MODULI SPACES OF CURVES AND COX RINGS

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1. Introduction

Let k be a field, X be a smooth projective geometrically irreducible k-variety and $\mathscr{C}$ be a smooth projective geometrically irreducible k-curve of genus $g_{\mathscr{C}}$. Let $\mathscr{K}_X$ be the canonical class of X. For every element y of the dual $\mathrm{NS}(X)^\vee$ of the Néron-Severi group of X let $\mathbf{Mor}(\mathscr{C}, X, y)$ denote the quasi-projective k-variety parametrizing the morphisms $f : \mathscr{C} \to X$ such that $[f_*\mathscr{C}] = y$. By [Deb01, §2.11], every irreducible component of $\mathbf{Mor}(\mathscr{C}, X, y)$ has dimension at least $(1 - g_{\mathscr{C}}) \dim(X) + \langle y\,,\, -\mathscr{K}_X\rangle$. The latter quantity will be refered to as the *expected dimension* of $\mathbf{Mor}(\mathscr{C}, X, y)$. It is a natural though difficult question to ask for the number and the dimension of the irreducible components of $\mathbf{Mor}(\mathscr{C}, X, y)$. Works addressing it for specific families of varieties include: [Cas04], [CS09], [dJS04], [HRS04], [HRS05], [KLO07], [KP01], [Per02], and [Tho98].

In this article, we study the question using the so-called Cox ring of X, restricting ourselves to a class of varieties whose Cox ring has an especially simple presentation. It is known, at least when $\mathscr{C}$ is rational, that the Cox ring of a toric variety X provides a useful description of the moduli spaces $\mathbf{Mor}(\mathscr{C}, X, y)$ ([Gue95, Bat02, Bou09b]). Toric varieties may be characterized by the fact that their Cox ring is a polynomial ring, hence are the simplest varieties from the viewpoint of the description of the Cox ring.

Here we consider varieties whose Cox ring may be presented by only one equation, which has moreover a kind of linearity property with respect to a certain subset of variables (see definitions 2.1 for more precision). We will call such varieties *linear intrinsic hypersurfaces* (the terminology *intrinsic hypersurface* is borrowed from [BH07]). Let $\mathbf{Mor}(\mathscr{C}, X, y)^\circ$ denote the open set of $\mathbf{Mor}(\mathscr{C}, X, y)$ consisting of those morphisms which do not factor through the boundary, that is, the union of the divisors of the sections used to present the Cox ring.

Our main result reads as follows (see theorem 2.4 for a more precise statement).

Theorem 1.1. *Let X be smooth projective $\mathbf{Q}$-variety which is a linear intrinsic hypersurface. Assume that certain rational combinatoric series derived from the equation of the Cox ring fulfills some explicit analytic properties. Let $\mathscr{C}$ be a smooth projective geometrically irreducible $\mathbf{Q}$-curve. For every $y \in \mathrm{NS}(X)^\vee$ lying in an explicit truncation of an explicit subcone of the dual of the effective cone, $\boldsymbol{Mor}(\mathscr{C}, X, y)^\circ$ is irreducible of the expected dimension and dense in $\boldsymbol{Mor}(\mathscr{C}, X, y)$.*

By *explicit* we mean explicit in terms of the data describing the Cox ring and the genus of the curve. A *truncation* of a polyedral cone $\mathcal{C}$ is a sub-polyhedron of $\mathcal{C}$ defined by a finite number of affine inequalities $\langle x\,,\,.\rangle \geqslant a$ where x lies in the dual of $\mathcal{C}$ and a is nonnegative. Let us stress that the needed properties of the combinatoric series alluded to in the above statement can be checked by a computer algebra system, once we have at our disposal an effective presentation of the Cox ring of X.

The basic strategy of the proof will be to count the number of points of the reduction of $\mathbf{Mor}(\mathscr{C}, X, y)$ modulo primes p with values in $\mathbf{F}_p$-extensions of large degree. We are thus reduced to a situation akin to the one encountered in the context of Manin's conjecture about the asymptotical behaviour of curves of bounded degree, and we apply technics similar to the ones used in [Bou03, Bou09a, Bou11]. The main difference is that in the present situation we fix the degree y and look at the asymptotic behaviour of the number of points with value in $\mathbf{F}_p$-extension of large degree whereas in the context of Manin's conjecture the $\mathbf{F}_p$-extension is fixed and the degree y becomes large. Our varieties are assumed to be defined over $\mathbf{Q}$ for the sake of simplicity and because all our examples of applications are, but by standard arguments the strategy could be applied over any field.

By the same method one can show the following theorem about toric varieties. We do not include the proof here, since it is really strongly similar to the one used in the case of linear intrinsic hypersurface, as well as technically easier, thanks to the fact that in the toric case the Cox ring has "no equation".

Theorem 1.2. *Let X be a smooth projective split toric variety. Let $\mathscr{C}$ be a smooth projective geometrically irreducible $\mathbf{Q}$-curve. For every y lying in an explicit truncation of the dual of the effective cone, $\mathbf{Mor}(\mathscr{C}, X, y)^\circ$ (the open set parametrizing those morphisms which do not factor through the complement of the open orbit) is irreducible of the expected dimension and dense in $\mathbf{Mor}(\mathscr{C}, X, y)$.*

In case $\mathscr{C} = \mathbf{P}^1$, the fact that $\mathbf{Mor}(\mathscr{C}, X, y)^\circ$ is irreducible of the expected dimension was proven in [Bou09b].

At the end of the article, we will give examples of linear intrinsic hypersurfaces for which theorem 1.1 applies for a "positive proportion" of y (see remark 2.6). One family of examples is drawn from [Bou11], and the other from Derenthal's list [Der06] of minimal resolution of singular del Pezzo surfaces whose Cox ring is presented by one equation. We will compare our results with those of [KLO07], whose authors deal with the case of blow-ups of projective spaces.

Here is an outline of the article. In the next section, after having introduced and defined the necessary objects, we state a more explicit version of our main result (theorem 2.4). In section 3, we recall some well-known facts about the connection between dimension and number of points in the reduction modulo a prime p. In section 4, we recall from [Bou09a] the expression of the number of points of $\mathbf{Mor}(\mathscr{C}, X, y)$ over a finite field in terms of a presentation of the Cox ring. Section 5 is devoted to some technical lemmas. In section 6, we prove the main theorem. In section 7, we give examples of applications.

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2. Statement of the result

Definitions and notations 2.1. Let X be a smooth projective $\mathbf{Q}$-variety whose geometric Picard group is free of finite rank with a trivial Galois action. Let $\mathrm{Eff}(X)$ be the effective cone of X. Assume moreover that X is a Mori dream space, that is, the Cox ring $\mathrm{Cox}(X)$ of X (*cf. e.g.* [Has04]) is finitely generated.

Let $(s_i)_{i\in\mathfrak{I}}$ be a finite family of non constant global sections generating the Cox ring. Let $\mathscr{E}_i$ be the divisor of s_i. The divisors $\{\mathscr{E}_i\}_{i\in\mathfrak{I}}$ span $\mathrm{Eff}(X)$. Later on, when considering a Mori dream space X, we shall always assume that such a family of sections has been chosen.

Let $\mathscr{C}$ be a smooth projective geometrically irreducible $\mathbf{Q}$-curve and $y \in \mathrm{NS}(X)^\vee$. We define a partition of $\mathbf{Mor}(\mathscr{C}, X, y)$ into locally closed subsets $\{\mathbf{Mor}(\mathscr{C}, X, y, \mathfrak{I}_*)\}_{\mathfrak{I}_*\subset\mathfrak{I}}$ as follows: a morphism φ lies in $\mathbf{Mor}(\mathscr{C}, X, y, \mathfrak{I}_*)$ if and only if $\mathfrak{I}_*$ is the set of indices $i \in \mathfrak{I}$ such that φ does not factor through $\mathscr{E}_i$. Thus $\mathbf{Mor}(\mathscr{C}, X, y, \mathfrak{I})$ is the open subset $\mathbf{Mor}(\mathscr{C}, X, y)^\circ$ of $\mathbf{Mor}(\mathscr{C}, X, y)$ parametrizing those morphisms $\mathscr{C} \to X$ which do not factor through the boundary $\underset{i\in\mathfrak{I}}{\cup} \mathscr{E}_i$.

Let $\mathscr{I}_{X,\mathfrak{I}}$ denote the kernel of the morphism $k[x_i]_{i\in\mathfrak{I}} \to \mathrm{Cox}(X)$ which maps x_i to s_i. It is homogeneous with respect to the natural $\mathrm{Pic}(X)$-grading on $k[x_i]$. Let $\mathscr{I}^{\mathrm{hom}}_{X,\mathfrak{I}}$ denote the set of the homogeneous elements of $\mathscr{I}_{X,\mathfrak{I}}$.

Definition 2.2. We retain notations 2.1. A Mori dream space X is said to be an *intrinsic hypersurface* if $\mathscr{I}_{X,\mathfrak{I}}$ is principal. Let I be a subset of $\mathfrak{I}$. The Mori dream space X is said to be a *linear intrinsic hypersurface* with respect to the pair $(\mathfrak{I}, I)$ if the classes of $\{\mathscr{E}_i\}_{i\in I}$ form a basis of $\mathrm{Pic}(X)$ and $\mathscr{I}_{X,\mathfrak{I}}$ is principal, with generator a linear form with respect to the variables $\{x_i\}_{i\notin I}$, whose coefficients are pairwise coprime monomials in the $\{x_i\}_{i\in I}$. Thus a generator of $\mathscr{I}_{X,\mathfrak{I}}$ may be written as

$$\sum_{j\in\mathfrak{I}\setminus I} \alpha_j \, x_j \prod_{i\in I} x_i^{b_{i,j}}, \quad (\alpha_j) \in k^{\mathfrak{I}\setminus I}, \quad (b_{i,j}) \in \mathbf{N}^{I\times(\mathfrak{I}\setminus I)} \tag{2.1}$$

where the sets $I_j = \{i \in I,\, b_{i,j} \neq 0\}$ are pairwise disjoint. The degree of the generator will be denoted by $\mathscr{D}_{\mathrm{tot}}$; thus $\mathscr{D}_{\mathrm{tot}}$ lies in $\mathrm{Pic}(X)$.

Examples of linear intrinsic hypersurfaces will be given in the last section of the paper.

Remark 2.3. If X is a linear intrinsic hypersurface with respect to $(\mathfrak{I}, I)$, one has $\dim(X) = [\mathfrak{I} \setminus I] - 1$. Moreover, by [BH07, Proposition 8.5], one has an adjunction formula allowing to compute the class of the canonical sheaf, namely

$$-\mathscr{K}_X = \sum_{i\in\mathfrak{I}} \mathscr{E}_i - \mathscr{D}_{\mathrm{tot}}. \tag{2.2}$$

Let X be a Mori dream space and $y \in \mathrm{NS}(X)^\vee$. Since the divisors $\{\mathscr{E}_i\}_{i\in\mathfrak{I}}$ span $\mathrm{Eff}(X)$, if y does not lie in $\mathrm{Eff}(X)^\vee$, $\mathbf{Mor}(\mathscr{C}, X, y, \mathfrak{I})$ is empty, that is, every morphism of degree y has its image contained in the boundary. We will not be interested in this kind of degeneracy and thus shall always assume that y lies in $\mathrm{Eff}(X)^\vee$. Let $\mathfrak{I}_*$ be a proper subset of $\mathfrak{I}$ such that $\underset{i\notin\mathfrak{I}_*}{\cap} \mathscr{E}_i$ is non empty. For every $\varphi \in \mathbf{Mor}(\mathscr{C}, X, y, \mathfrak{I}_*)$ and $j \in \mathfrak{I}_*$ such that $\mathscr{E}_j$ does not meet $\underset{i\notin\mathfrak{I}_*}{\cap} \mathscr{E}_i$, one must have $\varphi^*\mathscr{E}_j = 0$ hence $\langle y\,,\, \mathscr{E}_j\rangle = 0$. We will say that y satisfies the *degeneracy condition for* $\mathfrak{I}_*$ if the latter holds; it is thus a necessary condition for the non emptiness of $\mathbf{Mor}(\mathscr{C}, X, y, \mathfrak{I}_*)$.

We can now state a more explicit version of theorem 1.1.

Theorem 2.4. *We retain notations 2.1. Assume that X is a linear intrinsic hypersurface with respect to $(\mathfrak{I}, I)$ (cf. definition 2.2). Let $\mathscr{C}$ be a smooth projective geometrically irreducible $\mathbf{Q}$-curve of genus $g_{\mathscr{C}}$. Let $y \in \mathrm{Eff}(X)^\vee$ satisfying the numerical inequality*

$$\left\langle y\,,\, \frac{1}{[\mathfrak{I}\setminus I]-1} \sum_{j\in\mathfrak{I}\setminus I} \mathscr{E}_j - \mathscr{D}_{tot} \right\rangle \geqslant \mathrm{Sup}\left(1, \frac{4}{[\mathfrak{I}\setminus I]-1}\right) g_{\mathscr{C}} \,\dim(X). \tag{2.3}$$

(1) Assume that assumptions 5.7 hold for $(\mathfrak{I}, I)$. Then $\boldsymbol{Mor}(\mathscr{C}, X, y, \mathfrak{I})$ is irreducible of the expected dimension.

