomy
  - either almost surely infinitely often (recurrence),
  - or almost surely finitely many times (transience).

Example $\mathbb{X} = \mathbb{Z}^d$ with symmetric jumps on n.n.

Georg Pólya, Über eine Aufgabe der Wahrscheinlichkeitsrechnung betreffend die Irrfahrt im Straßennetz, Ann. Math. (1921).

The simple random walk on $\mathbb{X} = \mathbb{Z}^d$ is

- recurrent if $d \leq 2$,
- transient if $d \geq 3$.

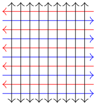
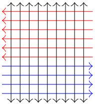
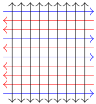
Proof by direct combinatorial estimates and Fourier analysis.

Distinctive property of the simple random walk on $\mathbb{Z}^d$: Abelian group of finite type generated by the support of the law of the simple random walk.

Why random walk are studied?

- Mathematical interest: simple models with three interwoven structures:
  - low-level algebraic structure conveying combinatorial information,
  - high-level algebraic structure conveying geometric information,
  - stochastic structure adapted to the two previous structures.
- Modelling transport phenomena in crystals (metals, semiconductors, ionic conductors, etc.)
- Intervening in all models described by differential equations involving a Laplacian (quantum mechanics, quantum field theory, statistical mechanics, etc.)
- Generalisable to non-commutative groups (random matrices, random dynamical systems, etc.)

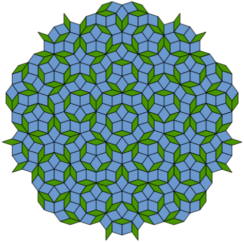
And when the graph is not a group?

Alternate lattice	Half-plane one-way	Random horizontal
		

- Hydrodynamic dispersion in porous rocks Matheron and Marsily (1980), numerical simulations Redner (1997), persistence Majumdar (2003).
- Propagation of information on networks (pathway signalling networks in genomics, neural system, world wide web, etc.)
- Differential geometry, Dirac operators, causal structures in quantum gravity.
- Random walks on semi-groupoids, failure of the reversibility condition.

And when the graph is not a group? (cont'd)

Quasi-periodic tilings of $\mathbb{R}^d$ of Penrose type



- Transport properties on quasi-periodic structures.
- Spectral properties of Schrödinger operators on quasi-periodic structures.
- Random walks on groupoids, non-random inhomogeneity.

Simple symmetric random walk on $\mathbb{X} = \mathbb{Z}^d$

- $\mathbb{X} = \mathbb{Z}^d$ considered as an Abelian group.

$$\mathbb{A} = \{\mathbf{e}_1, -\mathbf{e}_1, \ldots, \mathbf{e}_d, -\mathbf{e}_d\}, \quad \mathrm{card}\mathbb{A} = 2d.$$

- μ probability measure on $\mathbb{A} \Rightarrow$ probability measure on $\mathbb{X}$ with $\operatorname{supp}\mu = \mathbb{A}$.

  Symmetric: $\forall x \in \mathbb{A} : \mu(x) = \mu(-x)$.
  Uniform: $\forall x \in \mathbb{A} : \mu(x) \equiv \frac{1}{\mathrm{card}\mathbb{A}} = \frac{1}{2d}$.

- $\boldsymbol{\xi} = (\xi_n)_{n\in\mathbb{N}}$ i.i.d. sequence with $\xi_1 \sim \mu$.
- Define $X_0 = x \in \mathbb{X}$ and $X_{n+1} = X_n + \xi_{n+1}$. Then

$$P(x, y) = \mathbb{P}(X_{n+1} = y | X_n = x) = \mathbb{P}(\xi_{n+1} = y - x) = \mu(y - x).$$

- Simple (totally) symmetric random walk on $\mathbb{X}$ is the $\mathbb{X}$-valued Markov chain $\mathbf{X} = (X_n)_{n\in\mathbb{N}}$ with transition probability P and initial measure ϵ_x: $\mathrm{MC}(\mathbb{X}, P, \epsilon_x)$

