

From the curve to its Jacobian and back

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- 2 Period matrices and ThetaNullwerte
 - Period matrices
 - ThetaNullwerte
 - From the ThetaNullwerte to the Riemann matrix
 - From the Riemann matrix to the (quotients of) ThetaNullwerte
- 3 From the curve to its Jacobian
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 - Even characteristics
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Why do we care ?

CM method: CM-type + fundamental unit \rightsquigarrow lattice + polarization \rightsquigarrow period matrix \rightsquigarrow ThetaNullwerte \rightsquigarrow $\left\{ \begin{array}{l} \text{the curve over } \mathbb{C} \\ \text{invariants} \end{array} \right. \rightsquigarrow$ curve $/\mathbb{F}_q$.

AGM for point counting: curve $/\mathbb{F}_q \rightsquigarrow$ lift \rightsquigarrow quotients of ThetaNullwerte \rightsquigarrow canonical lift + info on Weil polynomial \rightsquigarrow Weil polynomial.

Other applications: fast computation of modular polynomials, class polynomials, isogenies ...

Caution: work over \mathbb{C} but try to show why it works in general.

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Definitions

Let C be a curve over $k \subset \mathbb{C}$ of genus $g > 0$.

The **Jacobian** of C is a torus $\text{Jac}(C) \simeq \mathbb{C}^g / \Lambda$ where

- the lattice $\Lambda = \Omega \mathbb{Z}^{2g}$,
- the matrix $\Omega = [\Omega_1, \Omega_2] \in M_{g, 2g}(\mathbb{C})$ is a **period matrix** and

-

$$\tau = \Omega_2^{-1} \Omega_1 \in \mathbb{H}_g = \{M \in \text{GL}_g(\mathbb{C}), {}^t M = M, \text{Im } M > 0\}$$

is a **Riemann matrix**.

Construction

- v_1, \dots, v_g be a k -basis of $H^0(C, \Omega^1)$,
- $\delta_1, \dots, \delta_{2g}$ be generators of $H_1(C, \mathbb{Z})$ such that $(\delta_j)_{1 \dots 2g}$ form a symplectic basis for the intersection pairing on C .

$$\Omega := [\Omega_1, \Omega_2] = \left[\int_{\delta_j} v_i \right]_{\substack{i=1, \dots, g \\ j=1, \dots, 2g}}.$$

- Magma (Vermeulen): can compute Ω for a hyperelliptic curve.
- Maple (Deconinck, van Hoeij) can compute Ω for any plane model.

Rem: there is a **polarization** j involved in the definition of Ω with Chern class

$$2i \left(\bar{\Omega} \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} {}^t \Omega \right)^{-1}.$$

Example

Ex: $E : y^2 = x^3 - 35x - 98 = (x - 7)(x - a)(x - \bar{a})$ which has complex multiplication by $\mathbb{Z}[\alpha]$ with $\alpha = \frac{-1 - \sqrt{-7}}{2}$ and $a = \frac{-7}{2} - \frac{\sqrt{-7}}{2}$.

$$\Omega = \left[2 \int_a^{\bar{a}} \frac{dx}{2y}, 2 \int_a^7 \frac{dx}{2y} \right] = c \cdot [\alpha, 1].$$

(Chowla, Selberg 67) formula gives

$$c = \frac{1}{8\pi\sqrt{7}} \cdot \Gamma\left(\frac{1}{7}\right) \cdot \Gamma\left(\frac{2}{7}\right) \cdot \Gamma\left(\frac{4}{7}\right)$$

with

$$\Gamma(x) = \int_0^{\infty} t^{x-1} \exp(-t) dt.$$

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Projective embedding

The polarization j comes from an **ample** divisor D on $\text{Jac}(C)$ (defined up to translation).

Theorem (Lefschetz, Mumford, Kempf)

*For $n \geq 3$, nD is **very ample**, i.e. one can embed $\text{Jac}(C)$ in a \mathbb{P}^{n^g-1} with a basis of sections of $\mathcal{L}(nD)$.*

For $n = 4$, the embedding is given by intersection of quadrics, whose equations are completely determined by the image of 0.

