Some special topics in Hilbert spaces

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Projections I

V vector space (not necessarily Hilbert).

- Let M, N subspaces of V. If

$$\forall v \in V, \exists! m \in M, \exists! n \in N : v = m + n,$$

  $V = M \oplus N$. Uniqueness of decomposition if $M \cap N = \{0, \}$.

- If $V = M \oplus N$, define $P : V \to V(M)$ by

$$V \ni v = m + n \mapsto Pv = P(m + n) = m \in M.$$

  Obviously $P^2 v = Pv = m$.

Definition

Linear operator $P : V \to V$ s.t. $P^2 = P$ is a **projection (on** M**)**.

Theorem

- *If $P : V \to V$ projection, then $V = \operatorname{im} P \oplus \ker P$.*
- *If M, N subspaces s.t. $V = M \oplus N$, then exists $P : V \to V$ with $\operatorname{im} P = M, \ker P = N$.*

- If $V = \mathbb{H}$ (orthogonality) and $\mathbb{M}$ closed subspace, $\mathbb{H} = \mathbb{M} \oplus \mathbb{M}^{\perp}$.
- If P projection on $\mathbb{M}$, writing unique decompositions $h = m + n$, $h' = m' + n'$,

$$\langle\, Ph \,|\, h' \,\rangle = \langle\, m \,|\, m' + n' \,\rangle = \langle\, m \,|\, m' \,\rangle = \langle\, m + n \,|\, m \,\rangle = \langle\, h \,|\, Ph' \,\rangle,$$

i.e. $P^* = P$. Self-adjoint projection called **orthoprojection**.

Projections III

Theorem

- *If P orthoprojection on $\mathbb{H}$, then* $\mathbb{H} = \operatorname{im} P \oplus \ker P$.
- *If $\mathbb{M}$ closed subspace of $\mathbb{H}$, then exists P orthoprojection with* $\operatorname{im} P = \mathbb{M}, \ker P = \mathbb{M}^{\perp}$.

Hence bijection between closed subspaces and orthoprojections.

Remark

If $\mathbb{M}$ not closed subspace, there is still orthoprojection associated with $\mathbb{H} = \overline{\mathbb{M}} \oplus \mathbb{M}^{\perp}$.

Definition

P_1, P_2 **orthogonal** othoprojections on $\mathbb{H}$ if $\operatorname{im} P_1 \perp \operatorname{im} P_2$.

Duality I

Definition

- **Algebraic dual** $\mathbb{H}^* = \{F : \mathbb{H} \to \mathbb{C}, F \text{ linear.}\}$
- **Topological dual** $\mathbb{H}' = \{F \in \mathbb{H}^* : F \text{ continuous.}\}$

Continuous $\Leftrightarrow$ bounded $\Leftrightarrow \|F\| := \sup_{h:\|h\|=1} |F(h)| < \infty$.
$g \in \mathbb{H}$ fixed: F_g defined by $F_g(h) = \langle\, g \mid h \,\rangle$ is in $\mathbb{H}'$.

Theorem (Fréchet-Riesz)

$$\forall F \in \mathbb{H}', \exists! g \in \mathbb{H} : F(\cdot) = \langle\, g \mid \cdot \,\rangle.$$

Duality II

Remark

- Map $g \mapsto J(g) = \langle\, g \mid \cdot \,\rangle$, identifies $\mathbb{H}$ and $\mathbb{H}'$ isometrically because $\|J(g)\| = \|g\|$.
- $J(\lambda g) = \overline{\lambda} J(g)$ anti-linear.
- $\mathbb{H}, \mathbb{H}'$ are **self-dual**, isomorphic as Banach, anti-isomorphic as Hilbert.

Orthonormal systems I

- System of vectors $(e_i)_{i\in I} \in \mathbb{H}$ is
  - **orthogonal** if $i, j \in I, i \neq j \Rightarrow \langle\, e_i \mid e_j \,\rangle = 0$,
  - **orthonormal** if $\langle\, e_i \mid e_j \,\rangle = \delta_{i,j}$.