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Combinatorial estimates for small d

Use of Stirling's formula $\lim_{n\to\infty} \frac{n!}{\sqrt{2\pi n} n^n \exp(-n)} = 1$, we get

$d = 1$: $P^{2n}(0,0) = \frac{1}{2^{2n}} \binom{2n}{n} \simeq \frac{c_1}{\sqrt{n}}$,

$d = 2$: $P^{2n}(0,0) = \frac{1}{4^{2n}} \sum_{j=0}^{n} \binom{2n}{j,j,n-j,n-j} = \frac{1}{4^{2n}} \sum_{j=0}^{n} \frac{(2n)!}{j!j!(n-j)!(n-j)!} \simeq \frac{c_2}{n}$.

Fourier estimates

- $\mathbb{P}(X_0 = x_0, X_1 = x_1, \ldots, X_n = x_n) = \mu(x_1 - x_0)\cdots\mu(x_n - x_{n-1})$;
- $P^n(x,y) := \sum_{x_1,\ldots x_{n-1}} \mathbb{P}(X_0 = x, X_1 = x_1, \ldots, X_n = y) = \mu^{*n}(y-x)$.
- For $\xi \sim \mu$ and μ uniform, $\chi(t) = \mathbb{E}\exp(i\langle\, t \,|\, \xi \,\rangle) = \sum_x \exp(i\langle\, t \,|\, x \,\rangle)\mu(x) = \frac{1}{d}\sum_{k=1}^d \cos(t_k)$.
- $P^{2n}(0,0) = \frac{1}{(2\pi)^d}\int_{[-\pi,\pi]^d} (\frac{1}{d}\sum_{k=1}^d \cos(t_k))^{2n} d^d t$.
- $\sum_n P^{2n}(0,0) = \frac{1}{(2\pi)^d}\int_{[-\pi,\pi]^d} (\frac{1}{1-(\frac{1}{d}\sum_{k=1}^d \cos(t_k))^2}) d^d t$.
- For small $\|t\|$: $\frac{1}{d}\sum_{k=1}^d \cos(t_k) = \frac{1}{d}\sum_{k=1}^d (1 - 2\sin^2\frac{t_k}{2}) \simeq 1 - \frac{\|t\|^2}{2d} \simeq \exp(-\frac{\|t\|^2}{2d})$.
- $P^{2n}(0,0) = \frac{2}{(2\pi)^d}\int_{[-\pi,\pi]^d} \exp(-\frac{\|t\|^2 n}{d}) d^d t \sim \frac{c_d}{n^{d/2}}$ as $n \to \infty$.

Transience for $d \geq 3$

Lemma

$$\sum_{n\in\mathbb{N}} P^{2n}(0,0) \quad \begin{cases} = \infty & d \leq 2 \\ < \infty & d \geq 3. \end{cases}$$

Corollary

The simple random walk on $\mathbb{Z}^d$ is transient for $d \geq 3$.

Proof.

By Borel-Cantelli lemma, $\mathbb{P}_0(X_n = 0, \text{i.o.}) = 0$. □

Recurrence for $d \leq 2$

$\sum_{n\in\mathbb{N}} P^{2n}(0,0) = \infty$ not enough to prove recurrence (not independence). Renewal argument:

- Define $r_n := \mathbb{P}_0(X_n = 0)$ (obviously $r_0 = 1$) and
- $\tau_0^\flat := \inf\{n \geq \flat : X_n = 0\}$ for $\flat \in \{0,1\}$; $f_n := \mathbb{P}_0(\tau_0^0 = 0)$.
- $r_n = \mathbb{P}_0(X_n = 0) = \sum_{j=1}^n \mathbb{P}_0(\tau_0^0 = j)\mathbb{P}_0(X_{n-j} = 0) \overset{f_0=0}{=} \sum_{j=0}^n f_j r_{n-j}$.
- $\hat{f}(s) = \sum_{n\in\mathbb{N}} f_n s^n$.
- $\hat{r}(s) = \sum_{n\in\mathbb{N}} r_n s^n = 1 + \sum_{n\geq 1}\sum_{j=0}^n f_j r_{n-j} s^n = 1 + \hat{f}(s)\hat{r}(s)$.