ThetaNullwert

A basis of sections of $\mathcal{L}(4D)$ is given by **theta functions** $\theta[\varepsilon](2z, \tau)$ with integer characteristics $[\varepsilon] = (\epsilon, \epsilon') \in \{0, 1\}^{2g}$ where

$$\theta \begin{bmatrix} \epsilon \\ \epsilon' \end{bmatrix} (z, \tau) = \sum_{n \in \mathbb{Z}^g} \exp \left(i\pi \left(n + \frac{\epsilon}{2} \right) \tau^t \left(n + \frac{\epsilon}{2} \right) + 2i\pi \left(n + \frac{\epsilon}{2} \right)^t \left(z + \frac{\epsilon'}{2} \right) \right).$$

When $\epsilon^t \epsilon' \equiv 0 \pmod{2}$, $[\varepsilon]$ is said **even** and one calls **ThetaNullwert**

$$\theta \begin{bmatrix} \epsilon \\ \epsilon' \end{bmatrix} (0, \tau) = \theta \begin{bmatrix} \epsilon \\ \epsilon' \end{bmatrix} (\tau) = \theta[\varepsilon](\tau) = \theta_{ab}$$

where the binary representations of a and b are ϵ, ϵ' .

Example

Let $q = \exp(\pi i \tau)$. There are 3 genus 1 ThetaNullwerte:

$$\theta_{00} = \theta \begin{bmatrix} 0 \\ 0 \end{bmatrix} (0, \tau) = \sum_{n \in \mathbb{Z}} q^{n^2},$$

$$\theta_{10} = \theta \begin{bmatrix} 1 \\ 0 \end{bmatrix} (0, \tau) = \sum_{n \in \mathbb{Z}} q^{(n + \frac{1}{2})^2},$$

$$\theta_{01} = \theta \begin{bmatrix} 0 \\ 1 \end{bmatrix} (0, \tau) = \sum_{n \in \mathbb{Z}} (-1)^n q^{n^2}.$$

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Case $g = 1$ Gauss, Cox 84, Dupont 07

- Let $z = \theta_{01}(\tau)^2 / \theta_{00}(\tau)^2$.
- Duplication formulae vs AGM formulae :

$$\left. \begin{aligned} \theta_{00}(2\tau)^2 &= \frac{\theta_{00}(\tau)^2 + \theta_{01}(\tau)^2}{2} \\ \theta_{01}(2\tau)^2 &= \theta_{00}(\tau) \cdot \theta_{01}(\tau) \\ \theta_{10}(2\tau)^2 &= \frac{\theta_{00}(\tau)^2 - \theta_{01}(\tau)^2}{2} \end{aligned} \right| \begin{aligned} a_n &= \frac{a_{n-1} + b_{n-1}}{2}, \\ b_n &= \sqrt{a_{n-1} \cdot b_{n-1}}, \end{aligned}$$

$$\Rightarrow \text{AGM}(\theta_{00}(\tau)^2, \theta_{01}(\tau)^2) = \lim \theta_{00}(2^n \tau)^2 = 1 \Rightarrow \text{AGM}(1, z) = \frac{1}{\theta_{00}(\tau)^2}.$$

$$\Rightarrow \theta_{10}(\tau)^2 = \sqrt{\theta_{00}(\tau)^4 - \theta_{01}(\tau)^4}.$$

- Transformation formula :

$$\theta_{00}(\tau)^2 = \frac{i}{\tau} \cdot \theta_{00} \left(\frac{-1}{\tau} \right)^2, \quad \theta_{10}(\tau)^2 = \frac{i}{\tau} \cdot \theta_{01} \left(\frac{-1}{\tau} \right)^2.$$

$$\Rightarrow AGM(\theta_{00}(\tau)^2, \theta_{10}(\tau)^2) = \frac{i}{\tau} \cdot \lim \theta_{00}(2^n \cdot \frac{-1}{\tau})^2 = \frac{i}{\tau} \cdot 1$$

$$\Rightarrow AGM(1, \sqrt{1 - z^2}) = \frac{i}{\tau} \cdot \frac{1}{\theta_{00}(\tau)^2}.$$

Proposition

$$\frac{i \cdot AGM(1, z)}{AGM(1, \sqrt{1 - z^2})} = \tau.$$

Difficulty: define the correct square root when the values are complex.

