Motivic height zeta functions

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1. Three counting problems

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$$H(x_0 : \cdots : x_n) = \mathrm{Max}\,(|x_i|) \text{ provided } x_k \in \mathbf{Z},\ \gcd(x_i) = 1.$$

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For k'/k an extension

$$\mathrm{Mor}_{k'}(\mathcal{C}, X, i, d) \stackrel{\mathrm{def}}{=} \{x \in \mathrm{Mor}_{k'}(\mathcal{C}, X),\ \deg((i \circ x)^* \mathcal{O}(1)) \leq d\}.$$

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(Grothendieck) $\exists$ a qu.-proj. k-variety $\mathfrak{Mor}_k(\mathcal{C}, X, i, d)$ s.t.

$$\forall k'/k, \quad Mor_{k'}(\mathcal{C}, X, i, d) = \mathfrak{Mor}_k(\mathcal{C}, X, i, d)(k')$$

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Problem

Describe the "asymptotic behaviour" of the moduli space $\mathfrak{Mor}_k(\mathcal{C}, X, i, d)$ when $d \to +\infty$.

Batyrev-Manin's program

From now on, we assume that :

- X is smooth and geometrically integral
- ω_X^{-1} is very ample
- $i : X \hookrightarrow \mathbf{P}^n_k$ is an anticanonical embedding $(i^*(\mathcal{O}(1)) = \omega_X^{-1})$
- In the arithmetic setting : $X(\mathbf{Q})$ is Zariski dense
- In both geometric settings : $X(k(\mathcal{C}))$ is Zariski dense

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Batyrev-Manin's program aims to precise (and solve !) the counting problem in the arithmetic and finite geometric settings.

An empirical formula in the arithmetic setting

$$n_{H_i}(d) \underset{d \to \infty}{\sim} C_{\text{P-B-T}}\, d \log(d)^{\mathrm{rk}(\mathrm{NS}(X))-1} \quad ?$$

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- $C_{\text{P-B-T}}$: a constant depending on X (Peyre, Batyrev-Tschinkel)
- In fact, we must often restrict the counting to a strict open Zariski subset of X in order to avoid accumulating subvarieties (e.g. exceptional divisors on del Pezzo surfaces).

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$$n_{H_i}(d) \underset{d\to\infty}{\sim} C_{\text{P-B-T}}\, d \log(d)^{\mathrm{rk}(\mathrm{NS}(X))-1} \quad ?$$

- Holds for large classes of varieties equipped with an algebraic group action, some complete intersections, some del Pezzo surfaces (work of Batyrev, Browning, de la Bretèche, Chambert-Loir, Manin, Peyre, Salberger, Tschinkel and many others, cf. Y. Tschinkel's lecture).

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- Still open for smooth cubic surfaces.
- Not true in general (counter-example by Batyrev and Tschinkel).

An empirical formula in the finite geometric case

$$\overline{\lim_{d\to+\infty}} \frac{n_{h_i}(d)}{(\#k)^d \, d^{\mathrm{rk}(\mathrm{NS}(X))-1}} = C_{\text{P-B-T}} \quad ?$$

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- Holds for smooth projective toric varieties (B.), generalized flag varieties (Peyre).

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- Holds for smooth projective toric varieties (B.), generalized flag varieties (Peyre).
- Batyrev and Tschinkel's counterexample still works in this setting.

A rough geometric analog of Batyrev-Manin's formula

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$\rho(V) = \#\{$ irreducible components of maximal dimension of $V\}$.

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This leads to :

An analog of B-M's empirical formula in the geometric setting

1. $\dim(\mathfrak{Mor}_k(\mathcal{C}, X, i, d)) - d$ bounded ?
2. $$\overline{\lim_{d\to+\infty}} \frac{\log \rho(\mathfrak{Mor}_k(\mathcal{C}, X, i, d))}{\log(d)} = \mathrm{rk}(\mathrm{NS}(X)) - 1 \quad ?$$

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- Holds for split toric varieties and split generalized flag varieties.

