CURVES OVER FINITE FIELDS

PAST, PRESENT, AND FUTURE

May 2021
Open problems in arithmetic statistics

$\mathbb{F}_q$ = finite field with $q$ elements

$C$ = smooth projective curve defined over $\mathbb{F}_q$ of genus $g$

Interested in:

1) distribution of $\# C(\mathbb{F}_q)$

2) $N_q(g) = \sup \{ \# C(\mathbb{F}_q) : C \text{ of genus } g \}$

3 main directions

- geometric: $f \times g$, let $g \to \infty$

- probabilistic: $f \times g$, let $g \to \infty$

- arithmetic: $C/\mathbb{Q}$ reduce mod $p$, $p \to \infty$
Geometric situation:

\[ N_g(q) \]

\[ | \# C(F_\ell^g) - (q+1) | \leq 2g \sqrt{q} \]

\[ q+1 - 2g \sqrt{q} \leq \# C(F_\ell^g) \leq q+1 + 2g \sqrt{q} \]

\[ \{ \# C(F_\ell^g) ; \text{C of genus } g \} \sim \]

\[ g < \chi \rightarrow \infty \]

Main term + lower order terms

Moments of L-functions
\( \chi_d \) quads, char & conductor \( d \)

\[ L \left( \frac{1}{2}, \chi_d \right) \sim d^\varepsilon \quad \text{as } d \text{ increases} \]

\[ \sum_{d \leq x} L \left( \frac{1}{2}, \chi_d \right) \rightarrow \text{moments} \]

**Example:** 4th moment

\[ \sum_{d \leq x} L \left( \frac{1}{2}, \chi_d \right)^4 \sim C \exp \left( \log x \right) + \text{term of order} + x^{3/4} + O \left( x^{1/2+\varepsilon} \right) \]

**Big problem:**

port results from function fields to number fields
Probabilistic situation

\[ G \subset S_n \text{ transitive subgroup} \]

\[ G\text{-statistics} : \]

\[ \# C(\mathbb{F}_q) \text{ where } \downarrow \text{ G-cover} \]

Malle's conjecture in function fields

Turkelli (2008): Malle over \( \mathbb{F}_q(t) \)

Ideas:

- Wright's paper on abelian extension (1989)

\[ \frac{21}{2^2} \times \frac{21}{2} \times \frac{21}{2} \]

- non-abelian groups \( G=S_3 \) (Wood, 2012)

- non-abelian Cohen-Lenstra
\[
\begin{aligned}
&\frac{\text{a}}{\text{b}} \\
&\# C(\mathbb{F}_q) \quad x_1 + \cdots + x_{q+1} \\
&\# C(\mathbb{F}_2) \\
&\# C(\mathbb{F}_2) \\
&\frac{\text{joint distribution}}{\# C(\mathbb{F}_q), \# C(\mathbb{F}_2)}
\end{aligned}
\]
Arithmetic situation

Big problem:
Prove Sato-Tate for abelian surfaces

Issue: Serve => need mer cont + cm properties
for all nontrivial irreducible reps of $Sp(4)$

Standard reps: paramodular conjecture

Need: infinitely many L-fns.

Other problems:

1. Ab var of div by Jacobians & curves & gen sg
   Are the S-T distributions the same?

2. Find all S-T gps of ab var of div in $d$?
   $d=1, 2, 3$?
Known (1975 - Meeden - Pati)

That may simple ab var \( \mathbb{Q} \) with exactly \( X \) \( \mathbb{F}_2 \) - point

\[ \text{Question: ordinary primes} \]

Ab. var \( X / \mathbb{Q} \)

reduce mod \( p \)

Prove that there are infinitely many \( p \)'s for which \( X (\text{mod} p) \) is ordinary.

What's the density of ordinary primes?

see Katz "Simple things we do not know"