Recurrence for $d \leq 2$ (cont'd)

- $\hat{r}(s) = \frac{1}{1-\hat{f}(s)}$. Monotone convergence theorem guarantees
  - $\lim_{s\uparrow 1} \hat{r}(s) = \hat{r}(1) = G^0(0,0) := \sum_{n\in\mathbb{N}} P^n(0,0)$.
  - $\lim_{s\uparrow 1} \hat{f}(s) = \hat{f}(1) = \mathbb{P}_0(\tau_0^0 < \infty)$.
- Hence $\hat{f}(1) = \mathbb{P}_0(\tau_0^0 < \infty) < 1 \Leftrightarrow \hat{r}(1) = \sum_{n\in\mathbb{N}} P^n(0,0) < \infty$.

Corollary

Symmetric r.w. on $\mathbb{Z}^d$ for $d \leq 2$ is recurrent.

Extensions

$\mathbb{A}$ can be not minimal (but always generating).

Example

- $\mathbb{A}_1 = \{e_1, -e_1, e_2, -e_2, e_1 - e_2, e_2 - e_1\}$.
- $\mathbb{A}_2 = \mathbb{Z}^2 \setminus \{\mathbf{0}\}$ (unbounded jumps).

Theorem (Chung and Fuchs (1951))

Random walk on $\mathbb{Z}^d$ is transient iff

$$\lim_{s \uparrow 1} \int_{[-\pi,\pi]^d} \mathrm{Re}\left(\frac{1}{1 - s\chi(t)}\right) d^d t < \infty.$$

Type of general random walks on $\mathbb{Z}^d$

$$\tilde{m}_k = \sum_{x \in \mathbb{Z}^d} \mu(x) \|x\|^k; \quad m = \sum_{x \in \mathbb{Z}^d} \mu(x) x.$$

Corollary

- $d = 1$: *If* $\tilde{m}_1 < \infty$ *and* $m = 0$ *then r.w. recurrent.*
- $d = 2$: *If* $\tilde{m}_2 < \infty$ *and* $m = 0$ *then r.w. recurrent.*
- $d \geq 3$: *always transient.*

Remark

If the r.w. has general heavy tails standard theorems not available.

What have we learnt so far?

- $\mathbb{Z}^d$ is not only a set but an (Abelian) group.
- R.w. is a compound object of 3 interwoven structures: combinatorial, geometric, stochastic mutually adapted.
- Space homogeneity and group structure $\Rightarrow$ brute force combinatorial estimates can be highly simplified by use of harmonic analysis (Fourier transform of μ^{*n}).

How can we generalise?

- Generalisation to non-commutative groups: the three interwoven structures and harmonic analysis survive. Very active domain (only marginally touched in these lectures).
- Weakening of the group structure to groupoid.
- Further weakening to semi-groupoid.

Very instructive to see why standard methods fail to treat these new structures.

Monoids
Algebraic combinatorial structure

- Alphabet $\mathbb{A} = \{E, N, W, S\}$.
- Define

$$\begin{aligned} \mathbb{A}^0 &= \{()\} \\ \mathbb{A}^n &= \{a = (a_1, \ldots, a_n), a_i \in \mathbb{A}\}, n \in \mathbb{N}, \\ \mathbb{A}^* &= \cup_{n \in \mathbb{N}} \mathbb{A}^n. \end{aligned}$$

- If $a \in \mathbb{A}^*$, then $\exists n \geq 0 : a \in \mathbb{A}^n$; length $|a| = n$. The empty word $e = ()$ has length $|()| = 0$.
- If $a, b \in \mathbb{A}^*$, then $c = a \circ b \in \mathbb{A}^{|a|+|b|} \subset \mathbb{A}^*$.

$$c = a \circ b = (a_1, \ldots, a_{|a|}, b_1, \ldots, b_{|b|}) = (\circ_{i=1}^{|a|} a_i) \circ (\circ_{j=1}^{|b|} b_j).$$

- $(\mathbb{A}^*, \circ)$ where $w \circ u$ is the concatenation of words w and u is a combinatorial monoid.