  If $I = \mathbb{N}$, system = sequence.
- If $(e_n)_{n\in\mathbb{N}}$ orhonormal sequence,
  - $\mathbb{H} \ni h \mapsto (c_n)_{n\in\mathbb{H}}$, with $c_n := c_n(h) = \langle\, e_n \mid h \,\rangle$, the sequence of **Fourier coefficients**.
  - $\sum_n c_n e_n = \sum_n \langle\, e_n \mid h \,\rangle e_n$, the **formal Fourier sequence**.

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Orthonormal systems II

Exercise

$(e_1, \ldots, e_n)$ orthonormal family in $\mathbb{H}$ ($\dim \mathbb{H} \geq n$), $(\lambda_1, \ldots, \lambda_n) \in \mathbb{C}^n$, $h \in \mathbb{H}$, $c_i(h) = \langle e_i \mid h \rangle$ for $i = 1, \ldots, n$.

$$\|h - \sum_{i=1}^{n} \lambda_i e_i\| = \|h\|^2 + \sum_{i=1}^{n} |\lambda_i - c_i(h)|^2 - \sum_{i=1}^{n} |c_i(h)|^2.$$

Remark

- Among all (λ_i), choice $\lambda_i = c_i(h)$ minimises $\operatorname{dist}(h, \mathbb{K})$, where $\mathbb{K} = \operatorname{vect}(e_1, \ldots, e_n)$.
- Vector $k = \sum_{i=1}^{n} c_i(h) e_i \in \mathbb{K}$ lies closer to h than any other vector of $\mathbb{K}$.
- In particular:
  $0 \leq \|h - k\|^2 = \|h\|^2 - \sum_{i=1}^{n} |c_i(h)|^2 \Rightarrow \sum_{i=1}^{n} |c_i(h)|^2 \leq \|h\|^2$.

Orthonormal systems III

Theorem

Let $(\epsilon_n)_n$ orthonormal sequence and (λ_n) complex sequence. Then

$$[\lim_{N\to\infty} \sum_{n=1}^{N} \lambda_n e_n = h \in \mathbb{H}] \Leftrightarrow [\sum_{n\in\mathbb{N}} |\lambda_n|^2 < \infty].$$

Corollary

$$\forall h \in \mathbb{H}, \sum_{n\in\mathbb{N}} \langle\, e_n \,|\, h \,\rangle e_n = g \in \mathbb{H}.$$

Is it true $g = h$?

Orthonormal systems IV

Theorem

Let $(e_n)_n$ orthonormal sequence in $\mathbb{H}$. The following are equivalent:

- $(e_n)_n$ *is complete (basis),*
- $\overline{\text{vect}}(e_n, n \in \mathbb{N}) = \mathbb{H}$,
- $\forall h \in \mathbb{H}, \|h\|^2 = \sum_{n\in\mathbb{N}} |\langle e_n \mid h \rangle|^2$,
- $\forall h \in \mathbb{H}, h = \sum_{n\in\mathbb{N}} \langle e_n \mid h \rangle e_n$,
- $\sum_{n\in\mathbb{N}} P_n \overset{s}{=} I_{\mathbb{H}}$, *where P_n is the orthogonal orthoprojection on $\mathbb{C}e_n$ and $\overset{s}{=}$ denotes strong convergence.*

Dirac's notation I

$\langle \cdot | \cdot \rangle$ denoted by angular bra(c)ket. Split into bra $\langle \cdot |$ (to represent vectors) and $| \cdot \rangle$ (to represent linear forms).

