On the automorphism group of algebraic curves in positive characteristic

Maria Montanucci
TECHNICAL UNIVERSITY OF DENMARK

Joint work with: Gábor Korchmáros, Pietro Speziali and Giovanni Zini.

Abstract

Algebraic curves over a finite field and their function fields have been a source of great fascination for number theorists and geometers alike, ever since the seminal work of Hasse and Weil in the 1930s and 1940s. Many important and fruitful ideas have arisen out of this area, where number theory and algebraic geometry meet.

Let $\mathcal{X}$ be a projective, geometrically irreducible, non-singular algebraic curve defined over an algebraically closed field $\mathbb{K}$ of positive characteristic p. Let $\mathbb{K}(\mathcal{X})$ be the field of rational functions on $\mathcal{X}$ (i.e. the function field of $\mathcal{X}$ over $\mathbb{K}$). The $\mathbb{K}$-automorphism group $Aut(\mathcal{X})$ of $\mathcal{X}$ is defined as the automorphism group of $\mathbb{K}(\mathcal{X})$ fixing $\mathbb{K}$ element-wise. The group $Aut(\mathcal{X})$ has a faithful action on the set of points of $\mathcal{X}$.

By a classical result, $Aut(\mathcal{X})$ is finite whenever the genus g of $\mathcal{X}$ is at least two. Furthermore it is known that every finite group occurs in this way, since, for any ground field $\mathbb{K}$ and any finite group G, there exists an algebraic curve $\mathcal{X}$ defined over $\mathbb{K}$ such that $Aut(\mathcal{X}) \cong G$.

This result raised a general problem for groups and curves, namely, that of determining the finite groups that can be realized as the $\mathbb{K}$-automorphism group of some curve with a given invariant. The most important such invariant is the *genus* g of the curve. In positive characteristic, another important invariant is the so-called *p-rank* of the curve, which is the integer $0 \leq \gamma \leq g$ such that the Jacobian of $\mathcal{X}$ has p^γ p-torsion points.

Several results on the interaction between the automorphism group, the genus and the p-rank of a curve can be found in the literature. A remarkable

example is the work of Nakajima (1987) who showed that the value of the p-rank deeply influences the order of a p-group of automorphisms of $\mathcal{X}$. Extremal examples with respect to Nakajima's bound are known (Korchmáros-Giulietti and Stichtenoth). The following open problem arised naturally:

Open Problem 1: How large can a d-group of automorphisms G of an algebraic curve $\mathcal{X}$ of genus $g \geq 2$ be when $d \neq p$ is a prime number? Is there a method to construct extremal examples as for the case $d = p$?

In his work Nakajima analyzed also the case of curves for which the p-rank is the largest possible (the so-called *ordinary curves*), namely $\gamma = g$, proving that they can have at most $84(g^2 - g)$ automorphisms. Since no extremal examples for this bound were found by Nakajima, also the following open problem arised naturally:

Open Problem 2: Is Nakajima's bound $|Aut(\mathcal{X})| \leq 84(g^2 - g)$, for an ordinary curve $\mathcal{X}$ of genus $g \geq 2$ sharp?

Hurwitz (1893) showed that if $\mathcal{X}$ is defined over $\mathbb{C}$ then $|Aut(\mathcal{X})| \leq 84(g - 1)$, which is known as the *Hurwitz bound*. This bound is sharp, i.e., there exist algebraic curves over $\mathbb{C}$ of arbitrarily high genus g whose automorphism group has order exactly $84(g - 1)$. Well-known examples are the Klein quartic and the Fricke-Macbeath curve.

Roquette (1970) showed that Hurwitz bound also holds in positive characteristic p, if p does not divide $|Aut(\mathcal{X})|$. A general bound in positive characteristic is $|Aut(\mathcal{X})| \leq 16g^4$ with one exception: the so-called Hermitian curve. This result is due to Stichtenoth (1973). The quartic bound $|Aut(\mathcal{X})| \leq 16g^4$ was improved by Henn (1978). Henn's result shows that if $|Aut(\mathcal{X})| > 8g^3$ then $\mathcal{X}$ is $\mathbb{K}$-isomorphic to one of 4 explicit exceptional curves, all having p-rank equal to zero. A third natural open problem arised as a consequence of this result:

Open Problem 3: Is it possible to find a (optimal) function $f(g)$ such that the existence of an automorphism group G of $\mathcal{X}$ with $|G| > f(g)$ implies that $\mathcal{X}$ has p-rank zero?

Henn's result clearly implies that $f(g) \leq 8g^3$. In this talk we will present our recent contributions to the 3 described open problems.