(2) Let $\mathfrak{I}_*$ be a proper subset of $\mathfrak{I}$ such that $\underset{i\notin\mathfrak{I}_*}{\cap} \mathscr{E}_i$ is non empty. Assume that y fulfills the degeneracy condition for $\mathfrak{I}_*$ and the numerical inequalities

$$\forall i \in I \setminus \mathfrak{I}_*, \quad \langle y\,,\, \mathscr{E}_i\rangle \geqslant g_{\mathscr{C}}. \tag{2.4}$$

In case $\mathfrak{I}\setminus I\subset\mathfrak{I}_*$, assume moreover that assumptions 5.7 hold for $(\mathfrak{I}_*,I)$ and that at least one of the inequalities in (2.4) *is strict. Then one has*

$$\dim(\boldsymbol{Mor}(\mathscr{C},X,y,\mathfrak{I}_*))<(1-g_{\mathscr{C}})\dim(X)+\langle y\,,\,-\mathscr{K}_X\rangle\,. \tag{2.5}$$

Remark 2.5. We still postpone the description of assumptions 5.7 which are the conditions on the combinatoric series alluded to in 1.1, and are somewhat technically cumbersome. It would be of course interesting to be able to drop them from the statement, or at least to both relax them and reinterprate them in a more conceptual way, on one hand to avoid the use of a computer algebra system, on the other hand to be able to cover more cases of linear intrinsic hypersurface (there are a few examples in Derenthal's list [Der06] for which the assumptions fail, see section 7).

Remark 2.6. The subcone of $\mathrm{Eff}(X)^\vee$ in the statement of theorem 1.1 is thus the dual of the cone generated by the effective cone and $\frac{1}{[\mathfrak{I}\setminus I]-1}\sum_{j\in\mathfrak{I}\setminus I}\mathscr{E}_j-\mathscr{D}_{\mathrm{tot}}$. Let us denote it by $\mathcal{C}_I$, and by $\widetilde{\mathcal{C}}$ the union of all the $\mathcal{C}_I$ for which X is a linear intrinsic hypersurface with respect to $(\mathfrak{I},I)$ and such that assumptions 5.7 hold.

The result will be optimal if $\widetilde{\mathcal{C}}$ coincides with the dual of the cone generated by the effective cone and $-\mathscr{K}_X$, hence with $\mathrm{Eff}(X)^\vee$ if $-\mathscr{K}_X$ lies in $\mathrm{Eff}(X)$ (thus in the toric case the result of theorem 1.2 is optimal). In the last section, we will give examples of linear intrinsic hypersurface for which either the result is optimal, or holds for a "positive proportion" of y, that is $\mathcal{C}_I$ is of maximal dimension for at least one choice of I.

3. Reduction modulo p

The following lemma is a standard application of Weil-Deligne.

Lemma 3.1. *Let X be a $\mathbf{Q}$-variety. If p is a prime and r a positive integer, we denote by $X(\mathbf{F}_{p^r})$ the set of $\mathbf{F}_{p^r}$-points of the reduction of X modulo p, which is well defined up to a finite number of primes. Assume that there exists an integer D and a positive integer N such that for almost all primes p one has*

$$\lim_{r\to+\infty}\quad p^{-r\,D}\,[X(\mathbf{F}_{p^r})]=N, \tag{3.6}$$

respectively

$$\lim_{r\to+\infty}\quad p^{-r\,D}\,[X(\mathbf{F}_{p^r})]=0. \tag{3.7}$$

Then in the first case one has $\dim(X)=D$, every irreducible component of X with dimension $\dim(X)$ is geometrically irreducible and there are N such irreducible components. In the second case one has $\dim(X)<D$.

4. Expression of the number of morphisms in terms of a presentation of the Cox ring

We retain notations 2.1; until otherwise specified X is only assumed to be a Mori dream space. If Y is a $\mathbf{Q}$-variety and p a prime number, we denote by Y_p the $\mathbf{F}_p$-variety obtained by reducing Y modulo p, which is well-defined up to a finite number of primes.

Let $\mathcal{C}_{\mathrm{inc}}$ be the class of subsets $\mathfrak{I}_*$ of $\mathfrak{I}$ such that $\underset{i\notin\mathfrak{I}_*}{\cap}\mathscr{E}_i\neq\varnothing$. Let $\mathrm{T}_{\mathrm{NS}}(X)\overset{\mathrm{def}}{=}\mathrm{Hom}(\mathrm{Pic}(X),\mathbf{G}_m)$. There exists a $\mathrm{T}_{\mathrm{NS}}(X)$-invariant morphism

$$\pi\;:\;\mathscr{T}_X\overset{\mathrm{def}}{=}\mathrm{Spec}(\mathrm{Cox}(X))\bigcap\bigcup_{\mathfrak{I}_*\in\mathcal{C}_{\mathrm{inc}}}\left\{\prod_{i\in\mathfrak{I}_*}x_i\neq 0\right\}\longrightarrow X \tag{4.8}$$

which makes $\mathscr{T}_X$ an X-torsor under $\mathrm{T}_{\mathrm{NS}}(X)$ (*cf.* [Bou11, §2.2]). For almost all primes p, this reduces to an X_p-torsor $\pi_p\;:\;\mathscr{T}_{X,p}\to X_p$ under $\mathrm{T}_{\mathrm{NS}}(X_p)$.

Let $\mathfrak{I}_* \in \mathcal{C}_{\text{inc}}$. We shall give below a formula for the number of points in the reduction of $\mathbf{Mor}(\mathscr{C}, X, y, \mathfrak{I}_*)$ modulo a prime p (formula (4.20)). First we have to introduce some definitions and notations. In order to motivate them, let us explain very sketchily how (4.20) is obtained. Using the torsor (4.8) and adaptating a proof of Cox who had previously addressed the toric setting ([Cox95]), one shows that the map $\varphi \in \mathbf{Mor}(\mathscr{C}, X, y, \mathfrak{I}_*) \mapsto \{(\varphi^*\mathscr{O}_X(\mathscr{E}_i), \varphi^* s_i)\}_{i\in\mathfrak{I}}$ induces a one-to-one correspondence between the set of points of $\mathbf{Mor}(\mathscr{C}, X, y, \mathfrak{I}_*)$ and the set of $\mathrm{T}_{\mathrm{NS}}(X)$-orbits of isomorphism classes of families $\{(\mathscr{L}_i, u_i)\}_{i\in\mathfrak{I}}$ where $\mathscr{L}_i$ is a line bundle on $\mathscr{C}$ and u_i a global section of $\mathscr{L}_i$, such that

- u_i is the zero section if and only if $i \notin \mathfrak{I}_*$;
- for every $(a_i) \in \mathbf{Z}^{\mathfrak{I}}$ such that $\sum a_i \mathscr{E}_i \sim 0$, the line bundle $\underset{i\in\mathfrak{I}}{\otimes} \mathscr{L}_i^{\otimes a_i}$ is trivial;
- for every $F \in \mathscr{I}^{\text{hom}}_{X,\mathfrak{I}}$, $F((u_i)_{i\in\mathfrak{I}}) = 0$;
- the sections $\{\prod_{i\in\mathfrak{K}} u_i\}_{\mathfrak{K}\in\mathcal{C}_{\text{inc}}}$ do not vanish simultaneously or, what amounts to the same regarding the first condition, the intersection of the supports of the divisors $\{\sum_{i\in\mathfrak{K}} \operatorname{div}(u_i)\}_{\substack{\mathfrak{K}\subset\mathfrak{I}_*\\ \mathfrak{K}\in\mathcal{C}_{\text{inc}}}}$ is empty.

We refer to [Bou09a, Théorème 1.11] for a more precise statement. Now let

$$\mathfrak{I}_*^\circ \overset{\text{def}}{=} \{i \in \mathfrak{I}_*,\ \mathscr{E}_i \cap \bigcap_{j\notin\mathfrak{I}_*} \mathscr{E}_j \neq \varnothing\}. \tag{4.9}$$

Thus $y \in \operatorname{Eff}(X)^\vee \cap \operatorname{NS}(X)^\vee$ satisfies the degeneracy condition for $\mathfrak{I}_*$ if for every $i \in \mathfrak{I}_* \setminus \mathfrak{I}_*^\circ$ one has $\langle y\,,\,\mathscr{E}_i\rangle = 0$. One sees that the last condition above on the data $\{(\mathscr{L}_i, u_i)\}$ is equivalent to the emptiness of the intersection of the supports of the divisors

$$\{\sum_{i\in\mathfrak{K}} \operatorname{div}(u_i)\}_{\substack{\mathfrak{K}\subset\mathfrak{I}_*^\circ\\ \mathfrak{K}\,\cup\,(\mathfrak{I}_*\setminus\mathfrak{I}_*^\circ)\in\mathcal{C}_{\text{inc}}}}. \tag{4.10}$$

We will perform a Möbius inversion to drop the conditions on the intersection of the supports.

Definitions and notations 4.1. Let $\mathfrak{I}_* \in \mathcal{C}_{\text{inc}}$. Let $\mu^\circ_{X,\mathfrak{I}_*^\circ} : \mathbf{N}^{\mathfrak{I}_*^\circ} \to \mathbf{Z}$ be defined recursively by

$$\forall\, \boldsymbol{n} \in \mathbf{N}^{\mathfrak{I}_*^\circ}, \qquad \sum_{0\leqslant \boldsymbol{m}\leqslant \boldsymbol{n}} \mu^\circ_{X,\mathfrak{I}_*^\circ}(\boldsymbol{m}) = \begin{cases} 1 & \text{if} \quad \underset{\substack{\mathfrak{K}\subset\mathfrak{I}_*^\circ\\ \mathfrak{K}\,\cup\,(\mathfrak{I}_*\setminus\mathfrak{I}_*^\circ)\in\mathcal{C}_{\text{inc}}}}{\operatorname{Inf}} \left(\sum_{i\in\mathfrak{K}} n_i\right) = 0 \\ 0 & \text{otherwise.} \end{cases} \tag{4.11}$$

Remark 4.2. For $i \in \mathfrak{I}_*^\circ$ one has $\mathfrak{I}_* \setminus \{i\} \in \mathcal{C}_{\text{inc}}$. Thus from the very definition, one sees that for $\boldsymbol{n} \in \mathbf{N}^{\mathfrak{I}_*^\circ}$ one has $\mu^\circ_{X,\mathfrak{I}_*^\circ}(\boldsymbol{n}) = 0$ as soon as $\sum_{i\in\mathfrak{I}_*^\circ} n_i = 1$.

For almost all primes p and every positive integer r, let $\mathscr{C}_{p,r} \overset{\text{def}}{=} \mathscr{C}_p \otimes \mathbf{F}_{p^r}$. Let $\mathscr{C}_{p,r}^{(0)}$ be the set of closed points of $\mathscr{C}_{p,r}$, $\operatorname{Div}(\mathscr{C}_{p,r})$ be the group of its divisors and $\operatorname{Div}_{\text{eff}}(\mathscr{C}_{p,r})$ be the monoid of its effective divisors. For $P \in \mathscr{C}_{p,r}^{(0)}$ and $\boldsymbol{n} \in \mathbf{N}^{\mathfrak{I}_*^\circ}$ we set

$$\mu_{X,\mathfrak{I}_*^\circ,p,r}((n_i\,P)_{i\in\mathfrak{I}_*^\circ}) \overset{\text{def}}{=} \mu^\circ_{X,\mathfrak{I}_*^\circ}(\boldsymbol{n}) \tag{4.12}$$

and by additivity we extend $\mu_{X,\mathfrak{I}_*^\circ,p,r}$ to a fonction $\operatorname{Div}_{\text{eff}}(\mathscr{C}_{p,r})^{\mathfrak{I}_*^\circ} \to \mathbf{Z}$, which is the unique fonction satisfying

$$\forall\, \boldsymbol{\mathcal{D}} \in \operatorname{Div}_{\text{eff}}(\mathscr{C}_{p,r})^{\mathfrak{I}_*^\circ}, \qquad \sum_{0\leqslant \boldsymbol{\mathcal{E}}\leqslant \boldsymbol{\mathcal{D}}} \mu_{X,\mathfrak{I}_*^\circ,p,r}(\boldsymbol{\mathcal{E}}) = \begin{cases} 1 & \text{if} \quad \underset{\substack{\mathfrak{K}\subset\mathfrak{I}_*^\circ\\ \mathfrak{K}\,\cup\,(\mathfrak{I}_*\setminus\mathfrak{I}_*^\circ)\in\mathcal{C}_{\text{inc}}}}{\operatorname{Inf}} \left(\sum_{i\in\mathfrak{K}} \mathcal{D}_i\right) = 0 \\ 0 & \text{otherwise.} \end{cases} \tag{4.13}$$

Definitions and notations 4.3. We retain notations 2.1 and 4.1. For $j \in \mathfrak{I} \setminus I$, write $\mathscr{E}_j = \sum_{i \in I} a_{i,j}\, \mathscr{E}_i$ with $(a_{i,j}) \in \mathbf{Z}^I$.