<table>
<tr><th>Usual notation</th><th>Dirac's notation</th></tr>
<tr><td colspan="2">Orhonormal basis</td></tr>
<tr><td>$(e_1, \ldots, e_n)$
$\psi = \sum_i \psi_i e_i$</td><td>$(| e_1 \rangle, \ldots | e_n \rangle)$
$| \psi \rangle = \sum_i \psi_i | e_i \rangle$</td></tr>
<tr><td colspan="2">$\langle \phi | \psi \rangle = \sum \overline{\phi}_i \psi_i$</td></tr>
<tr><td colspan="2">Duality $J : \mathbb{H} \to \mathbb{H}'$</td></tr>
<tr><td>$J : \phi \mapsto J(\phi)(\cdot) = \langle \phi | \cdot \rangle \in \mathbb{H}'$
$\langle \phi | \psi \rangle = J(\phi)(\psi)$</td><td>$J : | \phi \rangle \mapsto \langle \phi |$
$\langle \phi | \psi \rangle = \langle \phi || \psi \rangle$</td></tr>
<tr><td colspan="2">Self-adjoint operators $X = X^*$
$\langle \phi | X\psi \rangle = \langle X^* \phi | \psi \rangle = \langle X\phi | \psi \rangle$</td></tr>
</table>

Dirac's notation II

Usual notation	Dirac's notation
Orhonormal basis	
P_n rank 1 orthoprojection to $\mathbb{C}e_n$	$P_n = \vert e_n \rangle\langle e_n \vert$
$\psi = \psi_n e_n + \sum_{i \neq n} \psi_i e_i$	
$P_n \psi = \psi_n e_n$	$P_n \vert \psi \rangle = \vert e_n \rangle\langle e_n \vert \psi \rangle = \vert e_n \rangle \psi_n$
Spectral decomposition: $X = \sum_{x \in \text{spec}\, X} x M_x$	
$M_x =$ orhtoprojection to $\mathbb{C}\mathbf{u}_x$	$M_x = \vert \mathbf{u}_x \rangle\langle \mathbf{u}_x \vert$
$X\psi = \sum_x \langle \mathbf{u}_x \vert \psi \rangle \mathbf{u}_x$	$X \vert \psi \rangle = \sum_x \langle \mathbf{u}_x \vert \psi \rangle \vert \mathbf{u}_x \rangle$
Tensor product	
$\phi \otimes \psi$	$\vert \phi \rangle \otimes \vert \psi \rangle \equiv \vert \phi\psi \rangle$

Dirac's notation III

- If $\|\psi\| = 1$, then $\langle\, \psi \,|\, X\psi \,\rangle = \sum_{i,j} \overline{\psi}_i \psi_j \langle\, e_i \,|\, Xe_j \,\rangle$.
- Let $\rho = |\, \psi \,\rangle\langle\, \psi \,|$. Then

$$\begin{aligned}
\mathrm{tr}(\rho X) &= \sum_j \langle\, e_j \,|\, \rho X e_j \,\rangle \\
&= \sum_j \langle\, e_j \,|\, \psi \,\rangle\langle\, \psi \,|\, Xe_j \,\rangle \\
&= \sum_{i,j} \langle\, e_j \,|\, \psi \,\rangle\langle\, \psi \,|\, e_i \,\rangle\langle\, e_i \,|\, Xe_j \,\rangle \\
&= \sum_{i,j} \overline{\psi}_i \psi_j \langle\, e_i \,|\, Xe_j \,\rangle.
\end{aligned}$$

Density operators I

- Operator $\rho = |\,\psi\,\rangle\langle\,\psi\,|$ — with $\psi = \mathbb{H} \simeq \mathbb{C}^2$ — very special.
- In canonical basis, has matrix representation

$$\rho = \begin{pmatrix} |\psi_1|^2 & \psi_1\overline{\psi}_2 \\ \overline{\psi}_1\psi_2 & |\psi_2|^2 \end{pmatrix}, \text{with } \operatorname{tr}(\rho) = |\psi_1|^2 + |\psi_2|^2 = 1.$$

Diagonal elements of ρ = classical probability on $\{1, 2\}$.

- If $X = x_1|\,\epsilon_1\,\rangle\langle\,\epsilon_1\,| + x_2|\,\epsilon_2\,\rangle\langle\,\epsilon_2\,| = \begin{pmatrix} x_1 & \\ & x_2 \end{pmatrix}$ is classical observable,

$$\operatorname{tr}(\rho X) = x_1\rho_{11} + x_2\rho_{22},$$

i.e. non diagonal elements of ρ do not intervene.