Example

Example

Let $\mathbb{A} = \{E, N, W, S\}$ and $e = ()$, $u = NWSWE$, and $v = ESEEN$ be elements of $\mathbb{A}^*$. Then

$$\begin{aligned} e \circ v &= v = ()ESEEN = ESEEN, \\ v \circ e &= v = ESEEN() = ESEEN, \\ u \circ v &= NWSWE|ESEEN, \\ v \circ u &= ESEEN|NWSWE. \end{aligned}$$

Path space

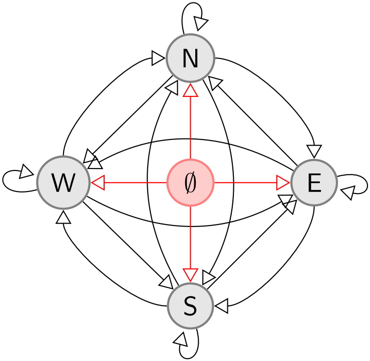
Definition

- $\mathbb{A}$ a finite alphabet.
- $\mathbb{K} := \mathbb{K}_{\mathbb{A}}$ the complete graph on $\mathbb{A}$ i.e. $\mathbb{K} = (\mathbb{K}^0, \mathbb{K}^1)$, where $\mathbb{K}^0 = \mathbb{A}$ the vertex set of $\mathbb{K}$ and $\mathbb{K}^1 = \mathbb{A} \times \mathbb{A}$ the edge set of $\mathbb{K}$.
- $\hat{\mathbb{K}} := \hat{\mathbb{K}}_{\mathbb{A}}$ the complete graph with source i.e. $\hat{\mathbb{K}} = (\hat{\mathbb{K}}^0, \hat{\mathbb{K}}^1)$, where $\hat{\mathbb{K}}^0 = \mathbb{A} \cup \{\emptyset\}$ the vertex set of $\hat{\mathbb{K}}$ and $\hat{\mathbb{K}}^1 = \mathbb{A} \times \mathbb{A} \cup \{\emptyset\} \times \mathbb{A}$ the edge set of $\hat{\mathbb{K}}$. The special vertex $\emptyset$ is the source of the graph.
- The path space $\mathsf{PS}(\hat{\mathbb{K}}_{\mathbb{A}})$ is isomorphic to $\mathbb{A}^*$.

The complete graphs $\mathbb{K}_{\mathbb{A}}$ and $\hat{\mathbb{K}}_{\mathbb{A}}$

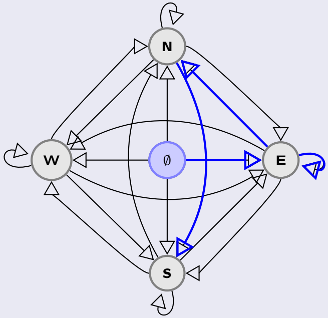
$$\mathbb{A} = \{E, N, W, S\}.$$

$\mathbb{K}_{\mathbb{A}}$
$\hat{\mathbb{K}}_{\mathbb{A}}$ $=$

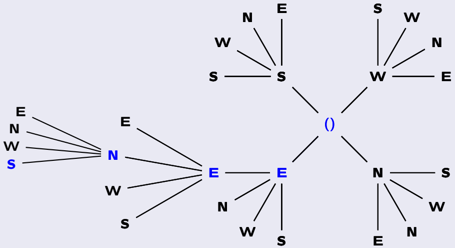


Path space tree

A path on $\hat{\mathbb{K}}_{\mathbb{A}}$



The path on $PS(\hat{\mathbb{K}}_{\mathbb{A}})$



The path on $\mathbb{A}^*$

()EENS

Groups of finite type

Algebraic geometrical structure

$$EW = WE = e, \; NS = SN = e,$$
$$E = a \Rightarrow W = a^{-1} \text{ and } N = b \Rightarrow S = b^{-1}.$$
$$\mathbb{A} = \{a, a^{-1}, b, b^{-1}\}; \; \text{RedWords}(\mathbb{A}) = \text{set of reduced words of } \mathbb{A}^*.$$

