Rationality of intersection points of a line and a quartic

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- points on the Jacobian are represented by the sum D of 3 points on the curve C.
- Choice of a good divisor at infinity $\rightsquigarrow$ condition $(*)$:

  There is a rational line ℓ^∞ which crosses the quartic C in four k-points $P_1^\infty, P_2^\infty, P_3^\infty, P_4^\infty$.

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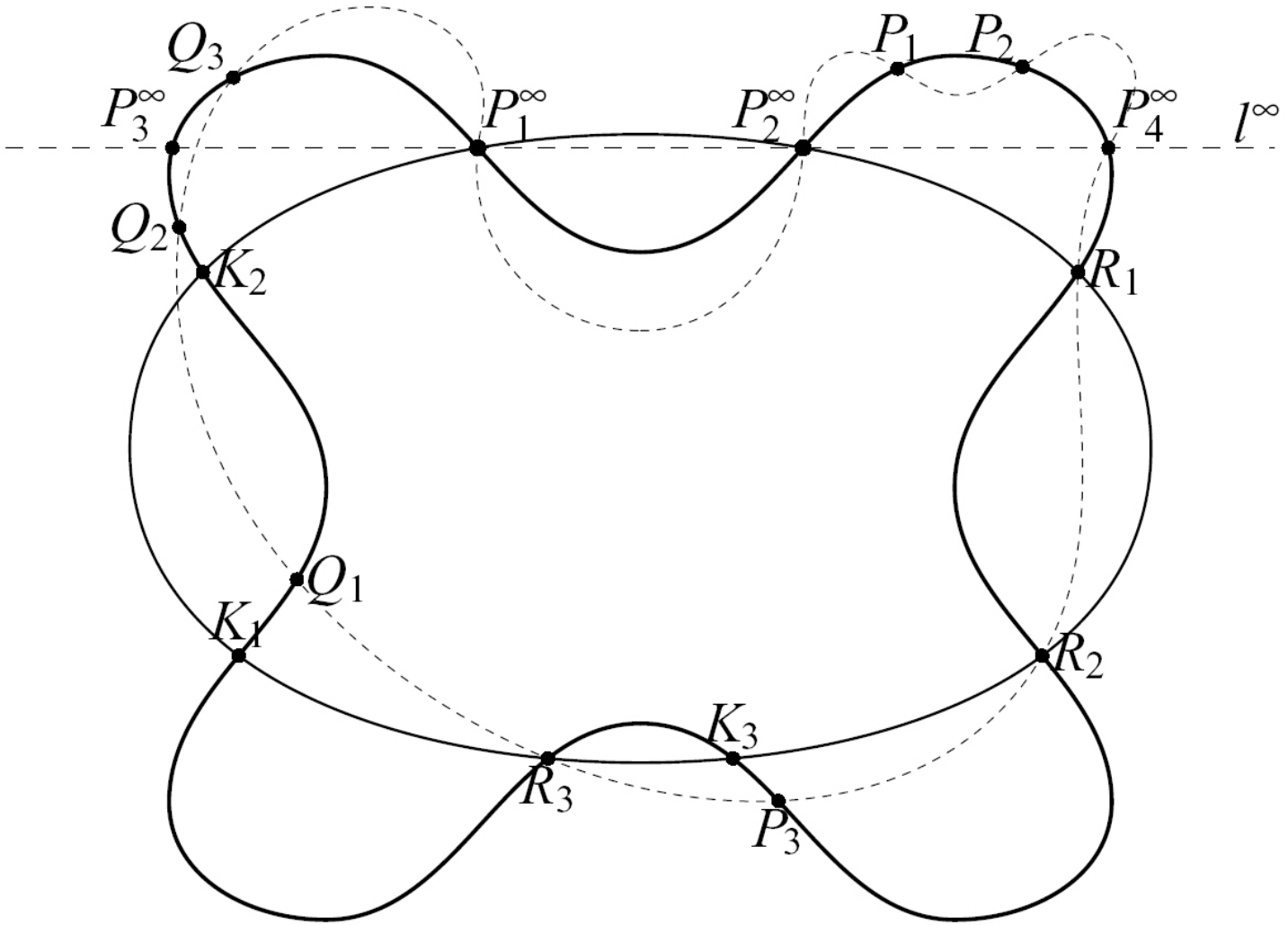
Why ? Because $(\ell^\infty \cdot C) \sim \kappa$, $(Q \cdot C) \sim 2\kappa$ and $(E \cdot C) \sim 3\kappa$ where κ is the canonical divisor on C.

A chord construction



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Remark: Quartics with a hyperflex form a sub-variety of codimension 1. So generically the two last cases do not occur.

Study of the condition $(*)$ over $\mathbb{F}_q$ (Oyono-R.)

Theorem

Let C be a smooth plane quartic over the finite field $\mathbb{F}_q$ with $q = p^n$ elements. If $q \geq 127$, then there exists a line ℓ which intersects C at rational points only.

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Let C be a smooth plane quartic over $\mathbb{F}_q$. If $q \geq 66^2 + 1$, then there exists a tangent to C which intersects C at rational points only.

Moreover, under a certain conjecture, the probability for a plane smooth quartic over a finite field to have a rational flex is about 0.63.

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1. Let $P \in C(k)$. Consider the separable geometric cover $\phi : C \to |\kappa - P| = \mathbb{P}^1$ of degree 3 induced by the linear system $|\kappa - P|$. This parametrizes lines through the point P.
2. Using effective Chebotarev's density theorem for function fields, one gets estimation on the number of completely split divisors in $|\kappa - P|$.

Proof for the tangent case

Let $T : C \to \mathrm{Sym}^2(C)$, $P \mapsto T_P(C) \cdot C - 2P$ be the tangential correspondence. We associate to it its correspondence curve

$$X_C = \{(P, Q) \in C \times C : Q \in T(P)\}$$

which is defined over k.

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Proposition (Aubry, Perret 95)

Let $X/\mathbb{F}_q$ be a geometrically irreducible curve of arithmetic genus π_X. Then

$$|\#X(\mathbb{F}_q) - (q+1)| \leq 2\pi_X\sqrt{q}.$$

In particular if $q \geq (2\pi_X)^2$ then X has a rational point.

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We prove that if X_C is not absolutely irreducible, then X_C is reducible and each component is isomorphic to C.Apply the previous bound to a component.

Remark: the only case where X_C is reducible is when C is geometrically isomorphic to the Klein quartic $x^3y + y^3z + z^3x = 0$ (in particular Conjecture 1 in the paper is false).

Computation of the arithmetic genus of X_C

If $p \neq 2$ and there is no hyperflex on C, one can use Riemann-Hurwitz formula with the degree 2-cover $\pi_1 : X_C \to C$ to get g_{X_C}.

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Remark : unfortunately, there is a mistake in a remark of our paper (corrected in the Arxiv version). We claimed that $\pi_{X_C} = 9$ when $p = 2$. This is due to a confusion between the degree of the dual curve and the degree of the dual map π_2 (which is inseparable in this case).

The case of flexes

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If this result is still true over $\bar{\mathbb{F}}_p$, using Chebotarev density theorem, one gets that the probability to have a plane quartic with a rational flex tends to

$$1 - \frac{1}{2!} + \frac{1}{3!} - \ldots - \frac{1}{24!} \approx 1 - \exp(-1) \approx 0.63$$

when q tends to infinity.

Harris' result over finite fields

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We conjecture that $p = 3$ is the only exception.