2. The motivic height zeta function

Classical height zeta functions

- In the arithmetic setting

$$\zeta_{X,H_i}(s) = \sum_{x \in X(\mathbf{Q})} H_i(x)^{-s}, \quad s \in \mathbf{C}$$

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$$\zeta_{X,h_i}(s) = Z_{X,h_i}(q^{-s}), \quad s \in \mathbf{C},$$

$$\text{where} \quad Z_{X,h_i}(T) = \sum_{x \in X(k(\mathcal{C}))} T^{h_i(x)} \quad \in \mathbf{Z}[[T]]$$

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analytical behaviour of the height ZF $\overset{\text{tauberian statements}}{\Longrightarrow}$ asymptotics for points of bounded height

The Grothendieck ring of varieties

Notation : $K_0(\mathrm{Var}_k)$

Generators : $[V]$, V a k-variety

Relations :

- $[V] = [V']$ if $V \xrightarrow{\sim} V'$,
- $[V] = [F] + [V \setminus F]$ if $F \hookrightarrow V$ is a closed immersion

Ring structure : $[V].[V'] \overset{\mathrm{def}}{=} [V \times V']$

The Grothendieck ring of motives

Notation : $K_0(\mathrm{ChMot}_k)$

Generators : $[M]$, M a Chow motive over k

Relations :

- $[M] = [M']$ if $M \xrightarrow{\sim} M'$,
- $[M] = [M'] + [M'']$ if $M = M' \oplus M''$

Ring structure : $[M].[M'] \stackrel{\text{def}}{=} [M \otimes M']$.

If k is finite, there is a ring morphism

$$\chi_{\#_k} : K_0(\mathrm{Var}_k) \longrightarrow \mathbf{Z}$$

such that for every k-variety V

$$\chi_{\#_k}([V]) = \# V(k).$$

Theorem (Gillet-Soulé, Guillen-Navarro Aznar, Bittner)

If $\mathrm{char}(k) = 0$, there is a unique ring morphism

$$\chi_{\mathrm{mot}} : K_0(\mathrm{Var}_k) \longrightarrow K_0(\mathrm{ChMot}_k)$$

such that for V smooth projective

$$\chi_{\mathrm{mot}}([V]) = \text{the class of the Chow motive of } V$$

For V a k-variety, let us denote $\chi_{\mathrm{mot}}([V])$ by $[V]$.

Definition of the motivic height zeta funcion

$$Z^{\mathrm{mot}}_{X,h_i}(T) = \sum_{d\geq 0} \left[\widetilde{\mathfrak{Mor}}_k(\mathcal{C}, X, i, d)\right] T^d$$

$$\in \begin{cases} K_0(\mathrm{Var}_k)[[T]] \\ \text{or} \\ K_0(\mathrm{ChMot}_k)[[T]] \text{ if } \mathrm{char}(k) = 0 \end{cases}$$

where

$$\widetilde{\mathfrak{Mor}}_k(\mathcal{C}, X, i, d) \stackrel{\mathrm{def}}{=} \mathfrak{Mor}_k(\mathcal{C}, X, i, d) \setminus \mathfrak{Mor}_k(\mathcal{C}, X, i, d-1)$$

parametrizes the morphisms of i-degree d.

The motivic height ZF specializes to the classical height ZF

$$Z^{\mathrm{mot}}_{X,h_i}(T) = \sum_{d\geq 0} \left[\widetilde{\mathfrak{Mor}}_k(\mathcal{C}, X, i, d)\right] T^d \quad \in K_0(\mathrm{Var}_k)[[T]]$$

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If k is finite,

$$\begin{aligned}\chi_{\#_k}\left(Z^{\mathrm{mot}}_{X,h_i}(T)\right) &= \sum_{d\geq 0} \#\widetilde{\mathfrak{Mor}}_k(\mathcal{C}, X, i, d)(k) T^d \quad \in \mathbf{Z}[[T]] \\ &= \sum_{d\geq 0} \#\{x \in X(k(\mathcal{C})), \quad h_i(x) = d\} T^d \\ &= Z_{X,h_i}(T).\end{aligned}$$