Let $\mathfrak{I}_* \in \mathcal{C}_{\mathrm{inc}}$. Let p be a sufficiently large prime and r a positive integer. Abusing notations, we let $\mathrm{Pic}^\circ(\mathscr{C}_{p,r})$ denote a set of representatives in $\mathrm{Div}(\mathscr{C}_{p,r})$ of $\mathrm{Pic}^\circ(\mathscr{C}_{p,r})$. We fix a degree 1 divisor $\mathfrak{d}_1$ in $\mathrm{Div}(\mathscr{C}_{p,r})$. For $y \in \mathrm{Eff}(X)^\vee \cap \mathrm{NS}(X)^\vee$ satisfying the degeneracy condition for $\mathfrak{I}_*$, $\boldsymbol{\mathcal{D}} \in \mathrm{Div}_{\mathrm{eff}}(\mathscr{C}_{p,r})^{\mathfrak{I}_*^\circ \cap I}$, $\boldsymbol{\mathcal{E}} \in \mathrm{Div}(\mathscr{C}_{p,r})^{\mathfrak{I}_*^\circ}$, $\boldsymbol{\mathfrak{E}} \in \mathrm{Pic}^\circ(\mathscr{C}_{p,r})^{I \setminus \mathfrak{I}_*}$ and $K_1, K_2 \subset \mathfrak{I}_* \setminus I$, let

$$\mathcal{N}(y, \mathfrak{I}_*, I, K_1, K_2, p, r, \boldsymbol{\mathfrak{E}}, \boldsymbol{\mathcal{D}}, \boldsymbol{\mathcal{E}}) \tag{4.14}$$

denote the cardinality of the set of the elements $(t_j)_{j \in \mathfrak{I}_* \setminus I}$ of the product

$$\prod_{j \in \mathfrak{I}_*^\circ \setminus I} H^\circ \left(\mathscr{C}_{p,r}, \mathscr{O}_{\mathscr{C}}(-\mathcal{E}_j + \sum_{i \in \mathfrak{I}_*^\circ \cap I} a_{i,j}\, (\mathcal{D}_i + \mathcal{E}_i) + \sum_{i \in I \setminus \mathfrak{I}_*} a_{i,j}\, (\mathfrak{E}_i + \mathfrak{d}_1 \, \langle y \,,\, \mathscr{E}_i \rangle) \right) \times \prod_{j \in \mathfrak{I}_* \setminus (\mathfrak{I}_*^\circ \cup I)} H^\circ\left(\mathscr{C}_{p,r}, \mathscr{O}_{\mathscr{C}}\right) \tag{4.15}$$

which satisfy

$$\forall j \in K_1, \quad t_j \neq 0, \qquad \forall j \in K_2, \quad t_j = 0 \tag{4.16}$$

and

$$\forall F \in \mathscr{I}_{X,\mathfrak{I}}^{\mathrm{hom}}, \quad F((s_{\mathcal{D}_i}\, s_{\mathcal{E}_i})_{i \in \mathfrak{I}_* \cap I}, 0, \dots, 0, (t_j\, s_{\mathcal{E}_j})_{j \in \mathfrak{I}_* \setminus I}, 0, \dots, 0) = 0 \tag{4.17}$$

(where we have set $s_{\mathcal{D}_i} = s_{\mathcal{E}_i} = s_{\mathcal{E}_j} = 1$ for $i \in I \cap (\mathfrak{I}_* \setminus \mathfrak{I}_*^\circ)$ and $j \in \mathfrak{I}_* \setminus (I \cup \mathfrak{I}_*^\circ)$).

We set, for every subset K of $\mathfrak{I}_* \setminus I$,

$$\mathcal{N}(y, \mathfrak{I}_*, I, K, p, r) \overset{\text{def}}{=} \sum_{\substack{\boldsymbol{\mathcal{E}} \in \mathrm{Div}_{\mathrm{eff}}(\mathscr{C}_{p,r})^{\mathfrak{I}_*^\circ} \\ \deg(\mathcal{E}_i) \leqslant \langle y \,,\, \mathscr{E}_i \rangle, \quad i \in \mathfrak{I}_*^\circ}} \mu_{X, \mathfrak{I}_*^\circ, p, r}(\boldsymbol{\mathcal{E}}) \sum_{\substack{\boldsymbol{\mathfrak{E}} \in \mathrm{Pic}^\circ(\mathscr{C}_{p,r})^{I \setminus \mathfrak{I}_*} \\ \boldsymbol{\mathcal{D}} \in \mathrm{Div}_{\mathrm{eff}}(\mathscr{C}_{p,r})^{\mathfrak{I}_*^\circ \cap I} \\ \deg(\mathcal{D}_i) = \langle y \,,\, \mathscr{E}_i \rangle - \deg(\mathcal{E}_i), \quad i \in \mathfrak{I}_*^\circ \cap I}} \mathcal{N}(y, \mathfrak{I}_*, I, \varnothing, K, p, r, \boldsymbol{\mathfrak{E}}, \boldsymbol{\mathcal{D}}, \boldsymbol{\mathcal{E}}). \tag{4.18}$$

Lemma 4.4. *We retain notations 2.1, 4.1 and 4.3. Let $y \in \mathrm{Eff}(X)^\vee \cap \mathrm{NS}(X)^\vee$ and $\mathfrak{I}_* \in \mathcal{C}_{inc}$. Assume that y satisfies the degeneracy condition for $\mathfrak{I}_*$. Then for almost all primes p and every positive integer r one has the relation*

$$[\boldsymbol{Mor}(\mathscr{C}, X, y, \mathfrak{I}_*)(\mathbf{F}_{p^r})] = \sum_{K \subset \mathfrak{I}_* \setminus I} (-1)^{|K|}\, \mathcal{N}(y, \mathfrak{I}_*, I, K, p, r). \tag{4.19}$$

Proof. For almost all primes p, there is a natural isomorphism $\mathrm{NS}(X) \xrightarrow{\sim} \mathrm{NS}(X_p)$ and $\mathbf{Mor}(\mathscr{C}, X, y, \mathfrak{I}_*)_p$ is isomorphic to $\mathbf{Mor}(\mathscr{C}_p, X_p, y, \mathfrak{I}_*)$. Thus from [Bou09a, §1.2 and §1.3] we have

$$[\mathbf{Mor}(\mathscr{C}, X, y, \mathfrak{I}_*)(\mathbf{F}_{p^r})] = \sum_{\substack{\boldsymbol{\mathcal{E}} \in \mathrm{Div}_{\mathrm{eff}}(\mathscr{C}_{p,r})^{\mathfrak{I}_*^\circ} \\ \deg(\mathcal{E}_i) \leqslant \langle y \,,\, \mathscr{E}_i \rangle, \quad i \in \mathfrak{I}_*^\circ}} \mu_{X, \mathfrak{I}_*^\circ, p, r}(\boldsymbol{\mathcal{E}}) \sum_{\substack{\boldsymbol{\mathfrak{E}} \in \mathrm{Pic}^\circ(\mathscr{C}_{p,r})^{I \setminus \mathfrak{I}_*} \\ \boldsymbol{\mathcal{D}} \in \mathrm{Div}_{\mathrm{eff}}(\mathscr{C}_{p,r})^{\mathfrak{I}_*^\circ \cap I} \\ \deg(\mathcal{D}_i) = \langle y \,,\, \mathscr{E}_i \rangle - \deg(\mathcal{E}_i), \quad i \in \mathfrak{I}_*^\circ \cap I}} \mathcal{N}(y, \mathfrak{I}_*, I, \mathfrak{I}_* \setminus I, \varnothing, p, r, \boldsymbol{\mathfrak{E}}, \boldsymbol{\mathcal{D}}, \boldsymbol{\mathcal{E}}) \tag{4.20}$$

hence the result by inclusion-exclusion. Strictly speaking in [Bou09a] only the case $\mathfrak{I}_* = \mathfrak{I}$ is extensively treated, but the general case may be addressed in the same way. ∎

Theorem 2.4 follows immediatly from lemmas 3.1 and 4.4 and the next proposition. Recall that we know from deformation theory that every irreducible component of $\mathbf{Mor}(\mathscr{C}, X, y)$ has dimension greater than or equal to the expected dimension $(1 - g_{\mathscr{C}}) \dim(X) + \langle y\,,\, -\mathscr{K}_X\rangle$.

Proposition 4.5. *We retain notations 2.1, 4.1 and 4.3. Assume that X is a linear intrinsic hypersurface with respect to $(\mathfrak{I}, I)$ (cf. definition 2.2). Let $\mathscr{C}$ be a smooth projective geometrically irreducible $\mathbf{Q}$-curve of genus $g_{\mathscr{C}}$. Let $y \in \mathrm{Eff}(X)^\vee$. Assume that y fulfills the numerical inequality*

$$\left\langle y\,,\, \frac{1}{[\mathfrak{I} \setminus I] - 1} \sum_{j \in \mathfrak{I}\setminus I} \mathscr{E}_j - \mathscr{D}_{tot} \right\rangle \geqslant \mathrm{Sup}\left(1, \frac{4}{[\mathfrak{I} \setminus I] - 1}\right) g_{\mathscr{C}}\, \dim(X). \tag{4.21}$$

(1) *For every non empty subset K of $\mathfrak{I} \setminus I$ one has*

$$\lim_{r\to\infty} \quad p^{-r\left[(1-g_{\mathscr{C}})\, \dim(X) + \langle y\,,\, -\mathscr{K}_X\rangle\right]}\, \mathfrak{N}(y, \mathfrak{I}, I, K, p, r) = 0. \tag{4.22}$$

(2) *Let $\mathfrak{I}_* \in \mathcal{C}_{inc}$ such that y satisfies the degeneracy condition for $\mathfrak{I}_*$. Let K be a subset of $\mathfrak{I}_* \setminus I$. Assume that $\mathfrak{I}_* \setminus I$ is a proper subset of $\mathfrak{I} \setminus I$ or that K is non empty. Assume moreover that y satisfies the numerical inequalities*

$$\forall i \in I \setminus \mathfrak{I}_*, \quad \langle y\,,\, \mathscr{E}_i\rangle \geqslant g_{\mathscr{C}}. \tag{4.23}$$

Then one has

$$\lim_{r\to\infty} \quad p^{-r\left[(1-g_{\mathscr{C}})\, \dim(X) + \langle y\,,\, -\mathscr{K}_X\rangle\right]}\, \mathfrak{N}(y, \mathfrak{I}_*, I, K, p, r) = 0. \tag{4.24}$$

(3) *Let $\mathfrak{I}_*$ be an element of $\mathcal{C}_{inc}$ such that $\mathfrak{I} \setminus I \subset \mathfrak{I}_*$ and y satisfies the degeneracy condition for $\mathfrak{I}_*$. Assume moreover that assumptions 5.7 hold for $(\mathfrak{I}_*, I)$ and that y satisfies the numerical inequalities*

$$\forall i \in I \setminus \mathfrak{I}_*, \quad \langle y\,,\, \mathscr{E}_i\rangle \geqslant g_{\mathscr{C}}. \tag{4.25}$$

In case $\mathfrak{I}_*$ is a proper subset of $\mathfrak{I}$, assume that at least one of the above inequalities is strict. Then one has

$$\lim_{r\to\infty} \quad p^{-r\,[(1-g_{\mathscr{C}})\, \dim(X) + \langle y\,,\, -\mathscr{K}_X\rangle]}\, \mathfrak{N}(y, \mathfrak{I}_*, I, \varnothing, p, r) = \begin{cases} 1 & \textit{if} \quad \mathfrak{I}_* = \mathfrak{I} \\ 0 & \textit{otherwise.} \end{cases} \tag{4.26}$$

Our task is now to prove proposition 4.5. The next section contains some preliminary technical lemmas which will be needed during the proof.

5. Technical lemmas

The following lemma is a very slight variation of [Bou11, Lemme 12], with an identical proof.