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Density operators II

- For $\rho = |\,\psi\,\rangle\langle\,\psi\,|$,

$$\mathbb{E}_\psi X = \langle\,\psi\,|\,X\psi\,\rangle = x_1|\psi_1|^2 + x_2|\psi_2|^2 = \mathrm{tr}(\rho X)$$

  i.e. ρ conveys same information as pure state ψ. While $\langle\,\psi\,|\,X\psi\,\rangle$ not linear in ψ, $\mathrm{tr}(\rho X)$ linear in ρ.

- For $\psi_1, \psi_2 \in \mathbf{S}_p$, let $\rho_1 = |\,\psi_1\,\rangle\langle\,\psi_1\,|$, $\rho_2 = |\,\psi_2\,\rangle\langle\,\psi_2\,|$, $\lambda \in [0,1]$, and $\rho = \lambda\rho_1 + (1-\lambda)\rho_2$.

- For $M(B)$, $B \in \mathcal{B}(\mathbb{R})$ sharp elementary observable:

$$\begin{aligned}\pi_M^\rho(B) &= \lambda\,\mathrm{tr}(\rho_1 M(B)) + (1-\lambda)\,\mathrm{tr}(\rho_2 M(B)) \\ &= \lambda\pi_M^{\psi_1}(B) + (1-\lambda)\pi_M^{\psi_2}(B),\end{aligned}$$

  but $\mathcal{M}_1(\mathcal{B}(\mathbb{R}))$ convex. Hence $\pi_M^\rho \in \mathcal{M}_1(\mathcal{B}(\mathbb{R}))$.

- $\rho = \lambda\rho_1 + (1-\lambda)\rho_2$ cannot be written $\rho = |\,\psi\,\rangle\langle\,\psi\,|$ any longer.

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Density operators III

Definition (States)

$$\mathbf{S} = \mathcal{D}(\mathbb{H}) := \{\rho \in \mathcal{B}_h(\mathbb{H}) : \rho \geq 0, \operatorname{tr} \rho = 1\}.$$

- $\rho \geq 0 \Leftrightarrow \forall h \in \mathbb{H}, \langle\, h \,|\, \rho h \,\rangle \geq 0 \Leftrightarrow \operatorname{spec} \rho \subseteq \mathbb{R}_+ \Leftrightarrow \exists a \in \mathcal{B}(\mathbb{H}) : \rho = a^* a$.
- $\mathcal{D}(\mathbb{H})$ is convex.
- $\operatorname{tr} \rho = 1 \Rightarrow \operatorname{spec} \rho \subseteq [0, 1]$.
- $\rho^2 \leq \rho$. $\rho^2 = \rho \Leftrightarrow \operatorname{spec} \rho = \{0, 1\}$ (projectors).
- Projections are in $\operatorname{extr} \mathbf{S} = \mathbf{S}_p$.

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Tensor products (for finite dimensional spaces) I

- $\mathbb{H}$ is self-dual. Hence
  - $\forall h \in \mathbb{H}, \exists! L(h) \in \mathbb{H}'$ such that $L(h)(g) = \langle\, h \,|\, g \,\rangle$.
  - $\forall F \in \mathbb{H}', \exists! v(F) \in \mathbb{H}$ such that $F(g) = \langle\, v(F) \,|\, g \,\rangle$.
  - $v(L(h)) = h; L(v(F)) = F$.
- $\mathbb{H}'$ is also a Hilbert space, hence vector $h \in \mathbb{H}$ can be seen
  - either as equivalent to the linear form $L(h)$,
  - or as a linear form acting on $\mathbb{H}'$, i.e. $\mathbb{H} \simeq (\mathrm{LF}(\mathbb{H}))'$.
- $\mathrm{BF}(\mathbb{H}_1, \mathbb{H}_2) := \{\beta : \mathbb{H}_1 \times \mathbb{H}_2 \to \mathbb{C}, \beta \text{ bilinear}\} =$ vector space.
- $(\mathrm{BF}(\mathbb{H}_1, \mathbb{H}_2))' := \{\tau : \mathrm{BF}(\mathbb{H}_1, \mathbb{H}_2) \to \mathbb{C}, \tau \text{ linear}\}$.
- $\mathbb{H}_1 \otimes \mathbb{H}_2$ will be identified with $(\mathrm{BF}(\mathbb{H}_1, \mathbb{H}_2))'$.