Examples

- $\Gamma = \langle \mathbb{A} \rangle$ the group $\mathbb{F}_2$ generated from RedWords($\mathbb{A}$) with relation set $\mathcal{R} = \emptyset$, hence called free.
- $\Gamma = \langle \mathbb{A} | aba^{-1}b^{-1} \rangle$ the group $\mathbb{Z}^2$ generated from RedWords($\mathbb{A}$) subject to the relation set: $\mathcal{R} = \{aba^{-1}b^{-1} = e\}$, hence Abelian.

Definition

If Γ is generated by RedWords($\mathbb{A}$) subject to the relations $\mathcal{R}$, write $\Gamma = \langle \mathbb{A} \,|\, \mathcal{R} \rangle$. $\langle \mathbb{A} \,|\, \mathcal{R} \rangle$ is a presentation of Γ. If there exists a presentation of Γ for which both $\mathbb{A}$ and $\mathcal{R}$ are finite, the group is termed of finite type.

Cayley graph

Definition

Let $\Gamma = \langle \mathbb{A} \,|\, \mathcal{R} \rangle$ be a finitely generated group. Assume $\mathbb{A}$ symmetric (i.e. $\mathbb{A}^{-1} = \mathbb{A}$) and $e \notin \mathbb{A}$. The associated Cayley graph $\mathrm{Cayley}(\Gamma, \mathbb{B}, \mathcal{R})$ is the graph $\mathbb{G} = (\mathbb{G}^0, \mathbb{G}^1)$ with

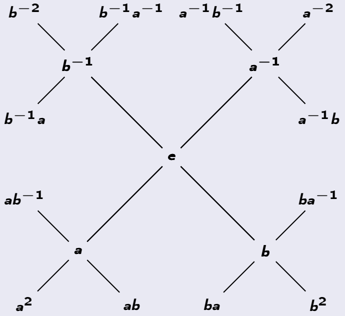
- vertex set $\mathbb{G}^0 := \Gamma$ and
- edge set $\mathbb{G}^1 := \{(x, y) \in \Gamma^2, x^{-1}y \in \mathbb{A}\}$.

Properties of the edge set $\mathbb{G}^1$:

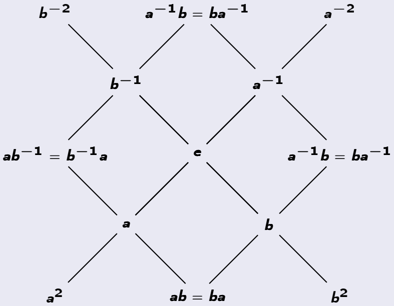
- $\mathbb{G}^1 := \{(x, y) \in \Gamma^2 : x^{-1}y = z \in \mathbb{A}\} = \{(x, xz) \in \Gamma^2 : z \in \mathbb{A}\}$. In particular, $e \notin \mathbb{A} \Rightarrow (x, x) \notin \mathbb{G}^1$ (no self-loops).
- $(x, y) \in \mathbb{G}^1 \Rightarrow x^{-1}y = z \in \mathbb{A} \Rightarrow y^{-1}x = z^{-1} \in \mathbb{A} \Rightarrow (y, x) \in \mathbb{G}^1$ i.e. $(\mathbb{G}^1)^{-1} = \mathbb{G}^1$. The graph $\mathbb{G}$ is undirected.
- $d_x^- = \mathrm{card}\{y \in \Gamma : (x, y) \in \mathbb{G}^1\} = \mathrm{card}\{(x, xz), z \in \mathbb{A}\} = \mathrm{card}\mathbb{A}$.

