On the automorphism group of algebraic curves in positive characteristic

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Preliminaries

Algebraic curves and birational invariants

- $K$: algebraically closed field of characteristic $p$
- $X \subseteq \mathbb{P}^r = \mathbb{P}^r(K)$: projective, geometrically irreducible, non-singular algebraic curve
- Algebraic function field $F/K$: $F = K(X)$
- $g = g(X) \geq 0$: genus of $X \to \text{Aut}(X)$ is infinite, if $g \leq 1$
- $\gamma = \gamma(X)$: $p$-rank (Hasse-Witt invariant) of $X \to 0 \leq \gamma \leq g$
- $\text{Aut}(X)$: (full) automorphism group of $X$ over $K$
Automorphism groups and quotient curves

$G := \text{finite automorphism group of } \mathcal{X}$

- $G$ acts faithfully on $\mathcal{X}$
- $G$ has a finite number of short orbits $\theta_1, ..., \theta_k$
- $\exists$ curve $\mathcal{Y}$ whose points are the $G$-orbits of $\mathcal{X}$
- $\mathcal{Y} := \mathcal{X}/G$ is called quotient curve of $\mathcal{X}$ by $G$
- $N_{\text{Aut}(\mathcal{X})}(G)/G \leq \text{Aut}(\mathcal{Y})$

Riemann-Hurwitz Formula:

$$2g(\mathcal{X}) - 2 = |G|(2g(\mathcal{Y}) - 2) + \text{Diff}(\mathcal{X}|\mathcal{Y})$$

Deuring-Shafarevic Formula: If $|G| = p^h$ then

$$\gamma(\mathcal{X}) - 1 = |G|(\gamma(\mathcal{Y}) - 1) + \sum_{i=1}^{k} (|G| - |\theta_i|)$$
How many automorphisms?

[Schmid (1938), Iwasawa-Tamagawa (1951), Roquette (1952), Garcia (1993)]

If \( g \geq 2 \) then \( \text{Aut}(\mathcal{X}) \) is a finite group

Classical Hurwitz bound (1892)

If \( p = 0 \) and \( g \geq 2 \) then \( |\text{Aut}(\mathcal{X})| \leq 84(g - 1) \)

Example: Klein quartic

\( \mathcal{K}: X^3 + Y + XY^3 = 0, \ g(\mathcal{K}) = 3, \ |\text{Aut}(\mathcal{K})| = |\text{PSL}(2, 7)| = 84(3 - 1) \)

- If \( \gcd(p, |\text{Aut}(\mathcal{X})|) = 1 \), then \( |\text{Aut}(\mathcal{X})| \leq 84(g - 1) \)
- If \( \gcd(p, |\text{Aut}(\mathcal{X})|) > 1 \), interesting behaviours can occur
Preliminaries

What if \( p \) divides \(|\text{Aut}(C)|\)?

- Hermitian curve \( \mathcal{H} : X^{q+1} = Y^q + Y, \ q = p^h \),
  \(|\text{Aut}(\mathcal{H})| = |\text{PGU}(3, q)| \geq 16g(\mathcal{H})^4 |

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**Stichtenoth (1973)**

If \( g = g(X) \geq 2 \) and \(|\text{Aut}(X)| \geq 16g^4 \) then \( X \) is the Hermitian curve \( \mathcal{H} \) (up to isomorphism). In particular \( \gamma(X) = 0 \).

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**Henn (1976)**

If \( g = g(X) \geq 2 \) and \(|\text{Aut}(X)| > 8g^3 \) then \( \gamma(X) = 0 \) and \( X \) is one of the following curves (up to isomorphism):

- \( \mathcal{Y} : Y^2 + Y + X^{2^k+1} = 0, \ p = 2, \ g = 2^{k-1} \) and \(|\text{Aut}(\mathcal{Y})| = 2^{2k+1}(2^k + 1) |\).

- The Roquette curve \( \mathcal{R} : Y^2 - (X^q - X) = 0 \) with \( p > 2, \ g = (q - 1)/2 \). Also \( \text{Aut}(\mathcal{R})/M \cong \text{PSL}(2, q), \text{PGL}(2, q) \), where \( q = p^r \) and \(|M| = 2|\).