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Tensor products (for finite dimensional spaces) II

- Define **simple tensor** $h_1 \otimes h_2$:

$$\forall \beta \in \mathsf{BF}(\mathbb{H}_1, \mathbb{H}_2), h_1 \otimes h_2(\beta) = \beta(h_1, h_2),$$

$$\mathbb{H}_1 \otimes \mathbb{H}_2 = \mathrm{vect}\{\tau = \sum_{i=1}^{n} \lambda_i h_{1,i} \otimes h_{2,i}\} \subseteq (\mathsf{BF}(\mathbb{H}_1 \otimes \mathbb{H}_2))'.$$

- Exercise: show that $\tau(\beta)$ is independent of representation of τ, where

$$\mathsf{BF}(\mathbb{H}_1, \mathbb{H}_2) \ni \beta \mapsto \tau(\beta) = \sum_{i=1}^{n} \lambda_i \beta(h_{1,i}, h_{2,i}).$$

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Tensor products (for finite dimensional spaces) III

Theorem

If $E \subset \mathbb{H}_1$ and $F \subset \mathbb{H}_2$ linearly independent sets of vectors then $\{e \otimes f, e \in E, f \in F\} \subset \mathbb{H}_1 \otimes \mathbb{H}_2$ is also linearly independent.

Proof.

- Let $\tau = \sum \lambda_i e_i \otimes f_i$, $e_i \in E$, $f_i \in F$.
- For arbitrary linear forms $L \in \mathbb{H}_1'$ and $M \in \mathbb{H}_2'$, consider bilinear form $\beta(e, f) = L(e)M(f)$.
- $\tau = 0 \Rightarrow \tau(\beta) = \sum \lambda_i e_i \otimes f_i(\beta) = \sum \lambda_i \beta(e_1, f_i) = M(\sum \lambda_i L(e_i) f_i) = 0$.
- True for every $M \in \mathbb{H}_2' \Rightarrow \sum \lambda_i L(e_i) f_i = 0$. But F independent set. Hence $\lambda_i L(e_i) = 0, \forall i, \forall L \in \mathbb{H}_1'$.
- E indepepent set $\Rightarrow \forall i, e_i \neq 0 \Rightarrow \forall i, \lambda_i = 0$.

□

Tensor products of operators

Corollary

If $(\epsilon_i), (\zeta_j)$ *orthonormal bases of* $\mathbb{H}_1, \mathbb{H}_2$,

$$\mathbb{H}_1 \otimes \mathbb{H}_2 \ni g = \sum_{i,j=1}^{i=d_1, j=d_2} c_{i,j} | \epsilon_i \rangle \otimes | \zeta_j \rangle = \sum_{j=1}^{d_2} | g_j \rangle \otimes | \zeta_j \rangle,$$

where $| g_j \rangle = \sum_{i=1}^{d_1} c_{i,j} | \epsilon_i \rangle$. *I.e.* $\mathbb{H}_1 \otimes \mathbb{H}_2 \simeq \mathbb{H}_1 \oplus \mathbb{H}_1 \cdots \oplus \mathbb{H}_1$ *(d_2 copies).*

Definition

If $X_i \in \mathcal{B}(\mathbb{H}_i)$, for $i = 1, 2$, then

$$(X_1 \otimes X_2)(h_1 \otimes h_2) = (X_1 h_1) \otimes (X_2 h_2)$$

(extended by linearity on $\mathbb{H}_1 \otimes \mathbb{H}_2$).