Lemma 5.1. *Let n be a positive integer, ρ be a real number, $\rho > 1$, and $(a_{\boldsymbol{d}}) \in \mathbf{C}^{\mathbf{N}^n}$. Assume that the series $F(\boldsymbol{t}) \overset{def}{=} \sum_{\boldsymbol{d}\in\mathbf{N}^n} a_{\boldsymbol{d}}\, \boldsymbol{t}^{\boldsymbol{d}}$ converges absolutely on a polydisc of multi-radius $(\rho^{-1+\nu}, \dots, \rho^{-1+\nu})$ with $\nu > 0$. For every positive real number η such that $\eta < \rho^{-1+\nu}$ denote by $||F||_\eta$ the quantity $\underset{|t_1|=\dots=|t_n|=\eta}{\mathrm{Sup}} |F(\boldsymbol{t})|$.*

Define $(b_{\boldsymbol{d}}) \in \mathbf{C}^{\mathbf{N}^n}$ by

$$\sum_{\boldsymbol{d}\in\mathbf{N}^n} b_{\boldsymbol{d}}\, \boldsymbol{t}^{\boldsymbol{d}} \overset{def}{=} \frac{F(\boldsymbol{t})}{(1 - \rho\, t_1) \dots (1 - \rho\, t_n)}. \tag{5.27}$$

Then for every $\varepsilon > 0$ such that $\varepsilon < \nu$ one has

$$\forall \boldsymbol{d} \in \mathbf{N}^n, \quad |b_{\boldsymbol{d}}| \leqslant \frac{1 + n\,\rho^{-\varepsilon}}{(1 - \rho^{-\varepsilon})^n}\, ||F||_{\rho^{-1+\varepsilon}}\, \rho^{|\boldsymbol{d}|} \tag{5.28}$$

and

$$\forall \boldsymbol{d} \in \mathbf{N}^n, \quad \left| b_{\boldsymbol{d}} - F(\rho^{-1}, \dots, \rho^{-1})\, \rho^{|\boldsymbol{d}|} \right| \leqslant \frac{\rho^{-\varepsilon}}{(1-\rho^{-\varepsilon})^n}\, ||F||_{\rho^{-1+\varepsilon}} \sum_{1 \leqslant i \leqslant n} \rho^{(1-\varepsilon)d_i + \sum_{j \neq i} d_j}. \tag{5.29}$$

Lemma 5.2. *Let p be a sufficiently large prime number. For $v \in \mathscr{C}_{p,r}^{(0)}$ denote by f_v the degree of the extension $\kappa_v/\mathbf{F}_{p^r}$, where κ_v is the residue field at v. Let $\theta : \mathbf{N}_{>0} \times \mathbf{N}_{>0} \to \mathbf{R}_{\geqslant 0}$ be an application such that there exists $C \geqslant 0$ and $\eta > 0$ satisfying*

$$\forall (r, f) \in \mathbf{N}_{>0} \times \mathbf{N}_{>0}, \quad \theta(r, f) \leqslant C\, p^{-r\, f\,(1+\eta)}. \tag{5.30}$$

Then

$$\lim_{r \to \infty} \prod_{v \in \mathscr{C}_{p,r}^{(0)}} [1 + \theta(r, f_v)] = 1. \tag{5.31}$$

Proof. Since

$$\left[\{v \in \mathscr{C}_{p,r}^{(0)},\ f_v = f\}\right] \leqslant \left[\mathscr{C}(\mathbf{F}_{p^{rf}})\right] = p^{r\,f} + \underset{f \to +\infty}{\mathcal{O}}\left(p^{\frac{rf}{2}}\right) \tag{5.32}$$

we have

$$[1 + \theta(r, f)]^{[\{v \in \mathscr{C}_{p,r}^{(0)},\, f_v = f\}]} \leqslant \exp\left[C\, p^{-r\,f\,\eta} + \underset{f \to +\infty}{\mathcal{O}}\left(p^{r\,f\,(-\frac{1}{2}+\eta)}\right)\right] \tag{5.33}$$

hence the result by dominated convergence. □

Lemma 5.3. *We retain notations 2.1 and 4.1. Let $\mathfrak{I}_* \in \mathcal{C}_{inc}$, $r \in \mathbf{N}$ and $\boldsymbol{d} \in \mathbf{R}^{\mathfrak{I}_*^\circ}$. We set*

$$\Theta(\mathfrak{I}_*, p, r, \boldsymbol{d}) \overset{def}{=} \sum_{\boldsymbol{\mathcal{E}} \in \mathrm{Div_{eff}}(\mathscr{C}_{p,r})^{\mathfrak{I}_*^\circ}} \left|\mu_{X, \mathfrak{I}_*^\circ, p, r}(\boldsymbol{\mathcal{E}})\right| p^{-r \sum_{i \in \mathfrak{I}_*^\circ} d_i \deg(\mathcal{E}_i)}. \tag{5.34}$$

Assume that for every $\boldsymbol{e} \in \{0,1\}^{\mathfrak{I}_*^\circ}$ such that $\mu^\circ_{X,\mathfrak{I}_*^\circ}(\boldsymbol{e}) \neq 0$ one has $\sum_{i \in \mathfrak{I}_*^\circ} d_i e_i > 1$. Then one has

$$\lim_{r \to +\infty} \Theta(\mathfrak{I}_*, p, r, \boldsymbol{d}) = 1. \tag{5.35}$$

Proof. $\Theta(\mathfrak{I}_*, p, r, \boldsymbol{d}) - 1$ is nonnegative and bounded from above by

$$-1 + \prod_{v \in \mathscr{C}_{p,r}^{(0)}} \sum_{\boldsymbol{e} \in \{0,1\}^{\mathfrak{I}_*^\circ}} \left|\mu^\circ_{X, \mathfrak{I}_*^\circ}(\boldsymbol{e})\right|\, p^{-r\, f_v \sum_{i \in \mathfrak{I}_*^\circ} d_i\, e_i} \tag{5.36}$$

and one can conclude thanks to lemma 5.2. □

Remark 5.4. By remark 4.2, the assumption on $\boldsymbol{d}$ holds for example if for all i one has $d_i > \frac{1}{2}$.

The following crucial lemma gives an estimate of the number of sections satisfying the equation of the Cox ring of a linear intrinsic hypersurface. It follows easily from [Bou11, Proposition 14], whose proof rests on elementary linear algebra and the Riemann-Roch theorem for curves.

Lemma 5.5. *We retain notations 2.1, 4.1 and 4.3. Assume that X is a linear intrinsic hypersurface with respect to $(\mathfrak{I}, I)$ (cf. definition 2.2). Let $\mathfrak{I}_* \in \mathcal{C}_{inc}$, $y \in \mathrm{Pic}(X)^\vee \cap \mathrm{Eff}(X)^\vee$ satisfying the degeneracy condition for $\mathfrak{I}_*$, $\boldsymbol{\mathfrak{E}} \in \mathrm{Pic}^\circ(\mathscr{C}_{p,r})^{I \setminus \mathfrak{I}_*}$, $\boldsymbol{\mathcal{E}} \in \mathrm{Div_{eff}}(\mathscr{C}_{p,r})^{\mathfrak{I}_*^\circ}$ and $\boldsymbol{\mathcal{D}} \in \mathrm{Div_{eff}}(\mathscr{C}_{p,r})^{\mathfrak{I}_*^\circ \cap I}$ such that for all $i \in \mathfrak{I}_*^\circ \cap I$ one has $\deg(\mathcal{D}_i) = \langle y\,,\, \mathscr{E}_i \rangle - \deg(\mathcal{E}_i)$. We set $\mathcal{D}_i = \mathcal{E}_i = \mathcal{E}_j = 0$ for $i \in I \setminus \mathfrak{I}_*^\circ$ and $j \in \mathfrak{I}_* \setminus (\mathfrak{I}_*^\circ \cup I)$.*

(1) Let $K \subset \mathfrak{I}_* \setminus I$. Let $(\alpha_j)_{j\in\mathfrak{I}_*\setminus(I\cup K)}$ be nonnegative real numbers such that $\sum \alpha_j = 1$. One has

$$\log_{p^r}[\mathcal{N}(\mathfrak{I}, I, \varnothing, K, p, r, \boldsymbol{\mathfrak{E}}, \boldsymbol{\mathfrak{D}}, \boldsymbol{\mathcal{E}})] \leqslant [\mathfrak{I}_* \setminus (I \cup K)] - 1 + \sum_{j\in\mathfrak{I}_*\setminus(I\cup K)} (1-\alpha_j)(\langle y \,,\, \mathscr{E}_j\rangle - \deg(\mathcal{E}_j)). \tag{5.37}$$

(2) Assume that $\mathfrak{I} \setminus I \subset \mathfrak{I}_*$. One has either

$$\begin{aligned}&\log_{p^r}[\mathcal{N}(\mathfrak{I}_*, I, \varnothing, \varnothing, p, r, \boldsymbol{\mathfrak{D}}, \boldsymbol{\mathcal{E}})]\\ &\leqslant [\mathfrak{I}\setminus I] - 1 + \left\langle y \,,\, -\mathscr{D}_{tot} + \sum_{j\in\mathfrak{I}\setminus I}\mathscr{E}_j\right\rangle - \sum_{j\in\mathfrak{I}\setminus I}\deg(\mathcal{E}_j) + \deg\left[\inf_{j\in\mathfrak{I}\setminus I}\Big(\sum_{i\in\mathfrak{I}_*\cap I} b_{i,j}(\mathcal{E}_i + \mathcal{D}_i) + \mathcal{E}_j\Big)\right]\end{aligned} \tag{5.38}$$

or

$$\log_{p^r}[\mathcal{N}(\mathfrak{I}_*, I, \varnothing, \varnothing, p, r, \boldsymbol{\mathfrak{D}}, \boldsymbol{\mathcal{E}})] \leqslant [\mathfrak{I}\setminus I] - 2 + \left(1 - \frac{1}{[\mathfrak{I}\setminus I]-1}\right)\sum_{j\in\mathfrak{I}\setminus I}(\langle y \,,\, \mathscr{E}_j\rangle - \deg(\mathcal{E}_j)). \tag{5.39}$$

(3) Assume that $\mathfrak{I} \setminus I \subset \mathfrak{I}_*$ and that there exists a numbering of $\mathfrak{I} \setminus I$ such that

$$\forall 1 \leqslant j \leqslant [\mathfrak{I}\setminus I] - 1, \quad \langle y \,,\, \mathscr{E}_j + \mathscr{E}_{j+1} - \mathscr{D}_{tot}\rangle \geqslant \deg(\mathcal{E}_j) + \deg(\mathcal{E}_{j+1}) + 2\,g_{\mathscr{C}} - 1. \tag{5.40}$$

Then one has

$$\begin{aligned}&\log_{p^r}[\mathcal{N}(\mathfrak{I}_*, I, \varnothing, \varnothing, p, r, \boldsymbol{\mathfrak{D}}, \boldsymbol{\mathcal{E}})]\\ &= ([\mathfrak{I}\setminus I]-1)(1-g_{\mathscr{C}}) + \left\langle y \,,\, -\mathscr{D}_{tot} + \sum_{j\in\mathfrak{I}\setminus I}\mathscr{E}_j\right\rangle - \sum_{j\in\mathfrak{I}\setminus I}\deg(\mathcal{E}_j) + \deg\left[\inf_{j\in\mathfrak{I}\setminus I}\left(\sum_{i\in\mathfrak{I}_*\cap I} b_{i,j}(\mathcal{E}_i + \mathcal{D}_i) + \mathcal{E}_j\right)\right].\end{aligned} \tag{5.41}$$

Next we introduce combinatoric series derived from the Cox ring equation of a linear intrinsic hypersurface. We retain notations 2.1 and assume that X is a linear intrinsic hypersurface with respect to $(\mathfrak{I}, I)$ (*cf.* definition 2.2). Let $\mathfrak{I}_*$ be an element of $\mathcal{C}_{\text{inc}}$ such that $\mathfrak{I}\setminus I \subset \mathfrak{I}_*$. We set, for $\boldsymbol{e} \in \mathbf{N}^{\mathfrak{I}_*^\circ}$,

$$F(\mathfrak{I}_*, I, \boldsymbol{e}, \rho, \boldsymbol{t}) \overset{\text{def}}{=} \sum_{\boldsymbol{d}\in\mathbf{N}^{\mathfrak{I}_*^\circ\cap I}} \rho^{\operatorname*{Inf}_{j\in\mathfrak{I}\setminus I}\left(e_j + \sum_{i\in\mathfrak{I}_*^\circ\cap I} b_{i,j}(d_i+e_i)\right)} \boldsymbol{t}^{\boldsymbol{d}} \in k[[\rho, (t_i)_{i\in\mathfrak{I}_*^\circ\cap I}]] \tag{5.42}$$

(where $e_j = 0$ for $j \notin I \cup \mathfrak{I}_*^\circ$) and

$$\widetilde{F}(\mathfrak{I}_*, I, \boldsymbol{e}, \rho, \boldsymbol{t}) \overset{\text{def}}{=} \left(\prod_{i\in\mathfrak{I}_*^\circ\cap I} 1 - t_i\right) F(\mathfrak{I}_*, I, \boldsymbol{e}, \rho, \boldsymbol{t}). \tag{5.43}$$

Remark 5.6. Recall that the sets $I_j = \{i \in I,\, b_{i,j} \neq 0\}$ were assumed to be pairwise disjoint. Let m be the lowest common multiple of the $b_{i,j}$'s which are positive. By partitioning $\mathbf{N}^{\mathfrak{I}_*^\circ\cap I}$ according to the various remainders of the d_i's modulo $m/b_{i,j}$ for $i \in I_j$, one sees easily that $F(\mathfrak{I}_*, I, \boldsymbol{e}, \rho, \boldsymbol{t})$ is rational, obtaining in fact an explicit formula, allowing computations by a computer algebra system. Arguing as in the proof of [Bou11, Proposition 57], one can in fact show that

$$\prod_{(i_j)\in\prod_{j\in\mathfrak{I}\setminus I} I_j}\left(1 - \rho^m \prod t_i^{m/b_{i_j,j}}\right)\prod_{i\in I}(1-t_i)\, F(\mathfrak{I}_*, I, \boldsymbol{e}, \rho, \boldsymbol{t}) \tag{5.44}$$

is a polynomial with explicitely bounded degrees in the t_i's, but this approach does not seem well-fitted to computational purposes.

