SYMMETRIC GROUPS AND LINEAR GROUPS

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We review some classical results.

1. Finite groups

Let G be a finite group, and let

$$\rho : G \to \mathrm{GL}(V) \tag{1.1}$$

be a morphism of groups, where V is a finite-dimensional vector space over $\mathbb{C}$ say, so that $\mathrm{GL}(V) \cong \mathrm{GL}_n(\mathbb{C})$ for some n.

It can happen that there exists a subspace $W \subset V$ that is invariant under G. This means that the matrices in $\rho(G)$ have an upper-triangular block structure after a suitable choice of basis. We are interested in the complement of this case:

Definition 1.1. V is called irreducible if there does not exist such an invariant $W \subset V$ that is non trivial.

Facts:

(i) There exists a *finite* number of such irreducible representations up to isomorphism. Their number is equal to the number of conjugacy classes of G.

(ii) Every representation is a direct sum of irreducible representations. More precisely, if $W \subset V$ is invariant, then there exists $W' \subset V$ such that $V = W \oplus W'$.

Example 1.2. Consider the symmetric group S_n. The corresponding number of irreducible representations then equals the number of partitions $p(n)$ of n, because the cycle structure of an element of S_n characterizes its conjugacy class.

Let $\pi = (\pi_1 \geq \cdots \geq \pi_k)$ be a partition of n. We get a corresponding Young tableau T whose *rows* are blocks of length π_i. We can label the blocks in this tableau as we like. Then we can construct a polynomial P_T by using the *columns* of T, by setting

$$P_T = \prod_{\text{columns } C} \text{VanderMonde } (x_i : i \in C) \tag{1.2}$$

So for example, for the tableau

$$\begin{array}{|c|c|}\hline 2 & 1 \\ \hline 4 & 3 \\ \hline 5 & \multicolumn{1}{c}{} \\ \cline{1-1}\end{array}$$

we get the polynomial $(x_2 - x_4)(x_2 - x_5)(x_4 - x_5) \times (x_1 - x_3)$. We can then construct the vector space

$$[\pi] = \langle P_T : T \text{ tables of the form } \pi\rangle, \tag{1.3}$$

which is called a Specht module. The irreducible representations of S_n are given by the $[\pi]$.

2. Other reductive groups

Apart from the finite groups, there are

(i) compact groups.
(ii) $\mathrm{GL}_n(\mathbb{C})$. This group allows infinitely many distinct irreducible representations. Given $\rho : \mathrm{GL}(V) \to \mathrm{GL}(V)$, we can look at the decomposition $V = S_\lambda U$ with $\lambda = (\lambda_1 \geq \cdots \geq \lambda_d)$ with $d = \dim(U)$. There is again a relation with partitions, and we will study this case more closely in a moment.

3. Complete reducibility

As above: given $W \subset V$ we want to find a decomposition

$$V = W \oplus W' \tag{3.1}$$

and more generally into irreducibles

$$V = W_1 \oplus \cdots \oplus W_k. \tag{3.2}$$

Canonically we have

$$V \cong \bigoplus_{\lambda \in Irrep(G)} E_\lambda \otimes \mathrm{Hom}_G(E_\lambda, V) \tag{3.3}$$

and after a choice of basis we have a non-canonical isomorphism

$$V \cong \bigoplus_{\lambda \in Irrep(G)} E_\lambda^{\oplus n_\lambda(V)} \tag{3.4}$$

The arrow from right to left in (3.3) sends $\sum e_\lambda \otimes f_\lambda$ to $\sum f_\lambda(e_\lambda)$.

This is interesting because it gives us some additional structure. We have that G embeds into $\mathrm{End}(V)$, and in the latter ring we can form A^G, the commutant of G. Therefore A^G acts on $F_\lambda = \mathrm{Hom}_G(E_\lambda, V)$ by postcomposition.

Theorem 3.1. *This action is irreducible. Moreover, the non-zero F_λ are pairwise inequivalent.*

4. Schur-Weyl duality

Let U be of finite dimension. Then S_n acts on $V = U^{\otimes n}$. Therefore we have

$$V = \bigoplus_{[\pi]} [\pi] \otimes F_\pi \tag{4.1}$$

and A^{S_n} acts on F_π. Given $g \in \mathrm{GL}(V)$, we can look at the morphism f_g that sends $v_1 \otimes \cdots \otimes v_n$ to $g(v_1) \otimes \cdots \otimes g(v_n)$.