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Quantum marginals I

Definition

- Let T trace class operator on $\mathbb{H}_1 \otimes \mathbb{H}_2$. The **partial trace** (w.r.t. $\mathbb{H}_2$) is the trace class operator $\operatorname{tr}_{\mathbb{H}_2} T$ on $\mathbb{H}_1$, defined by

$$\langle h_1 \mid \operatorname{tr}_{\mathbb{H}_2} T h_1' \rangle = \sum_j \langle h_1 \otimes \epsilon_{2,j} \mid T h_1' \otimes \epsilon_{2,j} \rangle,$$

  where $(\epsilon_{2,j})$ o.n.b. of $\mathbb{H}_2$.
- Let ρ be a density operator on $\mathbb{H}_1 \otimes \mathbb{H}_2$. Its partial trace $\operatorname{tr}_{\mathbb{H}_2} \rho$ on $\mathbb{H}_1$ is called the **(quantum) marginal** of ρ.

Remark

The density matrix $\rho \in \mathcal{D}(\mathbb{H}_1 \otimes \mathbb{H}_2)$ corresponds to joint probability; the quantum marginal corresponds to the marginal on $\mathbb{H}_1$ when the second part is integrated out.

Quantum marginals II

- $\mathbb{F} = \mathbb{G} \otimes \mathbb{H}$.
- $(|\, g_i \,\rangle)_{i \in I}$, $(|\, h_j \,\rangle)_{j \in J}$ onb on $\mathbb{G}$ and $\mathbb{H}$. Family $|\, g_i h_j \,\rangle := |\, g_i \,\rangle \otimes |\, h_j \,\rangle, i \in I, j \in J$ onb of $\mathbb{F}$.
- $|\, \phi \,\rangle \in \mathbb{F}$ ray, hence $\phi = \sum_{ij} W_{ij} |\, g_i h_j \,\rangle \Rightarrow \rho := |\, \phi \,\rangle\langle\, \phi \,| = \sum_{ij;kl} \overline{W}_{ij} W_{kl} |\, g_i h_j \,\rangle\langle\, g_k h_l \,|$.
- Define $\rho_1 := \mathrm{tr}_{\mathbb{H}}\, \rho$.

$$
\begin{aligned}
\langle\, g \,|\, \rho_1 g' \,\rangle &:= \sum_{m \in J} \langle\, g h_m \,|\, \rho g' h_m \,\rangle \\
&= \sum_{m \in J} \sum_{i,k \in I} \sum_{j,l \in J} \overline{W}_{ij} W_{kl} \langle\, g h_m \,|\, g_k h_l \,\rangle\langle\, g_i h_j \,|\, g' h_m \,\rangle \\
&= \sum_{i,k \in I} (W W^*)_{ki} \langle\, g \,|\, G^{ki} g' \,\rangle, \text{ where } G^{ki} = |\, g_k \,\rangle\langle\, g_i \,|.
\end{aligned}
$$

Quantum marginals III

- $\rho_1 = WW^* \Rightarrow \langle g \mid WW^* g \rangle = \|W^* g\|^2 \geq 0$.
- $\mathrm{tr}\, \rho_1 = \sum_{i\in I} (WW^*)_{ii} = \sum_{i\in I} \sum_{j\in J} W_{ij} \overline{W}_{ij} = \sum_{(i,j)\in I\times J} |W_{ij}|^2 = \langle \phi \mid \phi \rangle = 1$.
- Hence $\rho_1 \in \mathcal{D}(\mathbb{G})$.
- $\mathbb{G} = \mathbb{H} = \mathbb{C}^2$, $\phi = \frac{1}{\sqrt{2}}(|\, \epsilon_1 \epsilon_1 \,\rangle + |\, \epsilon_2 \epsilon_2 \,\rangle)$.
- $W = \begin{pmatrix} \frac{1}{\sqrt{2}} & \\ & \frac{1}{\sqrt{2}} \end{pmatrix} = W^*$.
- $\rho_1 = \begin{pmatrix} \frac{1}{2} & \\ & \frac{1}{2} \end{pmatrix}$. Hence $\rho_1^2 < \rho_1$. As a matter of fact, ρ_1 — marginal of a pure state — maximally disordered state!
- Another manifestation of the irreducibility of quantum randomness.