Let us write $\widetilde{F}(\mathfrak{I}_*, I, \boldsymbol{e}, \rho, \boldsymbol{t}) = \sum\limits_{\boldsymbol{d} \in \mathbf{N}^{\mathfrak{I}_*^\circ \cap I}} P(\mathfrak{I}_*, I, \boldsymbol{e}, \rho)_{\boldsymbol{d}}\ \boldsymbol{t}^{\boldsymbol{d}}$ where $P(\mathfrak{I}_*, I, \boldsymbol{e}, \rho)_{\boldsymbol{d}}$ is polynomial with respect to the variable ρ. We will need the following assumptions on the series $\widetilde{F}(\rho, \mathfrak{I}_*, I, \boldsymbol{e}, \boldsymbol{t})$.

Assumptions 5.7. *(1) For every sufficiently small $\eta > 0$, and every $\boldsymbol{d} \neq 0$, one has*

$$(1-\eta)\ |\boldsymbol{d}| \geqslant 1 + \eta + \deg_\rho P(\mathfrak{I}_*, I, 0, \boldsymbol{d}, \rho). \tag{5.45}$$

(2) For every $\boldsymbol{e} \in \{0,1\}^{\mathfrak{I}_*^\circ}$, one can write

$$\widetilde{F}(\mathfrak{I}_*, I, \boldsymbol{e}, \rho, \boldsymbol{t}) = \left(1 + \sum_{\boldsymbol{d} \neq 0} Q(\mathfrak{I}_*, I, \boldsymbol{e}, \rho)_{\boldsymbol{d}}\ \boldsymbol{t}^{\boldsymbol{d}}\right) R(\mathfrak{I}_*, I, \boldsymbol{e}, \rho, \boldsymbol{t}) \tag{5.46}$$

where $Q(\mathfrak{I}_*, I, \boldsymbol{e}, \rho)_{\boldsymbol{d}}$ is polynomial with respect to ρ and $R(\mathfrak{I}_*, I, \boldsymbol{e}, \rho, \boldsymbol{t})$ is polynomial with respect to ρ and $\boldsymbol{t}$. Moreover, for every sufficiently small $\eta > 0$, and every $\boldsymbol{d} \neq 0$, one has

$$(1-\eta)\ |\boldsymbol{d}| \geqslant 1 + \eta + \deg_\rho Q(\mathfrak{I}_*, I, \boldsymbol{e}, \rho)_{\boldsymbol{d}}. \tag{5.47}$$

(3) For $\boldsymbol{e} \in \mathbf{N}^{\mathfrak{I}_*^\circ}$, let $R(\mathfrak{I}_*, I, \boldsymbol{e}, \rho)_{\boldsymbol{d}}$ denote the coefficient of $\boldsymbol{t}^{\boldsymbol{d}}$ in $R(\mathfrak{I}_*, I, \boldsymbol{e}, \rho, \boldsymbol{t})$ and set

$$C(\mathfrak{I}_*, I, \boldsymbol{e}) \overset{def}{=} \operatorname*{Sup}_{\substack{\boldsymbol{d} \in \mathbf{N}^{\mathfrak{I}_*^\circ \cap I} \\ R(\mathfrak{I}_*, I, \boldsymbol{e}, \rho)_{\boldsymbol{d}} \neq 0}} \left[\deg_\rho R(\mathfrak{I}_*, I, \boldsymbol{e}, \rho)_{\boldsymbol{d}} - |\boldsymbol{d}|\right]. \tag{5.48}$$

Then one has

$$\forall \boldsymbol{e} \in \{0,1\}^{\mathfrak{I}_*^\circ} \setminus \{(0,\dots,0)\}, \quad \mu^\circ_{X,\mathfrak{I}_*^\circ}(\boldsymbol{e}) \neq 0 \Rightarrow C(\mathfrak{I}_*, I, \boldsymbol{e}) - |\boldsymbol{e}| \leqslant -2. \tag{5.49}$$

Remark 5.8. From (5.42) and (5.43) we see immediatly that for every $\boldsymbol{d}$, the ρ-polynomial $P(\mathfrak{I}_*, I, \boldsymbol{e}, \rho)_{\boldsymbol{d}}$ has at most $[\mathfrak{I}_*^\circ \cap I]$ non-zero coefficients, whose absolute values are bounded by $[\mathfrak{I}_*^\circ \cap I]$.

Moreover, if point 2 of the above assumptions holds, then for every $\boldsymbol{e} \in \{0,1\}^{\mathfrak{I}_*}$, letting $D(\mathfrak{I}_*, I, \boldsymbol{e})$ denote the degree with respect to $\boldsymbol{t}$ of $R(\mathfrak{I}_*, I, \boldsymbol{e}, \rho, \boldsymbol{t})$, it is straightforward to check that for every sufficiently small $\eta > 0$ the degree with respect to ρ of $P(\mathfrak{I}_*, I, \boldsymbol{e}, \rho)_{\boldsymbol{d}}$ is bounded by $|\boldsymbol{d}|\,(1-\eta) + C(\mathfrak{I}_*, I, \boldsymbol{e}) + \eta\, D(\mathfrak{I}_*, I, \boldsymbol{e})$.

Hence if point 1 and 2 of the above assumptions hold, one can check that, letting p denote a sufficiently large prime and setting

$$c(\mathfrak{I}_*, I, p, \eta) \overset{\text{def}}{=} [\mathfrak{I}_*^\circ \cap I]^2 \left[\prod_{i \in \mathfrak{I}_*^\circ \cap I} \frac{1}{1 - p^{-\frac{\eta}{2}}} - 1\right], \tag{5.50}$$

for every positive integers r and f and every sufficiently small $\eta > 0$, one has

$$\left|\left|-1 + \widetilde{F}(p^{rf}, \mathfrak{I}_*, I, 0, \boldsymbol{t})\right|\right|_{p^{r\left(-1+\frac{\eta}{2}\right)}} \leqslant c(\mathfrak{I}_*, p, \eta).p^{rf(-1-\eta)} \tag{5.51}$$

and for every $\boldsymbol{e} \in \{0,1\}^{\mathfrak{I}_*^\circ}$

$$\left|\left|\widetilde{F}(p^{rf}, \mathfrak{I}_*, I, \boldsymbol{e}, \boldsymbol{t})\right|\right|_{p^{r\left(-1+\frac{\eta}{2}\right)}} \leqslant c(\mathfrak{I}_*, I, p, \eta).p^{r\,f\,[C(\mathfrak{I}_*, I, \boldsymbol{e}) + \eta\, D(\mathfrak{I}_*, I, \boldsymbol{e})]}. \tag{5.52}$$

In particular, for every sufficiently small $\eta > 0$ there exists a positive integer R_η that for all $r \geqslant R_\eta$ and all f one has

$$\operatorname*{Inf}_{|t_1| = \cdots = |t_n| = p^{r\left(-1+\frac{\eta}{2}\right)}} \left|\widetilde{F}(p^{rf}, \mathfrak{I}_*, I, 0, \boldsymbol{t})\right| \geqslant \frac{1}{2}. \tag{5.53}$$

6. Proof of the main theorem

In this section we prove proposition 4.5, hence also theorem 2.4, as remarked before the statement of the proposition.

Let p be a sufficiently large prime number. Thanks to the Riemann hypothesis for abelian varieties, one has

$$[\mathrm{Pic}^{\circ}(\mathscr{C}_{p,r})] \underset{r\to+\infty}{\sim} p^{r\, g_{\mathscr{C}}}. \tag{6.54}$$

Let us write the Hasse-Weil zeta function of $\mathscr{C}_{p,r}$ as $\frac{P_{p,r}(t)}{(1-t)(1-p^r\,t)}$. Thanks to the Riemann hypothesis for curves, there exists a positive constant c depending only on $g_{\mathscr{C}}$ such that, for every t satisfying $|t|<p^{-\frac{r}{2}}$ one has

$$|P_{p,r}(t)| \leqslant 1+c\;|t|\;p^{\frac{r}{2}}. \tag{6.55}$$

From this one easily deduces, for every sufficiently small positive ε,

$$\limsup_{r\to+\infty} \left\| \frac{P_{p,r}(t)}{1-t} \right\|_{p^{r\,(-1+\varepsilon)}} \leqslant 1. \tag{6.56}$$

Let us denote by $\mathfrak{N}(p,r,d)$ the cardinality of $\{\boldsymbol{\mathfrak{D}}\in \mathrm{Div}_{\mathrm{eff}}(\mathscr{C}_{p,r}),\ \deg(\mathfrak{D})=d\}$. From lemma 5.1 we obtain for every sufficiently large r the inequality

$$\mathfrak{N}(p,r,d)\leqslant 2\,p^{r\,d}. \tag{6.57}$$

In the following, to ease reading, we shall almost systematically drop the indices y, $\mathfrak{I}_*$, I and p from the name of the previously introduced functions, since they may be assumed to remain fixed throughout the proof. We will also denote by $h_{p,r}$ the cardinality of $[\mathrm{Pic}^{\circ}(\mathscr{C}_{p,r})]$. Thanks to (6.54), if $\mathfrak{I}_*$ is a subset of $\mathfrak{I}$ such that (4.25) holds, we have

$$\begin{aligned} &\lim_{r\to+\infty} h_{p,r}^{[I\setminus\mathfrak{I}_*]}\, p^{-r\sum\limits_{i\in I\setminus\mathfrak{I}_*}\langle y\,,\mathscr{E}_i\rangle} \\ &\qquad = \begin{cases} 0 & \text{if at least one of the inequalities in (4.25) is strict} \\ 1 & \text{otherwise.}\end{cases} \end{aligned} \tag{6.58}$$

Let us prove point 1 of proposition 4.5. Here we have $\mathfrak{I}_*=\mathfrak{I}$. We set

$$\mathcal{N}_2(r,(\beta_j)_{j\in\mathfrak{I}\setminus I}) \overset{\mathrm{def}}{=} \sum_{\substack{\boldsymbol{\mathcal{E}}\in\mathrm{Div}_{\mathrm{eff}}(\mathscr{C}_{p,r})^{\mathfrak{I}}\\ \deg(\mathcal{E}_i)\leqslant\langle y\,,\mathscr{E}_i\rangle,\quad i\in\mathfrak{I}}} |\mu_{X,r}(\boldsymbol{\mathcal{E}})|\, p^{r\left[[\mathfrak{I}\setminus I]-2+\sum\limits_{j\in\mathfrak{I}\setminus I}(1-\beta_j)(\langle y\,,\mathscr{E}_j\rangle-\deg(\mathcal{E}_j))\right]} \prod_{i\in I}\mathfrak{N}(r,\langle y\,,\,\mathscr{E}_i\rangle-\deg(\mathcal{E}_i)). \tag{6.59}$$

Let K be a non-empty subset of $\mathfrak{I}\setminus I$ and $j_0\in K$. From (5.37) we deduce the inequality

$$\mathcal{N}(K,r)\leqslant \mathcal{N}_2(r,\left(\frac{1}{[\mathfrak{I}\setminus I]-1}\right)_{j\neq j_0},(1)_{j_0})\leqslant \mathcal{N}_2(r,\left(\frac{1}{[\mathfrak{I}\setminus I]-1}\right)_{j\in\mathfrak{I}\setminus I}). \tag{6.60}$$

Recall from remark 2.3 that $\dim(X) = [\mathfrak{I}\setminus I] - 1$ and $-\mathscr{K}_X = \sum_{i\in\mathfrak{I}} \mathscr{E}_i - \mathscr{D}_{\text{tot}}$. Thus, thanks to (6.57), one has for r large enough the inequality

$$\begin{aligned}&p^{-r\left[(1-g_{\mathscr{C}})\,\dim(X)+\langle y\,,\,-\mathscr{K}_X\rangle\right]}\,\mathcal{N}_2\Big(r,\Big(\frac{1}{[\mathfrak{I}\setminus I]-1}\Big)_{j\in\mathfrak{I}\setminus I}\Big)\\ &\leqslant 2^{[I]}\,p^{r\left[g_{\mathscr{C}}\,\dim(X)-1-\left\langle y\,,\,\sum_{j\in\mathfrak{I}\setminus I}\frac{1}{[\mathfrak{I}\setminus I]-1}\mathscr{E}_j-\mathscr{D}_{\text{tot}}\right\rangle\right]}\,\Theta\left(r,(1)_{i\in I},\left(1-\frac{1}{[\mathfrak{I}\setminus I]-1}\right)_{j\in\mathfrak{I}\setminus I}\right).\end{aligned}\tag{6.61}$$

Owing to (4.21), (6.54), lemma 5.3 and remark 5.4. we obtain

$$\lim_{r\to\infty}\; p^{-r\left[(1-g_{\mathscr{C}})\,\dim(X)+\langle y\,,\,-\mathscr{K}_X\rangle\right]}\,\mathcal{N}(K,r)=0,\tag{6.62}$$

thus proving point 1 of proposition 4.5.