3. The “main term” of the motivic height zeta function

The main term of the classical height zeta function
(arithmetic or finite geometric setting)

Standard tauberian statements lead to

An analytic version of Batyrev-Manin's empirical formula

$$\lim_{s\to 1}(s-1)^{\mathrm{rk}(\mathrm{NS}(X))}\zeta_{H_i}(s)=\frac{C_{\text{P-B-T}}}{(\mathrm{rk}(\mathrm{NS}(X))-1)!}\quad ?$$

Different versions of the Tamagawa number

We are now going to :

1. describe (the interesting part of) the constant $C_{\text{P-B-T}}$
2. define a motivic analog of it
3. study a motivic analog of BM's analytic formula

From now on, we assume that :

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1. In the arithmetic setting, X has a smooth model $\mathcal{X}$ over **Z**.
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Up to "easy" terms not discussed here, $C_{\text{P-B-T}}$ is the Tamagawa number $\tau(X)$ defined by :

The Tamagawa number of X

- In the arithmetic setting :

$$\tau(X) = \prod_{p \text{ prime}} (1 - p^{-1})^{\mathrm{rk}(\mathrm{NS}(X))} \frac{\#\mathcal{X}(\mathbf{F}_p)}{p^{\dim(X)}}.$$

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- In the finite geometric setting

$$\tau(X) = \prod_{x \text{ closed point of } \mathcal{C}} (1 - \#\kappa_x^{-1})^{\mathrm{rk}(\mathrm{NS}(X))} \frac{\#X(\kappa_x)}{(\#\kappa_x)^{\dim(X)}}.$$

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- In the geometric setting

$$\tau(X) = \;??$$

We need a notion of “eulerian motivic product”.

Motivic eulerian product

Notation :

$$\mathcal{M} = \begin{cases} K_0(\mathrm{Var}_k) \\ \text{or} \\ \chi_{\mathrm{mot}}(K_0(\mathrm{Var}_k)) \subset K_0(\mathrm{ChMot}_k) \text{ if } \mathrm{char}(k) = 0 \end{cases}$$

$$\mathbf{L} = [\mathbf{A}^1]$$

Kapranov zeta function

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Definition (Kapranov)

V a k-variety

$$Z_{V,\mathrm{Kap}}(T) = \sum_{n \geq 0} [\mathrm{Sym}^n(X)]\ T^n \in \mathcal{M}[[T]]$$

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If k is finite,

$$\chi_{\#_k}(Z_{V,\mathrm{Kap}}(T)) = \text{the usual Hasse-Weil zeta function } Z_{V,\mathrm{HW}}(T).$$

Hasse-Weil zeta functions

If $k = \mathbf{F}_q$, recall that

$$Z_{V,\mathrm{HW}}(T) = \exp\left(\sum_{d \geq 1} \# V\left(\mathbf{F}_{q^d}\right) \frac{T^d}{d}\right) = \prod_{d \geq 1} \left(1 - T^d\right)^{-\# V_{0,d}}$$

where

$$V_{0,d} = \{\text{irreducibles rational zero-cycles of degree } d \text{ on } V\}$$

For $d \geq 1$, define $\Phi_d(V) \in \mathcal{M}$ by

$$\sum_{d \geq 1} \Phi_d(V)\, T^d = T \frac{d}{dT} \log Z_{\mathrm{V,Kap}}(T)$$

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Examples : $\Phi_d(\mathbf{A}^1) = \mathbf{L}^d$, $\Psi_1(V) = [V]$

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$$\Phi_d(V) = \sum_{e|d} e\,\Psi_e(V).$$