- The Hermitian curve \( \mathcal{H} : X^{q+1} = Y^q + Y, \ q = p^h, \ p \) prime.

- The Suzuki curve \( \mathcal{S} : X^{q_0}(X^q + X) + Y^q + Y = 0 \), with \( p = 2, \ q_0 = 2^r \geq 2, \ q = 2q_0^2, \ g(\mathcal{S}) = q_0(q - 1) \), and \( \text{Aut}(\mathcal{S}) = Sz(q) \) (Suzuki group).
Preliminaries

The link between $\text{Aut}(\mathcal{X})$ and $\gamma(\mathcal{X})$

Theorem (Nakajima, 1987)

1 If $\mathcal{X}$ is ordinary, then $|\text{Aut}(\mathcal{X})| \leq 84(g^2 - g) \rightarrow$ no extremal examples provided!

2 Let $S$ be a $p$-subgroup of $\text{Aut}(\mathcal{X})$. Then

$$|S| \leq \begin{cases} 
g(\mathcal{X}) - 1, & \text{if } \gamma(\mathcal{X}) = 1, \\
4(\gamma(\mathcal{X}) - 1), & \text{if } \gamma(\mathcal{X}) \geq 2, \\
\max\{g(\mathcal{X}), 4p/(p - 1)^2 g(\mathcal{X})^2\}, & \text{if } \gamma(\mathcal{X}) = 0.
\end{cases}$$

3 If $|S| > \frac{2p}{p-1} g(\mathcal{X})$ then $\gamma(\mathcal{X}) = 0$.

• Open Problem 1: What if $S$ is a $d$-group where $d \neq p$ is a prime?

• Open Problem 2: Can Nakajima’s bound 1 be improved?

• Open Problem 3: Find an optimal $f(g)$ such that if $|\text{Aut}(\mathcal{X})| > f(g)$ then $\gamma(\mathcal{X}) = 0$ (clearly $f(g) \leq 8g^3$), e.g. can $f(g) \sim g^2$?
What if the classical Hurwitz bound does not hold?

Classification results

Let $G$ automorphism group of a curve $\mathcal{X}$ of genus $g \geq 2$. A consequence of the Riemann-Hurwitz Formula:

- If $G$ has more than 4 short orbits, then $|G| \leq 4(g - 1)$
- If $G = G_P$ and $p$ does not divide $|G|$, then $|G| \leq 4g + 2$

Exceptions to the classical Hurwitz bound, for a group $|G| > 84(g - 1)$, occur only in the following cases:

1. $G$ has two short orbits and both are non-tame; here $|G| \leq 16g^2$
2. $G$ has three short orbits with precisely one non-tame orbit; here $|G| \leq 24g^2$
3. $G$ has a unique short orbit which is non-tame; here $|G| \leq 8g^3$
4. $G$ has two short orbits and one short orbit is tame, one non-tame

→ IDEA: What about bounds for $|G|$ in Case 3? All the curves in Henn’s result satisfy case 4
Open Problem 1: \( d \)-group of automorphisms, \( d \neq p \) prime number

Our contributions to Open Problem 1

Let \( G \) be a \( d \)-group of automorphisms of a curve \( \mathcal{X} \) of genus \( g \geq 2 \).

1. How large is \( |G| \) with respect to \( g \)?
2. Structure in terms of generators and relations of extremal groups \( G \)
3. Is the bound sharp? Explicit construction of extremal examples \((\mathcal{X}, G)\)

Zomorrodian (1985-1987): the case \( \text{Char}(K) = 0 \)

\[ |G| \leq 9(g - 1) \] and the bound is sharp if and only if \( g - 1 = 3^k \) and \( g \geq 10 \)

- (Giulietti-Korchmáros 2010-2017, Stichtenoth 1973) Nakajima extremal curves

Our results:

- Zomorrodian’s result holds also when \( \text{Char}(K) = p \neq 0 \) and \( d \neq 2, p \)

For the interesting case \( d = 3 \):

- the group structure of \( G \) is uniquely determined
- two general methods to construct extremal examples \((\mathcal{X}, G)\).
Theorem (Korchmáros-M., 2020)

Let \( g(C) \geq 2 \). If \( G \) is a \( d \)-subgroup of \( \text{Aut}(X) \) with \( d \neq p \) and \( d \) odd then

\[
|G| \leq \begin{cases} 
9(g - 1), & \text{if } d = 3, \\
\frac{2d}{d-3}(g - 1), & \text{if } d > 3.
\end{cases}
\]

For \( d = 3 \) if equality holds then \( G \) is not abelian and \( g \neq 2 \).