Rem: one cannot get Ω from the ThetaNullwerte. But from the curve:

Theorem (Gauss, Cox 84)

If $E : y^2 = x(x - a^2)(x - b^2)$ then $[\omega_1, \omega_2] = \left[\frac{\pi}{AGM(a,b)}, \frac{i\pi}{AGM(a+b, a-b)} \right]$ is a period matrix relative to dx/y .

Use the same ingredients as above and, as first step, the Thomae's formulae

$$\omega_2 \cdot a = \pi \cdot \theta_{00}(\tau)^2, \quad \omega_2 \cdot b = \pi \cdot \theta_{01}(\tau)^2.$$

Case $g \geq 2$

Particular case: real Weierstrass points and $g = 2$ (Bost-Mestre 88).

General case (Dupont 07): under some (experimentally verified) conjectures.

Proposition

One can compute τ in terms of $\theta[\varepsilon](\tau)^2/\theta[0](\tau)^2$ in time

$$O(g^2 \cdot 2^g \cdot M(n) \cdot \log n)$$

for n digits of precision ($M(n)$ is the complexity of the binary multiplication).

Question: what about the period matrix ?

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The work of (Dupont 07)

Naive method: $O(M(n)\sqrt{n})$ for $g = 1$ and $O(n^{2+\epsilon})$ for $g = 2$.

New method: invert the AGM. Complexity for n bits of precision on the quotients

- $O(M(n) \log n)$ for $g = 1$,
- $O(n^{1+\epsilon})$ for $g = 2$ (conjectural algorithm).

Main idea for $g = 1$: let

$$f(z) = i \cdot \text{AGM}(1, z) - \tau \cdot \text{AGM}(1, \sqrt{1 - z^2}).$$

Then $f(\theta_{01}(\tau)^2/\theta_{00}(\tau)^2) = 0$. Do a Newton algorithm on f .

- can we get rid of the conjectures ?
- can we generalize to all genera ?
- can we compute the ThetaNullwerte alone ?

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Thomae's formulae

Let C be a hyperelliptic curve $C : y^2 = \prod_{i=1}^{2g+1} (x - \lambda_i)$.

Theorem (Thomae's formulae)

$$\theta[\varepsilon](\tau)^4 = \pm \left(\frac{\det \Omega_2}{\pi^g} \right)^2 \prod_{(i,j) \in I} (\lambda_i - \lambda_j)$$

with the choice of the basis of differentials $x^i dx/y$ (the set I depends on $[\varepsilon]$ and on the basis of $H_1(C, \mathbb{Z})$).

Proof: see (Fay 73) using a variational method.

Proof for the quotients:

- study the zeroes of the section

$$s_\varepsilon(P) = \theta[\varepsilon](\phi_{P_0}(P))$$

where $P_0 \in C$ and $\phi_{P_0}(P) = P - P_0 \in \text{Jac}(C)$.

- $c \cdot f(P) = \frac{s_\varepsilon(P)^2}{s_{\varepsilon'}(P)^2}$ for an explicit $f \in \mathbb{C}(C)$.
- $c = \frac{s_\varepsilon(P_1)^2}{s_{\varepsilon'}(P_1)^2 f(P_1)} = \frac{s_\varepsilon(P_2)^2}{s_{\varepsilon'}(P_2)^2 f(P_2)}$ for P_1, P_2 such that $\frac{s_\varepsilon(P_2)^2}{s_{\varepsilon'}(P_2)^2} = \frac{s_{\varepsilon'}(P_1)^2}{s_\varepsilon(P_1)^2}$.

Rem: work in progress by Cosset for non-integral $[\varepsilon]$.

Main result on $s_\varepsilon(P)$

Let C be a curve of genus $g > 0$, $P_0 \in C$.

Theorem

If $s_\varepsilon(P)$ is not identically zero, then $s_\varepsilon(P)$ has g zeroes P_1, \dots, P_g such that the divisor $D = P_1 + \dots + P_g$ is characterized by $D - gP_0 \sim \varepsilon + \kappa$ where κ is a constant depending only on the homology basis and on P_0 .

Geometrically, let $\Theta = \{z, \theta[0](z, \tau) = 0\} \subset \text{Jac}(C)$, L be the corresponding ample line bundle.

- Poincaré's formula $\Rightarrow (\phi_{P_0}(C) \cdot \Theta) = g$.
- s_ε is a section of the line bundle $\mathcal{L}_\varepsilon = \phi_{P_0}^* t_\varepsilon^* L = \mathcal{O}_C(P_1 + \dots + P_g)$.