Let us prove point 2 of proposition 4.5. Let $\mathfrak{I}_*$ be a subset of $\mathfrak{I}$ such that $\mathfrak{I}_*\setminus I$ is a non empty proper subset of $\mathfrak{I}\setminus I$ and $K\subset\mathfrak{I}_*\setminus I$. We have $\mathcal{N}(K,r)\leqslant\mathcal{N}(\varnothing,r)$. Arguing as above, we obtain the inequality

$$\begin{aligned}&p^{-r\left[(1-g_{\mathscr{C}})\,\dim(X)+\langle y\,,\,-\mathscr{K}_X\rangle\right]}\,\mathcal{N}(K,r)\\ &\leqslant 2^{[I\cap\mathfrak{I}_*^{\circ}]}\,p^{r\,g_{\mathscr{C}}\,\dim(X)-r\left\langle y\,,\,\sum\limits_{j\in\mathfrak{I}\setminus(I\cup\mathfrak{I}_*)}\mathscr{E}_j+\frac{1}{[\mathfrak{I}_*\setminus I]}\sum\limits_{j\in\mathfrak{I}_*\setminus I}\mathscr{E}_j-\mathscr{D}_{\text{tot}}\right\rangle}\,p^{-r\left\langle y\,,\,\sum\limits_{i\in I\setminus\mathfrak{I}_*}\mathscr{E}_i\right\rangle}\,h_{p,r}^{[I\setminus\mathfrak{I}_*]}\\ &\qquad\times\Theta\left(r,(1)_{i\in\mathfrak{I}_*\cap I},\left(1-\frac{1}{[\mathfrak{I}_*\setminus I]}\right)_{j\in\mathfrak{I}_*\setminus I}\right).\end{aligned}\tag{6.63}$$

Since $[\mathfrak{I}_*\setminus I]\leqslant[\mathfrak{I}\setminus I]-1$, thanks to (4.21), (6.58) and lemma 5.3, we obtain

$$\lim_{r\to\infty}\; p^{-r\left[(1-g_{\mathscr{C}})\,\dim(X)+\langle y\,,\,-\mathscr{K}_X\rangle\right]}\,\mathcal{N}(K,r)=0.\tag{6.64}$$

The case $\mathfrak{I}_*\setminus I=\mathfrak{I}\setminus I$, $K\neq\varnothing$ is similar. If $\mathfrak{I}_*\setminus I=\varnothing$, we have $\mathcal{N}(K,r)=1$ and the result is straightforward. Hence point 2 of proposition 4.5 is established.

Let us now show point 3 of proposition 4.5. We set

$$\varphi(\boldsymbol{\mathcal{D}},\boldsymbol{\mathcal{F}},\boldsymbol{\mathcal{G}})\overset{\text{def}}{=}\left\langle y\,,\,-\mathscr{D}_{\text{tot}}+\sum_{j\in\mathfrak{I}\setminus I}\mathscr{E}_j\right\rangle-\sum_{j\in\mathfrak{I}\setminus I}\deg(\mathcal{E}_j)+\deg\left[\inf_{j\in\mathfrak{I}\setminus I}\left(\sum_{i\in I\cap\mathfrak{I}_*^{\circ}}b_{i,j}(\mathcal{E}_i+\mathcal{D}_i)+\mathcal{E}_j\right)\right],\tag{6.65}$$

$$\mathcal{N}_0(r)\overset{\text{def}}{=}h_{p,r}^{[I\setminus\mathfrak{I}_*]}\sum_{\substack{\boldsymbol{\mathcal{D}}\in\operatorname{Div}_{\text{eff}}(\mathscr{C}_{p,r})^{I\cap\mathfrak{I}_*^{\circ}}\\ \deg(\mathcal{D}_i)=\langle y\,,\,\mathscr{E}_i\rangle,\quad i\in I\cap\mathfrak{I}_*^{\circ}}}p^{\,r\,[([\mathfrak{I}\setminus I]-1)(1-g_{\mathscr{C}})+\varphi(\boldsymbol{\mathcal{D}},0,0)]}\tag{6.66}$$

and

$$\mathcal{N}_1^*(r)\overset{\text{def}}{=}\sum_{\substack{\boldsymbol{\mathcal{E}}\in\operatorname{Div}_{\text{eff}}(\mathscr{C}_{p,r})^{\mathfrak{I}_*^{\circ}}\setminus\{(0,\dots,0)\}\\ \deg(\mathcal{E}_i)\leqslant\langle y\,,\,\mathscr{E}_i\rangle,\quad i\in\mathfrak{I}}}|\mu_{X,r}(\boldsymbol{\mathcal{E}})|\;h_{p,r}^{[I\setminus\mathfrak{I}_*]}\sum_{\substack{\boldsymbol{\mathcal{D}}\in\operatorname{Div}_{\text{eff}}(\mathscr{C}_{p,r})^{I\cap\mathfrak{I}_*^{\circ}}\\ \deg(\mathcal{D}_j)=\langle y\,,\,\mathscr{E}_i\rangle-\deg(\mathcal{E}_i),\quad i\in I\cap\mathfrak{I}_*^{\circ}}}p^{\,r\,[([\mathfrak{I}\setminus I]-1)(1-g_{\mathscr{C}})+\varphi(\boldsymbol{\mathcal{D}},\boldsymbol{\mathcal{F}},\boldsymbol{\mathcal{G}})]}.\tag{6.67}$$

Let us fix a numbering of $\mathfrak{I}\setminus I$ and set for $j_0\in\mathfrak{I}\setminus I$

$$\mathcal{N}_{1,j_0}(r)\overset{\text{def}}{=}\sum_{\substack{\boldsymbol{\mathcal{E}}\in\mathrm{Div}_{\mathrm{eff}}(\mathscr{C}_{p,r})^{\mathfrak{I}_*^{\circ}}\\ \deg(\mathcal{E}_i)\leqslant\langle y\,,\mathscr{E}_i\rangle,\quad i\in\mathfrak{I}_*^{\circ}\\ \left\langle y\,,\mathscr{E}_{j_0}+\mathscr{E}_{j_0+1}-\mathscr{D}_{\mathrm{tot}}\right\rangle\leqslant\deg(\mathcal{E}_{j_0})+\deg(\mathcal{E}_{j_0+1})+2\,g_{\mathscr{C}}-2}}|\mu_{X,r}(\boldsymbol{\mathcal{E}})|\,h_{p,r}^{[I\setminus\mathfrak{I}_*]}\sum_{\substack{\boldsymbol{\mathcal{D}}\in\mathrm{Div}_{\mathrm{eff}}(\mathscr{C}_{p,r})^{I\cap\mathfrak{I}_*^{\circ}}\\ \deg(\mathcal{D}_j)\leqslant\langle y\,,\mathscr{E}_i\rangle-\deg(\mathcal{E}_i),\quad i\in I\cap\mathfrak{I}_*^{\circ}}}p^{r[([\mathfrak{I}\setminus I]-1)+\varphi(y,\boldsymbol{\mathcal{D}},\boldsymbol{\mathcal{F}},\boldsymbol{\mathcal{G}})]}.\tag{6.68}$$

For $j_0,j_1\in\mathfrak{I}\setminus I$ with $j_0\neq j_1$, we have the relation

$$\mathscr{G}_{j_0}+\mathscr{G}_{j_1}-\mathscr{D}_{\mathrm{tot}}=\sum_{j\in\mathfrak{I}\setminus I}\mathscr{E}_j-([\mathfrak{I}\setminus I]-1)\,\mathscr{D}_{\mathrm{tot}}+\sum_{j\in\mathfrak{I}\setminus(I\cup\{j_0,j_1\})}(\mathscr{D}_{\mathrm{tot}}-\mathscr{E}_j).\tag{6.69}$$

By definition of $\mathscr{D}_{\mathrm{tot}}$, we have $\mathscr{D}_{\mathrm{tot}}-\mathscr{E}_j\in\mathrm{Eff}(X)$. Using (4.21) we obtain the inequality

$$\langle y\,,\,\mathscr{E}_{j_0}+\mathscr{E}_{j_1}-\mathscr{D}_{\mathrm{tot}}\rangle\geqslant 4\,g_{\mathscr{C}}\,\dim(X).\tag{6.70}$$

Thus from point 2 and 3 of lemma 5.5 we deduce the inequality

$$|\mathcal{N}(\varnothing,r)-\mathcal{N}_0(r)|\leqslant\mathcal{N}_1^*(r)+\mathcal{N}_2\left(r,\left(\frac{1}{[\mathfrak{I}\setminus I]-1}\right)_{j\in\mathfrak{I}\setminus I}\right)+\sum_{1\leqslant j\leqslant[\mathfrak{I}\setminus I]-1}\mathcal{N}_{1,j}(r).\tag{6.71}$$

We first show

$$\lim_{r\to+\infty}\quad p^{-r\left[\dim(X)(1-g_{\mathscr{C}})+\langle y\,,\,-\mathscr{K}_X\rangle\right]}\,\mathcal{N}_1^*(r)=0.\tag{6.72}$$

The quantity involved in (6.72) equals

$$\sum_{\substack{\boldsymbol{\mathcal{E}}\in\mathrm{Div}_{\mathrm{eff}}(\mathscr{C}_{p,r})^{\mathfrak{I}_*^{\circ}}\setminus\{(0,\dots,0)\}\\ \deg(\mathcal{E}_i)\leqslant\langle y\,,\mathscr{E}_i\rangle,\quad i\in\mathfrak{I}_*^{\circ}}}|\mu_{X,r}(\boldsymbol{\mathcal{E}})|\,p^{-r\sum\limits_{j\in\mathfrak{I}\setminus I}\deg(\mathcal{E}_j)}$$

$$\times\,p^{-r\left\langle y\,,\sum\limits_{i\in I}\mathscr{E}_i\right\rangle}\,h_{p,r}^{[I\setminus\mathfrak{I}_*]}\,a(r,\boldsymbol{\mathcal{E}},(\langle y\,,\,\mathscr{E}_i\rangle-\deg(\mathcal{E}_i))_{i\in I\cap\mathfrak{I}_*^{\circ}})\tag{6.73}$$

where, for $\boldsymbol{d}\in\mathbf{N}^{I\cap\mathfrak{I}_*^{\circ}}$, we have set

$$a(r,\boldsymbol{\mathcal{E}},\boldsymbol{d})\overset{\text{def}}{=}\sum_{\substack{\boldsymbol{\mathcal{D}}\in\mathrm{Div}_{\mathrm{eff}}(\mathscr{C}_{p,r})^{I\cap\mathfrak{I}_*^{\circ}}\\ \deg(\mathcal{D}_i)=d_i,\quad i\in I\cap\mathfrak{I}_*^{\circ}}}p^{r\deg\left[\inf\limits_{j\in\mathfrak{I}\setminus I}\left(\sum\limits_{i\in I\cap\mathfrak{I}_*^{\circ}}b_{i,j}(\mathcal{E}_i+\mathcal{D}_i)+\mathcal{E}_j\right)\right]}.\tag{6.74}$$

Setting $G(r,\boldsymbol{\mathcal{E}},\boldsymbol{t})\overset{\text{def}}{=}\sum_{\boldsymbol{d}\in\mathbf{N}^{I\cap\mathfrak{I}_*^{\circ}}}a(r,\boldsymbol{\mathcal{E}},\boldsymbol{d})\,\boldsymbol{t}^{\boldsymbol{d}}$ we have

$$G(r,\boldsymbol{\mathcal{E}},\boldsymbol{t})=\prod_{v\in\mathscr{C}_{p,r}^{(0)}}F(v(\boldsymbol{\mathcal{E}}),p^{r\,f_v},\boldsymbol{t}^{f_v})=\prod_{i\in I\cap\mathfrak{I}_*^{\circ}}\frac{P_{p,r}(t_i)}{(1-t_i)(1-p^r t_i)}\prod_{v\in\mathscr{C}_{p,r}^{(0)}}\widetilde{F}(v(\boldsymbol{\mathcal{E}}),p^{r\,f_v},\boldsymbol{t}^{f_v}).\tag{6.75}$$

For $r\geqslant 1$ and $\boldsymbol{\mathcal{E}}\in\mathrm{Div}_{\mathrm{eff}}(\mathscr{C}_{p,r})^{\mathfrak{I}}$ one thus may write

$$\widetilde{G}(r,\boldsymbol{\mathcal{E}},\boldsymbol{t})\overset{\text{def}}{=}\left(\prod_{i\in I\cap\mathfrak{I}_*^{\circ}}1-p^r\,t_i\right)G(r,\boldsymbol{\mathcal{E}},\boldsymbol{t})=\widetilde{G}(r,0,\boldsymbol{t})\prod_{\substack{v\in\mathscr{C}_{p,r}^{(0)}\\ v(\boldsymbol{\mathcal{E}})\neq 0}}\frac{\widetilde{F}(v(\boldsymbol{\mathcal{E}}),p^{r\,f_v},\boldsymbol{t}^{f_v})}{\widetilde{F}(0,p^{r\,f_v},\boldsymbol{t}^{f_v})}.\tag{6.76}$$

Let us show that for every sufficiently small $\eta > 0$ we have

$$\limsup_{r\to\infty} \quad \left|\left|\widetilde{G}(r,0,\boldsymbol{t})\right|\right|_{p^{-r(1+\eta)}} \leqslant 1. \tag{6.77}$$

We have indeed

$$\left|\left|\widetilde{G}(r,0,\boldsymbol{t})\right|\right|_{p^{-r(1+\eta)}} \leqslant \left(\frac{||P_{p,r}||_{p^{-r(1+\eta)}}}{1-p^{-r(1+\eta)}}\right)^{[I\cap\mathfrak{I}_*^\circ]} \prod_{v\in\mathscr{C}_{p,r}^{(0)}} \left|\left|\widetilde{F}(0,p^{\,r\,f_v},\boldsymbol{t}^{f_v})\right|\right|_{p^{-r(1+\eta)}}. \tag{6.78}$$