Remark: the bound is sharp for \( d \geq 5 \) (abelian groups)

Fermat curve \( F_d : x^d + y^d + 1 = 0 \) has genus \( (d - 1)(d - 2)/2 \), \( C_d \times C_d \cong G < \text{Aut}(F_d) \) of order \( d^2 = 2d(g - 1)/(d - 3) \):

\[
G = \{(x, y) \mapsto (\lambda x, \mu y) | \lambda^d = \mu^d = 1\}
\]

• known: \( G \) abelian then \( |G| \leq 4g + 4 \implies \) if \( G \) is extremal and \( d = 3 \) then \( G \) is non-abelian (interesting case)
Open Problem 1: $d$-group of automorphisms, $d \neq p$ prime number

Improvements of the bound for non-abelian groups

**Theorem (Korchmáros-M., 2020)**

Let $G$ be a non-abelian $d$-subgroup of $\text{Aut}(\mathcal{X})$. If $Z$ is an order $d$ subgroup of $Z(G)$ such that the quotient curve $\bar{\mathcal{X}} = \mathcal{X}/Z$ has genus at most 1 then $\bar{\mathcal{X}}$ is elliptic and

$$|G| \leq \frac{2d}{d-1} (g - 1)$$

apart from the case where $d = 3$ and $|G| = 9(g - 1)$.

- $g(\mathcal{X}/Z) \geq 2 \implies \mathcal{X}/Z$ is still extremal as $G/Z \leq \text{Aut}(\mathcal{X}/Z)$
- "Minimal" extremal examples are those for which $g(\mathcal{X}/Z) \leq 1$
- Interesting case: $d = 3$ ($d \geq 5$ $G$ is abelian)
- An Extremal 3-Zomorrodian curve is a curve $\mathcal{X}$ of genus $g \geq 2$ admitting $G \leq \text{Aut}(\mathcal{X})$ with $|G| = 9(g - 1)$
Open Problem 1: \( d \)-group of automorphisms, \( d \neq p \) prime number

**Minimal Extremal 3-Zomorrodian curves: structure of \( G \)**

Proposition (Korchmáros-M., 2020)

Let \( G \) be a Sylow 3-subgroup of a curve of an Extremal 3-Zomorrodian curve of elliptic type and genus \( g = 3^h + 1, \ h \geq 3 \). Then

- either \( Z(G) \cong C_3 \) or \( Z(G) \cong C_3 \times C_3 \),
- \( G \) has 3 short orbits \( \theta, \sigma, \Omega \) of sizes \( |G|/3 \), \( |G|/3 \) and \( |G|/9 \)
- \( G \) can be generated by 2 elements \( \Rightarrow [G : \Phi(G)] = 9 \);
- maximal subgroups of \( G \) are normal of index 3. Exactly one of them is either abelian or minimal non-abelian.

- Minimal non-abelian case: Qu Haipeng, Yang Sushan, Xu Mingyao, and An Lijian, Finite \( p \)-groups with a minimal non-abelian subgroup of index \( p \) (I), J. Algebra 358 (2012), 178-188.
Open Problem 1: $d$-group of automorphisms, $d \neq p$ prime number

**Elliptic type:** structure of $G$

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**Theorem (Korchmáros-M., 2020)**

If $|Z(G)| = 3$ then $G$ has no abelian maximal subgroups of index 3 and

- $|G| = 3^{2e}$ and $G = \langle s_1, s_2, s | s_1^{3e} = s_2^{3e-1} = 1, s^3 = s_1^{\delta 3^{e-1}}, [s_1, s] = s_2, [s_2, s] = s_2^{-3} s_1^{-3}, [s_2, s_1] = s_1^{3^{e-1}} \rangle$ where $\delta = 0, 1, 2$;

- $|G| = 3^{2e+1}$ and $G = \langle s_1, s_2, s | s_1^{3e} = s_2^{3e} = 1, s^3 = s_2^{\delta 3^{e-1}}, [s_1, s] = s_2, [s_2, s] = s_2^{-3} s_1^{-3}, [s_2, s_1] = s_1^{3^{e-1}} \rangle$ where $\delta = 0, 1, 2$.