Examples of Cayley graphs

Cayley($\mathbb{F}_2, \mathbb{A}, \emptyset$)



Cayley($\mathbb{Z}^2, \mathbb{A}, \{ab = ba\}$)



Random jumps
Probabilistic structure

Remark

- μ *probability measure on* $\mathbb{X}$ *with* $\operatorname{supp} \mu = \mathbb{A}$.
- $(X_n)_{n \in \mathbb{N}}$ *is* $MC(\mathbb{X}, P, \cdot)$ *(of a very special type) with*

$$P(x, y) = \mathbb{P}(X_{n+1} = y | X_n = x) = \mu(x^{-1}y), x, y \in \mathbb{X}.$$

- *Graph of the stochastic matrix* $=$ *Cayley graph of the underlying group.*
- *The combinatorial, geometric, and stochastic structures are mutually adapted.*

Constrained Cayley graphs

Definition

Let $\mathbb{A}$ finite be given (generating) and $\Gamma = \langle \mathbb{A} \,|\, \mathcal{R} \rangle$. Let $c : \Gamma \times \mathbb{A} \to \{0, 1\}$ be a choice function. Define the constrained Cayley graph $\mathbb{G} = (\mathbb{G}^0, \mathbb{G}^1) = \mathrm{Cayley}_c(\Gamma, \mathbb{B}, \mathcal{R})$ by

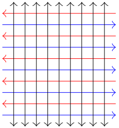
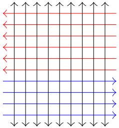
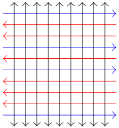
- $\mathbb{G}^0 = \Gamma$,
- $\mathbb{G}^1 = \{(x, xz) \in \Gamma^2 : z \in \mathbb{B}; c(x, z) = 1\}$.
- $d_x^- = \mathrm{card}\{y \in \Gamma : (x, y) \in \mathbb{G}^1\}$.

Properties of constrained Cayley graphs

- $0 \leq d_x^- \leq \text{card}\mathbb{A}$.
- If $d_x^- = 0$ for some x, then x is a sink. All graphs considered here have $d_x^- > 0$.
- If $d_x^- < \text{card}\mathbb{A}$ for some x then there are genuinely directed edges.
- The graph can fail to be transitive. All graphs considered here are transitive i.e. for all $x, y \in \mathbb{G}^0$, there exists a finite sequence $(x_0 = x, x_1, \ldots, x_n = y)$ with $(x_{i-1}, x_i) \in \mathbb{G}^1$ for all $i = 1, \ldots, n$.
- Algebraic structure of $\text{Cayley}_c(\Gamma, \mathbb{B}, \mathcal{R})$: a groupoid or a semi-groupoid.

Examples of semi-groupoids

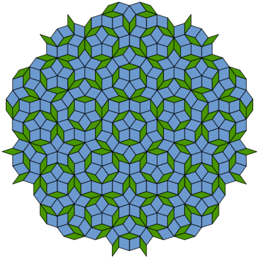
Vertex set $\mathbb{X} = \mathbb{Z}^2$, i.e. for all $x \in \mathbb{X}$, we write $x = (x_1, x_2)$; generating set $\mathbb{A} = \{\mathbf{e}_1, -\mathbf{e}_1, \mathbf{e}_2, -\mathbf{e}_2\}$.

Alternate lattice	Half-plane one-way	Random horizontal
		
$c(\mathbf{x}, \mathbf{e_2}) = c(\mathbf{x}, -\mathbf{e_2}) = 1$ $c(\mathbf{x}, \mathbf{e_1}) = 1, x_\mathbf{2} \in 2\mathbb{Z}$ $c(\mathbf{x}, -\mathbf{e_1}) = 1, x_\mathbf{2} + 1 \in 2\mathbb{Z}$	$c(\mathbf{x}, \mathbf{e_2}) = c(\mathbf{x}, -\mathbf{e_2}) = 1$ $c(\mathbf{x}, \mathbf{e_1}) = 1, x_\mathbf{2} < 0$ $c(\mathbf{x}, -\mathbf{e_1}) = 1, x_\mathbf{2} \geq 0$	$c(\mathbf{x}, \mathbf{e_2}) = c(\mathbf{x}, -\mathbf{e_2}) = 1$ $c(\mathbf{x}, \mathbf{e_1}) = \theta_{x_\mathbf{2}}$ $c(\mathbf{x}, -\mathbf{e_1}) = 1 - \theta_{x_\mathbf{2}}$