- **Riemann's theorem:** $\exists \kappa_0$ a theta characteristic such that

$$\mathrm{Sym}^{g-1} C - \kappa_0 = \Theta.$$

$\Rightarrow \mathcal{L}_0 = \phi_{P_0}^* L = \mathcal{O}_C(P_1^0 + \dots + P_g^0)$ with $P_1^0 + \dots + P_g^0 \sim \kappa_0 + P_0$.
Indeed $P_i^0 - P_0 \sim -(\sum_{j \neq i} P_j^0 - \kappa_0) \in -\Theta = \Theta$.

- **Canonical isomorphism:**

$$\phi_{P_0}^* : \mathrm{Pic}^0(\mathrm{Jac}(C)) = \widehat{\mathrm{Jac}(C)} \rightarrow \mathrm{Pic}^0(C) = \mathrm{Jac}(C)$$

is an isomorphism with inverse $-\phi_L$.

\Rightarrow

$$\begin{aligned} \phi_{P_0}^*(t_\varepsilon^* L \otimes L^{-1}) &= -\phi_L^{-1} \circ \phi_L(\varepsilon) = \mathcal{O}_C(\varepsilon) \\ &= \mathcal{O}_C(\sum P_i - P_0 - \kappa_0) = \mathcal{O}_C(\sum P_i - \kappa) \end{aligned}$$

where $\kappa = \kappa_0 - (g-1)P_0$.

Lemma

$s_\varepsilon \equiv 0 \iff \varepsilon + \kappa \sim D - gP_0$ with $D \in \mathbf{Sym}^g(C)$ and $i(D) > 0$.

$$\begin{aligned}
 s_\varepsilon \equiv 0 &\iff \forall P, P - P_0 - \varepsilon \in \Theta \\
 &\iff P - P_0 - D + gP_0 + \kappa_0 - (g-1)P_0 \in \mathbf{Sym}^{g-1}(C) - \kappa_0 \\
 &\iff P - D + 2\kappa_0 \in \mathbf{Sym}^{g-1}(C) \\
 &\iff D - P \in \mathbf{Sym}^{g-1}(C) \iff i(D) > 0
 \end{aligned}$$

Corollary

The zero divisor D of s_ε is completely determined by the equivalence $D - gP_0 \sim \varepsilon + \kappa$.

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Non hyperelliptic case genus 3

Let C be a smooth plane quartic.

Theorem (Weber 1876)

$$\left(\frac{\theta[\mathcal{E}](\tau)}{\theta[\mathcal{E}'](\tau)} \right)^4 = \frac{[b_i, b_j, b_{ij}][b_{ik}, b_{jk}, b_{ij}][b_j, b_{jk}, b_k][b_i, b_{ik}, b_k]}{[b_j, b_{jk}, b_{ij}][b_i, b_{ik}, b_{ij}][b_i, b_j, b_k][b_{ik}, b_{jk}, b_k]}$$

where the b_i, b_{ij} are linear equations of certain bitangents of C and $[b_i, b_j, b_k]$ is the determinant of the matrix of the coefficients of (once for all fixed) equations of the bitangents.

- Weber's proof uses $s_{\mathcal{E}}(P)$.
- **Question:** can we find a formula for a Thetanullwert alone like in the hyperelliptic case ?

Derivative of theta functions

When $\epsilon^t \epsilon' \equiv 1 \pmod{2}$, $[\epsilon]$ is said **odd** and we write $[\mu]$ instead.

Definition

The **theta gradient** (with odd characteristic $[\mu]$) is the vector

$$\nabla\theta[\mu] := \left(\frac{\partial\theta[\mu](z, \tau)}{\partial z_1}(0, \tau), \dots, \frac{\partial\theta[\mu](z, \tau)}{\partial z_g}(0, \tau) \right).$$

The **theta hyperplane** is the projective hyperplane

$$\nabla\theta[\mu] \cdot (X_1, \dots, X_g) = 0$$

of \mathbb{P}^{g-1} defined by a theta gradient.

We denote the matrix

$$J[\mu_1, \dots, \mu_g] := (\nabla\theta[\mu_1], \dots, \nabla\theta[\mu_g])$$

and $[\mu_1, \dots, \mu_g]$ its determinant (called **Jacobian Nullwerte**).