If $|Z(G)| = 9$ then $G$ has no abelian subgroups of index 3 and

- $G = \langle s_1, s_2, \beta, x | s_1^{3n} = s_2^{3n-1} = x^3 = 1, \beta^3 = x^2, [s_1, \beta] = s_2, [s_2, \beta] = s_2^{-3} s_1^{-3}, [s_1, s_2] = x, [x, s_1] = [x, s_2] = 1 \rangle$, for $|G| = 3^{2n+1}, e \geq 3$;

- $G = \langle s_1, s_2, \beta, x | s_1^{3n} = s_2^{3n} = x^3 = 1, \beta^3 = x^2, [s_1, \beta] = s_2, [s_2, \beta] = s_2^{-3} s_1^{-3}, [s_1, s_2] = x, [x, s_1] = [x, s_2] = 1 \rangle$, for $|G| = 3^{2n+2}, n \geq 2$.

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Can we construct infinite families of Extremal 3-Zomorrodian curves?
Open Problem 1: \( d \)-group of automorphisms, \( d \neq p \) prime number

Construction of Elliptic type Extremal 3-Zomorrodian curves for every \((g, |G|) = (3^h + 1, 3^{h+2})\)

- Elliptic curve \( \mathcal{E} : X^3 + Y^3 + Z^3 = 0 \) \((J(\mathcal{E}) = \text{Jacobian group})\)
- \( P = (-1, 0, 1) \) is an inflection point of \( \mathcal{E} \), and \( \bar{\alpha} : (X, Y, Z) \mapsto (X, \epsilon Y, Z) \) with \( \epsilon^3 = 1 \) primitive, is an order 3 automorphism of \( \mathcal{E} \) fixing \( P \)
- \( \bar{\alpha} \) has two more fixed points on \( \mathcal{E} \), namely \( P_1 = (-\epsilon, 0, 1) \) and \( P_2 = (-\epsilon^2, 0, 1) \)
  \( \implies \bar{\alpha} \notin J(\mathcal{E}) \)

**Theorem (Korchmáros-M. 2020)**

A 3-group \( \bar{G} \) of automorphisms of \( \mathcal{E} \) can be written up to conjugation as \( \bar{G} = \bar{H} \rtimes \langle \bar{\alpha} \rangle = \bar{H} \rtimes \bar{G}_P \) where \( \bar{H} = \bar{G} \cap J(\mathcal{E}) \) and \( \bar{G} \) can be generated by 2 elements
• let $\bar{G} = \bar{H} \rtimes \langle \bar{\alpha} \rangle \leq \text{Aut}(\mathcal{E})$ with $|\bar{G}| = 3^{h+1}$, $h \geq 2$

• Since $\bar{G}$ can be generated by 2 elements, $\bar{G}/\Phi(\bar{G})$ is elementary abelian of order 9

• since $\bar{H}$ is maximal, $\Phi(\bar{G}) \leq \bar{H}$

• $\theta_1 = \Phi(\bar{G})$–orbit containing $P$ $\implies |\theta_1| = 3^{h-1}$

• $\Phi(\bar{G})$ is a normal subgroup of $\bar{H}$, the $\bar{H}$-orbit $\theta$ containing $P$ is partitioned into three $\Phi(\bar{G})$-orbits which may be parameterized by $\Phi(\bar{G})$ together with its two cosets in $\bar{H}$
(Korchmáros-Nagy-Pace, 2014) If $Q \in \theta_2$ then the line through $P$ and $Q$ meets $E$ in a point $R \in \theta_3$.

$r$ has homogenous equation $mX - Y + mZ = 0$ for some $m \in K$.