If $k = \mathbf{F}_q$,

$$\chi_{\#k}\left(\Phi_d(V)\right) = \#V(\mathbf{F}_{q^d})$$

and

$$\chi_{\#k}\left(\Psi_d(V)\right) = \#V_{0,d}$$

We have (in $\mathcal{M} \otimes \mathbf{Q}[[T]]$)

$$Z_{V,\mathrm{Kap}}(T) = \exp\left(\sum_{d\geq 1} \Phi_d(V)\,\frac{T^d}{d}\right) = \prod_{d\geq 1}\left(1 - T^d\right)^{-\Psi_d(V)}$$

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where, for $M \in \mathcal{M} \otimes \mathbf{Q}$ and $P \in \mathcal{M} \otimes \mathbf{Q}[[T]]$

$$(1 + T\,P(T))^M$$

stands for

$$1 + M\,T\,P(T) + \frac{M\,(M-1)}{2}\,T^2\,P(T)^2 + \dots \quad \in \mathcal{M} \otimes \mathbf{Q}[[T]]$$

The relation

$$Z_{V,\mathrm{Kap}}(T) = \prod_{d \geq 1} \left(1 - T^d\right)^{-\Psi_d(V)}$$

may be viewed as a decomposition into a eulerian motivic product.

A motivic Tamagawa number

Recall that in the finite geometric setting ($k = \mathbf{F}_q$)

$$\tau(X) = \prod_{x \in \mathcal{C}^{(0)}} (1 - \#\kappa_x^{-1})^{\mathrm{rk}(\mathrm{NS}(X))} \frac{\#X(\kappa_x)}{(\#\kappa_x)^{\dim(X)}}.$$

$$= \prod_{d \geq 1} \left[(1 - q^{-d})^{\mathrm{rk}(\mathrm{NS}(X))} \frac{\#X(\mathbf{F}_{q^d})}{q^{d \dim(X)}} \right]^{\#\mathcal{C}_{0,d}}$$

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which in turn suggests to define in the geometric setting

$$\tau_{\mathrm{mot}}(X) = \prod_{d \geq 1} \left((1 - \mathbf{L}^{-d})^{\mathrm{rk}(\mathrm{NS}(X))} \frac{\Phi_d(X)}{\mathbf{L}^{d \dim(X)}} \right)^{\Psi_d(\mathcal{C})}$$

4. The motivic Tamagawa number

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Two questions :

1. Is it possible to give a meaning to the expression above, i.e. is $\tau_{\mathrm{mot}}(X)$ well defined ?

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1. Is it possible to give a meaning to the expression above, i.e. is $\tau_{\mathrm{mot}}(X)$ well defined ?
2. Does the geometric analog of B-M's analytic empirical formula

$$\left[(1 - \mathbf{L}\, T)^{\mathrm{rk}(\mathrm{NS}(X))} \; Z^{\mathrm{mot}}_{h_i, X}(T) \right] (\mathbf{L}^{-1}) = \tau_{\mathrm{mot}}(X)$$

hold ?

First question : is $\tau_{\mathrm{mot}}(X)$ well defined ?

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$$\prod_{d\geq 1}\left((1-\mathbf{L}^{-d})^{\mathrm{rk}(\mathrm{NS}(X))}\,\frac{\Phi_d(X)}{\mathbf{L}^{d\,\dim(X)}}\right)^{\Psi_d(\mathcal{C})}\qquad ?$$

We need to complete $\mathcal{M}$ with respect to a filtration : for example the one introduced by Kontsevich for the theory of motivic integration.