Theorem 4.1. *We have $A^{S_n} = \langle f_g : g \in \mathrm{GL}(V)\rangle$.*

Definition 4.2. We let $S_\pi U = \mathrm{Hom}_{S_n}([\pi], U^{\otimes n})$. (In fact $\mathrm{Hom}_{S_n}([\pi], \_^{\otimes n})$ is a functor, which is called the Schur functor.) It is an irreducible representation of $\mathrm{GL}(U)$ if it is non-zero.

By Schur–Weyl we then have

$$U^{\otimes n} = \bigoplus_{|\pi|=n} [\pi] \otimes S_\pi U, \tag{4.2}$$

where $[\pi]$ is a representation of S_n and where $S_\pi U$ is a representation of $\mathrm{GL}(U) = \mathrm{GL}_d(\mathbb{C})$.

Example 4.3. Let $\pi = (n)$. Then $P_T = x_1 \dots x_n$ for any T, so that $[\pi]$ is the trivial representation. We have $S_\pi U = \mathrm{Sym}^n(V) = \mathrm{Pol}_n(V^*)$.

Let $\pi = (1, \dots, 1)$. Then $P_T = \pm \prod (x_i - x_j)$ for any T, so that $[\pi]$ is the sign representation. We have $S_\pi U = \bigwedge^n V$.

Slogan: Representation of symmetric groups $\cong$ Representations of linear groups.

On the left hand side we can apply combinatorial methods, whereas on the right hand side we get covariants, Lie theory, geometry and analysis.

5. Tensor products

We have

$$[\pi] \otimes [\lambda] = \bigoplus_{\nu} K^{\nu}_{\pi,\lambda} [\nu] \tag{5.1}$$

where the $K^{\nu}_{\pi,\lambda}$ are called the Kronecker coefficients. We can then ask when these coefficients are non-zero. We have

$$K_{\pi,\lambda,\nu} = K^{\nu}_{\pi,\lambda} = \dim([\pi] \otimes [\lambda] \otimes [\nu])^{S_n} \tag{5.2}$$

Write

$$\begin{aligned} U^{\otimes n} &= \bigoplus_{\pi} [\pi] \otimes S_\pi U \\ V^{\otimes n} &= \bigoplus_{\lambda} [\lambda] \otimes S_\lambda V \\ U^{\otimes n} \otimes V^{\otimes n} &= \bigoplus_{\pi,\lambda} [\pi] \otimes [\lambda] \otimes S_\pi U \otimes S_\lambda V. \end{aligned} \tag{5.3}$$

The left hand side here equals $(U \otimes V)^{\otimes n}$, so that it is also isomorphic to $\bigoplus_\nu [\nu] \otimes S_\nu(U \otimes V)$. We can therefore rewrite

$$\begin{aligned} K_{\pi,\lambda,\nu} &= \mathrm{mult}(S_\pi(U) \otimes S_\lambda(V), S_\nu(U \otimes V)) \\ &= \mathrm{mult}(S_\pi(U) \otimes S_\lambda(V) \otimes S_\nu(W), S^n(U \otimes V \otimes W)). \end{aligned} \tag{5.4}$$

The Clebsch–Gordan problem is that of determining the decomposition of $S_\pi U \otimes S_{\pi'} U$, which at least includes $S_{\pi+\pi'} U$. This already gives us that

$$K_{\pi+\pi',\lambda+\lambda',\nu+\nu'} \geq K_{\pi,\lambda,\nu}, \tag{5.5}$$

if $K_{\pi',\lambda',\nu'} \neq 0$.

We can also decompose

$$S_\pi U \otimes S_{\pi'} U = \bigoplus_\lambda c^\lambda_{\pi,\pi'} S_\lambda U \tag{5.6}$$

where the $c^\lambda_{\pi,\pi'}$ are called the Littlewood–Richardson coefficients. By work of Knutson and Tao, these are non-zero if and only if certain linear forms $\ell_i(\pi, \pi', \lambda)$ are ≥ 0. In turn, this means that there exists $k > 0$ for which $c^{k\lambda}_{k\pi,k\pi'} > 0$.

The Littlewood–Richardson coefficients can now be considered as the limit of the Kronecker coefficients. So there exists ℓ_0 such that $K_{\lambda,\mu,\nu} \neq 0$ implies $\ell_0(\lambda, \mu, \nu) \geq 0$, and if equality holds, then we have $K_{\lambda,\mu,\nu} = c^{\nu'}_{\lambda',\mu'}$ for certain partitions λ', μ', ν'. And again there are ℓ_i such that $\ell_i(\lambda, \mu, \nu) \geq 0$ for all i if and only if there exists $k > 0$ such that $K_{k\lambda,k\mu,k\nu} \neq 0$ (from S_{nk}). But this time this is *not* equivalent to having $K_{\lambda,\mu,\nu} \neq 0$.