(inflectional) tangent to $E$ at $P$ is $i : X + Z = 0$. 
in $K(\mathcal{E}) = K(x, y)$ with $x^3 + y^3 + 1 = 0$ define $t = \frac{mx - y + m}{x + 1}$ then $(t) = Q + R - 2P$

let $w = \prod_{f \in \Phi(\bar{G})} f(t)$. Then $(w) = -2\theta_1 + \theta_2 + \theta_3$ and $\bar{g} \in \bar{G}$ acts on \{\theta_1, \theta_2, \theta_3\}.

\[
\begin{align*}
(1) \quad \bar{g}(w)/w &= \lambda, \text{ for some } \lambda \in K \\
(2) \quad (\bar{g}(w)/w) &= -2\theta_1 + \theta_2 + \theta_3 - (-2\theta_3 + \theta_1 + \theta_2) = -3\theta_1 + 3\theta_3
\end{align*}
\]

In any case

(key property) $\bar{g}(w)/w = v^3$, for some $v \in K(x, y)$
• We define
\[ \mathcal{X} : \begin{cases} x^3 + y^3 + 1 = 0, \\ z^3 = w \end{cases} \implies g(\mathcal{X}) = 3^h + 1 \]

• Also every \( \bar{g} \in \bar{G} \) can be lifted in three ways creating a group \( G \leq \text{Aut}(\mathcal{X}) \) of order \( 3|\bar{G}| = 3^{h+2} = 9(g - 1) \)

• Indeed for \( \bar{g} \in \bar{G} \) we define,
\[ g : (x, y, z) \mapsto (\bar{g}(x), \bar{g}(y), vz), \]
where \( v^3 = \bar{g}(w)/w \). Then
\[ g(z^3) = v^3z^3 = \frac{\bar{g}(w)}{w}w = \bar{g}(w) = g(w) \implies \mathcal{X} \text{ is preserved!} \]
Open Problem 1: \(d\)-group of automorphisms, \(d \neq p\) prime number

Explicit examples using MAGMA

- \(g = 10, \ |G| = 81\)
  \[
  \begin{cases}
  x^3 + y^3 + 1 = 0; \\
  z^3 = \frac{x}{y^2}.
  \end{cases}
  \]

- \(g = 28, \ |G| = 729\)
  \[
  \begin{cases}
  x^3 + y^3 + 1 = 0; \\
  z^3 = \frac{(y^{18} + 3y^{15} + 52y^{12} + 26y^9 + 52y^6 + 3y^3 + 1)}{(y^{17} + 3y^{14} + 5y^{11} + 5y^8 + 3y^5 + y^2)}x.
  \end{cases}
  \]

- \(g = 82, \ |G| = 2187\)
  \[
  \begin{cases}
  x^3 + y^3 + 1 = 0; \\
  z^3 = \frac{(y^{54} + 9y^{51} + 151y^{48} + 191y^{45} + 243y^{42} + 21y^{39} + 86y^{36} + 184y^{33} + y^{30} + 153y^{27} + y^{24} + 184y^{21} + 86y^{18} + 21y^{15} + 243y^{12} + 191y^9 + 151y^6 + 9y^3 + 1)}{(y^{53} + 9y^{50} + 261y^{47} + 258y^{44} + 138y^{41} + 146y^{38} + 206y^{35} + 24y^{32} + 12y^{29} + 12y^{26} + 24y^{23} + 206y^{20} + 146y^{17} + 138y^{14} + 258y^{11} + 261y^8 + 9y^5 + y^2)}x.
  \end{cases}
  \]
Open Problem 2: Automorphism groups of ordinary curves

Ordinary algebraic curves with many automorphisms

$\mathcal{X}$ is ordinary if $g(\mathcal{X}) = \gamma(\mathcal{X})$

- **Nakajima (1987):** $|Aut(\mathcal{X})| \leq 84(g(\mathcal{X}) - 1)g(\mathcal{X}) \rightarrow$ can this bound be improved?

**Theorem (Korchmáros-M., 2019)**

Let $\mathcal{X}$ be an ordinary curve of genus $g(\mathcal{X}) \geq 2$ defined over an algebraically closed field of odd characteristic $p$. If $G \leq Aut(\mathcal{X})$ is solvable then

$$|G| \leq 34(g(\mathcal{X}) + 1)^{3/2} < 68\sqrt{2}g(\mathcal{X})^{3/2}$$

- This is the best bound known for automorphism groups of ordinary curves

- **(Korchmáros-M.-Speziali, 2018)** Extremal example up to the constant term: a generalized Artin-Schreier extension of the Artin-Mumford curve


- $\implies$ Our bound cannot be improved!
Open Problem 2: Automorphism groups of ordinary curves

Why is the hypothesis $G$ solvable relevant/useful?