Kontsevich's topology

Notations :

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$$\text{Recall} \quad \mathbf{L} = [\mathbf{A}^1_k]$$

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$$\mathcal{M}_{\mathrm{loc}} = \begin{cases} K_0(\mathrm{Var}_k)\left[\mathbf{L}^{-1}\right] \\ \text{or} \\ \chi_{\mathrm{mot}}\left(K_0(\mathrm{Var}_k\left[\mathbf{L}^{-1}\right])\right) \subset K_0(\mathrm{ChMot}_k) \text{ if } \mathrm{char}(k) = 0 \end{cases}$$

Kontsevich's topology

For $d \in \mathbf{Z}$,

$$\mathcal{F}^d\mathcal{M}_{\mathrm{loc}} = \langle\ \mathbf{L}^{-i}[V], \quad V \text{ a variety }, \quad i - \dim(V) \geq d\ \rangle$$

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$$\widehat{\mathcal{M}} = \varprojlim \mathcal{M}_{\mathrm{loc}}/\mathcal{F}^d\mathcal{M}_{\mathrm{loc}}.$$

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Example :

$$\lim_{d \to +\infty} \mathbf{L}^{-d} = 0$$

The convergence of $\tau_{\mathrm{mot}}(X)$

When $k = \mathbf{F}_q$, the convergence of the eulerian product defining $\tau(X)$ follows from the asymptotic (consequence of Weil-Deligne)

$$\#X(\mathbf{F}_{q^d}) = q^{d\,\dim X} + \mathrm{rk}(\mathrm{NS}(X))\, q^{d\,(\dim X - 1)} + \underset{d\to\infty}{\mathcal{O}}\left(q^{d\,(\dim X - \frac{3}{2})}\right)$$

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Question

Does the motivic analog

$$\Phi_d(X) - \mathbf{L}^{d\,\dim(X)} - \mathrm{rk}(\mathrm{NS}(X))\,\mathbf{L}^{d\,(\dim(X)-1)} \in \mathcal{F}^{d(\frac{3}{2}-\dim X)}\mathcal{M},$$

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Affirmative answer $\Rightarrow$ $\tau_{\mathrm{mot}}(X)$ is well defined in $\widehat{\mathcal{M}} \otimes \mathbf{Q}$.

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This holds (and therefore $\tau_{\mathrm{mot}}(X)$ is well defined in $\widehat{\mathcal{M}}_{\mathbf{Q}}$) when

1. X is a split toric variety
2. X is a split generalized flag variety.

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Sketch of proof : use the cellular decomposition and the fact that $\Phi_d(\mathbf{A}^n) = \mathbf{L}^{n\,d}$.

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For more general X, I don't know.

Let

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$$\mathcal{F}^{d}_{\mathrm{Poinc}}\mathcal{M}_{\mathrm{loc}} = \{M \in \mathcal{M}_{\mathrm{loc}}, \quad \deg(\mathrm{Poinc}(M)) \leq -d\}$$

and

$$\widehat{\mathcal{M}}_{\mathrm{Poinc}} = \varprojlim \mathcal{M}_{\mathrm{loc}}/\mathcal{F}^{d}_{\mathrm{Poinc}}\mathcal{M}_{\mathrm{loc}}$$

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For general X, I don't know, however we have

Proposition

$$\Phi_d(X) - \mathbf{L}^{d\,\dim(X)} - \mathrm{rk}(\mathrm{NS}(X))\,\mathbf{L}^{d\,(\dim(X)-1)} \in \mathcal{F}^{d\,(3-2\,\dim X)}_{\mathrm{Poinc}}\mathcal{M}$$

Therefore $\tau_{\mathrm{mot}}(X)$ is well defined in $\widehat{\mathcal{M}}_{\mathrm{Poinc}} \otimes \mathbf{Q}$.

Proof : Observe that the MacDonald formula

$$\sum_{d\geq 0} \operatorname{Poinc}\left(\left[\operatorname{Sym}^d X\right]\right) T^d = \frac{\prod_{i \text{ odd}} (1+t^i T)^{b_i(X)}}{\prod_{i \text{ even}} (1-t^i T)^{b_i(X)}}$$

allows to compute explicitely $\operatorname{Poinc}(\Phi_d(X))$, and use the fact that under our assumptions

$$b_{2\dim X-1}(X) = 0$$

and

$$b_{2\dim X-2}(X) = \operatorname{rk}(\operatorname{NS}(X)).$$

Second question : does the motivic analog of B-M's analytic empirical formula hold ?