For all three lattices: $\forall x \in \mathbb{Z}^\mathbf{2}, d_\mathbf{x}^- = 3$.

Example of groupoid

- Choose integer $N \geq 2$; decompose $\mathbb{R}^N = E \oplus E^\perp$ with $\dim E = d$ and $\dim E^\perp = N - d$, $1 \leq d < N$.
- K the unit hypercube in $\mathbb{R}^N$.
- $\pi : \mathbb{R}^N \to E$ and $\pi^\perp : \mathbb{R}^N \to E^\perp$ projections.
- For generic orientation of E and $t \in E_\perp$ let $\mathcal{K}_t := \{x \in \mathbb{Z}^N : \pi^\perp(E + t) \in \ \pi^\perp(K)\}$.
- $\pi(\mathcal{K}_t)$ is a quasi-periodic tiling of $E \cong \mathbb{R}^d$ (of Penrose type).
- For generic orientations of E, points in $\mathcal{K}_t$ are in bijection with points of the tiling.
- $\mathbb{A} = \{\pm \mathbf{e}_1, \ldots, \pm \mathbf{e}_N\}$.
- $c(x, z) = \mathbb{1}_{\mathcal{K}_t \times \mathcal{K}_t}(x, x + z)$, $z \in \mathbb{A}$.



$\text{Cayley}_c(\mathbb{Z}^N, \mathbb{A})$

- $\text{Cayley}_c(\mathbb{Z}^N, \mathbb{A})$ is undirected (groupoid).
- d_x^- can be made arbitrarily large.

Results
For semi-groupoids

Theorem (Campanino and P. (2003))

The simple random walk

- *on the alternate 2-dimensional lattice is recurrent,*
- *on the half-plane one-way 2-dimensional lattice is transient,*
- *on the randomly horizontally directed 2-dimensional lattice, where $(\theta_{x_2})_{x_2 \in \mathbb{Z}}$ is an i.i.d. $\{0, 1\}$-distributed sequence of average 1/2, is transient for almost all realisations of the sequence.*

Triggered various developments: Guillotin and Le Ny (2007), Pete (2008), Pène (2009), Devulder and Pène (2011), de Loynes (2012).

Results (cont'd)

For semi-groupoids

Theorem (Campanino and P. (2012), ArXiV:1204.5297)

- $f : \mathbb{Z} \to \{0, 1\}$ *a Q-periodic function* ($Q \geq 2$): $\sum_{y=1}^{Q} f(y) = 1/2$.
- $(\rho_y)_{y \in \mathbb{Z}}$ *i.i.d. Rademacher sequence.*
- $(\lambda_y)_{y \in \mathbb{Z}}$ *i.i.d.* $\{0, 1\}$*-valued sequence such that* $\mathbb{P}(\lambda_y = 1) = \frac{c}{|y|^\beta}$ *for large* $|y|$.
- $\theta_y = (1 - \lambda_y) f(y) + \lambda_y \frac{1+\rho_y}{2}$.

1. *If* $\beta < 1$ *then the simple random walk is almost surely transient.*
2. *If* $\beta > 1$ *then the simple random walk is almost surely recurrent.*

Results

For groupoids

Theorem (de Loynes, thm 3.1.2 in PhD thesis (2012)[a])

[a] Available at http://tel.archives-ouvertes.fr/tel-00726504.

The simple random walk on (adjacent edges of) a generic Penrose tiling of the d-dimensional space is

- *recurrent, if $d \leq 2$, and*
- *transient, if $d \geq 3$.*