Case of Riemann-Mumford-Kempf singularity theorem

Let C be any curve of genus $g > 0$.

Theorem

Let ϕ be the *canonical map*

$$\phi : C \rightarrow \mathbb{P}^{g-1}, P \mapsto (\omega_1(P), \dots, \omega_g(P)).$$

Let D be an effective divisor of degree $g - 1$ on C such that $h^0(D) = 1$.
Then

$$\left(\frac{\partial \theta(z, \tau)}{\partial z_1}(D - \kappa_0, \tau), \frac{\partial \theta(z, \tau)}{\partial z_g}(D - \kappa_0, \tau) \right) \Omega_2^{-1} \iota(X_1, \dots, X_g) = 0$$

is an hyperplan of \mathbb{P}^{g-1} which cuts out the divisor $\phi(D)$ on the curve $\phi(C)$.

Rem: this can also be re-interpreted geometrically in terms of Gauss maps.

Generalization of Jacobi's derivative formula

$$\frac{d\theta \begin{bmatrix} 1 \\ 1 \end{bmatrix}}{dz}(z, \tau) = \pi \cdot \theta \begin{bmatrix} 0 \\ 0 \end{bmatrix}(0, \tau) \cdot \theta \begin{bmatrix} 0 \\ 1 \end{bmatrix}(0, \tau) \cdot \theta \begin{bmatrix} 1 \\ 0 \end{bmatrix}(0, \tau).$$

Theorem (Igusa 80)

Let $[\mu_1], \dots, [\mu_g]$ be distinct odd theta characteristics such that the function $[\mu_1, \dots, \mu_g](\tau)$ is contained in the \mathbb{C} -algebra $\mathbb{C}[\theta]$ generated by the functions $\theta[\varepsilon](\tau)$ for all even characteristics $[\varepsilon]$. Then

$$[\mu_1, \dots, \mu_g](\tau) = \pi^g \sum_{[\varepsilon_{g+1}], \dots, [\varepsilon_{2g+2}] \in \mathcal{S}} \pm \prod_{i=g+1}^{2g+2} \theta[\varepsilon_i](\tau),$$

where \mathcal{S} is the set of all $g + 2$ -tuples $\{[\varepsilon_{g+1}], \dots, [\varepsilon_{2g+2}]\}$ even theta characteristics such that $\{[\mu_1], \dots, [\mu_g], [\varepsilon_{g+1}], \dots, [\varepsilon_{2g+2}]\}$ forms a **fundamental system**.

Sketch of the proof of Weber's formula (Nart, R. unpublished)

Let $[\varepsilon], [\varepsilon']$ be two even characteristics in genus 3.

- create two fundamental systems of the form

$$\{[\mu_1], [\mu_2], [\mu_3], [\varepsilon], [\varepsilon_4], [\varepsilon_5], [\varepsilon_6], [\varepsilon_7]\}, \{[\mu'_1], [\mu'_2], [\mu'_3], [\varepsilon'], [\varepsilon_4], [\varepsilon_5], [\varepsilon_6], [\varepsilon_7]\}.$$

- $\#\mathcal{S} = 1$ and

$$\frac{[\mu_1, \mu_2, \mu_3]}{[\mu'_1, \mu'_2, \mu'_3]} = \frac{\theta[\varepsilon]}{\theta[\varepsilon']}.$$

- An odd 2-torsion point μ is given by $D - \kappa_0$ where D is a degree 2 divisor, support of a bitangent of equation $b_\mu = 0$.

-

$$[b_{\mu_1}, b_{\mu_2}, b_{\mu_3}] = \det(\Omega_2)^{-1} \cdot (\lambda_{\mu_1} \lambda_{\mu_2} \lambda_{\mu_3}) \cdot [\mu_1, \mu_2, \mu_3]$$

where λ_i are constants depending on the choice of scalar multiplier for b_i and of τ .

- use several quotients to get rid of the λ_i .