Does the series

$$(1-\mathbf{L}\,T)^{\mathrm{rk}(\mathrm{NS}(X))}\; Z^{\mathrm{mot}}_{X,h_i}(T)$$

converge in $\widehat{\mathcal{M}} \otimes \mathbf{Q}$ (or $\widehat{\mathcal{M}}_{\mathrm{Poinc}} \otimes \mathbf{Q}$) at $T = \mathbf{L}^{-1}$ to $\tau_{\mathrm{mot}}(X)$?

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Theorem (announced by Peyre)

This holds for a split flag variety.

Does the series $(1 - \mathbf{L}\, T)^{\mathrm{rk}(\mathrm{NS}(X))}\ Z^{\mathrm{mot}}_{X,h_i}(T)$ converge in $\widehat{\mathcal{M}} \otimes \mathbf{Q}$ (or $\widehat{\mathcal{M}}_{\mathrm{Poinc}} \otimes \mathbf{Q}$) at $T = \mathbf{L}^{-1}$ to $\tau_{\mathrm{mot}}(X)$?

Theorem (announced by Peyre)

This holds for a split flag variety.

- The proof relies on results by Kapranov on motivic Eisenstein series.
- In this case, $Z^{\mathrm{mot}}_{X,h_i}(T)$ is rational.
- In this case, the notion of eulerian motivic product is not really needed to define $\tau_{\mathrm{mot}}(X)$.

Does the series $(1-\mathbf{L}\,T)^{\mathrm{rk}(\mathrm{NS}(X))}\, Z^{\mathrm{mot}}_{X,h_i}(T)$ converge in $\widehat{\mathcal{M}}\otimes\mathbf{Q}$ (or $\widehat{\mathcal{M}}_{\mathrm{Poinc}}\otimes\mathbf{Q}$) at $T=\mathbf{L}^{-1}$ to $\tau_{\mathrm{mot}}(X)$?

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If $\mathrm{char}(k)=0$ and $\mathcal{C}=\mathbf{P}^1$ this holds for split toric varieties.

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Ingredients of the proof :

1. Cox's homogeneous coordinates on toric varieties, allowing a good description of the moduli space $\mathfrak{Mor}_k(\mathbf{P}^1, X, i, d)$
2. a "motivic counting" argument, relying on Denef and Loeser's construction associating a virtual motive to a first order ring formula ; this allows to compute the main part at $T = \mathbf{L}^{-1}$ of the height ZF.

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For the second point, we have to reinterprete $\Psi_d(X)$ as the virtual motive associated to a first order logic ring formula.

The construction of Denef and Loeser

For the sake of simplicity, assume $k = \mathbf{Q}$.

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To a first order ring formula φ in coefficients in $\mathbf{Q}$, such as

$$\varphi_0 \, : \, \exists y, \quad x = y^2 \wedge x \neq 0$$

is canonically associated a virtual motive $\chi(\varphi) \in \mathcal{M} \otimes \mathbf{Q}$ which counts, for every k finite with $\mathrm{char}(k) >> 0$ the number of points in $\varphi(k)$.

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$$\text{Example : } \varphi_0(k) = \{x \in k^*, \quad \exists y \in k, \quad x = y^2\}$$

$$\#\varphi_0(k) = \frac{k-1}{2} \text{ for } \mathrm{char}(k) > 2$$

$$\chi(\varphi_0) = \frac{[\mathbf{G}_m]}{2} = \frac{\mathbf{L}-1}{2}$$

An alternative definition of $\Psi_d(V)$

$$\left(V^d\right)^0 = \{(x_i) \in V^d, \quad x_i \neq x_j \text{ if } i \neq j\}$$

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$$\left(V^d\right)^0 / \mathfrak{S}_d = \left(\mathrm{Sym}^d\, V\right)^0 \subset \mathrm{Sym}^d\, V$$