Thanks to (5.51), (6.55) and lemma 5.2 we are done. Similarily, one shows that

$$\lim_{r\to\infty} \quad \widetilde{G}(r,0,(p^{-r},\dots,p^{-r})) = 1. \tag{6.79}$$

Now, owing to (5.52) and (5.53), we have for every sufficiently small $\eta > 0$ and every sufficiently large r the inequality

$$\left|\left|\widetilde{G}(r,\boldsymbol{\mathcal{E}},\boldsymbol{t})\right|\right|_{p^{r\left(-1+\frac{\eta}{2}\right)}} \leqslant \left|\left|\widetilde{G}(r,0,\boldsymbol{t})\right|\right|_{p^{r\left(-1+\frac{\eta}{2}\right)}} \prod_{\substack{v\in\mathscr{C}_{p,r}^{(0)}\\ v(\boldsymbol{\mathcal{E}})\neq 0}} 2\,c(\eta)\,p^{\,r f_v[C(v(\boldsymbol{\mathcal{E}}))+\eta\,D(v(\boldsymbol{\mathcal{E}}))]}. \tag{6.80}$$

Applying lemma 5.1 we obtain for all $\boldsymbol{d} \in \mathbf{N}^{I\cap\mathfrak{I}_*^\circ}$, every sufficiently large r and every $\boldsymbol{\mathcal{E}} \in \mathrm{Div}_{\mathrm{eff}}(\mathscr{C}_{p,r})^{\mathfrak{I}}$ the inequality

$$|a(r,\boldsymbol{\mathcal{E}},\boldsymbol{d})| \leqslant p^{r|\boldsymbol{d}|}\frac{1+[I\cap\mathfrak{I}_*^\circ]\,.p^{-r\,\eta}}{(1-p^{-r\eta})^{[I\cap\mathfrak{I}_*^\circ]}}\left|\left|\widetilde{G}(r,0,\boldsymbol{t})\right|\right|_{p^{r(-1+\frac{\eta}{2})}} \prod_{\substack{v\in\mathscr{C}_{p,r}^{(0)}\\ v(\boldsymbol{\mathcal{E}})\neq 0}} 2\,c(\eta)\,p^{r f_v[C(v(\boldsymbol{\mathcal{E}}))+\eta\,D(v(\boldsymbol{\mathcal{E}}))]}. \tag{6.81}$$

Thus $p^{-r\left[\dim(X)(1-g_{\mathscr{C}})+\langle y\,,\,-\mathscr{K}_X\rangle\right]}\mathfrak{N}_1^*(r)$ is bounded from above by

$$\frac{1+[I\cap\mathfrak{I}_*^\circ]\;p^{-r\,\eta}}{(1-p^{-r\eta})^{[I\cap\mathfrak{I}_*^\circ]}}\left|\left|\widetilde{G}(0,\boldsymbol{t})\right|\right|_{p^{r\left(-1+\frac{\eta}{2}\right)}} h_{p,r}^{[I\setminus\mathfrak{I}_*]}.p^{-r\sum_{i\in I\setminus\mathfrak{I}_*}\langle y\,,\,\mathscr{E}_i\rangle}$$
$$\times\left(-1+\prod_{v\in\mathscr{C}_{p,r}^{(0)}} 1+2\,c(\eta)\sum_{\boldsymbol{e}\in\{0,1\}^{\mathfrak{I}_*^\circ}\setminus(0,\dots,0)} |\mu_X^\circ(\boldsymbol{e})|\,p^{r f_v[C(\boldsymbol{e})+\eta\,D(\boldsymbol{e})-|\boldsymbol{e}|]}\right). \tag{6.82}$$

Thanks to point 3 of assumptions 5.7 and lemma 5.2, the last factor in (6.82) tends to 0 as r goes to $+\infty$. Hence, thanks to (6.77) and (6.58), (6.72) is proved.

Next we show that for $j_0 \in \mathfrak{I}\setminus I$ we have

$$\lim_{r\to+\infty} \quad p^{-r\left[\dim(X)(1-g_{\mathscr{C}})+\langle y\,,\,-\mathscr{K}_X\rangle\right]}\mathfrak{N}_{1,j_0}(r) = 0. \tag{6.83}$$

The quantity involved in (6.83) equals

$$p^{r\,\dim(X)\,g_{\mathscr{C}}} \sum_{\substack{\boldsymbol{\mathcal{E}}\in\mathrm{Div}_{\mathrm{eff}}(\mathscr{C}_{p,r})^{\mathfrak{I}_*^\circ}\\ \deg(\mathcal{E}_i)\leqslant\langle y\,,\,\mathscr{E}_i\rangle,\quad i\in\mathfrak{I}_*^\circ\\ \left\langle y\,,\,\mathscr{E}_{j_0}+\mathscr{E}_{j_0+1}-\mathscr{D}_{\mathrm{tot}}\right\rangle\leqslant\deg(\mathcal{E}_{j_0})+\deg(\mathcal{E}_{j_0+1})+2\,g_{\mathscr{C}}-2}} |\mu_{X,r}(\boldsymbol{\mathcal{E}})|\,p^{-r\sum\limits_{j\in\mathfrak{I}\setminus I}\deg(\mathcal{E}_j)}$$
$$\times\, h_{p,r}^{[I\setminus\mathfrak{I}_*]}\,p^{-r\left\langle y\,,\,\sum\limits_{i\in I}\mathscr{E}_i\right\rangle}\;a(r,\boldsymbol{\mathcal{E}},(\langle y\,,\,\mathscr{E}_i\rangle-\deg(\mathcal{E}_i))_{i\in I\cap\mathfrak{I}_*^\circ}). \tag{6.84}$$

Using the last inequality in the description of the summation domain and (6.70) we obtain

$$-\frac{1}{4}[\deg(\mathcal{E}_{j_0})+\deg(\mathcal{E}_{j_0+1})] \leqslant -\dim(X)\,g_{\mathscr{C}}+\frac{g_{\mathscr{C}}}{2}-\frac{1}{2} \tag{6.85}$$

and we see that (6.84) is bounded from above by

$$p^{-r\,\frac{g_{\mathscr{C}}}{2}}\sum_{\substack{\boldsymbol{\mathcal{E}}\in\mathrm{Div}_{\mathrm{eff}}(\mathscr{C}_{p,r})^{\mathfrak{I}_*^\circ}\\ \deg(\mathcal{E}_i)\leqslant\langle y\,,\mathscr{E}_i\rangle,\quad i\in\mathfrak{I}_*^\circ}}|\mu_{X,r}(\boldsymbol{\mathcal{E}})|\,p^{-r\left[\sum\limits_{j\in\mathfrak{I}\setminus I}\deg(\mathcal{E}_j)-\frac14[\deg(\mathcal{E}_{j_0})+\deg(\mathcal{E}_{j_0+1})]\right]}$$

$$\times\,h_{p,r}^{[I\setminus\mathfrak{I}_*]}\,p^{\left\langle y\,,\sum\limits_{i\in I}\mathscr{E}_i\right\rangle}\,a(r,\boldsymbol{\mathcal{E}},(\langle y\,,\,\mathscr{E}_i\rangle-\deg(\mathcal{E}_i))_{i\in I\cap\mathfrak{I}_*^\circ}).\qquad(6.86)$$

Now arguing as in the case of $\mathcal{N}_1^*(r)$ we see that $p^{-r\left[\dim(X)(1-g_{\mathscr{C}})+\langle y\,,-\mathscr{K}_X\rangle\right]}\mathcal{N}_{1,j_0}(r)$ is bounded from above by

$$p^{-r\frac{g_{\mathscr{C}}+1}{2}}.\frac{1+[I\cap\mathfrak{I}_*^\circ]\,p^{-r\,\eta}}{(1-p^{-r\eta})^{[I\cap\mathfrak{I}_*^\circ]}}\left|\left|\widetilde{G}(r,0,\boldsymbol{t})\right|\right|_{p^{r\left(-1+\frac{\eta}{2}\right)}}h_{p,r}^{[I\setminus\mathfrak{I}_*]}.p^{-r\,\sum_{i\in I\setminus\mathfrak{I}_*}\langle y\,,\mathscr{E}_i\rangle}$$
$$\times\prod_{v\in\mathscr{C}_{p,r}^{(0)}}1+2\,c(\eta)\sum_{\boldsymbol{e}\in\{0,1\}^{\mathfrak{I}_*^\circ}\setminus(0,\dots,0)}\left|\mu_{X,\mathfrak{I}}^\circ(\boldsymbol{e})\right|p^{rf_v\left[C(\boldsymbol{e})+\eta\,D(\boldsymbol{e})-|\boldsymbol{e}|+\frac14(e_{j_0}+e_{j_0+1})\right]}.\qquad(6.87)$$

Thanks to (6.77), point 3 of assumptions 5.7 lemma 5.2 and (6.58), (6.83) is proved.

Finally we show

$$\lim_{r\to+\infty}\quad p^{-r\left[\dim(X)(1-g_{\mathscr{C}})+\langle y\,,-\mathscr{K}_X\rangle\right]}\,\mathcal{N}_0(r)=\begin{cases}1&\text{if}\quad\mathfrak{I}_*=\mathfrak{I}\\0&\text{otherwise.}\end{cases}\qquad(6.88)$$

One has

$$\mathcal{N}_0(r)=p^{r\left[\dim(X)(1-g_{\mathscr{C}})+\left\langle y\,,-\mathscr{D}_{\mathrm{tot}}+\sum_{j\in\mathfrak{I}\setminus I}\mathscr{E}_j\right\rangle\right]}h_{p,r}^{[I\setminus\mathfrak{I}_*]}\,a(r,0,(\langle y\,,\,\mathscr{E}_i\rangle)_{i\in I\cap\mathfrak{I}_*^\circ}).\qquad(6.89)$$

Owing to lemma 5.1 we deduce

$$\left|p^{-r\left[\dim(X)(1-g_{\mathscr{C}})+\langle y\,,-\mathscr{K}_X\rangle\right]}\mathcal{N}_0(r)-h_{p,r}^{[I\setminus\mathfrak{I}_*]}\,p^{-r\sum\limits_{i\in I\setminus\mathfrak{I}_*}\langle y\,,\mathscr{E}_i\rangle}\widetilde{G}(r,0,(p^{-r},\dots,p^{-r}))\right|$$
$$\leqslant\frac{p^{-r\frac{\eta}{2}}}{(1-p^{-r\frac{\eta}{2}})^{[I\cap\mathfrak{I}_*^\circ]}}\left|\left|\widetilde{G}(r,0,\boldsymbol{t})\right|\right|_{p^{r\left(-1+\frac{\eta}{2}\right)}}h_{p,r}^{[I\setminus\mathfrak{I}_*]}p^{-r\sum\limits_{i\in I\setminus\mathfrak{I}_*}\langle y\,,\mathscr{E}_i\rangle}\sum_{i\in I\cap\mathfrak{I}_*^\circ}p^{-r\frac{\eta}{2}\langle y\,,\mathscr{E}_i\rangle}.\qquad(6.90)$$

Thanks to (6.77) and (6.58) the right hand side of (6.90) tends to 0 as r goes to $+\infty$. Thanks to (6.79), this concludes the proof of (6.88), and, owing to (6.71), this settles point 3 of proposition 4.5.

7. Examples

7.1. **A family of intrinsic quadrics.** There is a family of smooth projective varieties $(X_n)_{n\geqslant 3}$ satisfying the following property (*cf.* [Bou11, §4.3]; the third point below can easily be deduced from [*ibid.*, Remarque 44])

(1) $\dim(X_n)=n-1$
(2) the Cox ring of X_n may be generated by sections $\{s_i\}_{0\leqslant i\leqslant 2\,n}$, with divisors $\{\mathscr{E}_i\}_{0\leqslant i\leqslant 2\,n}$ and such that $(\mathscr{E}_0,\dots,\mathscr{E}_n)$ is a basis of $\mathrm{Pic}(X)$, the ideal of relations in the Cox ring is generated by $\sum_{1\leqslant i\leqslant n}s_is_{i+n}$ and for $i\in\{1,\dots,n\}$ we have $\mathscr{E}_{i+n}\sim-\mathscr{E}_i+\sum_{0\leqslant i'\leqslant n}\mathscr{E}_{i'}$
(3) the maximal subsets J of $\{0,\dots,n\}$ such that $\cap_{i\in J}\mathscr{E}_i\neq\varnothing$ are those of the shape $\{0,\dots,n\}\setminus\{i_0,i_1\}$ where i_0, i_1 are distinct elements of $\{1,\dots,n\}$.