Remarks

- For $g = 4$, $\#S = 2$ and for $g = 7$, $\#S = 960$.
- For $g \leq 5$ it is known that $[\mu_1, \dots, \mu_g]$ is in $\mathbb{C}[\theta]$. In general, it is not true but $[\mu_1, \dots, \mu_g]$ can be expressed as a quotient of two polynomials in the ThetaNullwerte. There is also a precise conjectural formula (Igusa 80).
- Could we directly invert the formula, i.e. express a ThetaNullwert in terms of Jacobian Nullwerte (at least for $g \leq 5$) ?
- (Nakayashiki 97, Enolski, Grava 06): Thomae's formula for $y^n = \prod_{i=1}^m (x - \lambda_i)^{n-1} \cdot \prod_{i=m+1}^{2m} (x - \lambda_i)$.
- a general theory exists (Klein vol.3 p.429, Matone-Volpato 07 over \mathbb{C} , Shepherd-Barron preprint 08 over any field). Their expressions involve determinants of bases of $H^0(C, \mathcal{L}(2K_C + \mu))$. But no formula or implementation has been done.

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Torelli theorem: classical versions

Let C/k be a curve of genus $g > 0$.

Theorem

C is uniquely determined up to k -isomorphism by $(\text{Jac}(C), j)$.

Corollary

C is uniquely determined up to \mathbb{C} -isomorphism by Ω or by the ThetaNullwerte.

From the Jacobian to its curve: hyperelliptic case

$$C : y^2 = x(x-1) \prod_{i=1}^{2g-1} (x - \lambda_i).$$

Idea: invert quotient Thomae's formulae (Mumford Tata II p.136, Takase 96, Koizumi 97)

$$\frac{\lambda_k - \lambda_l}{\lambda_k - \lambda_m} = i^c \cdot \frac{\theta[\varepsilon_1]^2 \cdot \theta[\varepsilon_2]^2}{\theta[\varepsilon_3]^2 \cdot \theta[\varepsilon_4]^2}, \quad c \in \{0, 1, 2, 3\}.$$

- For genus 1: $\lambda_1 = \theta_1^4 / \theta_0^4$.
- For genus 2 (Rosenhain formula):

$$\lambda_1 = -\frac{\theta_{01}^2 \theta_{21}^2}{\theta_{30}^2 \theta_{10}^2}, \quad \lambda_2 = -\frac{\theta_{03}^2 \theta_{21}^2}{\theta_{30}^2 \theta_{12}^2}, \quad \lambda_3 = -\frac{\theta_{03}^2 \theta_{01}^2}{\theta_{10}^2 \theta_{12}^2}.$$

- For genus 3 (Weng 01):

$$\lambda_1 = \frac{(\theta_{15}\theta_3)^4 + (\theta_{12}\theta_1)^4 - (\theta_{14}\theta_2)^4}{2(\theta_{15}\theta_3)^4}, \quad \lambda_2 = \frac{(\theta_4\theta_9)^4 + (\theta_6\theta_{11})^4 - (\theta_{13}\theta_8)^4}{2(\theta_4\theta_9)^4}, \dots$$

From the Jacobian to its curve : non hyperelliptic genus 3

(Weber 1876) shows how to find the **Riemann model**:

$$C : \sqrt{x(a_1x + a'_1y + a''_1z)} + \sqrt{y(a_2x + a'_2y + a''_2z)} + \sqrt{z(a_3x + a'_3y + a''_3z)} = 0$$

with

$$a_1 = i \frac{\theta_{41}\theta_{05}}{\theta_{50}\theta_{14}}, \quad a'_1 = i \frac{\theta_{05}\theta_{66}}{\theta_{33}\theta_{50}}, \quad a''_1 = -\frac{\theta_{66}\theta_{41}}{\theta_{14}\theta_{33}},$$

$$a_2 = i \frac{\theta_{25}\theta_{61}}{\theta_{36}\theta_{70}}, \quad a'_2 = i \frac{\theta_{61}\theta_{02}}{\theta_{57}\theta_{34}}, \quad a''_2 = \frac{\theta_{02}\theta_{25}}{\theta_{70}\theta_{57}},$$

$$a_3 = i \frac{\theta_{07}\theta_{43}}{\theta_{16}\theta_{52}}, \quad a'_3 = i \frac{\theta_{60}\theta_{20}}{\theta_{75}\theta_{16}}, \quad a''_3 = \frac{\theta_{20}\theta_{07}}{\theta_{52}\theta_{75}}.$$

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Torelli theorems: odd versions

Theorem (Grushevsky, Salvati Manni 04)

A generic abelian variety of dimension $g \geq 3$ is uniquely determined by its theta gradients.