- **First observation:** if $g(K) = 2$ then $|G| \leq 48$ (known), so the statement is true. We assume $g(K) \geq 3$.

- By contradiction: $(G, g(K))$ is a minimal counterexample, that is, $|G| > 34(g(K) + 1)^{3/2}$ and if $g(V) < g(K)$, $V$ is ordinary and $H \leq \text{Aut}(V)$ is solvable then $|H| \leq 34(g(V) + 1)^{3/2}$

- Since $G$ is solvable, it admits a minimal normal subgroup $S$ which is elementary abelian.

- **Two cases are treated separately:** either $S$ is a $p$-group, or it has order prime to $p$.

- In both cases we try to construct a quotient curve which is still ordinary and gives a contradiction to the minimality of $(G, g(K))$. 

Our bound is sharp (up to the constant term)

\[ q = p^h, \ h \geq 1 \text{ and } K = \overline{\mathbb{F}}_q. \]  

For \((m, p) = 1\) let

\[ \mathcal{Y} : y^q + y = x^m + 1/x^m \]

and \( F = K(\mathcal{Y}) \) its function field. Let \( t = x^{m(q-1)} \). \( F|K(t) \) is not Galois

**Theorem (Korchmáros-M.-Speziali, 2018)**

The Galois closure of \( F|K(t) \) is \( L = K(x, y, z) \) where \( y^q + y = x^m + 1/x^m \) and \( z^{a} + z = x^{m} \). Also

- \( g(L) = (q - 1)(q^m - 1) \), \( \gamma(L) = (q - 1)^2 \),
- \( \text{Aut}(L) \) contains a subgroup \( Q \rtimes U \) of index 2 where \( Q \) is an elementary abelian normal subgroup of order \( q^2 \) and \( U \) is a cyclic of order \( m(q - 1) \),
- if \( m = 1 \), \( L \) is ordinary and \( |\text{Aut}(\mathcal{X})| > 2g^{3/2} \).

- (M.-Zini, 2018): infinite family of extremal examples (Generalized Artin-Mumford curves) \( \mathcal{X}_{L_1, L_2} : L_1(x) \cdot L_2(y) = 1 \), where \( L_1 \) and \( L_2 \) are linearized polynomials.
Open Problem 2: Automorphism groups of ordinary curves

Large automorphism groups of ordinary curves

Natural questions:

• What if $p = 2$ and $G$ is solvable?
• What if $p$ is odd but $G$ is not solvable?

**Theorem (M.-Speziali, 2019)**

Let $\mathcal{X}$ be an ordinary curve of even genus $g(\mathcal{X}) \geq 2$ defined over an algebraically closed field of odd characteristic 2. If $G \leq Aut(\mathcal{X})$ is solvable then

$$|G| \leq 35(g(\mathcal{X}) + 1)^{3/2}$$

**Theorem (M.-Speziali, 2019)**

Let $\mathcal{X}$ be an ordinary curve of even genus $g(\mathcal{X}) \geq 2$ defined over an algebraically closed field of odd characteristic $p$. If $G \leq Aut(\mathcal{X})$ is not solvable then

$$|G| \leq 822g(\mathcal{X})^{7/4}$$

• A general and sharp refinement of Nakajima’s bound is still an open problem!
Open Problem 3: large automorphism groups imply $p$-rank zero

The third open problem: improving Henn’s result

If $G \leq \text{Aut}(\mathcal{X})$ is such that $|G| > 84(g(\mathcal{X}) - 1)$ then one of the following occurs:

1. $G$ has two short orbits and both are non-tame; here $|G| \leq 16g^2$
2. $G$ has three short orbits with precisely one non-tame orbit; here $|G| \leq 24g^2$
3. $G$ has a unique short orbit which is non-tame; here $|G| \leq 8g^3$
4. $G$ has two short orbits and one short orbit is tame, one non-tame (if $|G| \geq 8g^3$ then $G$ is known and $\gamma(\mathcal{X}) = 0$).

Open Problem 3

Is it possible to find a (optimal) function $f(g)$ such that the existence of an automorphism group $G$ of $\mathcal{X}$ with $|G| > f(g)$ implies that $\mathcal{X}$ has $p$-rank zero?