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$$\left(V^d\right)^0 / \mathfrak{S}_d = \left(\mathrm{Sym}^d\, V\right)^0 \subset \mathrm{Sym}^d\, V$$

$\forall k$ finite, $\mathrm{char}(k) >> 0$, there is a natural bijection

$$\{x \in \left(\mathrm{Sym}^d\, V\right)^0(k), \quad \mathrm{Dec}(x) = \langle\ \sigma\ \rangle, \quad \sigma \text{ a } d\text{-cycle of } \mathfrak{S}_d\}$$

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The set at the top may be viewed as the set of k-points of a first order ring formula $\psi_d(V)$ with coefficients in $\mathbf{Q}$.

Proposition

The virtual motive of this formula coincides with $\Psi_d(V)$.

Proof : One has to show that, in $\mathcal{M} \otimes \mathbf{Q}[[T]]$

$$Z_{V,\mathrm{Kap}}(T) = \prod_{d \geq 1} \left(1 - T^d\right)^{-\chi(\psi_d(V))}$$

This is achieved by "motivic counting".

Proof : One has to show that, in $\mathcal{M} \otimes \mathbf{Q}[[T]]$

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For example, the equality for the T^2-coefficient reads

$$\left[\mathrm{Sym}^2(V)\right] = \frac{1}{2}([V]^2 - [V]) + [V] + \chi(\psi_2(V))$$

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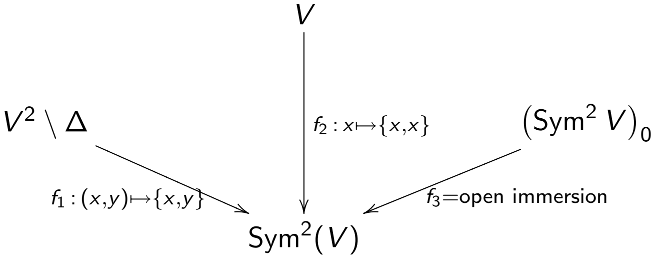
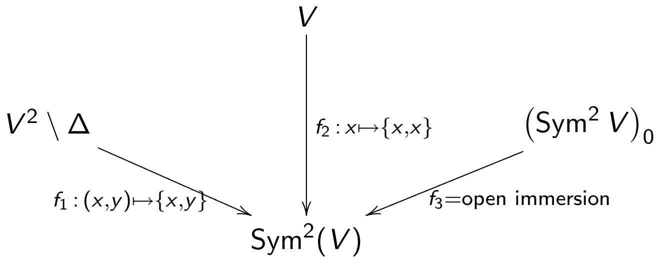
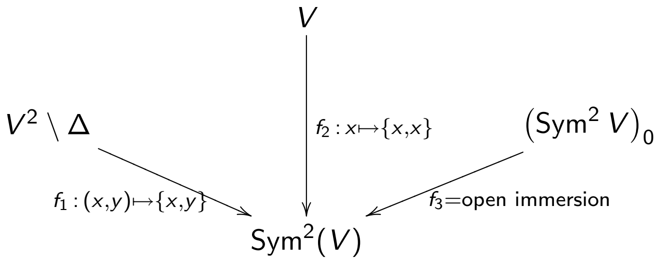




$\forall k$ finite, $\mathrm{char}(k) >> 0$,

$$\mathrm{Sym}^2(V)(k) = f_1(X^2 \setminus \Delta(k)) \bigsqcup f_2(X(k)) \bigsqcup f_3(\psi_2(V)(k))$$



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These facts imply

$$\left[\mathrm{Sym}^2(V)\right] = \frac{1}{2}([V]^2 - [\Delta_V]) + [V] + \chi(\psi_2(V)) \quad \text{Q.E.D.}$$

Similar counting arguments are used to compute the main part at $T = \mathbf{L}^{-1}$ of the motivic height zeta function of a split toric variety.