Hence X is a linear intrinsic hypersurface with respect to $(\{0,\dots,2n\},\{0,\dots,n\})$. It is straightforward to check that $\frac{1}{n-1}\sum_{1\leqslant i\leqslant n}\mathscr{E}_{i+n}-\mathscr{D}_{\text{tot}}$ lies in $\text{Eff}(X)$. Moreover, we have to show that assumptions 5.7 are satisfied for every $\mathfrak{I}_*\subset\{0,\dots,2n\}$ such that $\{n+1,\dots,2n\}\subset\mathfrak{I}_*$ and $\cap_{i\notin\mathfrak{I}_*}\mathscr{E}_i\neq\varnothing$. In view of the third point above, the necessary arguments are contained in the proof of [*ibid.*, Théorème 47]. Thus we obtain

Theorem 7.1. *For every $n\geqslant 3$, for every y lying in a trunction of $\text{Eff}(X_n)^\vee$, $\boldsymbol{Mor}(\mathscr{C},X_n,y,\mathfrak{I})$ is irreducible of the expected dimension and dense in $\boldsymbol{Mor}(\mathscr{C},X_n,y)$.*

7.2. **Minimal resolution of singular del Pezzo surfaces.** If X is an intrinsic hypersurface with a chosen set of generating sections $\{s_i\}_{i\in\mathfrak{I}}$, let us call a subset $I\subset\mathfrak{I}$ admissible if X is a linear intrinsic hypersurface with respect to $(\mathfrak{I},I)$. For I admissible denote by $\mathcal{C}_I$ the dual of the cone generated by the effective cone and $\frac{1}{[\mathfrak{I}\setminus I]-1}\sum_{j\in\mathfrak{I}\setminus I}\mathscr{E}_j-\mathscr{D}_{\text{tot}}$ and $\widetilde{\mathcal{C}}$ the union of the cone $\mathcal{C}_I$ when I ranges over the admissible subsets of $\mathfrak{I}$.

Derenthal has classified in [Der06] all the singular del Pezzo surfaces of degree at least 3 whose minimal resolution is an intrinsic hypersurface, giving in each case an explicit presentation of the Cox ring. There are 21 such surfaces, and among these it turns out that there are 20 for which there exists at least one admissible subset I (the exception being one of the cubic surfaces with a $\mathbf{D}_4$ singularity). Among them we are interested in those for which there exists at least one admissible subset I such that $\mathcal{C}_I$ is of maximal dimension (that is to say those for which, once the ad hoc assumptions on the combinatoric series are satisfied, there is a "positive proportion" of y for which theorem 2.4 guarantees that $\mathbf{Mor}(\mathscr{C},X,y)$ is irreducible of the expected dimension, see remark 2.6). It turns out that the 20 surfaces may be divided into three classes, according to the three following possibilities:

(1) For each subset $\{i_1,i_2,i_3\}$ of $\mathfrak{I}_*$ with cardinality 3, $\mathscr{E}_{i_1}\cap\mathscr{E}_{i_2}\cap\mathscr{E}_{i_3}$ is empty. In this case there are several choices of admissible I and for each of them $\mathcal{C}_I$ has maximal dimension (5 surfaces).
(2) There is exactly one subset $\{i_1,i_2,i_3\}$ of $\mathfrak{I}_*$ with cardinality 3, such that $\mathscr{E}_{i_1}\cap\mathscr{E}_{i_2}\cap\mathscr{E}_{i_3}$ is a point, and moreover $I=\mathfrak{I}_*\setminus\{i_1,i_2,i_3\}$ is admissible. In this case the only admissible subset for which $\mathcal{C}_I$ has maximal dimension is $\mathfrak{I}_*\setminus\{i_1,i_2,i_3\}$ (6 surfaces).
(3) There is exactly one subset $\{i_1,i_2,i_3\}$ of $\mathfrak{I}_*$ with cardinality 3, such that $\mathscr{E}_{i_1}\cap\mathscr{E}_{i_2}\cap\mathscr{E}_{i_3}$ is a point, and moreover $I=\mathfrak{I}_*\setminus\{i_1,i_2,i_3\}$ is not admissible. In this case there are no admissible subsets for which $\mathcal{C}_I$ has maximal dimension[1](9 surfaces).

For each of the surfaces in the first two classes, we use Maple to check whether the assumptions on the combinatoric series hold. This happens to be the case for each of them except one (the other cubic surface with a $\mathbf{D}_4$ singularity). For each of the 10 remaining surfaces, we estimate the "proportion" of those y for which theorem 2.4 guarantees that $\mathbf{Mor}(\mathscr{C},X,y)$ is irreducible of the expected dimension by computing the ratio $\frac{\text{Vol}_{-\mathscr{K}_X}(\widetilde{\mathcal{C}})}{\text{Vol}_{-\mathscr{K}_X}(\text{Eff}(X)^\vee)}$, where $\text{Vol}_{-\mathscr{K}_X}$ is the volume of the intersection with the affine hyperplane $\langle .,-\mathscr{K}_X\rangle=1$. Using the description of the surface as a blowing-up of the projective plane, we also compute the ratio $\frac{\text{Vol}_{-\mathscr{K}_X}(\mathcal{C}_{\text{KLO}})}{\text{Vol}_{-\mathscr{K}_X}(\text{Eff}(X)^\vee)}$ where $\mathcal{C}_{\text{KLO}}$ is a subcone $\mathcal{C}_{\text{KLO}}$ of $\text{Eff}(X)^\vee$ described in [KLO07] and such that, according to the main theorem of *ibid.*, for every $y\in\text{NS}(X)^\vee\cap\mathcal{C}_{\text{KLO}}$,

[1]For some surfaces in the third class, it is certainly possible to show that $\mathbf{Mor}(\mathscr{C},X,y)$ is irreducible of the expected dimension for a positive proportion of y, by using a similar strategy and a counting lemma akin to the one used in [Bou11, §5] to prove the geometric Manin's conjecture for the sextic del Pezzo surface with an $\mathbf{A}_2$ singularity (which belongs to the third class).

$\mathbf{Mor}(\mathbf{P}^1, X, y)$ is irreducible of the expected dimension. The authors of *ibid.* use deformation-theoretic arguments and it seems very likely that similar arguments could yield the same result in higher genus, upon replacing $\mathcal{C}_{\mathrm{KLO}}$ by an adequate truncation. The results of the computations, for which we benefited from [Fra09], are presented in table 1. Each surface is identified by its degree and the type of the singularity[2]. We give generators of $\mathcal{C}_{\mathrm{KLO}}^{\vee}$ in terms of the boundary divisors, for which we use the notations of [Der06]. Note that the desingularization of the sextic with a $\mathbf{A}_1$ singularity is isomorphic to the variety X_3 of theorem 7.1.

degree	singularities	$\frac{\mathrm{Vol}(\widetilde{\mathcal{C}})}{\mathrm{Vol}(\mathrm{Eff}(X)^{\vee})}$	generators of $\mathcal{C}_{\mathrm{KLO}}^{\vee}$	$\frac{\mathrm{Vol}(\mathcal{C}_{\mathrm{KLO}})}{\mathrm{Vol}(\mathrm{Eff}(X)^{\vee})}$	$\frac{\mathrm{Vol}(\mathcal{C}_{\mathrm{KLO}})}{\mathrm{Vol}(\widetilde{\mathcal{C}})}$
6	$\mathbf{A}_1$	1	E_1, E_2, E_3, E_4	1	1
5	$\mathbf{A}_1$	23/36	$E_2, E_3, E_4, E_5, E_1 - E_5$	1/4	≈ 0.391
5	$\mathbf{A}_2$	3/8	$E_2, E_3, E_4, E_5, E_1 - 2\,E_5$	3/14	≈ 0.571
4	$3\mathbf{A}_1$	31/72	$E_3, E_5, E_6, E_7, E_9,$ $E_2 - E_3 - 3\,E_6$	1/28	≈ 0.083
4	$\mathbf{A}_2 + \mathbf{A}_1$	65/288	$E_1, E_4, E_6, E_7, E_9,$ $E_2 - E_4 - 3\,E_9$	3/80	≈ 0.166
4	$\mathbf{A}_3$	3/32	$E_2, E_3, E_4, E_5, E_8,$ $E_1 - E_2 - 4\,E_5 - E_8$	1/70	≈ 0.152
4	$\mathbf{A}_3 + \mathbf{A}_1$	1/8	$E_3, E_4, E_5, E_6, E_7,$ $A_1 - E_3 - 3\,E_4 - 6\,E_5 - E_6$	1/35	≈ 0.228
3	$2\mathbf{A}_2 + \mathbf{A}_1$	2567/23760	$E_1, E_2, E_5, E_6, E_7, E_{10},$ $E_3 - E_1 - 3\,E_6 - E_7 - 2\,E_{10}$	1/686	≈ 0.014
3	$\mathbf{A}_3 + 2\,\mathbf{A}_1$	181/3888	$E_1, E_2, E_3, E_5, E_6, E_8,$ $E_4 - E_1 - E_3 - 3\,E_5 - 4\,E_8$	2/2205	≈ 0.019
3	$\mathbf{A}_4 + \mathbf{A}_1$	5/288	$E_1, E_3, E_4, E_6, E_7, E_8,$ $A - E_1 - E_3 - 3\,E_4 - 6\,E_6 - 3\,E_8$	1/441	≈ 0.131

TABLE 1

Of course, since the authors of [KLO07] address the case of general blowing-ups of projective space, their method covers a wide range of varieties which are not justiciable of our approach. Nevertheless, one notes that for the examples in table 1, where both methods apply, our numerical constraints are weaker than theirs (one can check that this is also true in general for toric varieties which are blowing-ups of a projective space). One hopes that Cox rings might prove helpful for the understanding of the geometry of the moduli spaces of morphisms, at least when they have a sufficiently simple presentation. A similar philosophy prevails in the context of Manin's conjecture about the asymptotic behaviour of rational points/curves of bounded height/degree.

REFERENCES

[Bat02] Victor Batyrev. Communication at the AIM workshop Rational and integer points on higher dimensional varieties, Palo Alto. 2002.

[BH07] Florian Berchtold and Jürgen Hausen. Cox rings and combinatorics. *Trans. Amer. Math. Soc.*, 359(3):1205–1252 (electronic), 2007.

[Bou03] David Bourqui. Fonction zêta des hauteurs des variétés toriques déployées dans le cas fonctionnel. *J. Reine Angew. Math.*, 562:171–199, 2003.

[2] Knowing that the minimal resolution of each surface in the table is an intrinsic hypersurface, and having ruled out the case of the **D**4 singularity in degree 3, these informations determine completely the isomorphism class of the surface, according to the results of [Der06]

[Bou09a] David Bourqui. Comptage de courbes sur le plan projectif éclaté en trois points alignés. *Ann. Inst. Fourier (Grenoble)*, 59(5):1847–1895, 2009.

[Bou09b] David Bourqui. Produit eulérien motivique et courbes rationnelles sur les variétés toriques. *Compos. Math.*, 145(6):1360–1400, 2009.

[Bou11] David Bourqui. La conjecture de Manin géométrique pour une famille de quadriques intrinsèques. *Manuscr. Math.*, 135(1-2):1–41, 2011.

[Cas04] Ana-Maria Castravet. Rational families of vector bundles on curves. *Internat. J. Math.*, 15(1):13–45, 2004.

[Cox95] David A. Cox. The functor of a smooth toric variety. *Tohoku Math. J. (2)*, 47(2):251–262, 1995.

[CS09] Izzet Coskun and Jason Starr. Rational curves on smooth cubic hypersurfaces. *Int. Math. Res. Not. IMRN*, (24):4626–4641, 2009.

[Deb01] Olivier Debarre. *Higher-dimensional algebraic geometry*. Universitext. Springer-Verlag, New York, 2001.

[Der06] Ulrich Derenthal. Singular Del Pezzo surfaces whose universal torsors are hypersurfaces. `arXiv:math/0604194v1`, 2006.

[dJS04] Aise Johan de Jong and Jason Starr. Cubic fourfolds and spaces of rational curves. *Illinois J. Math.*, 48(2):415–450, 2004.

[Fra09] Matthias Franz. Convex - a Maple package for convex geometry - version 1.1.3, 2009. 2009.

[Gue95] Martin A. Guest. The topology of the space of rational curves on a toric variety. *Acta Math.*, 174(1):119–145, 1995.

[Has04] Brendan Hassett. Equations of universal torsors and Cox rings. In *Mathematisches Institut, Georg-August-Universität Göttingen: Seminars Summer Term 2004*, pages 135–143. Universitätsdrucke Göttingen, Göttingen, 2004.

[HRS04] Joe Harris, Mike Roth, and Jason Starr. Rational curves on hypersurfaces of low degree. *J. Reine Angew. Math.*, 571:73–106, 2004.

[HRS05] Joe Harris, Mike Roth, and Jason Starr. Curves of small degree on cubic threefolds. *Rocky Mountain J. Math.*, 35(3):761–817, 2005.

[KLO07] Bumsig Kim, Yongnam Lee, and Kyungho Oh. Rational curves on blowing-ups of projective spaces. *Michigan Math. J.*, 55(2):335–345, 2007.

[KP01] Bumsig Kim and Rahul Pandharipande. The connectedness of the moduli space of maps to homogeneous spaces. In *Symplectic geometry and mirror symmetry (Seoul, 2000)*, pages 187–201. World Sci. Publ., River Edge, NJ, 2001.

[Per02] Nicolas Perrin. Courbes rationnelles sur les variétés homogènes. *Ann. Inst. Fourier (Grenoble)*, 52(1):105–132, 2002.

[Tho98] Jesper Funch Thomsen. Irreducibility of $\overline{M}_{0,n}(G/P,\beta)$. *Internat. J. Math.*, 9(3):367–376, 1998.