• we already see that if $|\text{Aut}(\mathcal{X})| > 24g^2$ then either Case 3 or 4 occurs.

• $\rightarrow$ Natural idea: improve the bounds in 3 and/or 4 to obtain (up to finite exceptions) a function $f(g) = cg^2$ for some constant $c$.
Theorem (M., 2021)

Let $G \leq \text{Aut}(\mathcal{X})$, where $g = g(\mathcal{X}) \geq 2$ and $\mathcal{X}$ is defined over an algebraically closed field of characteristic $p > 0$.

1. If $G$ satisfies Case 3 then $|G| \leq 336g(\mathcal{X})^2$.

2. If $|G| \geq 60g^2$ and Case 3 is satisfied then $\gamma(\mathcal{X})$ is positive and congruent to zero modulo $p$.

3. If $|G| \geq 900g^2$ then Case 4 is satisfied. If $\gamma(\mathcal{X}) \neq 0$ then $g(\mathcal{X})$ is odd.

Furthermore, if for $P, R \in O_1$ (non-tame short orbit) one has $g(\mathcal{X}/G_P^{(1)}) = 0$ and $G_{P,R}$ is either a $p$-group or a prime to $p$ group then $\gamma(\mathcal{X}) = 0$.

Work in progress: Is it true that if $|G| \geq 900g^2$ then $\gamma(\mathcal{X}) = 0$?
Open Problem 3: large automorphism groups imply $p$-rank zero

Sketch of the proof of the first item

• By contradiction $|G| > 336g^2$

• Let $O := P^G$ be the unique short orbit of $G$

• [Case 1: $O = \{P\}$] Thus, $G = G_P$. Let $X_1 := X/G_P^{(1)}$

• If $X_1$ is not rational $\rightarrow |G| = |G_P| = |G_P^{(1)} \times H| \leq g(4g + 2) < 5g^2$, a contradiction

• Let $X_1$ be rational. Thus, $G_P = G_P^{(1)} \times H$. If $\alpha \in H$ then $\alpha$ induces an automorphism $\alpha'$ on $X_1$

• Since every automorphism of a rational function field whose order is prime to $p$ has exactly 2 fixed places $\rightarrow \alpha'$ fixes a place $Q \neq P$

• This implies that $Q^G$ is short and $Q^G \neq O$, a contradiction

• This shows that if $G = G_P$ and Case 3 is satisfied then $|G| < 5g^2$
Open Problem 3: large automorphism groups imply $p$-rank zero

Sketch of the proof of the first item

• [Case 2: $O \supset \{P\}$]

• $g(\mathcal{X}/G_P) = 0$ and either $\gamma(\mathcal{X}) = 0$ or $\gamma(\mathcal{X}) > 0$ and $G_P$, $G_P^{(1)}$ have the same two (non-tame) short orbits

• First aim: To prove that the case $\gamma = \gamma(\mathcal{X}) > 0$ is impossible

• If $\gamma > 0$ then $G_P$ has 2 short orbits $O_1 = \{P\}$ and $O_2$

• $O = \{P\} \cup O_2$

• Since $G_P$ acts transitively on $O_2 = O \setminus \{P\} \rightarrow G$ acts 2-transitively on $O$

• Idea: Use the complete list of finite 2-transitive groups to exclude the case $\gamma > 0$

• Second aim: the case $\gamma = 0$ is not possible from the Deuring-Shafarevic formula
Open Problem 3: large automorphism groups imply $p$-rank zero

Examples: Curves satisfying case 4

- **Example 1:** GK Curve:

  \[ C_n : Y^{n^3+1} + (X^n + X)(\sum_{i=0}^{n} (-1)^{i+1} X^{i(n-1)} n^{n+1}) = 0 \]

  \[ |Aut(C_n)| = (n^3 + 1)n^3(n - 1) \sim 4g^2 \]

- **Example 2:** Skabelund curves

  \[ \tilde{S} : \begin{cases} 
  y^q + y = x^{q_0}(x^q + x), \\
  t^m = x^q + x 
  \end{cases} \]

  where $q = 2q_0^2 = 2^{2s+1}$ and $m = q - 2q_0 + 1$

  \[ |Aut(\tilde{S})| = m(q^2 + 1)q^2(q - 1) \sim 4g^2 \]
Thank you

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