Theorem (Caporaso, Sernesi 03)

A general curve C of genus $g \geq 3$ is uniquely determined by its theta hyperplanes.

Rem: the second result is not a corollary of the first.

Hyperelliptic case: genus 2 example (Guàrdia 01,07)

Let $[\mu_1], \dots, [\mu_6]$ be the odd characteristics. Then C admits a **symmetric** model

$$y^2 = x \left(x - \frac{[\mu_1, \mu_3]}{[\mu_2, \mu_3]} \right) \left(x - \frac{[\mu_1, \mu_4]}{[\mu_2, \mu_4]} \right) \left(x - \frac{[\mu_1, \mu_5]}{[\mu_2, \mu_5]} \right) \left(x - \frac{[\mu_1, \mu_6]}{[\mu_2, \mu_6]} \right).$$

Remarks:

- his theory of symmetric models has nice invariants, nice reduction properties.
- he (also in Shimura's book 98 p.192) shows how to find algebraic differentials: $\Omega_2 = \frac{1}{2i\pi\theta[\varepsilon]} J(\mu_1, \dots, \mu_g)$ is such that $(dz_1, \dots, dz_g)\Omega_2$ are algebraic over k if τ comes from an abelian variety A defined over k .

Non hyperelliptic curves of genus 3: Guàrdia 09

Refinement of Riemann model: a smooth plane quartic over k is k -isomorphic to

$$\sqrt{\frac{[b_7 b_2 b_3][b_7 b'_2 b'_3]}{[b_1 b_2 b_3][b'_1 b'_2 b'_3]}} X_1 X'_1 + \sqrt{\frac{[b_1 b_7 b_3][b_7 b'_1 b'_3]}{[b_1 b_2 b_3][b'_1 b'_2 b'_3]}} X_2 X'_2 + \sqrt{\frac{[b_1 b_2 b_7][b_7 b'_1 b'_2]}{[b_1 b_2 b_3][b'_1 b'_2 b'_3]}} X_3 X'_3 = 0$$

where X_i, X'_i are the equations of the bitangents b_i, b'_i .

Ex: Take $A = E^3$ where E has CM by $\sqrt{-19}$ + the unique indecomposable principal polarization. Then $A = \text{Jac}(C)$ where

$$\begin{aligned} C : \quad & U^4 + 2U^3V - 2U^3W + (6 - 3i\sqrt{19})U^2V^2 + 18U^2VW + (6 + 3i\sqrt{19})U^2W^2 \\ & + (5 - 3i\sqrt{19})UV^3 + (15 + 3i\sqrt{19})UV^2W + (-15 + 3i\sqrt{19})UVW^2 \\ & + (-5 - 3i\sqrt{19})UW^3 + \frac{1}{2}(3 - 3i\sqrt{19})V^4 + (12 + 4i\sqrt{19})V^3W - 30V^2W^2 \\ & + (12 - 4i\sqrt{19})VW^3 + \frac{1}{2}(3 + 3i\sqrt{19})W^4 = 0. \end{aligned}$$

C descends over \mathbb{Q} as

$$C : x^4 + (1/9)y^4 + (2/3)x^2y^2 - 190y^2 - 570x^2 + (152/9)y^3 - 152x^2y - 1083 = 0$$

Summary

	$g = 1$	$g = 2$	$g \geq 3$ h.	$g = 3$ n.h.	$g > 3$ n.h.
$\theta \rightarrow \tau$	fast	fast conj.	fast conj.	fast conj.	fast conj.
$\tau \rightarrow \theta$	algo fast quotient	algo fast quot.	algo	algo	algo
$C \rightarrow \Omega$	fast	(free) algo	algo	algo	plane model
$C \rightarrow \theta$	fast	algo	algo	algo quot.	theory
$\theta \rightarrow C$	fast	fast	fast	fast	?
$\nabla\theta \rightarrow C$	fast	fast	fast	fast	?

algo: there exists an algorithm but slow.

fast (conj.): there exists a fast (conjectural) algorithm.

quot.: for the quotient of ThetaNullwerte.

theory: the theory is done but no implementation has been done.

?: nothing is done.